New Solution of Diffusion-Advection Equation for Cosmic-Ray Transport Using Ultradistributions

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Abstract

In this paper we exactly solve the diffusion-advection equation for cosmic-ray transport. With this purpose we use the Theory of Ultradistributions of J. Sebastiao e Silva, to give a general solution for this equation. From this solution, we obtain several approximations as limiting cases of various situations of physical and astrophysical interest. One of them involves Solar cosmic-rays' diffusion.

Keywords: cosmic rays; diffusion; ultradistributions; fractional derivatives

1 Introduction

1.1 The problem at hand

Fractional derivatives constitute a rather old subject, although not as familiar as the integer-order counterparts [1, 2]. Fractional derivatives have recently been used in regards to many physical problems [for a small sample, see for instance [3, 4, 5, 6]] and to hydrology [7]. Fractional derivatives have been recently applied to model super-diffusion of particles in astrophysical scenarios [8, 9]. There is a considerable evidence emerging from data gathered by spacecrafts showing that the transport of energetic particles in the turbulent heliospheric medium is super-diffusive [11, 12]. An interesting work to be mentioned, in a different vein, is that of [10]

People employ fractional space derivatives so as to model anomalous diffusion or dispersion. Here, a particle plume spreads at a rate that is not the one of a classical Brownian model. If a fractional derivative takes the place of the second derivative in a diffusion or dispersion equation, this results in enhanced diffusion, or super-diffusion. In the case of a constant coefficients, one dimensional advection-dispersion equation, analytical solutions can be found by recourse to Fourier transform methods [7]. Many other problems require, instead, a treatment with variable coefficients [16].

In astrophysics, great activity revolves around the development of super diffusive models for the transport of electrons and protons in the heliosphere [13, 14, 15]. This sort of transport displays a power-law growth of the mean square displacement of the diffusing particles, $\langle \Delta x^2 \rangle \propto t^{\alpha}$, with $\alpha > 1$ (see, for instance, [17]). The special case $\alpha = 2$ is called ballistic. The limit case $\alpha \to 1$ is that of normal diffusion, described by a Gaussian propagator. Particles associated with violent solar events diffuse in the solar wind, a turbulent scenario that can be taken as statistically homogeneous at large distances from the sun [11]. This entails that the propagator $P(x, x', t, t')$, describing the probability of finding a particle that has been injected at (x', t') at the space time location (x, t), depends solely on the differences $x - x'$ and $t - t'$. In the super diffusive regime the propagator $P(x, x', t, t')$ is not Gaussian, and is characterized by power-law tails, emerging as the solution a non local diffusive process, governed by an integral equation. This equation can be cast as a diffusion one, in which the Laplacian is replaced by a term involving fractional derivatives [18]. See also [19, 20, 21, 22, 23, 24, 25], and references therein. An interesting step towards a more accurate analytical treatment of this problem was recently provided by Litvinenko and Effenberger (LE) in [8].

1.2 Ultradistributions

A series of papers [26, 27, 28, 29, 30] show that the Ultradistribution theory of Sebastiao e Silva [31, 32, 33] permits a significant advance in the treatment of quantum field theory. In particular, with the use of the convolution of Ultradistributions, one can show that it is possible to define a general product of distributions (a product in a ring with divisors of zero) that sheds new light on the question of the divergences in Quantum Field Theory. Furthermore, Ultradistributions of Exponential Type (UET) are adequate to describe Gamow States and exponentially increasing fields in Quantum Field Theory [35, 36, 37].

Other papers ([38, 39, 40]) demonstrated that Ultradistributions of Exponential type provide an adequate framework for a consistent treatment of string and string field theories. In particular, a general state of the closed string is represented by UET of compact support, and as a consequence the string field is a linear combination of UET of compact support. Moreover, five recent papers ([41, 42, 43, 44, 45]) show that Ultradistributions can be used to develop in a consistent way the so-called Non-Extensive Statistical Mechanics, allowing for an adequate definition of q-Fourier and q-Laplace transforms, and for the removal of divergences of this theory.

Ultradistributions also have the advantage of being representable by means of analytic functions. In general, they are easy to work with and, as we shall see, have interesting properties. One of those properties is that Schwartz's tempered distributions are canonical and continuously injected into Ultradistributions Another interesting property is that the space of UET is reflexive under the operation of Fourier transform (in a similar way of tempered distributions of Schwartz)

1.3 Our goal

In this paper we wish to show that Ultradistributions provide an adequate tool for a consistent treatment of a fractional differential diffusion-advection equation.

A more conventional treatment of this equation is given in [9]. The present treatment is of a much more general character.

This paper is organized as follows: In section 2, we summarize a set of mathematical concepts, while, in section 3 we formulate the problem to be addressed herein. We obtain in Section 4 a general solution of the fractional diffusion-advection equation, our main result . In section 5, we discuss the so-called weak diffusion approximation and in section 6 we analyze an important change of variables. Some conclusions are drawn in Section 7. The appendices A, B, C and D give the fundamentals of the mathematical theory used in this work.

2 Some basic ideas related to Hilbert spaces

The main task in distribution theory is to reinterpret functions as linear functionals acting on a space of test functions. While standard functions act by integration against a test function, many other linear functionals do not arise in this way. Precisely, these are the *generalized functions*. One has a panoply of possible choices for the space of test functions, which leads to distinct spaces of distributions. The basic space of test functions is that of smooth functions with compact support, which yields standard distributions. Employing the space of smooth, rapidly decreasing test functions gives instead the so-called tempered distributions. They are very important because they have a well-defined Fourier transform. While every tempered distribution is a distribution in the normal sense, the converse is not true. Generally, the larger the space of test functions, the more restrictive the notion of distribution.

A nuclear space is a topological vector space with many of the nice properties of finite-dimensional vector spaces. The topology on them can be construed by recourse to a family of semi-norms whose unit balls-radii decrease rapidly. Vector spaces whose elements are *smooth* tend to be nuclear spaces. An example of nuclear space is provided by the set of smooth functions on a compact manifold. All finite-dimensional vector spaces are nuclear. There are no Banach spaces that are nuclear, except for the finite-dimensional ones. A large part of the theory of nuclear spaces was developed by Alexander Grothendieck (see [46]).

A rigged Hilbert space, or Guelfand's triplet, is a construction designed to connect the distribution and square-integrable aspects of functional analysis. Rigged spaces were introduced to study spectral theory and bring together bound states and continuous spectra. The concept of rigged Hilbert space provides for such a desideratum an abstract functional-analytic framework. More precisely, an equipped Hilbert space consists of a Hilbert space $\mathbf{H}$ plus a subspace $\mathbf{\Phi}$ which is endowed with a finer topology. This is one for which the natural inclusion

$$\mathbf{\Phi} \subseteq \mathbf{H}, \tag{2.1}$$

is continuous. One can assume that $\mathbf{\Phi}$ is dense in $\mathbf{H}$ for the Hilbert norm. We need also the inclusion of dual spaces $\mathbf{H}^*$ in $\mathbf{\Phi}^*$. The latter, dual to $\mathbf{\Phi}$ in its test function topology, is realized as a space of distributions or generalized functions, and the linear functionals on the subspace $\mathbf{\Phi}$ of type $\phi \rightarrow < \nu \rightarrow, \phi >$ for ν in $\mathbf{H}$, are faithfully represented as distributions. Now, via application of the Riesz representation theorem one identifies $\mathbf{H}^*$ with $\mathbf{H}$. Therefore, the definition of rigged Hilbert space can be given in terms of a double inclusion

$$\mathbf{\Phi} \subseteq \mathbf{H} \subseteq \mathbf{\Phi}^*. \tag{2.2}$$

The most important illustrations are those for which $\mathbf{\Phi}$ is a nuclear space. This captures an abstract expression of the idea that $\mathbf{\Phi}$ consists of test functions and $\mathbf{\Phi}^*$ of the corresponding distributions. For more details, see Appendices A and B.

3 Formulation of the Problem

The authors of [8] have proposed the equation:

$$\frac{\partial f(x,t)}{\partial t} = \kappa \frac{\partial^\lambda f(x,t)}{\partial |x|^\lambda} + a \frac{\partial f(x,t)}{\partial x} + \delta(x) \quad t > 0 \tag{3.1}$$

where $t > 0$, for the distribution function $f(x,t)$. In their specific case, f refers to solar cosmic-ray transport. They used the following definition of fractional derivative (see [17]):

$$\frac{\partial^\lambda f}{\partial |x|^\lambda} = \frac{1}{\pi} \sin\left(\frac{\pi\lambda}{2}\right) \Gamma(\lambda+1) \int\limits_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{\lambda+1}} \, d\xi. \tag{3.2}$$

To solve (3.1) the authors use the Green function given by

$$\frac{\partial \mathcal{G}(x,t)}{\partial t} = \kappa \frac{\partial^\lambda \mathcal{G}(x,t)}{\partial |x|^\lambda} + \delta(x)\delta(t). \tag{3.3}$$

Using this Green function, the solution of (3.1), with the initial condition $f(x,0) = 0$, can be written as

$$f(x,t) = \int_0^t \mathcal{G}(x + at', t')\, dt'. \tag{3.4}$$

The solution to the above problem is well posed, except for one major problem: the fractional derivative used is not defined for $\lambda = 1$ and *does not coincide for this value of* λ *with the usual derivative defined by Newton and Leibniz.*

We will solve in this paper this serious problem by recourse to a definition of fractional derivative *valid for all values of* λ*, both real or complex*, and matching things for $\lambda \in \mathcal{N}$ ($\mathcal{N}$= the set of natural numbers), with the usual derivative defined by Newton and Leibniz. To achieve this goal we use the definition given in [26] for distributions of exponential type and extended in our Appendix C to ultradistributions of exponential type.

An interesting property of this fractional derivative is that it unifies in a single operation the operations of derivation and indefinite integration, for any real or complex value of λ.

4 General Solutions

To solve (3.1) we divide the problem into two parts:
$1)x \geq 0$

$$\frac{\partial f(x,t)}{\partial t} = \kappa \frac{\partial^\lambda f(x,t)}{\partial x^\lambda} + a \frac{\partial f(x,t)}{\partial x} + \delta(x), \tag{4.1}$$

$$\frac{\partial \mathcal{G}(x,t)}{\partial t} = \kappa \frac{\partial^\lambda \mathcal{G}(x,t)}{\partial x^\lambda} + \delta(x)\delta(t). \tag{4.2}$$

and
$2)x < 0$

$$\frac{\partial f(x,t)}{\partial t} = \kappa \frac{\partial^\lambda f(x,t)}{\partial (-x)^\lambda} + a \frac{\partial f(x,t)}{\partial x} + \delta(x), \tag{4.3}$$

$$\frac{\partial \mathcal{G}(x,t)}{\partial t} = \kappa \frac{\partial^\lambda \mathcal{G}(x,t)}{\partial (-x)^\lambda} + \delta(x)\delta(t). \tag{4.4}$$

Our solution will be valid for all values of λ such that $|e^{\kappa(-ik)^\lambda}| \leq |e^{\kappa k}|$. For the remaining possible values of λ, the solution is obtained via analytic prolongation. In fact, for these last values, the solutions of the equations above become exponentially growing ones, which forces one to i) appeal to ultradistributions of exponential type and ii) extend these equations to the complex plane. Thus, we have for 1)

$$\frac{\partial f(z,t)}{\partial t} = \kappa \frac{\partial^\lambda f(z,t)}{\partial z^\lambda} + a \frac{\partial f(z,t)}{\partial z} + \delta(z), \tag{4.5}$$

$$\frac{\partial \mathcal{G}(z,t)}{\partial t} = \kappa \frac{\partial^\lambda \mathcal{G}(z,t)}{\partial z^\lambda} + \delta(z)\delta(t), \tag{4.6}$$

and for 2)

$$\frac{\partial f(z,t)}{\partial t} = \kappa \frac{\partial^\lambda f(z,t)}{\partial (-z)^\lambda} + a \frac{\partial f(z,t)}{\partial z} + \delta(z), \tag{4.7}$$

$$\frac{\partial \mathcal{G}(z,t)}{\partial t} = \kappa \frac{\partial^\lambda \mathcal{G}(z,t)}{\partial (-z)^\lambda} + \delta(z)\delta(t). \tag{4.8}$$

Using now the complex Fourier transformation we can obtain the solution to our four equations. For 1) one has

$$f(z,t) = \frac{1}{2\pi} \oint\limits_{\Gamma} \{H[\Im(z)]H[-\Re(k)] - H[-\Im(z)]H[\Re(k)]\}H[\Im(k)] \times$$
$$\frac{e^{[\kappa(-ik)^\lambda - iak]t} - 1}{\kappa(-ik)^\lambda - iak} e^{-ikz}\, dk \tag{4.9}$$

$$\mathcal{G}(z,t) = \frac{H(t)}{2\pi} \oint\limits_{\Gamma} \{H[\Im(z)]H[-\Re(k)] - H[-\Im(z)]H[\Re(k)]\}H[\Im(k)] \times$$
$$e^{\kappa(-ik)^\lambda t} e^{-ikz}\, dk, \tag{4.10}$$

and for 2)

$$f(z,t) = -\frac{1}{2\pi} \oint\limits_{\Gamma} \{H[\Im(z)]H[-\Re(k)] - H[-\Im(z)]H[\Re(k)]\}H[-\Im(k)] \times$$
$$\frac{e^{[\kappa(ik)^\lambda - iak]t} - 1}{\kappa(-ik)^\lambda - iak} e^{-ikz}\, dk \tag{4.11}$$

$$\mathcal{G}(z,t) = -\frac{H(t)}{2\pi} \oint\limits_{\Gamma} \{H[\Im(z)]H[-\Re(k)] - H[-\Im(z)]H[\Re(k)]\}H[-\Im(k)] \times$$
$$e^{\kappa(ik)^\lambda t} e^{-ikz}\, dk. \tag{4.12}$$

We pass now to find explicit expressions for equations (4.9)-(4.12).

Case 1) $x \geq 0$

Expanding $e^{\kappa(-ik)^{\lambda}t}$ in power series we obtain, for (4.10),

$$\mathcal{G}(z,t) = \frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\oint\limits_{\Gamma}\{H[\Im(z)]H[-\Re(k)] - H[-\Im(z)]H[\Re(k)]\}\times$$

$$H[\Im(k)](-ik)^{\lambda n}e^{-ikz}\,dk. \tag{4.13}$$

Each term of the sum in (4.13) is a tempered ultradistribution (see Appendix A). We go then to the real axis and evaluate the cut along it. Thus,

$$\mathcal{G}(x,t) = \frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\oint\limits_{\Gamma}H[\Im(k)](-ik)^{\lambda n}e^{-ikx}\,dk. \tag{4.14}$$

Eq. (4.14) can be cast in the following equivalent form

$$\mathcal{G}(x,t) = \frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}e^{-i\frac{\pi}{2}\lambda n}\int\limits_{-\infty}^{\infty}(k+i0)^{\lambda n}e^{-ikx}\,dk, \tag{4.15}$$

or

$$\mathcal{G}(x,t) = \frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}e^{-i\frac{\pi}{2}\lambda n}\int\limits_{0}^{\infty}\left(k^{\lambda n}e^{-ikx} + e^{i\pi\lambda n}k^{\lambda n}e^{ikx}\right)\,dk. \tag{4.16}$$

The integrals given in (4.16) have been calculated in [47]. One has

$$\mathcal{G}(x,t) = \frac{iH(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\Gamma(\lambda n+1)\left[\frac{e^{i\pi\lambda n}}{(x+i0)^{\lambda n+1}} - \frac{e^{-i\pi\lambda n}}{(x-i0)^{\lambda n+1}}\right]. \tag{4.17}$$

By recourse to (3.4) we have for f

$$f(x,t) = \frac{i}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n}{n!}\Gamma(\lambda n+1)\int\limits_{0}^{t}\left[\frac{e^{i\pi\lambda n}t'^n}{(x+at'+i0)^{\lambda n+1}} - \right.$$

$$\left.\frac{e^{-i\pi\lambda n}t'^n}{(x+at'-i0)^{\lambda n+1}}\right]\,dt'. \tag{4.18}$$

Using (D.1,) we obtain the general solution for $x \geq 0$

$$f(x,t) = \frac{i}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\Gamma(\lambda n+1)B(1,n+1)\times$$

$$\left[\frac{e^{i\pi\lambda n}}{(x+i0)^{\lambda n+1}}F\left(\lambda n+1,n+1;n+2;-\frac{at}{x+i0}\right)-\right.$$
$$\left.\frac{e^{-i\pi\lambda n}}{(x-i0)^{\lambda n+1}}F\left(\lambda n+1,n+1;n+2;-\frac{at}{x-i0}\right)\right]. \tag{4.19}$$

Case 2) $x < 0$

From (4.12), and expanding $e^{(ik)^\lambda}$ in power series, $\mathcal{G}$ adopts the form

$$\mathcal{G}(z,t) = -\frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\oint\limits_{\Gamma}\{H[\Im(z)]H[-\Re(k)]-H[-\Im(z)]H[\Re(k)]\}\times$$
$$H[-\Im(k)](ik)^{\lambda n}e^{-ikz}\,dk. \tag{4.20}$$

Each term of the sum in (4.20) is, again, a tempered ultradistribution. Thus, proceeding as in the case $x \geq 0$, we obtain

$$\mathcal{G}(x,t) = -\frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\oint\limits_{\Gamma}H[-\Im(k)](ik)^{\lambda n}e^{-ikx}\,dk, \tag{4.21}$$

or

$$\mathcal{G}(x,t) = \frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}e^{i\frac{\pi}{2}\lambda n}\int\limits_{-\infty}^{\infty}(k-i0)^{\lambda n}e^{-ikx}\,dk. \tag{4.22}$$

Eq. (4.22) can be rewritten as

$$\mathcal{G}(x,t) = \frac{H(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}e^{i\frac{\pi}{2}\lambda n}\int\limits_{0}^{\infty}\left(k^{\lambda n}e^{-ikx}+e^{-i\pi\lambda n}k^{\lambda n}e^{ikx}\right)\,dk. \tag{4.23}$$

Using [47] we then have

$$\mathcal{G}(x,t) = \frac{iH(t)}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\Gamma(\lambda n+1)\left[\frac{1}{(x+i0)^{\lambda n+1}}-\frac{1}{(x-i0)^{\lambda n+1}}\right]. \tag{4.24}$$

As it should, $\mathcal{G}$ vanishes for $x \geq 0$. From (3.4), we can write f as

$$f(x,t) = \frac{i}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n}{n!}\Gamma(\lambda n+1)\int_0^t\left[\frac{t'^n}{(x+at'+i0)^{\lambda n+1}} - \frac{t'^n}{(x+at'-i0)^{\lambda n+1}}\right]\,dt', \tag{4.25}$$

and, according to (D.1), f is finally given by

$$f(x,t) = \frac{i}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n t^n}{n!}\Gamma(\lambda n+1)B(1,n+1)\times \left[\frac{1}{(x+i0)^{\lambda n+1}}F\left(\lambda n+1, n+1; n+2; -\frac{at}{x+i0}\right) - \frac{1}{(x-i0)^{\lambda n+1}}F\left(\lambda n+1, n+1; n+2; -\frac{at}{x-i0}\right)\right]. \tag{4.26}$$

5 A useful approximation

Following [8], we shall now consider a weak diffusion approximation. Within this approximation, we can treat κ as a small parameter and expand f up to order one [8]. Thus, we can write for $x \geq 0$

$$f(x,t) = f_0(x,t) + f_1(x,t), \tag{5.1}$$

where f_0 is given by

$$f_0(x,t) = \frac{it}{2\pi}\left[(x+i0)^{-1}F\left(1,1;2;-\frac{at}{x+i0}\right) - (x-i0)^{-1}F\left(1,1;2;-\frac{at}{x-i0}\right)\right]. \tag{5.2}$$

By recourse to [50], we can express $F(1,1;2;z)$ in terms of elementary functions, i.e.,

$$F(1,1;2;-z) = \frac{1}{z}\ln(1+z), \tag{5.3}$$

and obtain for f_0 the expression

$$f_0(x,t) = \frac{1}{a}[H(-x) - H(-x-at) = \frac{1}{2a}[Sgn(x+at) - Sgn(x)]. \tag{5.4}$$

For f_1 we have

$$f_1(x,t) = \frac{i\kappa t^2}{4\pi}\Gamma(\lambda+1)\left[\frac{e^{i\pi\lambda}}{(x+i0)^{\lambda+1}}F\left(\lambda+1,2;3;-\frac{at}{x+i0}\right) - \right.$$
$$\left.\frac{e^{-i\pi\lambda}}{(x-i0)^{\lambda+1}}F\left(\lambda+1,2;3;-\frac{at}{x-i0}\right)\right]. \tag{5.5}$$

Using again the result (D.7) one has

$$F(\lambda+1,2;3;z) = \frac{2}{\lambda(\lambda-1)z^2}\left[1+\frac{\lambda z-1}{(1-z)^\lambda}\right],$$

f_1 and then

$$f_1(x,t) = \frac{i\kappa\Gamma(\lambda-1)}{2\pi a^2}\left\{e^{i\pi\lambda}\left[\frac{1}{(x+i0)^{\lambda-1}} - \frac{x+\lambda at}{(x+at+i0)^\lambda}\right] - \right.$$
$$\left. e^{-i\pi\lambda}\left[\frac{1}{(x-i0)^{\lambda-1}} - \frac{x+\lambda at}{(x+at-i0)^{\lambda-1}}\right]\right\}. \tag{5.6}$$

Thus, we have for f in the weak diffusion approximation

$$f(x,t) = \frac{1}{2a}[Sgn(x+at) - Sgn(x)] +$$
$$\frac{i\kappa\Gamma(\lambda-1)}{2\pi a^2}\left\{e^{i\pi\lambda}\left[\frac{1}{(x+i0)^{\lambda-1}} - \frac{x+\lambda at}{(x+at+i0)^\lambda}\right] - \right.$$
$$\left. e^{-i\pi\lambda}\left[\frac{1}{(x-i0)^{\lambda-1}} - \frac{x+\lambda at}{(x+at-i0)^{\lambda}}\right]\right\}. \tag{5.7}$$

For $x > 0$ f this becomes simplified and one has

$$f(x,t) = \frac{\kappa}{a^2\Gamma(2-\lambda)}\left[\frac{1}{x^{\lambda-1}} - \frac{x+\lambda at}{(x+at)^\lambda}\right]. \tag{5.8}$$

We can now distinguish two limiting cases. The first one is the asymptotic situation $x >> at$. In this instance

$$f(x,t) = \frac{1}{2\Gamma(-\lambda)}\frac{\kappa t^2}{x^{\lambda+1}}. \tag{5.9}$$

The second case is $0 < x << at$. For it we have

$$f(x,t) = \frac{1}{\Gamma(2-\lambda)} \frac{\kappa}{a^2} x^{1-\lambda}. \tag{5.10}$$

For $x < 0$ we have for f_0 the same expression obtained for the case $x \geq 0$, and thus, for f_1,

$$f_1(x,t) = \frac{i\kappa t^2}{4\pi} \Gamma(\lambda+1) \left[\frac{1}{(x+i0)^{\lambda+1}} F\left(\lambda+1, 2; 3; -\frac{at}{x+i0}\right) - \right.$$
$$\left. \frac{1}{(x-i0)^{\lambda+1}} F\left(\lambda+1, 2; 3; -\frac{at}{x-i0}\right) \right]. \tag{5.11}$$

As a consequence, we have for f

$$f(x,t) = \frac{1}{2a}[Sgn(x+at) - Sgn(x)] +$$
$$\frac{i\kappa\Gamma(\lambda-1)}{2\pi a^2} \left[\frac{1}{(x+i0)^{\lambda-1}} - \frac{1}{(x-i0)^{\lambda-1}} \right.$$
$$\left. - \frac{x+\lambda at}{(x+at+i0)^{\lambda}} + \frac{x+\lambda at}{(x+at-i0)^{\lambda}} \right]. \tag{5.12}$$

For $x + at < 0$, (5.2) adopts the form

$$f(x,t) = \frac{\kappa}{a^2\Gamma(2-\lambda)} \left[\frac{1}{|x|^{\lambda-1}} + \frac{x+\lambda at}{|x+at|^{\lambda}} \right]. \tag{5.13}$$

When $x << -at$, (5.13) transforms into

$$f(x,t) = \frac{1}{2\Gamma(-\lambda)} \frac{\kappa t^2}{|x|^{\lambda+1}}. \tag{5.14}$$

Another special situation arises when $x < 0$, $x + at > 0$, and $x << -at$. In this case, from (5.12) we deduce the following expression for f:

$$f(x,t) = \frac{1}{a} + \frac{1}{\Gamma(2-\lambda)} \frac{\kappa}{a^2} |x|^{1-\lambda}. \tag{5.15}$$

6 Change of frame

Assume that, in the solar wind rest frame, the particles' transport is represented by the fractional-diffusion equation (FDE) without advection term ($a = 0$ in (3.1)). The shock front, started at $x_0 = -V_{sh}t_0$, moves with constant speed V_{sh}. It is considered as highly localized in the x-coordinate) and constitutes the source of the particles. Then we face an FDE with a uniformly moving Dirac's delta source of the form $\delta(x - V_{sh}t)$. So as to have a stationary delta source, we require performing a suitable coordinates-change, reformulating our task in a reference frame where the shock front is stationary. We also modify the time-origin so that the source begins being active at $t = 0$. In such a modified reference frame, the transport equation acquires an advection term with velocity $a = V_{sh}$, and a stationary source $\delta(0)$, that begins at $t = 0$. After solving the diffusion-advection equation in this frame, one expresses the solution in terms of the original coordinates associated with the solar wind rest frame. Such step is briefly described by the 3 correspondences $a \to v_{sh}$, $t \to t + t_0$, and $x \to x - v_{sh}t$. Consequently, Eqs. (4.19) and (4.26) acquire the form, for $x \geq 0$,

$$f(x,t) = \frac{i}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n (t+t_0)^{n+1}}{n!}\Gamma(\lambda n+1)\mathcal{B}(1,n+1)\times$$
$$\left[\frac{e^{i\pi\lambda n}}{(x-v_{sh}t+i0)^{\lambda n+1}}F\left(\lambda n+1,n+1;n+2;-\frac{v_{sh}(t+t_0)}{x-v_{sh}t+i0}\right)-\right.$$
$$\left.\frac{e^{-i\pi\lambda n}}{(x-v_{sh}t-i0)^{\lambda n+1}}F\left(\alpha n+1,n+1;n+2;-\frac{v_{sh}(t+t_0)}{x-v_{sh}t-i0}\right)\right]. \quad (6.1)$$

And for $x < 0$:

$$f(x,t) = \frac{i}{2\pi}\sum_{n=0}^{\infty}\frac{\kappa^n (t+t_0)^{n+1}}{n!}\Gamma(\lambda n+1)\mathcal{B}(1,n+1)\times$$
$$\left[\frac{1}{(x-v_{sh}t+i0)^{\lambda n+1}}F\left(\lambda n+1,n+1;n+2;-\frac{v_{sh}(t+t_0)}{x-v_{sh}t+i0}\right)-\right.$$
$$\left.\frac{1}{(x-v_{sh}t-i0)^{\lambda n+1}}F\left(\alpha n+1,n+1;n+2;-\frac{v_{sh}(t+t_0)}{x-v_{sh}t-i0}\right)\right]. \quad (6.2)$$

Thus, in the weak diffusion approach of (5.7) and (5.12), we have for $x \geq 0$

$$f(x,t) = \frac{1}{2v_{sh}}[Sgn(x+v_{sh}t_0) - Sgn(x-v_{sh}t)]-$$

$$\frac{i\kappa\Gamma(\lambda-1)}{2\pi V_{sh}^2}\left\{(x+(\lambda-1)v_{sh}t+v_{sh}t_0)\left[\frac{e^{i\pi\lambda}}{(x+v_{sh}t_0+i0)^\lambda}--\right.\right.$$
$$\left.\left.\frac{e^{-i\pi\lambda}}{(x+v_{sh}t_0-i0)^\lambda}\right]+\frac{e^{-i\pi\lambda}}{(x-v_{sh}t-i0)^{\lambda-1}}-\frac{e^{i\pi\lambda}}{(x-v_{sh}t+i0)^{\lambda-1}}\right\}. \tag{6.3}$$

For $x < 0$ we have

$$f(x,t)=\frac{1}{2v_{sh}}[Sgn(x+v_{sh}t_0)-Sgn(x-v_{sh}t)]-$$
$$\frac{i\kappa\Gamma(\lambda-1)}{2\pi V_{sh}^2}\left\{(x+(\lambda-1)v_{sh}t+v_{sh}t_0)\left[\frac{1}{(x+v_{sh}t_0+i0)^\lambda}--\right.\right.$$
$$\left.\left.\frac{1}{(x+v_{sh}t_0-i0)^\lambda}\right]+\frac{1}{(x-v_{sh}t-i0)^{\lambda-1}}-\frac{1}{(x-v_{sh}t+i0)^{\lambda-1}}\right\}. \tag{6.4}$$

7 Conclusions

By recourse to ultradistributions, we have provided here an explicit analytical solution for an advection-diffusion equation (ADE) involving fractional derivatives. First, we devised a generalized treatment for these derivatives that includes the normal case.

We also found the exact solution for the ADE both in the x-configuration space and in the associated k-space, that are related via a Fourier transform. Our solution allows us to obtain in a unified and systematic fashion all the different approximations that were introduced in [8], each one in a distinct manner. We achieve in this way a great degree of generality.

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Appendix A

Distributions of Exponential Type

For the benefit of the reader, we present here a brief description of the main properties of Tempered Ultradistributions and of Ultradistributions of Exponential Type.

Notations. The notations are almost textually taken from ref[32]. Let $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) be the real (respectively complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, ..., x_n)$ (resp $z = (z_1, z_2, ..., z_n)$). We shall use the following notations:

(i) $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$; $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$

(ii)$x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, ..., x_n \geqq 0$

(iii)$x \cdot y = \sum\limits_{j=1}^{n} x_j y_j$

(iV)$| x |= \sum\limits_{j=1}^{n} | x_j |$

Consider the set of n-tuples of natural numbers $\mathbb{N}^n$. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, ..., p_n)$, where p_j is a natural number, $1 \leqq j \leqq n$. $p + q$ denote $(p_1 + q_1, p_2 + q_2, ..., p_n + q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, ..., p_n \geqq q_n$. x^p means $x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$. We denote by $| p |= \sum\limits_{j=1}^{n} p_j$ and by D^p we understand the differential operator $\partial^{p_1+p_2+...+p_n}/\partial {x_1}^{p_1} \partial {x_2}^{p_2} ... \partial {x_n}^{p_n}$

For any natural number k we define $x^k = x_1^k x_2^k ... x_n^k$ and $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \partial x_2^k ... \partial x_n^k$

The space $\boldsymbol{\mathcal{H}}$ of test functions such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any natural numbers p and q is defined (ref.[32]) by means of the countably set of norms:

$$\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p,\, x} e^{p|x|} \left|D^q \hat{\phi}(x)\right| \quad , \quad p = 0, 1, 2, ... \tag{A.1}$$

According to reference[48] $\boldsymbol{\mathcal{H}}$ is a $\boldsymbol{\mathcal{K}}\{\mathbf{M_p}\}$ space with:

$$M_p(x) = e^{(p-1)|x|} \quad , \quad p = 1, 2, ... \tag{A.2}$$

$\boldsymbol{\mathcal{K}}\{e^{(p-1)|x|}\}$ complies condition $(\mathcal{N})$ of Guelfand (ref.[49]). It is a countable Hilbert and nuclear space:

$$\boldsymbol{\mathcal{K}}\{e^{(p-1)|x|}\} = \boldsymbol{\mathcal{H}} = \bigcap_{p=1}^{\infty} \boldsymbol{\mathcal{H}}_p \tag{A.3}$$

where $\mathcal{H}_p$ is obtained by completing $\mathcal{H}$ with the norm induced by the scalar product:

$$< \hat{\phi}, \hat{\psi} >_p = \int\limits_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q \overline{\hat{\phi}}(x) D^q \hat{\psi}(x) \; dx \quad ; \quad p = 1, 2, ... \tag{A.4}$$

where $dx = dx_1 \; dx_2 ... dx_n$

If we take the conventional scalar product:

$$< \hat{\phi}, \hat{\psi} >= \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x) \hat{\psi}(x) \; dx \tag{A.5}$$

then $\mathcal{H}$, completed with (A.5), is the Hilbert space $\mathbf{H}$ of square integrable functions.

By definition, the space of continuous linear functionals defined on $\mathcal{H}$ is the space $\mathbf{\Lambda}_\infty$ of the distributions of the exponential type (ref.[32]).

The Fourier transform of a distribution of exponential type $\hat{F}$ is given by (see [31, 32]):

$$F(k) = \int\limits_{-\infty}^{\infty} H[\Im(k)] H[\Re(x) - H[-\Im(k)] H[-\Re(x)] \hat{F}(x) e^{ikx} \; dx =$$

$$H[\Im(k)] \int\limits_{0}^{\infty} \hat{F}(x) e^{ikx} - H[-\Im(k)] \int\limits_{-\infty}^{0} \hat{F}(x) e^{ikx} \tag{A.6}$$

where F is the corresponding tempered ultradistribution (see the next subsection).

The triplet

$$\mathfrak{H} = (\mathcal{H}, \mathbf{H}, \mathbf{\Lambda}_\infty) \tag{A.7}$$

is a Rigged Hilbert Space (or a Guelfand's triplet [49]).

Moreover, we have: $\mathcal{H} \subset \mathcal{S} \subset \mathbf{H} \subset \mathcal{S}' \subset \mathbf{\Lambda}_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (ref[34]).

Any Rigged Hilbert Space $\mathfrak{G} = (\mathbf{\Phi}, \mathbf{H}, \mathbf{\Phi}')$ has the fundamental property that a linear and symmetric operator on $\mathbf{\Phi}$, which admits an extension to a self-adjoint operator in $\mathbf{H}$, has a complete set of generalized eigenfunctions in $\mathbf{\Phi}'$ with real eigenvalues.

Tempered Ultradistributions

The Fourier transform of a function $\hat{\phi} \in \boldsymbol{\mathcal{H}}$ is

$$\phi(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x) \; e^{iz\cdot x} \; dx \tag{A.8}$$

Here $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We call $\mathfrak{H}$ the set of all such functions.

$$\mathfrak{H} = \mathcal{F}\{\boldsymbol{\mathcal{H}}\} \tag{A.9}$$

It is a $\boldsymbol{\mathcal{Z}}\{\mathbf{M_p}\}$ countably normed and complete space (ref.[48]), with:

$$M_p(z) = (1 + |z|)^p \tag{A.10}$$

$\mathfrak{H}$ is a nuclear space defined with the norms:

$$\|\phi\|_{pn} = \sup_{z \in V_n} (1 + |z|)^p |\phi(z)| \tag{A.11}$$

where $V_k = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n :| \; Imz_j \; |\leqq k, 1 \leqq j \leqq n\}$

We can define the habitual scalar product:

$$< \phi(z), \psi(z) >= \int\limits_{-\infty}^{\infty} \phi(z)\psi_1(z) \; dz = \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x) \; dx \tag{A.12}$$

where:

$$\psi_1(z) = \int\limits_{-\infty}^{\infty} \hat{\psi}(x) \; e^{-iz\cdot x} \; dx$$

and $dz = dz_1 \; dz_2 ... dz_n$

By completing $\mathfrak{H}$ with the norm induced by (A.12) we obtain the Hilbert space of square integrable functions.

The dual of $\mathfrak{H}$ is the space $\boldsymbol{\mathcal{U}}$ of tempered ultradistributions (ref.[31, 32]). Namely, a tempered ultradistribution is a continuous linear functional defined on the space $\mathfrak{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis.

The set $\mathfrak{U} = (\mathfrak{H}, H, \boldsymbol{\mathcal{U}})$ is also a Rigged Hilbert Space.

Moreover, we have: $\mathfrak{H} \subset \boldsymbol{\mathcal{S}} \subset \mathsf{H} \subset \boldsymbol{\mathcal{S}}' \subset \boldsymbol{\mathcal{U}}$.

$\boldsymbol{\mathcal{U}}$ can also be characterized in the following way (ref.[32]): let $\boldsymbol{\mathcal{A}}_\omega$ be the space of all functions $F(z)$ such that:

A) $F(z)$ is analytic on the set $\{z \in \mathbb{C}^n : |Im(z_1)| > p, |Im(z_2)| > p, ..., |Im(z_n)| > p\}$.

B) $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, ..., |Im(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $F(z)$.

Let $\boldsymbol{\Pi}$ be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then $\boldsymbol{\mathcal{U}}$ is the quotient space:

C) $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{A}}_\omega / \boldsymbol{\Pi}$

By a pseudo-polynomial we denote a function of z of the form
$\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$ with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \boldsymbol{\mathcal{A}}_\omega$

Due to these properties it is possible to represent any ultradistribution as (ref.[32]):

$$F(\phi) = < F(z), \phi(z) > = \oint_{\Gamma} F(z)\phi(z) \; dz \tag{A.13}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup ...\Gamma_n$ and where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$).

Formula (A.13) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "Dirac Formula" for ultradistributions (ref.[31]):

$$F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)} \; dt \tag{A.14}$$

where the "density" $f(t)$ is the cut of $F(z)$ along the real axis and satisfy:

$$\oint_{\Gamma} F(z)\phi(z) \; dz = \int_{-\infty}^{\infty} f(t)\phi(t) \; dt \tag{A.15}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (A.15) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $F(z)$ is bounded by a power of z (ref.[32]):

$$|F(z)| \leq C|z|^p \tag{A.16}$$

where C and p depend on F.

The representation (A.15) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ do not alter the ultradistribution:

$$\oint_{\Gamma}\{F(z)+P(z)\}\phi(z)\ dz = \oint_{\Gamma} F(z)\phi(z)\ dz + \oint_{\Gamma} P(z)\phi(z)\ dz$$

But:

$$\oint_{\Gamma} P(z)\phi(z)\ dz = 0$$

as $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing),

$$\therefore \quad \oint_{\Gamma}\{F(z)+P(z)\}\phi(z)\ dz = \oint_{\Gamma} F(z)\phi(z)\ dz \tag{A.17}$$

The inverse Fourier transform of (A.6) is given by:

$$\hat{F}(x) = \frac{1}{2\pi}\oint_{\Gamma} F(k)e^{-ikx}\ dk = \int_{-\infty}^{\infty} f(k)e^{-ikx}\ dx \tag{A.18}$$

Appendix B

Ultradistributions of Exponential Type

Consider the Schwartz space of rapidly decreasing test functions $\mathcal{S}$. Let Λ_j be the region of the complex plane defined as:

$$\Lambda_j = \{z \in \mathbb{C} : |\Im(z)| < j : j \in \mathbb{N}\} \tag{B.1}$$

According to ref.[31, 33] be the space of test functions $\hat{\phi} \in \mathfrak{V}_j$ is constituted by the ser of all entire analytic functions of $\mathcal{S}$ for which

$$||\hat{\phi}||_j = \max_{k\leq j}\left\{\sup_{z\in\Lambda_j}\left[e^{(j|\Re(z)|)}|\hat{\phi}^{(k)}(z)|\right]\right\} \tag{B.2}$$

is finite.
The space $\mathfrak{Z}$ is then defined as:

$$\mathfrak{Z} = \bigcap_{j=0}^{\infty}\mathfrak{V}_j \tag{B.3}$$

It is a complete countably normed space with the topology generated by the set of semi-norms $\{\|\cdot\|_j\}_{j\in\mathbb{N}}$. The topological dual of $\mathfrak{Z}$, denoted by $\mathfrak{B}$, is by definition the space of ultradistributions of exponential type (ref.[31, 33]). Let $\mathfrak{S}$ be the space of rapidly decreasing sequences. According to ref.[49] $\mathfrak{S}$ is a nuclear space. We consider now the space of sequences $\mathfrak{P}$ generated by the Taylor development of $\hat{\phi}\in\mathfrak{Z}$

$$\mathfrak{P}=\left\{\mathfrak{Q}:\mathfrak{Q}\left(\hat{\phi}(0),\hat{\phi}^{'}(0),\frac{\hat{\phi}^{''}(0)}{2},...,\frac{\hat{\phi}^{(n)}(0)}{n!},...\right):\hat{\phi}\in\mathfrak{Z}\right\} \tag{B.4}$$

The norms that define the topology of $\mathfrak{P}$ are given by:

$$\|\hat{\phi}\|_p^{'}=\sup_n\frac{n^p}{n}|\hat{\phi}^n(0)| \tag{B.5}$$

$\mathfrak{P}$ is a subspace of $\mathfrak{S}$ and as consequence is a nuclear space. The norms $\|\cdot\|_j$ and $\|\cdot\|_p^{'}$ are equivalent, the correspondence

$$\mathfrak{Z}\Longleftrightarrow\mathfrak{P} \tag{B.6}$$

is an isomorphism and therefore $\mathfrak{Z}$ is a countably normed nuclear space. We define now the set of scalar products

$$<\hat{\phi}(z),\hat{\psi}(z)>_n=\sum_{q=0}^{n}\int\limits_{-\infty}^{\infty}e^{2n|z|}\overline{\hat{\phi}^{(q)}}(z)\hat{\psi}^{(q)}(z)\;dz=$$

$$\sum_{q=0}^{n}\int\limits_{-\infty}^{\infty}e^{2n|x|}\overline{\hat{\phi}^{(q)}}(x)\hat{\psi}^{(q)}(x)\;dx \tag{B.7}$$

This scalar product induces the norm

$$\|\hat{\phi}\|_n^{''}=[<\hat{\phi}(x),\hat{\phi}(x)>_n]^{\frac{1}{2}} \tag{B.8}$$

The norms $\|\cdot\|_j$ and $\|\cdot\|_n^{''}$ are equivalent, and therefore $\mathfrak{Z}$ is a countably hilbertian nuclear space. Thus, if we call now $\mathfrak{Z}_p$ the completion of $\mathfrak{Z}$ by the norm p given in (B.8), we have:

$$\mathfrak{Z}=\bigcap_{p=0}^{\infty}\mathfrak{Z}_p \tag{B.9}$$

where

$$\mathfrak{Z}_0 = H \tag{B.10}$$

is the Hilbert space of square integrable functions.
As a consequence the triplet

$$\mathfrak{U} = (\mathfrak{Z}, H, \mathfrak{B}) \tag{B.11}$$

is also a Guelfand's triplet.

$\mathfrak{B}$ can also be characterized in the following way (refs.[31],[33]): let $\mathfrak{C}_\omega$ be the space of all functions $\hat{F}(z)$ such that:

A) $\hat{F}(z)$ is an analytic function for $\{z \in \mathbb{C} : |Im(z)| > p\}$.

B)- $\hat{F}(z)e^{-p|\Re(z)|}/z^p$ is a bounded continuous function in $\{z \in \mathbb{C} : |Im(z)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $\hat{F}(z)$.

Let $\mathfrak{N}$ be: $\mathfrak{N} = \{\hat{F}(z) \in \mathfrak{C}_\omega : \hat{F}(z) \text{ is entire analytic}\}$. Then $\mathfrak{B}$ is the quotient space:

C)- $\mathfrak{B} = \mathfrak{C}_\omega/\mathfrak{N}$

Due to these properties it is possible to represent any ultradistribution of exponential type as (ref.[31, 33]):

$$\hat{F}(\hat{\phi}) = < \hat{F}(z), \hat{\phi}(z) > = \oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz \tag{B.12}$$

where the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $Im(z) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $\hat{F}(z)$).

Formula (B.12) will be our fundamental representation for a ultradistribution of exponential type. The "Dirac Formula" for ultradistributions of exponential type is(ref.[31, 33]):

$$\hat{F}(z) \equiv \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\hat{f}(t)}{t - z}\ dt \equiv \frac{\cosh(\lambda z)}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t - z)\cosh(\lambda t)}\ dt \tag{B.13}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz = \int\limits_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t)\ dt \tag{B.14}$$

(B.13) should be used carefully. While $\hat{F}(z)$ is analytic function on Γ, the density $\hat{f}(t)$ is in general singular, so that the right hand side of (B.14) should be interpreted again in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $\hat{F}(z)$ is bounded by a exponential and a power of z (ref.[31, 33]):

$$|\hat{F}(z)| \leq C|z|^p e^{p|\mathfrak{R}(z)|} \tag{B.15}$$

where C and p depend on $\hat{F}$.

The representation (B.12) implies that the addition of any entire function $\hat{G}(z) \in \mathfrak{H}$ to $\hat{F}(z)$ does not alter the ultradistribution:

$$\oint_\Gamma \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\ dz = \oint_\Gamma \hat{F}(z)\hat{\phi}(z)\ dz + \oint_\Gamma \hat{G}(z)\hat{\phi}(z)\ dz$$

But:

$$\oint_\Gamma \hat{G}(z)\hat{\phi}(z)\ dz = 0$$

as $\hat{G}(z)\hat{\phi}(z)$ is an entire analytic function,

$$\therefore \quad \oint_\Gamma \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\ dz = \oint_\Gamma \hat{F}(z)\hat{\phi}(z)\ dz \tag{B.16}$$

Another very important property of $\mathfrak{B}$ is that $\mathfrak{B}$ is reflexive under the Fourier transform:

$$\mathfrak{B} = \mathcal{F}_c\left\{\mathfrak{B}\right\} = \mathcal{F}\left\{\mathfrak{B}\right\} \tag{B.17}$$

where the complex Fourier transform $F(k)$ of $\hat{F}(z) \in \mathfrak{B}$ is given by:

$$F(k) = H[\mathfrak{I}(k)] \int_{\Gamma_+} \hat{F}(z)e^{ikz}\ dz - H[-\mathfrak{I}(k)] \int_{\Gamma_-} \hat{F}(z)e^{ikz}\ dz =$$

$$\oint_\Gamma \{H[\mathfrak{I}(k)H[\mathfrak{R}(z)] - H[-\mathfrak{I}(k)H[-\mathfrak{R}(z)]\}\hat{F}(z)e^{ikz}\ dz =$$

$$H[\mathfrak{I}(k)] \int_0^\infty \hat{f}(x)e^{ikx}\ dx - H[-\mathfrak{I}(k)] \int_{-\infty}^0 \hat{f}(x)e^{ikx}\ dx \tag{B.18}$$

Here Γ_+ is the part of Γ with $\Re(z) \geq 0$ and Γ_- is the part of Γ with $\Re(z) \leq 0$ Using (B.18) we can interpret Dirac's Formula as:

$$F(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{s-k}\, ds \equiv \mathcal{F}_c\left\{\mathcal{F}^{-1}\{f(s)\}\right\} \tag{B.19}$$

The inverse Fourier transform corresponding to (B.19) is given by:

$$\hat{F}(z) = \frac{1}{2\pi} \oint_{\Gamma} \{H[\Im(z)]H[-\Re(k)] - H[-\Im(z)]H[\Re(k)]\}F(k)e^{-ikz}\, dk \tag{B.20}$$

The treatment for ultradistributions of exponential type defined on $\mathbb{C}^n$ is similar to the case of one variable. Thus let Λ_j be given as

$$\Lambda_j = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : |\Im(z_k)| \leq j \quad 1 \leq k \leq n\} \tag{B.21}$$

and

$$||\hat{\phi}||_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{j\left[\sum\limits_{p=1}^{n} |\Re(z_p)|\right]} \left|D^{(k)}\hat{\phi}(z)\right| \right] \right\} \tag{B.22}$$

where $D^{(k)} = \partial^{(k_1)}\partial^{(k_2)} \cdots \partial^{(k_n)} \quad k = k_1 + k_2 + \cdots + k_n$

$\mathfrak{B}^n$ is characterized as follows. Let $\mathfrak{C}^n_\omega$ be the space of all functions $\hat{F}(z)$ such that:

A′) $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C}^n : |Im(z_1)| > p, |Im(z_2)| > p, ..., |Im(z_n)| > p\}$.

B′) $\hat{F}(z)e^{-\left[p\sum\limits_{j=1}^{n}|\Re(z_j)|\right]}/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, ..., |Im(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $\hat{F}(z)$.

Let $\mathfrak{N}^n$ be: $\mathfrak{N}^n = \left\{\hat{F}(z) \in \mathfrak{C}^n_\omega : \hat{F}(z)\right.$ is entire analytic function at minus in one of the variables $z_j \quad 1 \leq j \leq n\}$ Then $\mathfrak{B}^n$ is the quotient space:

C′) $\mathfrak{B}^n = \mathfrak{C}^n_\omega/\mathfrak{N}^n$ We have now

$$\hat{F}(\hat{\phi}) = < \hat{F}(z), \hat{\phi}(z) > = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z)\, dz \tag{B.23}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup ...\Gamma_n$ and where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for

$\mathrm{Im}(z_j) < -\zeta$, $-\zeta < -p$. (Again the path Γ surrounds all the singularities of $\hat{F}(z)$). The n-dimensional Dirac's Formula is now

$$\hat{F}(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)} \, dt \tag{B.24}$$

and the "density" $\hat{f}(t)$ is such that

$$\oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) \, dz = \int_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t) \, dt \tag{B.25}$$

The modulus of $\hat{F}(z)$ is bounded by

$$|\hat{F}(z)| \leq C|z|^p e^{\left[p \sum_{j=1}^{n} |\Re(z_j)|\right]} \tag{B.26}$$

where C and p depend on $\hat{F}$.

Appendix C

Fractional derivative

According to [26] the fractional derivative of a distribution of exponential type $\hat{F}(x)$ is given by

$$\frac{d^\lambda \hat{F}(x)}{dx^\lambda} = \frac{1}{2\pi} \oint_{\Gamma} (-ik)^\lambda F(k) e^{-ikx} \, dk + \oint_{\Gamma} (-ik)^\lambda a(k) e^{-ikx} \, dk \tag{C.1}$$

Where $a(k)$ is entire analytic and rapidly decreasing. If $\lambda = -1$, d^λ/dx^λ is the inverse of the derivative (an integration). In this case the second term of the right side of (C.1) gives a primitive of $\hat{f}(x)$. Using Cauchy's theorem the additional term is

$$\oint_{\Gamma} \frac{a(k)}{k} e^{-ikx} \, dk = 2\pi a(0) \tag{C.2}$$

Of course, an integration should give a primitive plus an arbitrary constant. Analogously when $\lambda = -2$ (a double iterated integration) we have

$$\oint_{\Gamma} \frac{a(k)}{k^2} e^{-ikx} \, dk = \gamma + \delta x \tag{C.3}$$

where γ and δ are arbitrary constants.

For a ultradistribution of exponential type we have for the fractional derivative:

$$\frac{\partial^\lambda \hat{F}(z)}{\partial z^\lambda} = \frac{1}{2\pi}\oint_\Gamma \{H[\Im(z)H[-\Re(k)] - H[-\Im(z)H[\Re(k)]\}(-ik)^\lambda F(k)e^{-ikz}\,dk +$$

$$\oint_\Gamma \{H[\Im(z)H[-\Re(k)] - H[-\Im(z)H[\Re(k)]\}(-ik)^\lambda a(k)e^{-ikz}\,dk \tag{C.4}$$

where $a(k) \in \mathbf{Z}$. This fractional derivative behaves similarly to the above-defined for distributions of exponential type.

Unlike all other definitions of fractional derivative, (C.1) and (C.4) are defined for all values of λ, real o complex. Furthermore, are the only known definitions that unify derivation and integration in a single operation.

Appendix D

Some useful formulas related to the Hypergeometric Function

According to the result given in [51] we can obtain:

$$\int_0^t \frac{t'^n}{(x + at' \pm i0)^{\lambda n+1}}\,dt' = \frac{t^{n+1}}{(x \pm i0)^{\lambda n+1}} B(1, n+1)\times$$

$$F\left(\lambda n + 1, n+1, n+2; -\frac{at}{x \pm i0}\right). \tag{D.1}$$

Using the transformation formula given in [52] for the hypergeometric function

$$F(\lambda+1, 2; 3; z) = \frac{2\Gamma(1-\lambda)}{\Gamma(2-\lambda)}(-1)^{\lambda+1} z^{-\lambda-1} F\left(\lambda+1, \lambda-1; \lambda; \frac{1}{z}\right) +$$

$$\frac{2\Gamma(\lambda-1)}{\Gamma(\lambda+1)} z^{-2} F\left(2, 0; 2-\lambda; \frac{1}{z}\right), \tag{D.2}$$

with the particular value

$$F(a, 0; c; z) = 1. \tag{D.3}$$

we obtain the expression:

$$F(\lambda + 1, 2; 3; z) = \frac{2\Gamma(1-\lambda)}{\Gamma(2-\lambda)}(-1)^{\lambda+1} z^{-\lambda-1} F\left(\lambda + 1, \lambda - 1; \lambda; \frac{1}{z}\right) +$$

$$\frac{2\Gamma(\lambda-1)}{\Gamma(\lambda+1)} z^{-2} \tag{D.4}$$

Now by recourse to the transformation formula [53] we have:

$$F\left(\lambda + 1, \lambda - 1; \lambda; \frac{1}{z}\right) = \left(1 - \frac{1}{z}\right)^{-\lambda} F\left(-1, 1; \lambda; \frac{1}{z}\right), \tag{D.5}$$

or equivalently:

$$F\left(\lambda + 1, \lambda - 1; \lambda; \frac{1}{z}\right) = \frac{z^{\lambda}}{(z-1)^{\lambda}} \left(\frac{\lambda z - 1}{\lambda z}\right). \tag{D.6}$$

Thus, we get, finally,

$$F(\lambda + 1, 2; 3; z) = \frac{2}{\lambda(\lambda - 1) z^2} \left[1 + \frac{\lambda z - 1}{(1-z)^{\lambda}}\right]. \tag{D.7}$$