

# Semidistributivity and Whitman Property in Implication Zroupoids

Juan M. CORNEJO and Hanamantagouda P. SANKAPPANAVAR

## Abstract

In 2012, the second author introduced and studied in [San12] the variety  $\mathcal{I}$  of implication zroupoids that generalize De Morgan algebras and  $\vee$ -semilattices with 0. An algebra  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$ , where  $\rightarrow$  is binary and 0 is a constant, is called an *implication zroupoid* ( $\mathcal{I}$ -zroupoid, for short) if  $\mathbf{A}$  satisfies:  $(x \rightarrow y) \rightarrow z \approx [(z' \rightarrow x) \rightarrow (y \rightarrow z)]'$ , where  $x' := x \rightarrow 0$ , and  $0'' \approx 0$ . Let  $\mathcal{I}$  denote the variety of implication zroupoids and  $\mathbf{A} \in \mathcal{I}$ . For  $x, y \in \mathbf{A}$ , let  $x \wedge y := (x \rightarrow y')$  and  $x \vee y := (x' \wedge y')$ . In an earlier paper we had proved that if  $\mathbf{A} \in \mathcal{I}$ , then the algebra  $\mathbf{A}_{mj} = \langle A, \vee, \wedge \rangle$  is a bisemigroup. In this paper we generalize the notion of semi-distributivity from lattices to bisemigroups and prove that, for every  $\mathbf{A} \in \mathcal{I}$ , the bisemigroup  $\mathbf{A}_{mj}$  is semidistributive. Secondly, we generalize the Whitman Property from lattices to bisemigroups and prove that the subvariety  $\mathcal{MEJ}$  of  $\mathcal{I}$ , defined by the identity:  $x \wedge y \approx x \vee y$ , satisfies the Whitman Property.

## 1 Introduction

Bernstein [Be34] gave a system of axioms, in 1934, for Boolean algebras in terms of implication only. The second author of this paper extended Bernstein's theorem to De Morgan algebras in [San12] by showing that the varieties of De Morgan algebras, Kleene algebras, and Boolean algebras are term-equivalent, to varieties whose defining axioms use only the implication  $\rightarrow$  and the constant 0. The primary role played by the identity (I):  $(x \rightarrow y) \rightarrow z \approx [(z' \rightarrow x) \rightarrow (y \rightarrow z)]'$ , where  $x' := x \rightarrow 0$ , which occurs as an axiom in the definition of each of those new varieties led him, in 2012, to introduce a new (equational) class of algebras called "Implication zroupoids" in [San12].

An algebra  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$ , where  $\rightarrow$  is binary and 0 is a constant, is called an *implication zroupoid* ( $\mathcal{I}$ -zroupoid, for short) if  $\mathbf{A}$  satisfies:

- (I)  $(x \rightarrow y) \rightarrow z \approx [(z' \rightarrow x) \rightarrow (y \rightarrow z)]'$ , where  $x' := x \rightarrow 0$ , and
- (I<sub>0</sub>)  $0'' \approx 0$ .

Let  $\mathcal{I}$  denote the variety of implication zroupoids. These algebras generalize De Morgan algebras and  $\vee$ -semilattices with zero. For more details on the motivation leading to these algebras, we refer the reader to [San12] (or the relevant papers mentioned at the end of this paper).

The investigations into the (complex) structure of the lattice of subvarieties of  $\mathcal{I}$ , begun in [San12], have continued in [CS16a], [CS16b], [CS17a], [CS17b], [CS18a], [CS18b], [CS19], [GSV19] and [CS20]. The present paper is a sequel to this series of papers and is devoted to making further contributions to the theory of implication zroupoids.

Throughout this paper we use the following definitions:

$$(M) \quad x \wedge y := (x \rightarrow y')' \quad \text{and} \quad (J) \quad x \vee y := (x' \wedge y')'.$$

The relation  $\leq$  is defined as follows:  $x \leq y$  if and only if  $x \wedge y = x$ . We note that, in general, it is not a partial order on  $\mathcal{I}$ . However,  $\mathcal{I}_{2,0}$  is the maximal subvariety of  $\mathcal{I}$  in which the relation  $\leq$  is a partial order (see [CS16a]).

With each  $\mathbf{A} \in \mathcal{I}$ , we associate the following algebra:

$$\mathbf{A}^{\text{mj}} := \langle A, \wedge, \vee, 0 \rangle \qquad \mathbf{A}_{\text{mj}} := \langle A, \wedge, \vee \rangle.$$

It was proved in [CS17b, Corollary 4.6] that if  $\mathbf{A}$  is an implication zroupoid, then  $\mathbf{A}_{\text{mj}}$  is a bisemigroup (i.e., an algebra with two binary operations which are both associative.)

**Theorem 1.1** [CS17b, Corollary 4.6] *If  $\mathbf{A} \in \mathcal{I}$  then  $\langle A, \wedge \rangle$  and  $\langle A, \vee \rangle$  are semigroups.*

Two of the important subvarieties of  $\mathcal{I}$  are:  $\mathcal{I}_{2,0}$  and  $\mathcal{MC}$  which are defined relative to  $\mathcal{I}$ , respectively, by the following identities, where  $x \wedge y := (x \rightarrow y')'$ :

$$(\mathcal{I}_{2,0}) \quad x'' \approx x,$$

$$(\mathcal{MC}) \quad x \wedge y \approx y \wedge x.$$

Members of the variety  $\mathcal{I}_{2,0}$  are called *involutive*, and members of  $\mathcal{MC}$  are called *meet-commutative*. An algebra  $\mathbf{A} \in \mathcal{I}$  is symmetric if  $\mathbf{A}$  is both involutive and meet-commutative. Let  $\mathcal{S}$  denote the variety of symmetric  $\mathcal{I}$ -zroupoids. In other words,  $\mathcal{S} = \mathcal{I}_{2,0} \cap \mathcal{MC}$ .

The notions of meet-semidistributivity, join-semidistributivity and semidistributivity for lattices were first defined, in 1961, by Jónsson [Jo61], who proved that free lattices are semidistributive. These notions have been investigated in group theory and in semigroup theory also. For example, It was shown by Shiryaev [Sh85] that meet-semidistributivity and distributivity are equivalent for the lattice of subgroups of a group; on the other hand, join-semidistributivity and distributivity are distinct. In 1999, [JoJo99] showed that each of meet-semidistributivity and join-semidistributivity is equivalent to distributivity on the lattice of inverse subsemigroups containing all the idempotents of an inverse semigroup.

It is clear that the notion of semidistributivity extends naturally from lattices to bisemigroups.

**Definition 1.2** *A bisemigroup is*

- *meet-semidistributive if it satisfies the following conditions:*
  - (M<sub>1</sub>)  $x \wedge y \approx x \wedge z$  implies  $x \wedge (y \vee z) \approx x \wedge y$  (left meet-semidistributive law),
  - (M<sub>2</sub>)  $x \wedge y \approx z \wedge y$  implies  $(x \vee z) \wedge y \approx x \wedge y$  (right meet-semidistributive law)
- *join-semidistributive if it satisfies the following conditions:*
  - (J<sub>1</sub>)  $x \vee y \approx x \vee z$  implies  $x \vee (y \wedge z) \approx x \vee y$  (left join-semidistributive law),
  - (J<sub>2</sub>)  $x \vee y \approx z \vee y$  implies  $(x \wedge z) \vee y \approx x \vee y$  (right join-semidistributive law).
- *semidistributive if it is both meet-semi-distributive and join-semi-distributive.*

It is easy to see that semilattices, viewed as bisemilattices, where the two binary operations are the same, are semidistributive; and De Morgan algebras are clearly semidistributive. The variety  $\mathcal{I}$  of implication zroupoids contains both the varieties of  $\vee$ -semilattices with 0 and De Morgan algebras. Moreover, every implication zroupoid  $\mathbf{A}$  gives rise to the bisemigroup  $\mathbf{A}_{mj}$ . So, it is natural to ask whether  $\mathbf{A}_{mj}$  is semidistributive, for an I-zroupoid  $\mathbf{A}$ . The purpose of this note is to answer this question in the positive by proving that  $\mathbf{A}_{mj}$  is semidistributive for every I-zroupoid  $\mathbf{A}$ .

## 2 Preliminaries

We refer the reader to the books [BD74], [BS81] and [R74] for the concepts and results assumed in this paper.

We now present some preliminary results that will be useful later.

**Lemma 2.1** [San12, Theorem 8.15] *The following identities are equivalent in the variety  $\mathcal{I}$ :*

- (a)  $0' \rightarrow x \approx x$ ,
- (b)  $x'' \approx x$ ,
- (c)  $(x \rightarrow x')' \approx x$ ,
- (d)  $x' \rightarrow x \approx x$ .

**Lemma 2.2** [CS17a, Lemma 3.4] *Let  $\mathbf{A}$  be an I-zroupoid. Then  $\mathbf{A}$  satisfies:*

- (a)  $(x \rightarrow y) \rightarrow z \approx [(x \rightarrow y) \rightarrow z]''$ ,
- (b)  $(x \rightarrow y)' \approx (x'' \rightarrow y)'$ .

**Lemma 2.3** *Let  $\mathbf{A} \in \mathcal{I}_{2,0}$ . Then  $\mathbf{A}$  satisfies:*

- (1)  $(x \rightarrow 0') \rightarrow y \approx (x \rightarrow y') \rightarrow y$ ,
- (2)  $0 \rightarrow x' \approx x \rightarrow 0'$ ,
- (3)  $0 \rightarrow (x \rightarrow y) \approx x \rightarrow (0 \rightarrow y)$ ,
- (4)  $(y \rightarrow x) \rightarrow y \approx (0 \rightarrow x) \rightarrow y$ ,
- (5)  $(x \rightarrow y')' \rightarrow z \approx x \rightarrow (y \rightarrow z)$ ,
- (6)  $0 \rightarrow (x \rightarrow y')' \approx 0 \rightarrow (x' \rightarrow y)$ ,
- (7)  $[x \rightarrow (x' \rightarrow y)]' \approx x' \rightarrow (0 \rightarrow y)'$ .

**Proof** Item (1) can be found in [CS17b, Lemma 2.7]. Item (2) is proved in [San12]. The proofs of items (3), (4), and (6) can be found in [CS16a]. Items (5) and (7) are proved, respectively, in [CS17b] and [CS20].  $\square$

### 3 Semidistributivity of involutive I-zroupoids

In this section we prove that if  $\mathbf{A} \in \mathcal{I}_{2,0}$ , then the bisemigroup  $\mathbf{A}_{mj}$  is semi-distributive. This result will play an important role in the proof of the main Theorem (Theorem 4.2) in the next section. To this end, we need the following lemmas.

**Lemma 3.1** *Let  $\mathbf{A} \in \mathcal{I}_{2,0}$  and  $a, b \in A$  such that  $a \rightarrow b' = a \rightarrow c'$ . Then*

- (1)  $(a' \rightarrow b) \rightarrow a' = (a' \rightarrow c) \rightarrow a'$ ,
- (2)  $0 \rightarrow (a' \rightarrow b)' = 0 \rightarrow (a' \rightarrow c)'$ ,
- (3)  $a' \rightarrow (b \rightarrow 0')' = a' \rightarrow (c \rightarrow 0')'$ ,
- (4)  $(b' \rightarrow c) \rightarrow a' = c \rightarrow a'$ ,
- (5)  $a \rightarrow (b' \rightarrow c)' = a \rightarrow b'$ .

**Proof**

- (1)  $(a' \rightarrow b) \rightarrow a' \stackrel{2.3(4)}{=} (0 \rightarrow b) \rightarrow a' \stackrel{2.3(2) \text{ and } x'' \approx x}{=} (b' \rightarrow 0') \rightarrow a' \stackrel{(I)}{=} [(a'' \rightarrow b') \rightarrow (0' \rightarrow a')]' = [(a \rightarrow b') \rightarrow (0' \rightarrow a')]' \stackrel{hyp}{=} [(a \rightarrow c') \rightarrow (0' \rightarrow a')]' = [(a'' \rightarrow c') \rightarrow (0' \rightarrow a')]' \stackrel{(I)}{=} (c' \rightarrow 0') \rightarrow a' \stackrel{2.3(2) \text{ and } x'' \approx x}{=} (0 \rightarrow c) \rightarrow a' \stackrel{2.3(4)}{=} (a' \rightarrow c) \rightarrow a'$ .
- (2)  $0 \rightarrow (a' \rightarrow b)' \stackrel{2.3(6) \text{ and } x'' \approx x}{=} 0 \rightarrow (a'' \rightarrow b') = 0 \rightarrow (a \rightarrow b') \stackrel{hyp}{=} 0 \rightarrow (a \rightarrow c') = 0 \rightarrow (a'' \rightarrow c') \stackrel{2.3(6) \text{ and } x'' \approx x}{=} 0 \rightarrow (a' \rightarrow c)'$ .
- (3)  $a' \rightarrow (b \rightarrow 0')' \stackrel{2.3(2)}{=} a' \rightarrow (0 \rightarrow b')' \stackrel{2.3(7)}{=} [a \rightarrow (a' \rightarrow b)]' = [a'' \rightarrow (a' \rightarrow b)]' = [(a' \rightarrow 0) \rightarrow (a' \rightarrow b)]' \stackrel{(I)}{=} [((a' \rightarrow b)'' \rightarrow a') \rightarrow (0 \rightarrow (a' \rightarrow b)')]'' = ((a' \rightarrow b) \rightarrow a') \rightarrow (0 \rightarrow (a' \rightarrow b)')' \stackrel{(1)}{=} ((a' \rightarrow c) \rightarrow a') \rightarrow (0 \rightarrow (a' \rightarrow b)')' \stackrel{(2)}{=} ((a' \rightarrow c) \rightarrow a') \rightarrow (0 \rightarrow (a' \rightarrow c)')' = [((a' \rightarrow c)'' \rightarrow a') \rightarrow (0 \rightarrow (a' \rightarrow c)')]'' \stackrel{(I)}{=} [(a' \rightarrow 0) \rightarrow (a' \rightarrow c)]' = [a \rightarrow (a' \rightarrow c)]' \stackrel{2.3(7)}{=} a' \rightarrow (0 \rightarrow c')' \stackrel{2.3(2)}{=} a' \rightarrow (c \rightarrow 0')'$ .
- (4)  $(b' \rightarrow c) \rightarrow a' \stackrel{(I)}{=} [(a'' \rightarrow b') \rightarrow (c \rightarrow a')]' = [(a \rightarrow b') \rightarrow (c \rightarrow a')]' \stackrel{hyp}{=} [(a \rightarrow c') \rightarrow (c \rightarrow a')]' = [(a'' \rightarrow c') \rightarrow (c \rightarrow a')]' \stackrel{(I)}{=} (c' \rightarrow c) \rightarrow a' \stackrel{2.1(d)}{=} c \rightarrow a'$ .
- (5)  $a \rightarrow (b' \rightarrow c)' = a'' \rightarrow (b' \rightarrow c)' = (a' \rightarrow 0) \rightarrow (b' \rightarrow c)' \stackrel{(I)}{=} [((b' \rightarrow c)'' \rightarrow a') \rightarrow (0 \rightarrow (b' \rightarrow c)')]'' = [((b' \rightarrow c) \rightarrow a') \rightarrow (0 \rightarrow (b' \rightarrow c)')]'' \stackrel{(4)}{=} [(c \rightarrow a') \rightarrow (0 \rightarrow (b' \rightarrow c)')]'' \stackrel{2.3(6)}{=} [(c \rightarrow a') \rightarrow (0 \rightarrow (b \rightarrow c'))]' \stackrel{2.3(3)}{=} [(c \rightarrow a') \rightarrow (b \rightarrow (0 \rightarrow c'))]' \stackrel{2.3(5)}{=} [(c \rightarrow a') \rightarrow ((b \rightarrow 0')' \rightarrow c)]' = [(c'' \rightarrow a') \rightarrow ((b \rightarrow 0')' \rightarrow c)]' \stackrel{(I)}{=} (a' \rightarrow (b \rightarrow 0')') \rightarrow c' \stackrel{(3)}{=} (a' \rightarrow (c \rightarrow 0')') \rightarrow c' \stackrel{(I)}{=} [(c'' \rightarrow a') \rightarrow ((c \rightarrow 0')' \rightarrow c)]' \stackrel{2.3(5)}{=} [(c'' \rightarrow a') \rightarrow (c \rightarrow (0 \rightarrow c'))]' \stackrel{2.3(3)}{=} [(c'' \rightarrow a') \rightarrow (0 \rightarrow (c \rightarrow c'))]' = [(c'' \rightarrow a') \rightarrow (0 \rightarrow (c'' \rightarrow c'))]' \stackrel{2.1(d)}{=} [(c'' \rightarrow a') \rightarrow (0 \rightarrow c')]' \stackrel{(I)}{=} (a' \rightarrow 0) \rightarrow c' = a \rightarrow c' \stackrel{hyp}{=} a \rightarrow b'$ .

□

**Theorem 3.2** *Let  $\mathbf{A} \in \mathcal{I}_{2,0}$ . Then  $\mathbf{A}$  is meet-semidistributive.*

**Proof** First we will show that  $\mathbf{A}$  satisfies the condition  $(M_1)$ . By hypothesis, we have that  $\mathbf{A} \models x \wedge y \approx x \wedge z$ . Let  $a, b, c \in A$ . Now, observe that  $a \rightarrow b' = (a \rightarrow b')'' = (a \wedge b)' = (a \wedge c)' = (a \rightarrow c')'' = a \rightarrow c'$ . Hence, by Lemma 3.1 (5), we have

$$(3.1) \quad a \rightarrow (b' \rightarrow c)' = a \rightarrow b'.$$

Therefore,  $a \wedge (b \vee c) \stackrel{\text{def of } \vee}{=} a \wedge (b' \wedge c')' \stackrel{\text{def of } \wedge}{=} a \wedge (b' \rightarrow c'')'' = a \wedge (b' \rightarrow c) \stackrel{\text{def of } \wedge}{=} (a \rightarrow (b' \rightarrow c)')' \stackrel{(3.1)}{=} (a \rightarrow b')' \stackrel{\text{def of } \wedge}{=} a \wedge b$ .

Next, we desire to check that  $\mathbf{A}$  satisfies  $(M_2)$ . Let us assume then that  $\mathbf{A} \models x \wedge y \approx z \wedge y$ , and let  $a, b, c \in A$ . Hence,  $(a \vee c) \wedge b \stackrel{\text{def of } \vee \text{ and } \wedge}{=} ((a' \rightarrow c'')'' \rightarrow b')' \stackrel{(I_{2,0})}{=} ((a' \rightarrow c) \rightarrow b')' \stackrel{(I)}{=} ((b'' \rightarrow a') \rightarrow (c \rightarrow b')'')'' \stackrel{\text{def of } \wedge}{=} ((b'' \rightarrow a') \rightarrow (c \wedge b))'' \stackrel{hyp}{=} ((b'' \rightarrow a') \rightarrow (a \wedge b))'' \stackrel{\text{def of } \wedge}{=} ((b'' \rightarrow a') \rightarrow (a \rightarrow b')'')'' \stackrel{(I)}{=} ((a' \rightarrow a) \rightarrow b')' \stackrel{2.1(d)}{=} (a \rightarrow b')' \stackrel{\text{def of } \wedge}{=} a \wedge b$ . Then  $\mathbf{A}$  is meet-semidistributive. □

**Theorem 3.3** *Let  $\mathbf{A} \in \mathcal{I}_{2,0}$ . Then  $\mathbf{A}_{m_j}$  is semidistributive.*

**Proof**

By Theorem 3.2,  $\mathbf{A}_{m_j}$  is meet-semi-distributive. It is easy to see that  $(J_1)$  and  $(J_2)$  follow from  $(M_1)$  and  $(M_2)$  in view of [CS17a, Theorem 7.1]. Hence  $\mathbf{A}$  is semidistributive. □

## 4 Semidistributivity of I-zroupoids

In this section, we prove the main theorem of this paper. For this, we need one more crucial result proved in [CS17b].

**Theorem 4.1 (Transfer Theorem)** [CS17b]

*Let  $t_i(\bar{x}), i = 1, \dots, 6$ , be terms, where  $\bar{x}$  denotes the sequence  $\langle x_1, \dots, x_n \rangle$ ,  $x_i$  being variables. Let  $\mathcal{V}$  be a subvariety of  $\mathcal{I}$ .*

*If*

$$\mathcal{V} \cap \mathcal{I}_{2,0} \models (t_1(\bar{x}) \rightarrow t_2(\bar{x})) \rightarrow t_3(\bar{x}) \approx (t_4(\bar{x}) \rightarrow t_5(\bar{x})) \rightarrow t_6(\bar{x}),$$

*then*

$$\mathcal{V} \models (t_1(\bar{x}) \rightarrow t_2(\bar{x})) \rightarrow t_3(\bar{x}) \approx (t_4(\bar{x}) \rightarrow t_5(\bar{x})) \rightarrow t_6(\bar{x}).$$

We are now ready to present our first main result of this paper. It was quite surprising to the authors that this result holds.

**Theorem 4.2** *Let  $\mathbf{A} \in \mathcal{I}$ . Then  $\mathbf{A}_{m_j}$  is semidistributive.*

**Proof** Apply Theorem 3.3 and Theorem 4.1. □

The following corollaries are immediate.

**Corollary 4.3** *If  $\mathbf{A}$  is a symmetric I-zroupoid, then  $\mathbf{A}_{m_j}$  is semidistributive.*

An I-zroupoid  $\mathbf{A}$  is called an implication semigroup if  $\mathbf{A}$  satisfies the associative identity:  
 $x \rightarrow (y \rightarrow z) \approx (x \rightarrow y) \rightarrow z$ .

A complete description of the lattice of subvarieties of the variety of implication semigroups is given in [GSV19].

**Corollary 4.4** *If  $\mathbf{A}$  is an implication semigroup, then  $\mathbf{A}_{mj}$  is semidistributive.*

In [CS20], we generalized the notion of Birkhoff systems to Birkhoff bisemigroups. Recall from [CS20] that a bisemigroup  $\mathbf{A}$  is a Birkhoff bisemigroup if  $\mathbf{A}$  satisfies the Birkhoff identity:

$$(BR) \ x \wedge (x \vee y) \approx x \vee (x \wedge y).$$

The following corollary is immediate from Theorem 4.2 and one of main results of [CS20].

**Corollary 4.5** *If  $\mathbf{A} \in \mathcal{I}$ , then  $\mathbf{A}_{mj}$  is a semidistributive, Birkhoff bisemigroup.*

## 5 Whitman Property

It is a well-known result in lattice theory, proved by P. Whitman [Wh41] that every free lattice satisfies the following property (W):

$$x \wedge y \leq z \vee u \Rightarrow x \leq z \vee u \text{ or } y < z \vee u \text{ or } x \wedge y \leq z \text{ or } x \wedge y \leq u.$$

(W) is now known as Whitman Property.

As a matter of fact, (W) is one of four conditions used by P. M. Whitman [Wh41] to characterize free lattices. The Whitman condition was studied, for example, in [FN95] and [D77]. Also, B. Jonsson and J. E. Kiefer [JoKi62] investigated finite lattices which satisfy semidistributivity and the Whitman Property (W). Since the variety  $\mathcal{I}$  satisfies semidistributivity, it was only natural to wonder if  $\mathcal{I}$  or any of its subvarieties satisfies (W).

The following algebra shows that (W) fails in  $\mathcal{I}$  (for  $x=2, y=3, z=1, u=4$ ):

$\rightarrow$ :	0	1	2	3	4	5	6
0	5	5	5	5	5	5	5
1	2	2	2	5	2	5	2
2	1	1	2	3	6	5	6
3	4	6	2	3	4	5	6
4	3	3	5	3	3	5	3
5	0	1	2	3	4	5	6
6	6	6	2	3	6	5	6

We were surprised to find that, indeed, there are subvarieties of  $\mathcal{I}$  that do satisfy (W).

Let  $\mathcal{MEJ}$  (“meet equals join”) denote the subvariety of  $\mathcal{I}$  defined by the identity:  
 $x \wedge y \approx x \vee y$ . Some properties of  $\mathcal{MEJ}$  and its relationships with other known subvarieties of  $\mathcal{I}$  are investigated in [CS20a]. We just mention here that the variety of implication semigroups mentioned at the end of the previous section is, in fact, a subvariety of  $\mathcal{MEJ}$ , as proved in [CS20a].

**Lemma 5.1** *Let  $\mathbf{A} \in \mathcal{MEJ}$ . Then  $\mathbf{A}$  satisfies:*

- (1)  $(x \rightarrow y')' \approx (x' \rightarrow y'')''$ ,
- (2)  $0 \rightarrow [(x \rightarrow y) \rightarrow z] \approx (x \rightarrow y) \rightarrow z$ ,
- (3)  $0 \approx 0'$ ,
- (4)  $0 \rightarrow [x \rightarrow (y \rightarrow z)] \approx (x \rightarrow y')' \rightarrow z$ ,
- (5)  $[0 \rightarrow \{x \rightarrow (y \rightarrow z)\}] \leq z$ .

**Proof** Let  $a, b, c \in A$ . Then

- (1)  $(a \rightarrow b')' \stackrel{\text{def of } \wedge}{=} a \wedge b \stackrel{x \wedge y \approx x \vee y}{=} a \vee b \stackrel{\text{def of } \vee}{=} (a' \wedge b')' \stackrel{\text{def of } \wedge}{=} (a' \rightarrow b'')''$ .
- (2)  $0 \rightarrow [(a \rightarrow b) \rightarrow c] \stackrel{(I)}{=} 0 \rightarrow [(c' \rightarrow a) \rightarrow (b \rightarrow c)']' \stackrel{0 \approx 0''}{=} 0'' \rightarrow [(c' \rightarrow a) \rightarrow (b \rightarrow c)']'$   
 $\stackrel{2.2(a)}{=} [0'' \rightarrow ((c' \rightarrow a) \rightarrow (b \rightarrow c)')]' \stackrel{(1)}{=} [0''' \rightarrow ((c' \rightarrow a) \rightarrow (b \rightarrow c)'')]''' \stackrel{0 \approx 0''}{=} [0' \rightarrow ((c' \rightarrow a) \rightarrow (b \rightarrow c)'')]'''$   
 $\stackrel{2.1(a) \text{ and } 4.1}{=} [((c' \rightarrow a) \rightarrow (b \rightarrow c)'')]''' \stackrel{2.2(a)}{=} ((c' \rightarrow a) \rightarrow (b \rightarrow c)')'$   
 $\stackrel{(I)}{=} (a \rightarrow b) \rightarrow c$ .
- (3)  $0' = 0 \rightarrow 0 \stackrel{0 \approx 0''}{=} 0 \rightarrow 0'' = 0 \rightarrow (0' \rightarrow 0) \stackrel{(2)}{=} 0' \rightarrow 0 = 0'' \stackrel{0 \approx 0''}{=} 0$ .
- (4)  $0 \rightarrow (a \rightarrow (b \rightarrow c)) \stackrel{(3)}{=} 0' \rightarrow (a \rightarrow (b \rightarrow c)) \stackrel{4.1}{=} 0' \rightarrow (a'' \rightarrow (b \rightarrow c)) \stackrel{(3)}{=} 0 \rightarrow (a'' \rightarrow (b \rightarrow c))$   
 $= 0 \rightarrow ((a' \rightarrow 0) \rightarrow (b \rightarrow c)) \stackrel{(2)}{=} a'' \rightarrow (b \rightarrow c) \stackrel{2.3(5) \text{ and } 4.1}{=} (a'' \rightarrow b')' \rightarrow c \stackrel{4.1}{=} (a \rightarrow b')' \rightarrow c$ .
- (5)  $[0 \rightarrow (a \rightarrow (b \rightarrow c))] \wedge c \stackrel{\text{def of } \wedge}{=} [[0 \rightarrow (a \rightarrow (b \rightarrow c))] \rightarrow c']' \stackrel{(4)}{=} [[(a \rightarrow b')' \rightarrow c] \rightarrow c']'$   
 $\stackrel{4.1}{=} [[(a \rightarrow b')' \rightarrow c''] \rightarrow c']' \stackrel{2.3(1)}{=} [[(a \rightarrow b')' \rightarrow 0'] \rightarrow c']' \stackrel{(3)}{=} [(a \rightarrow b)'' \rightarrow c']' \stackrel{(1)}{=} [(a \rightarrow b')'' \rightarrow c'']'' \stackrel{4.1}{=} (a \rightarrow b')' \rightarrow c \stackrel{(4)}{=} 0 \rightarrow (a \rightarrow (b \rightarrow c))$ .

□

**Lemma 5.2** *Let  $\mathbf{A} \in \mathcal{MEJ}$ . Let  $a, b, c, d \in A$  such that  $[(a \rightarrow b')' \rightarrow (c \rightarrow d)'']' = (a \rightarrow b)'$ . Then:*

- (1)  $(a \rightarrow b') \rightarrow (c' \rightarrow d) = a' \rightarrow b$
- (2)  $0 \rightarrow (a' \rightarrow b) \leq d$ .

**Proof**

- (1)  $(a \rightarrow b') \rightarrow (c' \rightarrow d) \stackrel{2.2(a)}{=} (a \rightarrow b') \rightarrow (c' \rightarrow d)'' \stackrel{5.1(1)}{=} (a \rightarrow b') \rightarrow (c'' \rightarrow d)'''$   
 $\stackrel{4.1}{=} (a \rightarrow b') \rightarrow (c \rightarrow d)' \stackrel{2.2(a)}{=} [(a \rightarrow b') \rightarrow (c \rightarrow d)']'' \stackrel{5.1(1)}{=} [(a \rightarrow b')' \rightarrow (c \rightarrow d)'']'''$   
 $\stackrel{4.1}{=} [(a \rightarrow b')' \rightarrow (c \rightarrow d)'']' \stackrel{\text{hyp}}{=} (a \rightarrow b')' \stackrel{5.1(1)}{=} (a' \rightarrow b)'' \stackrel{4.1}{=} a' \rightarrow b$ .
- (2)  $0 \rightarrow (a' \rightarrow b) \stackrel{(1)}{=} 0 \rightarrow ((a \rightarrow b') \rightarrow (c' \rightarrow d)) \leq d$  by Lemma 5.1 (5) with  $x := a \rightarrow b', y := c', z := d$ .

□

We now present our second main result of the paper.

**Theorem 5.3** *The variety  $\mathcal{MEJ}$  satisfies the Whitman Property (W).*

**Proof** Let  $\mathbf{A} \in \mathcal{MEJ}$  and  $a, b, c, d \in A$  such that  $a \wedge b \leq c \vee d$ . Since the identity  $x \wedge y \approx x \vee y$  holds in  $\mathcal{MEJ}$  we have that  $a \wedge b \leq c \wedge d$ . Then  $[(a \rightarrow b)'] \rightarrow (c \rightarrow d)'' \stackrel{def \ of}{=} (a \wedge b) \wedge (c \wedge d)$   $\stackrel{a \wedge b \leq c \wedge d}{=} a \wedge b \stackrel{def \ of}{=} (a \rightarrow b)'$ . Hence, by Lemma 5.2 (2),

$$(5.1) \quad 0 \rightarrow (a' \rightarrow b) \leq d.$$

Therefore,  $a \wedge b \stackrel{def \ of}{=} (a \rightarrow b)' \stackrel{5.1(1)}{=} (a' \rightarrow b)'' \stackrel{4.1}{=} a' \rightarrow b \stackrel{5.1(2)}{=} 0 \rightarrow (a' \rightarrow b) \stackrel{(5.1)}{\leq} d$ . □

It should be remarked here that the relation  $\leq$  is not, in general, a partial order on algebras in  $\mathcal{I}$ . In fact, it was shown in [CS16a] that the variety  $\mathcal{I}_{2,0}$  is the maximal subvariety of  $\mathcal{I}$  with respect to the property that  $\leq$  is a partial order in it. Thus we can state the following corollary where the relation  $\leq$  is indeed a partial order.

The following corollary is immediate from Corollary 4.5 and Theorem 5.3.

**Corollary 5.4** *If  $\mathbf{A} \in \mathcal{MEJ} \cap \mathcal{I}_{2,0}$ , then  $\mathbf{A}_{mj}$  is a semidistributive, Birkhoff bisemigroup which satisfies Whitman Property (W).*

## 6 Concluding Remarks

It is well-known that the variety of semilattices is congruence-semi-distributive. A proof is given in [Pa64]. It is also well-known that the variety of De Morgan algebras is congruence-distributive. Since the variety of implication zroupoids contains both the variety of  $\vee$ -semilattices with 0 and the variety of De Morgan algebras, the following problem arises naturally.

**PROBLEM 1:** Is the variety of implication zroupoids congruence-semidistributive?

Although the classes of meet semi-distributive bisemigroups, join semi-distributive bisemigroups, and semi-distributive bisemigroups are natural extensions of the corresponding classes of lattices, they do not, to the best of our knowledge, seem to have been investigated in the literature so far. The main result of this paper certainly seem to warrant such an investigation. To facilitate such a study, we mention the following open problem which naturally arises.

**PROBLEM 2** Investigate the quasivarieties of meet semi-distributive bisemigroups, join semi-distributive bisemigroups and semi-distributive bisemigroups.

Let  $\mathbf{A} = \langle A, \vee, \wedge \rangle$  be a bisemigroup.  $\mathbf{A}$  is  $\wedge$ -idempotent if  $\mathbf{A}$  satisfies the  $\wedge$ -idempotent identity:  $x \wedge x \approx x$ .  $\vee$ -idempotent bisemigroups are defined dually. A bisemigroup  $\mathbf{A}$  is idempotent if it is both  $\wedge$ -idempotent and  $\vee$ -idempotent.  $\mathbf{A}$  is left [right] mj-distributive if  $\mathbf{A} \models x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$  [ $(y \vee z) \wedge x \approx (y \wedge x) \vee (z \wedge x)$ ]. Left [right] jm-distributive bisemigroups are defined dually.  $\mathbf{A}$  is mj-distributive if it is both left mj-distributive and right mj-distributive. jm-distributive bisemigroups are defined dually.  $\mathbf{A}$  is distributive if it is both mj-distributive and jm-distributive.

We conclude this paper with the following (easy) observation.



**Theorem 6.1** *Let  $\mathbf{A}$  be a left  $mj$ -distributive,  $\vee$ -idempotent bisemigroup. Then  $\mathbf{A}$  is meet-semidistributive. In particular, every distributive, idempotent bisemigroup is semi-distributive.*

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### Compliance with Ethical Standards:

**Conflict of Interest:** The first author declares that he has no conflict of interest. The second author declares that he has no conflict of interest.

**Ethical approval:** This article does not contain any studies with human participants or animals performed by any of the authors.

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JUAN M. CORNEJO  
Departamento de Matemática  
Universidad Nacional del Sur  
Alem 1253, Bahía Blanca, Argentina  
INMABB - CONICET

[jmcornejo@uns.edu.ar](mailto:jmcornejo@uns.edu.ar)

HANAMANTAGOUDA P. SANKAPPANAVAR  
Department of Mathematics  
State University of New York  
New Paltz, New York 12561  
U.S.A.

[sankapph@newpaltz.edu](mailto:sankapph@newpaltz.edu)