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A NEW DEFINITION OF A FRACTIONAL DERIVATIVE OF LOCAL TYPE

PAULO M. GUZMÁN, GUILLERMO LANGTON, LUCIANO M. LUGO MOTTA BITTENCURT, JULIAN MEDINA, JUAN E. NÁPOLES VALDES

Abstract. In this paper we present a new definition of a local fractional derivative non-conformable and we obtain the main properties of the same, equivalent to the classic derivative of integer order.

1. Introduction

The idea of fractional calculus is as old as traditional calculus. The history of fractional calculus dates back to more than 300 years ago, and the original question which led to the name fractional calculus was: what does $\frac{d^n f}{dx^n}$ mean if $n = \frac{1}{2}$. Since then, several mathematicians contributed to the development of fractional calculus: Riemann, Liouville, Caputo, Grunwald, Letnikov, etc. (see [4], [6] and [12]). Until recently, research on fractional calculus was confined to the field of mathematics but, in the last two decades, many applications of fractional calculus appeared in various fields of engineering, applied sciences, economy, etc. (cf. [5], [7], [9] and [16]). As a result, fractional calculus has become an important topic for researchers in various fields. Further some recent work about fractional derivatives are [2], [8] and [15]. The paper [5] is devoted to research fractional derivatives and integrals obtained by iterating conformable integrals. They obtained left- and right-fractional conformable integrals. With a standard fractional procedure we found left- and right-fractional conformable derivatives in the sense of Riemann-Liouville and Caputo.

Among the inconsistencies of the existing fractional derivatives D^α are:

1) Most of the fractional derivatives except Caputo-type, do not satisfy $D^\alpha(1) = 0$, if α is not a natural number.

2) All fractional derivatives do not satisfy the familiar Product Rule for two functions $D^\alpha(fg) = gD^\alpha(f) + fD^\alpha(g)$.

3) All fractional derivatives do not satisfy the familiar Quotient Rule for two functions $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$ with $g \neq 0$.

4) All fractional derivatives do not satisfy the Chain Rule for composite functions $D^\alpha(f \circ g)(t) = D^\alpha(f(g))D^\alpha g(t)$.

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5) The fractional derivatives do not have a corresponding "calculus".

6) All fractional derivatives do not satisfy the Indices Rule $D^{\alpha}D^{\beta}(f) = D^{\alpha+\beta}(f)$.

However, in [1] the authors define a new well-behaved simple fractional derivative called the conformable fractional derivative, depending just on the basic limit definition of the derivative (cf. also [13] and [14]). Namely, for a function $f : (0, +\infty) \to \mathbb{R}$ the conformable fractional derivative of order $0 < \alpha \leq 1$ of f at $t > 0$ was defined by

$$T_{\alpha}f(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

As a consequence of the above definition, the authors proved that many of the previous inadequacies are overcome. The adjective conformable may or may not be appropiate here, since this was initially referred to as a conformable fractional derivative $D^{\alpha}f(t)$, when $\alpha \to 1$ satisfies $D^{\alpha}f(t) \to f'(t)$; i.e., when $\alpha \to 1$, $D^{\alpha}f(t)$ preserves the angle of the tangent line to the curve, while in our definition, as we shall see later, this angle is not conserved.

The purpose of this paper is to generalize the results obtained in this paper and to introduce a new definition of fractional derivative non-conformable as a natural extension of the well-known definition of derivative of a function in a point, in particular show that the inadequacies 1) -4) are overcome. In future works we will complete the study of this new fractional derivative non-conformable constructing a theoretical body similar to the traditional calculus.

2. New Fractional Derivative

In this section, we give our new definition of a non-conformable fractional derivative of a function in a point t and obtain several results that are close resemblance of those found in classical calculus.

Definition 2.1. *Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by $N_1^{\alpha}f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon e^{t^{-\alpha}}) - f(t)}{\varepsilon}$ for all $t > 0$, $\alpha \in (0, 1)$. If f is $\alpha-$differentiable in some $(0, a)$, and $\lim_{t \to 0^+} N_1^{(\alpha)}f(t)$ exists, then define $N_1^{(\alpha)}f(0) = \lim_{t \to 0^+} N_1^{(\alpha)}f(t)$.*

As a consequence of the above definition, we obtain the following result known in classical calculus.

Theorem 2.2. *If a function $f : [0, +\infty) \to \mathbb{R}$ is N-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$ then f is continuous at t_0.*

Proof. Since $f(t_0 + \varepsilon e^{t_0^{-\alpha}}) - f(t_0) = \frac{f(t_0+\varepsilon e^{t_0^{-\alpha}}) - f(t_0)}{\varepsilon}\varepsilon$. Then

$$\mathop{lim}_{\varepsilon \to 0} \left(f(t_0 + \varepsilon e^{t_0^{-\alpha}}) - f(t_0) \right) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon e^{t_0^{-\alpha}}) - f(t)}{\varepsilon} \lim_{\varepsilon \to 0} \varepsilon$$

Let $k = \varepsilon e^{t_0^{-\alpha}}$ then $k \to 0$ if $\varepsilon \to 0$, so we have

$$\lim_{\varepsilon\to 0}\left(f(t_0+\varepsilon e^{t_0^{-\alpha}})-f(t_0)\right)=\lim_{k\to 0}\left(f(t_0+k)-f(t_0)\right)=N_1^{\alpha}f(t_0)\lim_{\varepsilon\to 0}\varepsilon=0.$$

From this we have the continuity of f at t_0. □

Theorem 2.3. *Let f and g be N-differentiable at a point $t>0$ and $\alpha\in(0,1]$. Then*

a) $N_1^{\alpha}(af+bg)(t)=aN_1^{\alpha}(f)(t)+bN_1^{\alpha}(g)(t)$.
b) $N_1^{\alpha}(t^p)=e^{t^{-\alpha}}pt^{p-1},\ p\in\mathbb{R}$.
c) $N_1^{\alpha}(\lambda)=0,\ \lambda\in\mathbb{R}$.
d) $N_1^{\alpha}(fg)(t)=fN_1^{\alpha}(g)(t)+gN_1^{\alpha}(f)(t)$.
e) $N_1^{\alpha}(\frac{f}{g})(t)=\frac{gN_1^{\alpha}(f)(t)-fN_1^{\alpha}(g)(t)}{g^2(t)}$.
f) If, in addition, f is differentiable then $N_1^{\alpha}(f)=e^{t^{-\alpha}}f'(t)$.
g) Being f differentiable and $\alpha=n$ integer, we have $N_1^{n}(f)(t)=e^{t^{-n}}f'(t)$.

Proof. a) Let $H(t)=(af+bg)(t)$ then $N_1^{\alpha}H(t)=\lim_{\varepsilon\to 0}\frac{H(t+\varepsilon e^{t^{-\alpha}})-H(t)}{\varepsilon}$ and from this we have the desired result.

b) It is sufficient to develop in power series $\left(t+\varepsilon e^{t^{-\alpha}}\right)^p$, in this way we have two cases:

I) If $p\in\mathbb{N}$ we obtain

$$N_1^{\alpha}(t^n)=\lim_{\varepsilon\to 0}\frac{(t+\varepsilon e^{t^{-\alpha}})^n-t^n}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{t^n+n\varepsilon t^{n-1}e^{t^{-\alpha}}+...-t^n}{\varepsilon}=nt^{n-1}e^{t^{-\alpha}}.$$

II) If $p\in\mathbb{R}$ we obtain in the same way

$$N_1^{\alpha}(t^p)=\lim_{\varepsilon\to 0}\frac{(t+\varepsilon e^{t^{-\alpha}})^p-t^p}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{t^p+p\varepsilon t^{p-1}e^{t^{-\alpha}}+\frac{p(p-1)t^{p-2}\left(\varepsilon e^{t^{-\alpha}}\right)^2}{1.2}+...-t^p}{\varepsilon}=pt^{p-1}e^{t^{-\alpha}}.$$

In both cases we obtain the desired result.

c) Easily follows from definition.

d) From definition we have

$$\begin{aligned}
N_1^{\alpha}(fg)(t) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{t^{-\alpha}})g(t+\varepsilon e^{t^{-\alpha}})-f(t)g(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{t^{-\alpha}})g(t+\varepsilon e^{t^{-\alpha}})-f(t)g(t+\varepsilon e^{t^{-\alpha}})+f(t)g(t+\varepsilon e^{t^{-\alpha}})-f(t)g(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{\left(f(t+\varepsilon e^{t^{-\alpha}})-f(t)\right)g(t+\varepsilon e^{t^{-\alpha}})}{\varepsilon}+\lim_{\varepsilon\to 0}\frac{\left(g(t+\varepsilon e^{t^{-\alpha}})-g(t)\right)f(t)}{\varepsilon}\\
&= fN_1^{\alpha}(g)(t)+gN_1^{\alpha}(f)(t)
\end{aligned}$$

e) In a similar way to the previous one we have

$$N_1^{\alpha}(\frac{f}{g})(t) = \lim_{\varepsilon\to 0}\frac{\frac{f(t+\varepsilon e^{t^{-\alpha}})}{g(t+\varepsilon e^{t^{-\alpha}})}-\frac{f(t)}{g(t)}}{\varepsilon}.$$

But

$$\begin{aligned}\frac{f(t+\varepsilon e^{t^{-\alpha}})}{g(t+\varepsilon e^{t^{-\alpha}})}-\frac{f(t)}{g(t)} &= \frac{f(t+\varepsilon e^{t^{-\alpha}})}{g(t+\varepsilon e^{t^{-\alpha}})}-\frac{f(t)}{g(t)}\frac{g(t+\varepsilon e^{t^{-\alpha}})}{g(t+\varepsilon e^{t^{-\alpha}})}=\frac{f(t+\varepsilon e^{t^{-\alpha}})g(t)-f(t)g(t+\varepsilon e^{t^{-\alpha}})}{g(t)g(t+\varepsilon e^{t^{-\alpha}})}\\ &= \frac{f(t+\varepsilon e^{t^{-\alpha}})g(t)-f(t)g(t+\varepsilon e^{t^{-\alpha}})-f(t)g(t)+f(t)g(t)}{g(t)g(t+\varepsilon e^{t^{-\alpha}})}\\ &= \frac{\left(f(t+\varepsilon e^{t^{-\alpha}})-f(t)\right)g(t)-\left(g(t+\varepsilon e^{t^{-\alpha}})-g(t)\right)f(t)}{g(t)g(t+\varepsilon e^{t^{-\alpha}})}.\end{aligned}$$

From this last expression we obtain the expected result.

f) $N_1^{\alpha}f(t)=\lim\limits_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}=\lim\limits_{k\to 0}\frac{f(t+k)-f(t)}{k}e^{t^{-\alpha}}$ with $k=\varepsilon e^{t^{-\alpha}}$ so that $N_1^{\alpha}f(t)=e^{t^{-\alpha}}f'(t)$.

g) $N_1^{n}(f)(t)=\lim\limits_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{t^{-n}})-f(t)}{\varepsilon}=\lim\limits_{\varepsilon\to 0}\frac{f(t+k)-f(t)}{k}e^{t^{-n}}$ with $k=\varepsilon e^{t^{t^{-n}}}$ and from here you get the desired result. □

Remark. *If $N_1^{\alpha}f(t)$ exists for $t>0$ then f is differentiable at t and $f'(t)=e^{-t^{-\alpha}}N_1^{\alpha}f(t)$.*

Theorem 2.4. *Let α,β be positive constants such that $0<\alpha,\beta<1$ and f be a function (non-constant) twice differentiable on an interval $(0,+\infty)$. Then*

$$N_1^{\alpha+\beta}f(t)\neq N_1^{\alpha}\left(N_1^{\beta}f(t)\right) \tag{2.1}$$

Proof. Follows easily from definition. □

An application of this result and Theorem 2 are the followings.

Theorem 2.5. *Let $f:[0,+\infty)\to\mathbb{R}$ be twice differentiable on $(0,+\infty)$ and $0<\alpha,\beta\leq 1$ such that $0<\alpha+\beta\leq 2$. Then $N_1^{\alpha}\left(N_1^{\beta}f(t)\right)=\left[N_1^{\alpha+\beta}f'(t)-\beta t^{1-\beta}N_1^{\alpha+\beta}f(t)\right]e^{t^{\alpha}+t^{\beta}-t^{-(\alpha+\beta)}}$.*

Proof. From the definition we have

$$N_1^{\alpha}\left(e^{t^{-\beta}}f'(t)\right)=e^{t^{-\alpha}}\left(e^{t^{-\beta}}f'(t)\right)'=e^{t^{-\alpha}}\left(e^{t^{-\beta}}f'(t)+f'(t)(-\beta)t^{-\beta-1}e^{t^{-\beta}}\right)=$$

$$e^{t^{-\alpha}+t^{-\beta}}\left(f'(t)-\beta t^{-\beta-1}f'(t)\right)=e^{t^{-\alpha}+t^{-\beta}}e^{-t^{-(\alpha+\beta)}}\left(e^{t^{-(\alpha+\beta)}}f'(t)-\beta t^{-\beta-1}f'(t)e^{t^{-(\alpha+\beta)}}\right)=$$

$$=\left[N_1^{\alpha+\beta}f'(t)-\beta t^{1-\beta}N_1^{\alpha+\beta}f(t)\right]e^{t^{\alpha}+t^{\beta}-t^{-(\alpha+\beta)}}.$$

□

Remark. *Although* $N_1^{\alpha} f(t) N_1^{\beta} f(t) \neq N_1^{\alpha+\beta} f(t)$*; i.e., this deviates from the behaviour of* α *an integer derivatives, non-conmutativity offers a richness that is interesting to explore. If* f *is a derivable function we have the following differential equations* $(\lambda \in \mathbb{R})$*:*

I) $N_1^{\alpha} y(t) + \lambda y(t) = 0$*, with solution* $y(t) = e^{\lambda \int^t \frac{ds}{e^{s^{-\alpha}}}}$.

II) $N_1^{\beta}\left(N_1^{\alpha} y(t)\right) = 0$*, from this we have* $y(t) = C_1 \int^t \frac{ds}{e^{s^{-\alpha}}} + C_2$*. This solution is independent from* β*; i.e., the order of the last derivation does not influence the general solution.*

III) $N_1^{\alpha} y(t) + p(t) y(t) = q(t)$*, from where* $y(t) = e^{-\int^t \frac{p(s)ds}{e^{s^{-\alpha}}}} \left[\int^t \frac{q(s)}{e^{s^{-\alpha}}} e^{\int^t \frac{p(s)ds}{e^{s^{-\alpha}}}} ds + C\right]$.

While there are very similar cases to the case where α *and* β *they are integers, the second gives us a significant difference with the known theory.*

Theorem 2.6. *Let* $f, h : [0, +\infty) \to \mathbb{R}$ *be functions such that* N_1^{α} *exists for* $t > 0$*, if* f *is differentiable on* $(0, +\infty)$ *and* $N_1^{\alpha} f(t) = e^{t^{-\alpha}} h(t)$*. Then* $h(t) = f'(t)$ *for all* $t > 0$.

Remark. *Failure to comply with the Semigroup Law may seem deceptive, but it is one of the essential characteristics of fractional derivatives (see [17] and [18]).*

2.1. **N-fractional derivative of certain functions.** Directly from property (f) of the previous theorem we have the following result.

Theorem 2.7. *We have*

a) $N_1^{\alpha}(1) = 0$.

b) $N_1^{\alpha}(e^{ct}) = c e^{ct} . e^{t^{-\alpha}}$.

c) $N_1^{\alpha}(\sin bt) = b e^{t^{-\alpha}} \cos bt$.

d) $N_1^{\alpha}(\cos bt) = -b e^{t^{-\alpha}} \sin bt$.

Remark. *It is clear that this theorem can be extended for any differentiable function.*

3. The Chain Rule

Now we will present the equivalent result, for N_1^{α}, of the well-known chain rule of classic calculus and that is basic in the Second Method of Liapunov, for the study of stability and thus we overcome the deficiency 4) indicated at the beginning of the work.

Theorem 3.1. *Let* $\alpha \in (0, 1]$*,* g *N-differentiable at* $t > 0$ *and* f *differentiable at* $g(t)$ *then* $N_1^{\alpha}(f \circ g)(t) = f'(g(t)) N_1^{\alpha} g(t)$.

Proof. We prove the result following a standard limit-approach. Firts case, if the function g is constant in a neighborhood of $a > 0$ then $N_1^{\alpha} f \circ g)(t) = 0$. If g is not a constant in a neighborhood of $a > 0$ we can find an $t_0 > 0$ such that $g(x_1) \neq g(x_2)$ for any $x_1, x_2 \in (a - t_0, a + t_0)$. Now, since g is continuous at a, for ε sufficiently small, we have

$$\begin{aligned}
N_1^{\alpha}(f\circ g)(a) &= \lim_{\varepsilon\to 0}\frac{f(g(t+\varepsilon e^{a^{-\alpha}}))-f(g(a))}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(g(a+\varepsilon e^{a^{-\alpha}}))-f(g(a))}{g(a+\varepsilon e^{a^{-\alpha}})-g(a)}\frac{g(a+\varepsilon e^{a^{-\alpha}})-g(a)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(g(a+\varepsilon e^{a^{-\alpha}}))-f(g(a))}{g(a+\varepsilon e^{a^{-\alpha}})-g(a)}\lim_{\varepsilon\to 0}\frac{g(a+\varepsilon e^{a^{-\alpha}})-g(a)}{\varepsilon}\\
&= \lim_{k\to 0}\frac{f(g(a+\varepsilon e^{a^{-\alpha}}))-f(g(a))}{g(a+\varepsilon e^{a^{-\alpha}})-g(a)}\lim_{\varepsilon\to 0}\frac{g(a+\varepsilon e^{a^{-\alpha}})-g(a)}{\varepsilon}.
\end{aligned}$$

Making

$$\varepsilon_1 = g(a+\varepsilon e^{a^{-\alpha}})-g(a)$$

in the first factor we have

$$\lim_{\varepsilon\to 0}\frac{f(g(a+\varepsilon e^{a^{-\alpha}}))-f(g(a))}{g(a+\varepsilon e^{a^{-\alpha}})-g(a)} = \lim_{\varepsilon_1\to 0}\frac{f(g(a)+\varepsilon_1)-f(g(a))}{\varepsilon_1}$$

from here

$$\begin{aligned}
N_1^{\alpha}(f\circ g)(a) &= \lim_{\varepsilon_1\to 0}\frac{f(g(a)+\varepsilon_1)-f(g(a))}{\varepsilon_1}\lim_{\varepsilon\to 0}\frac{g(a+\varepsilon e^{a^{-\alpha}})-g(a)}{\varepsilon}\\
&= f'(g(a))N_1^{\alpha}g(a).
\end{aligned}$$

□

Finally, in this section we present some examples of the above Chain Rule.

Example 3.2. *Let* $h(t)=\sin^2 t$.

Let's calculate the derivative by two paths:

i) *By the Chain Rule*

$$\begin{aligned}
N_1^{\alpha}\left[\sin^2 t\right] &= \left(\sin^2 t\right)' N_1^{\alpha}(\sin t)\\
&= 2e^{t^{-\alpha}}\sin t\cos t.
\end{aligned}$$

ii) *From Theorem 2*

$$h(t)=\sin^2 t=\left(\frac{e^{it}-e^{-it}}{2i}\right)^2=\frac{e^{2it}+e^{-2it}}{-4}$$

$$\begin{aligned} N_1^{\alpha}\left[\sin^2 t\right] &= N_1^{\alpha}\left[\frac{e^{2it}+e^{-2it}-2}{-4}\right] \\ &= -\frac{1}{4}\left[N_1^{\alpha}\left(e^{2it}\right)+N_1^{\alpha}\left(e^{-2it}\right)-2N_1^{\alpha}(1)\right] \\ &= \frac{ie^{t^{-\alpha}}}{2}\left[e^{-2it}-e^{2it}\right] \\ &= e^{t^{-\alpha}}\frac{e^{2it}-e^{-2it}}{2i} \\ &= e^{t^{-\alpha}}\sin 2t \\ &= 2e^{t^{-\alpha}}\sin t\cos t. \end{aligned}$$

Analogously, we have

$$\begin{aligned} N_1^{\alpha}\left[\cos^2 t\right] &= \left(\cos^2 t\right)' N_1^{\alpha}(\cos t) \\ &= -2e^{t^{-\alpha}}\sin t\cos t. \end{aligned}$$

Example 3.3. *Let $h(t) = \sin^n t$.*

i) *By the Chain Rule*

$$\begin{aligned} N_1^{\alpha}\left[\sin^n t\right] &= n\sin^{n-1} t\cos t e^{t^{-\alpha}} \\ &= ne^{t^{-\alpha}}\sin^{n-1} t\cos t. \end{aligned}$$

ii) *By Theorem 2*

$$\begin{aligned} N_1^{\alpha}\left[\sin^n t\right] &= \sin t N_1^{\alpha}\left[\sin^{n-1} t\right] + e^{t^{-\alpha}}\sin^{n-1} t\cos t \\ \sin t N_1^{\alpha}\left[\sin^{n-1} t\right] &= \sin^2 t N_1^{\alpha}\left[\sin^{n-2} t\right] + e^{t^{-\alpha}}\sin^{n-1} t\cos t \\ \sin^2 t N_1^{\alpha}\left[\sin^{n-2} t\right] &= \sin^3 t N_1^{\alpha}\left[\sin^{n-3} t\right] + e^{t^{-\alpha}}\sin^{n-1} t\cos t \\ \vdots \quad &\vdots \quad \vdots \\ \sin^{n-1} t N_1^{\alpha}\left[\sin t\right] &= \sin^n t N_1^{\alpha}\left[\sin^{n-n} t\right] + e^{t^{-\alpha}}\sin^{n-1} t\cos t \\ N_1^{\alpha}\left[\sin^n t\right] &= ne^{t^{-\alpha}}\sin^{n-1} t\cos t. \end{aligned}$$

Example 3.4. *Let $r(t) = \sin t^2$, this is a differentiable function. So, from Theorem 2 f) we have $N_1^{\alpha}[r(t)] = e^{t^{-\alpha}} r'(t) = e^{t^{-\alpha}} 2t\cos t^2$.*

By the Chain Rule we obtain:

$$\begin{aligned} N_1^{\alpha}\left[r \circ g\right](t) &= r'\left[g\left(t\right)\right] N_1^{\alpha} g\left(t\right) \\ &= \left(\sin t^2\right)' N_1^{\alpha}\left(t^2\right) \\ &= e^{t^{-\alpha}} 2t \cos t^2. \end{aligned}$$

4. Final Remarks

It is natural to ask what relation has $N_1^{\alpha}f(t)$ to the derivative defined by Abdejjawad in [1]. In the case that f is a derivable function, $T_{\alpha}f(0) = 0$ while $N_1^{\alpha}f(t) \to \infty$, on the other hand, if $t \to \infty$ we obtain that in general $T_{\alpha}f(t) \to \infty$ while $N_1^{\alpha}f(t) \to f'(t)$, i.e., our derivative returns the classical derivative when $t \to \infty$ which ensures that if f is derivable, the asymptotic properties of f are inherited, which is of vital importance in the Qualitative Theory of Differential Equations.

In [3] the authors present the following definition of local fractional derivative using kernels.

Definition 2.1. Let $k : [a, b] \to \mathbb{R}$ be a continuous nonnegative map such that $k(t) \neq 0$, whenever $t > a$. Given a function $f : [a, b] \to \mathbb{R}$ and $\alpha \in (0, 1)$ a real, we say that f is α-differentiable at $t > a$, with respect to kernel k, if the limit

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon k(t)^{1-\alpha}) - f(t)}{\varepsilon} \tag{4.1}$$

exists. The α-derivative at $t = a$ is defined by

$$f^{(\alpha)}(a) := \lim_{t \to a^+} f^{(\alpha)}(t)$$

if the limit exists.

Then, they state and do not demonstrate, because they affirm that it is trivial, **Theorem 2.2** where they obtain that

$$f^{(\alpha)}(t) = k(t)^{1-\alpha} f'(t), \ t > a \tag{4.2}$$

if f is differentiable for $t > a$.

In the Conclusion the authors state that *"some of the existent notions about local fractional derivative are very close related to the usual derivative function. In fact, the α-derivative of a function is equal to the first-order derivative, multiplied by a continuous function. Also, using formula (3), most of the results concerning α-differentiation can be deduced trivially from the ordinary ones. In the authors' opinion, local fractional calculus is an interesting idea and deserves further research, but definitions like (2) are not the best ones and a different path should be followed"*.

This is exactly what we have done.

By the other hand, the fractional derivative present here is local by nature, hence any comparison with classical fractional derivatives is erroneous, because we are considering mathematical objects of different kinds. This local and the derivatives of Caputo, Riemann-Liouville, etc. are global (see [9] for a comparison of this type).

On the other hand, consider the derivative c) in Theorem 5, in the classical sense we have $\mathrm{D}^{\alpha}(\sin bt) = b^{\alpha}\sin(bt + \frac{\pi}{4})$ but what's up if $\alpha = \frac{1}{2n}$, $n \in \mathbb{N}$, and $b = -1$? However using our definition, we have no problem $N_1^{\frac{1}{2}}(\sin(-t)) = -e^{t^{-\frac{1}{2}}}\cos t$ with $n = 1$. In addition, there are functions like $f(t) = e^{-t^{-\alpha}}$ whose fractional derivative in the classical sense is very difficult to calculate, if not impossible, while using our definition is very easy, so we have $N_1^{\alpha}(e^{-t^{-\alpha}}) = \alpha t^{-\alpha-1}$.

Consider the very simple differential equation $D^{\alpha}y + 3x^{-\frac{3}{2}}y = x^{-\frac{3}{2}}$. If one has to solve it using D^{α} as the Caputo or Riemann-Liouville definition, then must use either the Laplace transform or the fractional power series technique. By other hand, if D^{α} is our definition, easily we obtain that $y = -\frac{1}{3}\left(1 - e^{-6e^{-\frac{1}{\sqrt{x}}}}\right)$ is a particular solution with $\alpha = \frac{1}{2}$.

We would like to add an additional application of our fractional derivative to solve ordinary differential equations. Thus, consider the following linear first-order differential equation:

$$y' + \alpha t^{-\alpha-1}y = e^{t^{-\alpha}}\beta t^{\beta-1}. \tag{4.3}$$

It is clear that this equation can be written this way:

$$\left(N_1^{\alpha}y\right)e^{-t^{-\alpha}} + \left(N_1^{\alpha}e^{-t^{-\alpha}}\right)y = N_1^{\alpha}\left[t^{\beta}\right],$$

from where:

$$N_1^{\alpha}\left[y(t)e^{-t^{-\alpha}}\right] = N_1^{\alpha}\left[t^{\beta}\right].$$

According to Corollary 4 of [10], we easily obtain that:

$$y(t)e^{-t^{-\alpha}} = t^{\beta} + C,$$

where we get the general solution of (4):

$$y(t) = \left(t^{\beta} + C\right)e^{t^{-\alpha}}. \tag{4.4}$$

The general solution of equation (4) is known as:

$$y(t) = Ce^{t^{-\alpha}} + \beta e^{t^{-\alpha}}\int t^{\beta-1}e^{t^{-\alpha}+t^{-\beta}}dt. \tag{4.5}$$

It is easy to check the advantages of (5) over (6), apart from the fact that the latter's integral does not seem very easy to calculate, even for simple values of α and β.

In the Second Lyapunov Method, the Chain Rule is vital to calculate the total derivative of the Lyapunov Function. Using classical fractional derivatives, this is a problem that is not solved (see [11]), however, using our Theorem 7 it is easy to verify that difficulty is overcome, in a future work we will present concrete results in this direction. Nevertheless, we want to advance something in this direction, be it the Generalized Liénard System:

$$N_1^{\alpha}x(t) = y(t) - F(x(t)),$$
$$N_1^{\alpha}y(t) = -g(x(t)), \tag{4.6}$$

as a natural generalization of the classical Liénard system, with $F(x) = \int_0^x f(r)dr$, and f and g are continuous functions such that $f : \mathbb{R} \to \mathbb{R}_+$, and $g : \mathbb{R} \to \mathbb{R}$ with $xg(x) > 0$ for $x \neq 0$. The system (4) is equivalent to the equation $N_1^{2\alpha}x(t) + N_1^{\alpha}\left[F(x(t))\right] + g(x(t))$. We consider the following Lyapunov Function

$$V(x, y) = G(x) + \frac{y^2}{2}. \tag{4.7}$$

With $G(x) = \int_0^x g(s)ds$. We calculate the fractional derivative of (8) along the system (7):

$$N_1^{\alpha}V(x(t), y(t)) = N_1^{\alpha}\left[G(x(t))\right] + N_1^{\alpha}\left[\frac{y^2(t)}{2}\right].$$
$$N_1^{\alpha}V(x(t), y(t)) = g(x(t))N_1^{\alpha}x(t) + y(t)N_1^{\alpha}y(t).$$

From this we have

$$N_1^{\alpha}V(x(t), y(t)) = -g(x(t))F(x(t)).$$

Under conditions previously imposed on f and g, we have that V is a positive definite function and its derivative throughout the system (7) is non-positive, from this we have the stability according to Lyapunov of the trivial solution of the system (7).

Finally, we would like to point out that a limitation of our definition is that it assumes that the variable $t > 0$. Thus, the following open problem arises naturally: *if this condition can be overcome for some kinds of functions and if so, what are these functions?*

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Guillermo Langton, Luciano M. Lugo Motta Bittencurt
UNNE, FaCENA, Av. Libertad 5450, (3400) Corrientes, Argentina

Paulo M. Guzmán, Julian Medina
UNNE, Fac. de Ingeniería, Av. Las Heras 727, (3500) Resistencia, Chaco, Argentina

Juan E. Nápoles Valdes
UNNE, FaCENA, Av. Libertad 5450, (3400) Corrientes, Argentina; UTN, FRRE, French 414, (3500) Resistencia, Chaco, Argentina
E-mail address: jnapoles@exa.unne.edu.ar; jnapoles@frre.utn.edu.ar