Path Integral Framework for Characterizing and Controlling Decoherence Induced by Nonstationary Environments on a Quantum Probe

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Reliable processing of quantum information is a milestone to achieve for the deployment of quantum technologies. Uncontrolled, out-of-equilibrium sources of decoherence need to be characterized in detail for designing the control of quantum devices to mitigate the loss of quantum information. However, quantum sensing of such environments is still a challenge due to their nonstationary nature that in general can generate complex high-order correlations. We introduce a path integral framework to characterize nonstationary environmental fluctuations by a quantum probe. We find the solution for the decoherence decay of nonstationary, generalized Gaussian processes that induce pure dephasing. This dephasing when expressed in a suitable basis, based on the nonstationary noise eigenmodes, is defined by the overlap of a generalized noise spectral density and a filter function that depends on the control fields. This result thus extends the validity of the similar general expression for the dephasing of open quantum systems coupled to stationary noise processes to out-of-equilibrium environments. We show physical insights for a broad subclass of nonstationary noise processes that are local in time, in the sense that the noise correlation functions contain memory based on constraints of the derivatives of the fluctuating noise paths. Spectral and non-Markovian properties are discussed together with implementations of the framework to treat paradigmatic environments that are out of equilibrium, e.g., due to a quench and a pulsed noise. We show that our results provide tools for probing the spectral and time-correlation properties, and for mitigating decoherence effects of out-of-equilibrium—nonstationary—environments.

I. INTRODUCTION

The progress on controlling quantum systems has lead to the development of quantum technologies [1–4]. In order to deploy these technologies quantum information needs to be reliably processed. However, both the storage and processing of quantum information in quantum devices suffer from decoherence, the loss of quantum information as a function of time, that distorts the encoded information [5–16]. Uncontrolled sources of decoherence are ubiquitously present in the environment of a quantum system, and methods for mitigating their effects have been extensively explored [12,17–27]. There is no universal optimal solution for protecting against decoherence, as detailed control to the system has to be tailored based on the specific

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noise source [15,27–39]. A key required input for finding the optimal strategies for decoupling the environmental effects is detailed knowledge of the noise spectral properties [35,40–44]. Most of the approaches for controlling and characterizing decoherence effects have been developed for stationary noise fluctuations [19,20,28,29,32,35, 40,45–48]. Developing methods for controlling and characterizing nonstationary environmental fluctuations is a prerequisite to exploit the full extent of quantum technologies at atomic and nanoscales, where the environmental systems are intrinsically of the many-body type and are out of equilibrium [13,16,49–64].

Dynamical decoupling noise spectroscopy is a promising tool for characterizing fluctuating environments [35,36,40,45,65]. It is based on applying time-dependent control pulses to probe the noise spectral properties. Several quantum sensing methods have been designed to reconstruct the noise spectrum generated by semi-classical and quantum fluctuating sources [35,37,42,62, 66–71]. However, it remains open how to interpret the extracted noise spectrum probed by a quantum sensor when it is coupled to a complex and unknown environment

[44,61,72,73]. More importantly, there are no universal methods for determining the properties of the natural, out-of-equilibrium environments that lead to nonstationary, non-Markovian noise fluctuation processes [49,51,55,61, 63,64,67,74].

To deal with these outstanding problems, we here implement a path integral framework to describe the decoherence process on a controlled 1/2-spin, quantum-probe coupled to nonstationary noise sources. We set the path integral framework for the most general pure dephasing coupling with fluctuating fields that can be described by nonstationary, generalized Gaussian processes, and find the solution for the decoherence decay. Our results generalize the universal formula [19,20] for the dephasing decay that depends only on the overlap of a noise spectral density and a filter function that depends on the control fields. Specifically, we demonstrate its extension to the case of nonstationary Gaussian environments.

This generalization allows us to implement two important applications for the deployment of quantum technologies, dynamical decoupling noise spectroscopy for quantum sensing operations and mitigating by control methods the decoherence effects of out-of-equilibrium environments. Moreover, we provide simple interpretations of nonstationary noise spectrums and time-dependent correlations for a broad subclass of local-in-time noise processes and simple criteria to distinguish Markovian from non-Markovian noise dynamics. We also show how to implement our framework with two paradigmatic nonstationary noise processes, one derived from a quenched environment where excitations suddenly start spreading over a large number of degrees of freedom [13,50,56,64,74] and the other from noise that acts near a point in time [75–78]. Overall we introduce a tool to characterize and control decoherence effects of out-of-equilibrium environments, providing avenues of quantum information processing for the deployment of quantum technologies.

Our manuscript is organized as follows. In Sec. II we describe the quantum-probe interacting with fluctuating fields. In Sec. III we introduce the path integral framework to determine the qubit-probe dephasing induced by a general, nonstationary Gaussian noise in the time and frequency domains. In Sec. IV we define a generalized noise spectral density and control filter function in a suitable nonstationary, noise eigenmode basis, and show that in this basis the dephasing is given by the overlap between them. Based on this noise eigenmode basis, we show how to implement (i) quantum sensing of the generalized noise spectrum and (ii) optimal dynamical decoupling sequences to protect the quantum system against decoherence. In Sec. V we consider a broad subclass of nonstationary Gaussian noise processes that are local in time to provide more direct physical meanings of parameters that characterize these nonstationary noise processes within this path integral framework. In Sec. VI we apply the presented tools to two paradigmatic out-of-equilibrium environments: noise acting near a point in time by an analogy to a quantum harmonic oscillator and a quench on the environment that suddenly starts an Ornstein-Uhlenbeck diffusion process. Lastly, in Sec. VII we conclude with some remarks.

II. QUBIT-PROBE INTERACTING WITH FLUCTUATING FIELDS

We consider a qubit system with spin $S = 1/2$ as a quantum probe experiencing pure dephasing due to the interaction with its environment (bath) [27,79]. In the system rotating frame of reference, the Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}_{SB} + \mathcal{H}_B, \quad \mathcal{H}_{SB} = S_z \mathbf{g} \cdot \mathbf{B}, \tag{1}$$

where $\mathcal{H}_B$ is the bath Hamiltonian and $\mathcal{H}_{SB}$ is a general pure dephasing system-bath interaction Hamiltonian. The spin operator S_z in the z axis is the qubit-probe operator, the bath operators are represented by an n-dimensional vector $\mathbf{B} = [B_1, \ldots, B_n]$, and $\mathbf{g} = [g_1, \ldots, g_n]$ are the system-bath coupling strengths. Note that index i of components $g_i B_i$ of the system-bath interaction labels properties of the qubit-bath coupling network morphology. For example, it can label different components of a hyperfine interaction tensor [80,81], different spins on the environment [31,34,82], or spatial directions as in the case of anisotropic molecular diffusion [83–85]. Figure 1(a) shows a schematic representation of this interaction. This type of interaction is encountered in a wide range of systems, as for example nuclear spin systems in liquid and solid state NMR [31,34, 35,86,87], dephasing induced by spin bearing molecules in liquid and gas state NMR [84,88–92], the hyperfine interaction of electron spins in diamonds [36,80,81,93,94], electron spins in quantum dots [95–98], donors in silicon [99], electron spins in nanoscale nuclear spin baths [54], superconducting qubits [40], trapped ultracold atoms [45,100], etc.

By using an interaction representation with respect to the evolution of the isolated environment, we eliminate the bath Hamiltonian $\mathcal{H}_B$. The system-bath Hamiltonian becomes

$$\mathcal{H}_{SB}^{(B)}(t) = S_z \mathbf{g} \cdot (e^{-i\mathcal{H}_B t} \mathbf{B} e^{i\mathcal{H}_B t}). \tag{2}$$

Since $\mathcal{H}_B$ does not commute with $\mathcal{H}_{SB}$, the effective system-bath interaction $\mathcal{H}_{SB}^{(B)}$ is time dependent and the qubit-probe experiences a fluctuating coupling to the bath. We use the semiclassical field approximation to replace the quantum environment operator $e^{-i\widehat{\mathcal{H}}_B t} \mathbf{B} e^{i\widehat{\mathcal{H}}_B t}$ with a classical stochastic function $\mathbf{B}(t)$ [Figs. 1(a) and 1(b)]. The corresponding self-correlation function is thus given by $\langle B_i(t) B_j(t') \rangle = \frac{1}{2} \text{Tr}[\rho_e \{B_i(t), B_j(t')\}]$, where $\langle \cdot \rangle$ is the stochastic mean value of the semiclassical stochastic fields.
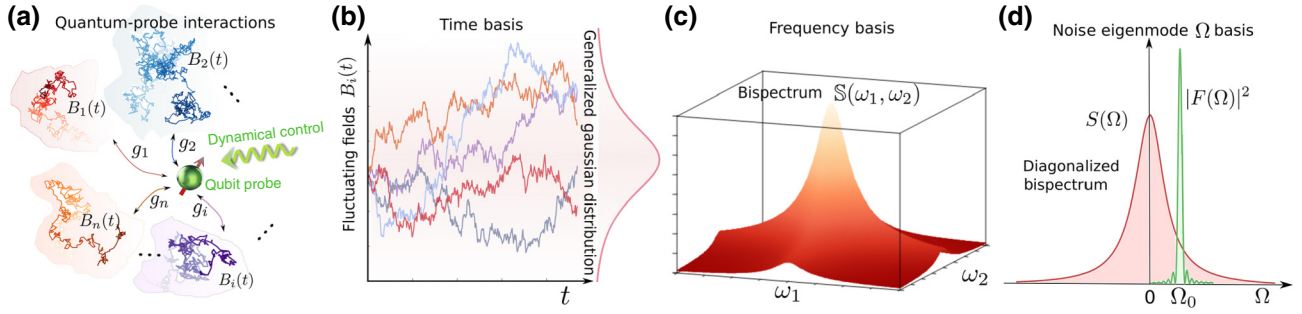
FIG. 1. Representation of the qubit-probe interaction with fluctuating fields derived from the path integral framework. (a) Schematic representation of a qubit-probe interacting with the fluctuating fields $B_i(t)$ that generate pure dephasing. The system-bath coupling strengths g_i are dynamically controlled. The index i labels properties of the qubit-bath coupling network morphology. (b) Representation of the fluctuating field paths $B_i(t)$ that give a generalized Gaussian distribution that is sensed by the quantum probe as a function of time. The fluctuating fields are described in the time basis $|t\rangle$. (c) Frequency basis $|\omega\rangle$ representation of the fluctuating field correlation functions, which provides a bidimensional, nonstationary noise spectrum $\mathbb{S}(\omega_1, \omega_2) = (\omega_1|\mathbb{G}|\omega_2)$. (d) Noise eigenmode Ω-basis representation of the nonstationary, generalized noise spectrum $S(\Omega)$ (red solid line), obtained after diagonalization of the noise spectrum in the frequency basis. A generalized filter function $F(\Omega)$ in the noise eigenmode Ω basis is shown with a green solid line. Here, it mainly probes a single noise eigenmode Ω_0.

The right-hand side of the equation is the quantum correlation function of the bath operators $B_{i/j}$, determined by the trace Tr over the bath degrees of freedom with ρ_e the density matrix of the environment and $\{\cdot\}$ denoting the anticommutator. This approximation, typically called the weak coupling approximation, represents well a time-dependent quantum correlation of the SB interaction up to second order [19,20,37,67,79,101,102] that properly describes a wide range of experimental setups [3,31,35,36,40,42,54, 62,65,67–69].

The evolution operator of a dynamically controlled qubit probe combined with this fluctuating field interaction is $e^{-i\phi[\mathbf{B},\mathbf{f}]S_z}$, where $\phi[\mathbf{B},\mathbf{f}]$ is the accumulated phase by the probe during the evolution from an initial time t_0 to a final time t_f, starting from an initial state polarized in the X-Y plane. Specifically,

$$\phi[\mathbf{B},\mathbf{f}] = \int dt \sum_{i=1}^{n} f_i(t)B_i(t), \qquad (3)$$

where $\int dt \equiv \int_{-\infty}^{+\infty} dt$ and the vector $\mathbf{f}(t) = [f_1(t), \ldots, f_n(t)]$ is a controlled modulation of each bath interaction term driven by dynamical control techniques during $t \in [t_0, t_f]$, being null outside this time interval [17–20,27,28]. For example, for a free evolution $t \in [t_0, t_f]$, $f_i(t) = g_i \Theta(t - t_0)\Theta(t_f - t)$, where Θ is the Heaviside step function, while for a dynamical decoupling sequence of π pulses applied to the qubit, $f_i(t)$ switches between $\pm g_i$ at the position of every pulse [27,28]. Note that we explicitly write the dependence on $\mathbf{B}$ and $\mathbf{f}$ of the qubit-probe phase, to highlight dependence of the phase on the stochastic fluctuating field that we then describe with the path integral approach and the control fields, respectively.

In this article we consider the general pure dephasing system-bath interaction Hamiltonian of Eq. (1). However, the path integral framework discussed below to describe nonstationary, general Gaussian noise that induces dephasing by this interaction can be extended to a dissipation interaction based on replacing the spin-probe operator S_z by S_x or S_y. For this dissipation interaction, the phase of Eq. (3) represents a population phase of a coherence exchange [19,20,102]. This adaptation describes interactions found, for example, in chemical quantum solvation dynamics with a time-dependent, structurally changing solvent [55,103,104] and nonequilibrium response of nanosystems and atomic systems coupled to driven Caldeira-Leggett baths [51,105,106].

III. PATH INTEGRAL FRAMEWORK FOR THE QUBIT-PROBE DEPHASING

In this section, we introduce the general path integral framework for describing the decoherence effects on the qubit probe, induced by a nonstationary, general Gaussian noise that describes an out-of-equilibrium environment. Under this framework, we define the generalized noise spectrum for nonstationary environments in terms of the inverse of the kernel operator that determines the probability of the noise field paths. We then show the implementation of this framework to stationary, general Gaussian noise processes, and recover the known expression for the induced dephasing.

A. Dephasing induced by nonstationary, general Gaussian noise

We use path integrals to calculate the decoherence effects on the qubit probe, induced by the fluctuating field

$\mathbf{B}(t)$. The notation $\phi[\mathbf{B}, \mathbf{f}]$ of the accumulated phase in Eq. (3) shows the functional dependence of ϕ with the evolution path followed by the fluctuating field $\mathbf{B}(t)$, where the qubit probe is driven by the control fields $\mathbf{f}(t)$. Since the evolved phase involves real square-integrable functions that conform a Hilbert space, we use a bra-ket notation to simplify the calculations and interpretations. A function $f(t)$ is associated with the ket $|f\rangle$, and similarly the multi-dimensional function $\mathbf{f}(t)$ to the ket $|\mathbf{f}\rangle$. The inner product of this space is given by

$$(\mathbf{f}|\mathbf{B}) \equiv \int dt \sum_{i=1}^{n} f_i(t) B_i(t). \quad (4)$$

We also introduce the time basis $\{|t\rangle : t \in \mathbb{R}\}$ of the Hilbert space of scalar functions of time such that $(t|f) = f(t)$ is a scalar function over the real numbers $\mathbb{R}$. Therefore, $(t|\mathbf{f}) = [f_1(t), \dots, f_n(t)]$ is a vector function in $\mathbb{R}^n$. This is analogous to the position basis $\{|\mathbf{x}\rangle : \mathbf{x} \in \mathbb{R}^n\}$ in quantum mechanics. Then the qubit-probe phase of Eq. (3) becomes

$$\phi[\mathbf{B}, \mathbf{f}] = (\mathbf{f}|\mathbf{B}), \quad (5)$$

considering that $\mathbf{f}(t) = (t|\mathbf{f})$ and $\mathbf{B}(t) = (t|\mathbf{B})$.

The qubit dephasing decay is given by the ensemble average $\langle e^{i\phi[\mathbf{B}, \mathbf{f}]} \rangle$ taken over all possible field path fluctuations of $\mathbf{B}$. Figure 1(b) shows a schematic representation of the paths of the field fluctuations. Based on path integral theory, this ensemble average is

$$\langle e^{i\phi[\mathbf{B}, \mathbf{f}]} \rangle = \int \mathcal{D}\mathbf{B} \exp\{-\mathcal{A}[\mathbf{B}]\} \exp\{-i(\mathbf{f}|\mathbf{B})\}, \quad (6)$$

where $\int \mathcal{D}\mathbf{B}$ is a functional integral with the probability distribution of $\mathbf{B}$ given by $\exp\{-\mathcal{A}[\mathbf{B}]\}$, and $\mathcal{A}[\mathbf{B}]$ is the action, in analogy with quantum path integrals in imaginary time [107].

We consider the most general nonstationary Gaussian noise process, for which the action is

$$\mathcal{A}[\mathbf{B}] = \frac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}) = \frac{1}{2} \int dt \int dt' \, \mathbf{B}^\dagger(t)\mathbb{D}(t, t')\mathbf{B}(t'), \quad (7)$$

where "$\dagger$" denotes Hermitian conjugation and $\mathbb{D}(t, t') = (t|\mathbb{D}|t')$ is a $n \times n$ matrix-valued function of t and t' that gives the kernel of the operator $\mathbb{D}$ in terms of times. The kernel operator $\mathbb{D}$ must be *positive definite* and *real* for the path integral to be well defined, and for the action to define a probability density $\exp\{-\mathcal{A}[\mathbf{B}]\}$. We assume that $\mathbb{D}$ is *Hermitian* without loss of generality, such that $\mathbb{D}(t, t') = \mathbb{D}^\dagger(t', t)$ (see Appendix A).

The dephasing decay of the qubit probe can be solved exactly using path integrals for this general nonstationary

Gaussian noise process as [107]

$$\langle e^{i\phi[\mathbf{B}, \mathbf{f}]} \rangle = \exp\left\{-\frac{1}{2}(\mathbf{f}|\mathbb{G}|\mathbf{f})\right\}$$
$$= \exp\left\{-\frac{1}{2} \int dt \int dt' \, \mathbf{f}^\dagger(t)\mathbb{G}(t, t')\mathbf{f}(t')\right\}, \quad (8)$$

where the operator $\mathbb{G}$ defines the self-correlation functions of the stochastic process

$$(t|\mathbb{G}|t')_{ij} = \mathbb{G}_{ij}(t, t') = \langle B_i(t) B_j(t') \rangle. \quad (9)$$

The operator $\mathbb{G}$ is defined by the inverse of the kernel operator

$$\mathbb{G} = \mathbb{D}^{-1}, \quad (10)$$

or, equivalently, by the expression

$$(t_1|\mathbb{D}\,\mathbb{G}|t_2) = \int dt\, \mathbb{D}(t_1, t)\mathbb{G}(t, t_2) = \mathbb{I}\delta(t_1 - t_2), \quad (11)$$

where $\delta(t)$ is the Dirac delta and $\mathbb{I}$ is an identity matrix of dimension $n \times n$.

Using the bra-ket notation, one can also solve Eq. (8) in the frequency basis of scalar functions $\{|\omega) = \int (dt/\sqrt{2\pi}) e^{-i\omega t} |t\rangle : \omega \in \mathbb{R}\}$, rather than performing the calculation using the time basis $\{|t\rangle : t \in \mathbb{R}\}$. This frequency basis yields the Fourier representation of the ensemble average of the qubit-probe signal:

$$\langle e^{i\phi[\mathbf{B}, \mathbf{f}]} \rangle = \exp\left\{-\frac{1}{2}(\mathbf{f}|\mathbb{G}|\mathbf{f})\right\}$$
$$= \exp\left\{-\frac{1}{2} \int d\omega_1 \int d\omega_2 \, \mathbf{F}^\dagger(\omega_1)\mathbb{S}(\omega_1, \omega_2)\mathbf{F}(\omega_2)\right\}. \quad (12)$$

Here $\mathbb{S}(\omega_1, \omega_2) = (\omega_1|\mathbb{G}|\omega_2)$ defines a spectral density of two frequency dimensions equivalent to the double Fourier transform of the correlation functions $\mathbb{G}(t, t')$ [Fig. 1(c)], and $\mathbf{F}(\omega) = (\omega|\mathbf{f}) = [F_1(\omega), \dots, F_n(\omega)]$ is a filter function defined by the Fourier transform of

$$\mathbf{f}(t) = \int \frac{d\omega}{\sqrt{2\pi}} \mathbf{F}(\omega) e^{i\omega t}. \quad (13)$$

Equations (8) and (12) are the qubit dephasing solutions for general—nonstationary—Gaussian noise processes derived from the presented path integral framework. Using the frequency basis, Eq. (10) becomes

$$(\omega_1|\mathbb{D}\,\mathbb{G}|\omega_2) = \int d\omega\, \mathbb{D}(\omega_1, \omega)\mathbb{S}(\omega, \omega_2) = \mathbb{I}\delta(\omega_1 - \omega_2), \quad (14)$$

where $\mathbb{D}(\omega, \omega') = (\omega|\mathbb{D}|\omega')$. Therefore, this shows that the noise generated by the nonstationary environment is

completely determined by the bispectrum $\mathbb{S}(\omega_1, \omega_2)$ that is defined by the inverse of the kernel operator $\mathbb{D}$ that describes the probability density of the field paths. Similar bispectra, functions of two frequencies, were shown to be useful tools to characterize *stationary, non-Gaussian* noise processes [42,67]. Here, we show that they are also useful tools to characterize *nonstationary, Gaussian* noise processes.

B. Dephasing induced by stationary, general Gaussian noise

For stationary noise, the kernel $\mathbb{D}(t, t')$ and the correlation functions $\mathbb{G}(t, t')$ are invariant under a time translation; therefore, they depend only on the time difference $t - t'$. We can consider then that $\mathbb{D}(t, t') = \mathbb{D}(t - t')$ and $\mathbb{G}(t, t') = \mathbb{G}(t - t')$, and that the two-dimensional Fourier transform of $\mathbb{G}(t, t')$ is then block diagonal in the frequency basis $(\omega|\mathbb{G}|\omega') = \mathbb{S}(\omega)\delta(\omega - \omega')$. Similarly, the Fourier transform of $\mathbb{D}(t, t')$ is $(\omega|\mathbb{D}|\omega') = \mathbb{D}(\omega)\delta(\omega - \omega')$. Equation (12) then becomes

$$\langle e^{i\phi[\mathbf{B},\mathbf{f}]} \rangle = \exp\left[-\frac{1}{2} \int d\omega\, \mathbf{F}^{\dagger}(\omega)\mathbb{S}(\omega)\mathbf{F}(\omega) \right]. \quad (15)$$

According to Eq. (10), the inverse of the kernel operator that describes the probability density of Eq. (7) defines the noise spectrum for stationary noise

$$\mathbb{S}(\omega) = \mathbb{D}(\omega)^{-1} \quad (16)$$

when expressed in the frequency basis. Thus, the noise spectrum of stationary noise processes is further simplified with respect to Eq. (14) to the multiplicative inverse of the operator $\mathbb{D}(\omega)$ in this basis, i.e., it is the inverse for each frequency mode.

Equation (15) expresses the dephasing decay induced by a stationary noise in terms of an integral over only one frequency, in contrast with Eq. (12) that integrates an overlap between the filter function and the spectral density of a nonstationary noise over two different frequencies. As a result, we here recover the universal formula for the qubit dephasing under the weak coupling approximation mainly considered for the $n = 1$ case, i.e., when the dephasing is given by $\langle e^{i\phi[\mathbf{B},\mathbf{f}]} \rangle = \exp[-\frac{1}{2} \int d\omega\, S(\omega)|F(\omega)|^2]$ [19,20,29, 30,32,34,108,109]. This formula for the dephasing is typically used for designing optimal control [20,28,34] and for dynamical decoupling noise spectroscopy [35,36,40,41,45, 65,110,111].

IV. NOISE SPECTRAL DENSITY ON THE NOISE EIGENMODE BASIS

In this section, we first introduce the noise eigenmode basis that defines the proper basis for nonstationary noise processes, to generalize the universal formula for the

dephasing of open quantum systems that depends on the overlap between a noise spectral density and a qubit-control filter function. We then show how stationary noise processes are described in this noise eigenmode basis. Based on the eigenmode representation, we discuss two important applications of this dephasing generalization to nonstationary environments: we show how to implement dynamical decoupling noise spectroscopy for quantum sensing, and optimized control methods to mitigate decoherence effects.

A. Nonstationary noise modes

The kernel operator $\mathbb{D}$ is in general *nonstationary*, and therefore the frequency basis does not *diagonalize it*, leading to the bispectrum $\mathbb{S}(\omega_1, \omega_2)$ of Eq. (12). For this reason, in general the elements of the frequency basis $\{|\omega\rangle : \omega \in \mathbb{R}\}$ are not the *eigenfunctions* of $\mathbb{D}$ for nonstationary noise processes. As the kernel operator $\mathbb{D}$ is Hermitian, we can introduce the basis $\{|\Omega\rangle : \Omega \in I\}$ that diagonalizes it,

$$\mathbb{D}|\Omega\rangle = D(\Omega)|\Omega\rangle, \quad (17)$$

where I is a set that indexes the elements of the eigenmode basis and $D(\Omega) \in \mathbb{R}_{>0}$ are the corresponding eigenvalues. This basis is normalized to satisfy the completeness relation $\mathbb{I} = \int d\Omega\, |\Omega\rangle(\Omega|$, where $\int d\Omega = \int_I d\Omega$ represents an integral when the spectrum is continuous or a sum when it is discrete. Note that $|\Omega\rangle$ are *vectors* in the same space as $|\mathbf{B}\rangle$, and they represent the nonstationary modes of the noise fluctuations $(t|\mathbf{B}) = \mathbf{B}(t) = \int d\Omega\, b_\Omega (t|\Omega)$, where $(t|\Omega)$ is a vector-valued time-dependent function and $b_\Omega = (\Omega|\mathbf{B})$ are scalars.

The correlation operator $\mathbb{G}$ is then diagonal in this basis $\mathbb{G}|\Omega\rangle = [D(\Omega)]^{-1}|\Omega\rangle$, as $\mathbb{I}|\Omega\rangle = \mathbb{G}\mathbb{D}|\Omega\rangle = \mathbb{G}|\Omega\rangle D(\Omega)$ following Eq. (10) and considering that $\{|\Omega\rangle : \Omega \in I\}$ is a basis. As a result, we can now define a *generalized noise spectrum* for *nonstationary* Gaussian noise processes as the eigenvalues of the correlation operator

$$S(\Omega) = [D(\Omega)]^{-1}, \quad (18)$$

where $\mathbb{G}|\Omega\rangle = S(\Omega)|\Omega\rangle$. The attenuation argument of the dephasing decay of Eq. (8) is then

$$(\mathbf{f}|\mathbb{G}|\mathbf{f}) = \int d\Omega\, S(\Omega)|(\Omega|\mathbf{f})|^2 \quad (19)$$

in this noise eigenmode basis. This Ω basis then provides a natural way to define a *generalized filter function for these nonstationary noise processes* as $F(\Omega) = (\Omega|\mathbf{f})$, to obtain

$$\langle e^{i\phi[\mathbf{B},\mathbf{f}]} \rangle = \exp\left[-\frac{1}{2} \int d\Omega\, S(\Omega)|F(\Omega)|^2 \right]. \quad (20)$$

The *generalized noise spectrum* $S(\Omega)$ for a *nonstationary Gaussian noise* is thus the *multiplicative inverse* of the *eigenvalues of the kernel operator* $\mathbb{D}$, indexed by the noise eigenmode parameter Ω.

Note that, as Ω is a parameter, in general its physical meaning depends specifically on the nonstationary process, and on the parametrization of the functional behavior $S(\Omega)$, as it is always possible to reparametrize the eigenmodes by a different parameter. Conversely, the eigenvalues of the operator $\mathbb{D}$ and their corresponding noise eigenspaces are always independent of the parametrization. By identifying the basis that diagonalizes the kernel operator $\mathbb{D}$, we map the bidimensional spectral density that depends on two frequencies [Fig. 1(c)] into a monoparametrical spectral density based on the nonstationary noise eigenmodes [Fig. 1(d)]. This sets one of the main results of this article, where Eq. (20) is a generalization *for the qubit-probe dephasing* for *nonstationary Gaussian noise* that is only determined by the overlap between the generalized spectral density of the environment $S(\Omega)$ and the generalized filter function $|F(\Omega)|^2$ determined by the control on the qubit probe. This result thus extends the validity of the universal formula for stationary pure dephasing noise processes of open quantum systems within the weak coupling approximation to nonstationary noise processes [19,20,29,30,32,34,35,108,109].

B. Stationary noise modes

In the stationary noise case, according to Eq. (15), the frequency dimension is separable from the morphological dimension of the qubit-bath coupling network represented by the vector indices i of $\mathbf{B}$ and $\mathbf{F}$ in Eq. (9), where we obtain $\mathbb{D}|\omega) = \mathbb{D}(\omega)|\omega)$. The noise eigenmodes in the stationary case are defined by the eigenvectors $\mathbf{b}_m(\omega) \in \mathbb{R}^n$ of $\mathbb{D}(\omega)$, with $m = 1, \ldots, n$, such that $\mathbb{D}(\omega)\mathbf{b}_m(\omega) = D_m(\omega)\mathbf{b}_m(\omega)$ with $D_m(\omega) \in \mathbb{R}_{>0}$. Therefore, the eigenmodes of the kernel operator are $|\omega, m) = \mathbf{b}_m(\omega)|\omega)$, and it acts on them as $\mathbb{D}|\omega, m) = D_m(\omega)|\omega, m)$. We consider the eigenmode basis index as $\Omega = (\omega, m)$ and use the notation $D(\Omega) = D_m(\omega)$. The dephasing of Eq. (20) then becomes

$$\langle e^{i\phi[\mathbf{B}, \mathbf{f}]} \rangle = \exp\left[-\frac{1}{2} \sum_m \int d\omega \, S_m(\omega)|F_m(\omega)|^2 \right], \quad (21)$$

where $S_m(\omega) = D_m^{-1}(\omega)$ are the eigenspectrums of the bath fluctuations on the qubit probe. The index m for example defines the principal axes of anisotropic diffusion tensors in magnetic resonance imaging [83–85].

While the choice of the frequency ω as the relevant parameter to describe the noise eigenmodes for the stationary case seems natural, its selection is also arbitrary as the choice of Ω in general. For example, one can keep the diagonal form of Eq. (21) but use, instead of the angular frequency ω, the frequency $\nu = \omega/(2\pi)$. One can also

use a different reparamerization like $\lambda = \omega^{1/3}$, or replace the complex exponentials with trigonometric functions and describe the noise eigenmodes with two parameters (E, π), with $E = \omega^2$ and π the parity of the trigonometric functions (see Appendix B).

C. Dynamical decoupling noise spectroscopy of nonstationary environments

An important application of the presented framework, which leads to the generalized picture for the qubit-probe dephasing based on Eqs. (12) and (20), is that it can be used to probe the noise spectral properties for nonstationary, general Gaussian noise processes. Several methods have been designed to probe the noise spectrum of stationary noise processes based on Eq. (15), mainly for the $n = 1$ case, i.e., when the dephasing is given by $\langle e^{i\phi[\mathbf{B}, \mathbf{f}]} \rangle = \exp[-\frac{1}{2}\int d\omega \, S(\omega)|F(\omega)|^2]$ [35,36,40, 41,45,65,110,111]. Control sequences generate filter functions that can allow only specific frequency components of the spectral density to produce dephasing on the qubit-probe system. The width of these "pass band" filters can be made arbitrarily narrow. Therefore, the spectral density can be reconstructed by performing a series of measurements with different filter functions to scan the noise spectrum. This procedure, termed dynamical decoupling noise spectroscopy [35], can be performed either by using continuous fields [45,110,111] or sequences of pulses [35,40,65].

The noise eigenmodes $|\Omega)$ that diagonalize the kernel operator $\mathbb{D}$ are thus the natural basis to probe the nonstationary noise spectrum based on Eq. (20). In the simplest case of stationary noise, this basis becomes $|\omega, m)$, and therefore, using a modulating function $f_m(t) = g_m e^{-i\omega_0 t} \propto (t|\omega_0)$ driven by control fields, one can probe the frequency modes of the noise spectrum for each morphological eigenmode m by scanning ω_0. For such control modulations, the filter function $|F_m(\omega)|^2 \propto \delta(\omega - \omega_0)$ senses single-frequency modes according to Eq. (21), and leads to the so-called continuous-wave noise spectroscopy [45,110–114]. However, for nonstationary noise processes, the frequency basis in general cannot probe selectively the noise eigenmodes. In this case, the eigenvectors $|\Omega)$ define the proper basis to probe the single noise eigenmodes according to Eq. (17). Designing filter functions $F(\Omega)$ such that $|F(\Omega)|^2 \propto \delta(\Omega - \Omega_0)$ [Fig. 1(d)] based on finding the control modulation functions that satisfy $\mathbf{f}_{\Omega_0}(t) = (t|\Omega_0)$, one can probe the noise eigenspectrum by scanning it by changing Ω_0 based on Eq. (20) as $(\mathbf{f}|\mathbb{G}|\mathbf{f}) \propto S(\Omega_0)$. Therefore, if the Ω basis is known, dynamical decoupling noise spectroscopy approaches designed to scan the spectral density in the frequency domain for stationary noise processes [35,40,45] can now be adapted to probe nonstationary noise processes using the Ω domain.

Alternatively, if information about the Ω basis is not known, dynamical decoupling noise spectroscopy to scan

multifrequency spectral densities can be implemented based on the general form derived in Eq. (12). Again, qubit noise spectroscopy control methods for estimating high-order noise spectra (so-called polyspectra) have been developed for stationary non-Gaussian noise processes [42,67]. The method is based on using dynamical control approaches based on frequency comb control modulations to design multidimensional filter functions to probe the polyspectra via repetition of suitable pulse sequences. Based on Eq. (12), this technique can now be straightforwardly adapted to estimate the nonstationary bispectrum $\mathbb{S}(\omega_1, \omega_2)$ that induces the qubit-probe dephasing. Then, once the bispectrum is estimated, it can then be diagonalized to determine its eigenmode basis $|\Omega)$ and eigenspectrum $S(\Omega)$.

D. Mitigating decoherence effects of nonstationary noise processes

The reduction of decoherence effects on a qubit probe coupled to nonstationary—out-of-equilibrium—noise processes is another important application of the presented framework. The key result for attaining this goal is the extension of the universal expression for the qubit dephasing induced by nonstationary, general Gaussian noise processes based on the overlap between a noise spectral density and a control filter function as demonstrated with Eq. (20). This extension allows implementing optimal control methods that were developed for mitigating decoherence induced by stationary noise processes on open quantum systems. These methods are based on finding an optimal control filter function $\mathbf{F}_{\text{opt}}(\omega) = (\omega|\mathbf{f}_{\text{opt}})$ that minimizes the overlap between the noise spectral density and the control filter function in Eq. (15), and then implementing it experimentally in the time basis as $\mathbf{f}_{\text{opt}}(t) = (t|\mathbf{f}_{\text{opt}})$ [12,29,30,32].

These control strategies can now be applied to nonstationary noise processes by designing an optimal filter $F_{\text{opt}}(\Omega)$ that, analogously to the stationary case, minimizes the overlap with the nonstationary Gaussian noise spectrum $S(\Omega)$ in Eq. (20). This is only possible due to the introduction of the generalized nonstationary Gaussian noise spectrum based on determining the noise eigenmodes $\mathbb{G}|\Omega) = S(\Omega)|\Omega)$, and is one of the main results of this paper. As described in Sec. IV C, $S(\Omega)$ can be inferred so as to determine the proper modulation control $|\mathbf{f}_{\text{opt}})$ that provides the optimal filter $(\Omega|\mathbf{f}_{\text{opt}}) = F_{\text{opt}}(\Omega)$. This control can then be expressed in the time basis as $\mathbf{f}_{\text{opt}}(t) = (t|\mathbf{f}_{\text{opt}})$ for its experimental implementation, as one does to decouple stationary environments. We have therefore shown here how the presented generalized path integral framework can be used for implementing dynamical control methods to mitigate decoherence effects induced by nonstationary, general Gaussian noise processes.

V. NONSTATIONARY NOISE PROCESSES LOCAL IN TIME

In this section we consider a broad subclass of nonstationary Gaussian noise processes that are local in time, as they allow more direct interpretations of the physical meaning that provides the path integral approach for the noise processes. We show that the dephasing for this type of nonstationary noise processes can be described by a differential operator based on constraints to the derivatives of the fluctuating field paths. We also show how these constraints are reflected on the functional behavior of the nonstationary noise spectrums. Therefore, the qubit-probe dephasing can be obtained by solving ordinary differential equations rather than integral equations, as is the case for non-local-in-time noise processes. This picture allows us to obtain a simpler expression for the noise spectrum of local-in-time, stationary noise processes as the inverse of the differential operator. In this case the differential operator is determined by a matrix polynomial of the frequency modes. We also show how this description for nonstationary, local-in-time noise processes provides conditions for differentiating Markovian noise processes from non-Markovian noise processes. Moreover, we introduce a generalized Markovian process that includes all the derivatives of **B** in the stochastic process that fully describes local-in-time nonstationary processes. The state of **B** is not only determined by the *probability distribution* of **B**, but by the *joint probability distribution* of **B** and its derivatives. While these local-in-time noise processes can in general be non-Markovian, in our generalized framework they appear as a natural extension of Markovian noise processes, and maintain many of their properties while being able to model a greater variety of environments.

A. Local-in-time framework

We here consider a broad subclass of nonstationary Gaussian noise processes that are local in time. The kernel that defines the probability distribution for the possible paths that can take a local-in-time fluctuation $\mathbf{B}(t)$ satisfies $\mathbb{D}(t, t') = 0$ when $t \neq t'$. The most general kernel operator satisfying this condition can then be expressed as an expansion series

$$\mathbb{D}(t, t') = \sum_{k=0}^{\infty} \mathbb{C}_k(t)\delta^{(k)}(t - t'), \qquad (22)$$

where $\delta^{(k)}$ denotes the kth time derivative of the Dirac delta function and $\mathbb{C}_k(t)$ are time dependent matrices. When the sum is finite, the most general action satisfying these conditions can be written as

$$\mathcal{A}[\mathbf{B}] = \int dt \sum_{k=0}^{N} \sum_{l=0}^{k} \mathbf{B}^{\dagger(k)}(t)\mathbb{D}_{k,l}(t)\mathbf{B}^{(l)}(t), \qquad (23)$$

where $\mathbf{B}^{(k)}(t)$ is the kth time derivative of $\mathbf{B}(t)$, N is the highest derivative order that contributes to the action, and $\mathbb{D}_{k,l}(t)$ are $n \times n$ matrices. We call this kind of nonstationary noise process *local in time*. The fact that these local-in-time noise processes are described by a kernel operator $\mathbb{D}(t, t')$ that vanishes for $t \neq t'$ indicates that long-time correlations of the process do not exist, and therefore the correlation functions $\mathbb{G}_{ij}(t, t') = \langle B_i(t)B_j(t') \rangle$ decay with $|t' - t|$.

We demonstrate in Appendix C that any local-in-time process can be described by an action of the form

$$
\begin{aligned}
\mathcal{A}[\mathbf{B}] &= \frac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}) \\
&= \int dt \sum_{k=0}^{N} \mathbf{B}^{\dagger(k)}(t)\mathbb{D}_k^H(t)\mathbf{B}^{(k)}(t) \\
&\quad + \int dt \sum_{k=1}^{N} \mathbf{B}^{\dagger(k)}(t)\mathbb{D}_k^A(t)\mathbf{B}^{(k-1)}(t),
\end{aligned}
\tag{24}
$$

where $\mathbb{D}_k^H(t)$ are $n \times n$ real symmetric (Hermitian) matrices and $\mathbb{D}_k^A(t)$ are real antisymmetric (anti-Hermitian) matrices. These matrices define constraints on the values that the fluctuating field $\mathbf{B}$ and its derivatives can take. For example, the larger $|\mathbb{D}_k^H|$ is, the smaller the derivative $\mathbf{B}^{(k)}$ must be or, equivalently, the slower $\mathbf{B}^{(k-1)}$ may change. The matrix norm $|\mathbb{D}_k^H|$ thus limits how fast $\mathbf{B}^{(k-1)}$ can lose information of its previous state. In particular, for $N > 1$, information can now be stored in the derivatives of $\mathbf{B}$, and it is this information storage that leads to these processes being *non-Markovian.* Based on Eq. (22), to achieve to Eq. (24) we have introduced the differential operator

$$
\begin{aligned}
\mathbb{D}(t) &= \sum_{k=0}^{N} \overleftarrow{\partial}_t^k \mathbb{D}_k^H(t)\, \partial_t^k \\
&\quad + \frac{1}{2}\sum_{k=1}^{N}\left[\overleftarrow{\partial}_t^k \mathbb{D}_k^A(t)\partial_t^{k-1} - \overleftarrow{\partial}_t^{k-1}\mathbb{D}_k^A(t)\partial_t^k\right],
\end{aligned}
\tag{25}
$$

which gives the kernel operator for local-in-time noise processes, where $\overleftarrow{\partial}_t$ denotes leftwise differentiation, such that $f(t)\overleftarrow{\partial}_t g(t) = f'(t)g(t)$. Note that if $\mathbf{B}$ is a one-dimensional process, $\mathbb{D}_k^A = 0$ for all k, since there are no nonzero real anti-Hermitian 1×1 matrices (scalars).

The inverse relation of Eq. (11) for local-in-time processes becomes

$$
\mathbb{D}(t)\mathbb{G}(t, t') = \delta(t - t')\mathbb{I}
\tag{26}
$$

under these assumptions. This relation shows that the time-dependent operator $\mathbb{D}(t)$ can be diagonalized by simply solving a set of *ordinary differential equations*. This
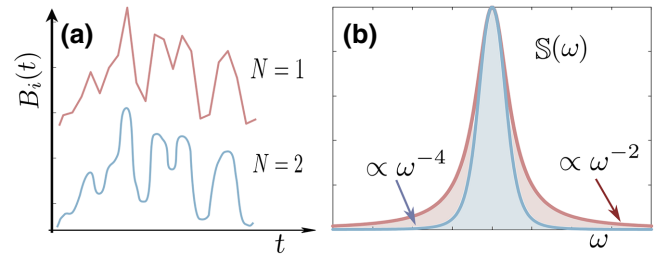

FIG. 2. Schematic comparison between a Markovian noise for $N = 1$ and a non-Markovian noise with $N = 2$ for one-dimensional, local-in-time stationary noise processes with different constraints on the differential operator $\mathbb{D}$. (a) Examples of single realizations of field fluctuations $B_i(t)$ for both cases. The differential operator has constraints up to the first derivative for $N = 1$, where the field fluctuation is continuous but not differentiable. For $N = 2$, the field fluctuation and its derivative are continuous. (b) The corresponding noise spectrums $\mathbb{S}(\omega)$. For $N = 1$, the spectral density is a Lorentzian and, for $N = 2$, the inverse of a quartic polynomial, with power-law tails $\propto \omega^{-2}$ and $\propto \omega^{-4}$ for large frequencies, respectively.

is in contrast to the case of nonlocal-in-time noise processes, where it is necessary to solve *integral equations* to obtain the noise eigenmodes. The autocorrelation functions $\mathbb{G}(t, t')$ are the Green functions of the local-in-time differential operator $\mathbb{D}(t)$. Therefore, a local-in-time noise process can be purely described by the constraints $\mathbb{D}_k^{H/A}(t)$ on the derivatives of the fluctuating field $\mathbf{B}(t)$, as seen in Eq. (24), providing more simple physical meanings for the form of the kernel operator given in Eq. (25). A direct implication of this kernel form is that it describes fluctuating field paths that are differentiable up to order $N - 1$. This means that $\mathbf{B}$ and its first $N - 1$ derivatives must be continuous functions (see Fig. 2 for an example) so that the action does not diverge and they contribute to the propagator (see Appendix D for a demonstration). Another implication of this kernel form is that its characterization only requires to estimate the matrices $\mathbb{D}_k^{H/A}(t)$, which are $2N + 1$ functions of $\mathbb{R} \to \mathbb{R}^{n \times n}$. Conversely, a general non-local-in-time process requires estimating the kernel $\mathbb{D}(t_1, t_2)$, which is a function $\mathbb{R}^2 \to \mathbb{R}^{n \times n}$.

The kernel operator of these local-in-time fluctuating field processes is described in the frequency basis as

$$
\begin{aligned}
\mathbb{D}(\omega_1, \omega_2) &= (\omega_1|\mathbb{D}|\omega_2) \\
&= \sum_{k=0}^{N} \omega_1^k \omega_2^k \mathbb{D}_k^H(\omega_1 - \omega_2) \\
&\quad + \frac{1}{2}\sum_{k=1}^{N} i(\omega_1^k \omega_2^{k-1} + \omega_1^{k-1}\omega_2^k)\mathbb{D}_k^A(\omega_1 - \omega_2),
\end{aligned}
\tag{27}
$$

where $\mathbb{D}_k^{H/A}(\omega_1 - \omega_2) = \int (dt/2\pi)\mathbb{D}_k^{H/A}(t)e^{it(\omega_1 - \omega_2)}$ are the Fourier transforms of the corresponding time-dependent constraints $\mathbb{D}_k^{H/A}(t)$ on the kth derivatives of the field fluctuations. Then, according to Eq. (14), its inverse gives the bispectrum $\mathbb{S}(\omega_1, \omega_2)$. Based on Eq. (27), this spectrum is similar to a polynomial of ω_1 and ω_2, except that in the nonstationary case, the "coefficients" $\mathbb{D}_k^{H/A}(\omega_1 - \omega_2)$ depend on the frequencies, and are determined by the temporal dependence of the constraints on the kth derivatives of the field fluctuations. The nonstationary evolution of the constraints is manifested for these local-in-time processes by functions that satisfy $\mathbb{D}(\omega_1, \omega_2) = \mathbb{D}(\omega_2, \omega_1)^\dagger$, and therefore have reflection-conjugation symmetry about the diagonal axis $\omega_1 = \omega_2$ in frequency space. In stationary processes as we see below, the $\mathbb{D}_k^{H/A}(\omega_1 - \omega_2)$ become proportional to a $\delta(\omega_1 - \omega_2)$ function; thus, the widths of these functions $\mathbb{D}_k^{H/A}(\omega_1 - \omega_2)$ show the degree of nonstationarity of a process.

An alternative interpretation of the physical properties of these processes comes from employing a time discretization of the time evolution (see Appendix E). The argument for the integral in Eq. (24) depends only on the local time t, i.e., the kernel only correlates the fluctuating field values and its derivatives instantaneously. However, the correlations imposed by the kernel on the field derivatives effectively correlate the field at different times locally. Replacing the field derivatives in Eq. (24) with finite time differences, the discrete version of the kernel correlates the value of the fluctuating field at the nearest $2N + 1$ times steps close to the time t, i.e., $\mathbf{B}(t \pm k\Delta t)$ with $k = -N, \dots, N$ and Δt the time discretization step. Thus, this shows how a kernel operator that is local in time correlates the fluctuating field values on a limited correlation time length, therefore introducing memory of the fluctuating field process $\mathbf{B}(t)$ on infinitesimally near times determined by the constraints on the field derivatives. These processes are in general non-Markovian, as discussed in Sec. V C, yet the long-time correlations do not exist.

B. Stationary noise spectrum as the inverse of the differential operator

For stationary noise processes that are local in time, the differential operator $\mathbb{D}$ is time independent:

$$\mathbb{D} = \sum_{k=0}^{N}(-1)^k \mathbb{D}_k^H \partial_t^{2k} + \sum_{k=1}^{N}(-1)^k \mathbb{D}_k^A \partial_t^{2k-1}, \quad (28)$$

where we have used $\overleftarrow{\partial_t} = -\partial_t$ (see Appendix C). The path integral framework then allows us to express the noise spectrum $\mathbb{S}(\omega)$ in terms of the operators $\mathbb{D}_k^{H,A}$ that define the probability density of field paths. Note that the $\mathbb{D}_k^{H,A}$ now define constant constraints on the derivatives of the

fluctuating field $\mathbf{B}(t)$. Figure 2(a) shows typical field fluctuations with constraints up to the derivative of the field ($N = 1$) and up to the second derivative of the field ($N = 2$). Based on Eq. (16), these constraints then define the noise spectrum as

$$\mathbb{S}(\omega) = [\mathbb{D}(\omega)]^{-1} = \left[\sum_{k=0}^{N}\mathbb{D}_k^H \omega^{2k} + i\sum_{k=1}^{N}\mathbb{D}_k^A \omega^{2k-1}\right]^{-1}, \quad (29)$$

where the matrix polynomial $\mathbb{D}(\omega)$ is the expression in the frequency basis of the differential operator $\mathbb{D}$ of a local-in-time stationary process. This equation is obtained from Eq. (27), where $\mathbb{D}_k^{H/A}(\omega_1 - \omega_2) \propto \delta(\omega_1 - \omega_2)$ for stationary processes. The polynomic functional behavior of the predicted noise spectrum is consistent with several experimental observations [35,93,98,115]. As examples, *white noise* corresponds to $N = 0$, a *Lorentzian* spectrum corresponds to $N = 1$, representing Ornstein-Uhlenbeck one-dimensional noise processes, and a spectrum given by the inverse of a quartic polynomial corresponds to $N = 2$ with a power-law tail $\propto \omega^{-4}$ [Fig. 2(b)]. This result thus provides a simple physical meaning for the power-law exponents dependence of the noise spectrum based on conditions $\mathbb{D}_k^H$ and $\mathbb{D}_k^A$ imposed on the derivatives of $\mathbf{B}(t)$ that the probability density of the possible paths can take.

An important application of Eq. (29) is that it facilitates the noise spectroscopy characterization, as the noise spectrum of local-in-time processes is defined by the $2N + 1$ *constant* $n \times n$ matrices $\mathbb{D}_k^{H/A}$. Therefore, the hypothesis of locality in time allows the implementation of *parametric estimation* to determine noise spectra, rather than using nonparametric estimation as described in Sec. IV C for more general noise processes. Therefore, noise spectroscopy methods for local-in-time stationary noise processes can thus reduce the amount of measurements required to reconstruct the noise spectrum.

C. Conditions for Markovian and non-Markovian noise processes

In this subsection, we state the conditions for the kernel operator $\mathbb{D}$ to represent a Markovian process. We show that non-Markovian, local-in-time processes can always be mapped to a Markovian process, which is described by the value of the field and its first $N - 1$ derivatives. We show that local-in-time processes are a generalization of Markovian processes and that those processes cannot generate long-time correlations. The probability distribution of $\mathbf{B}$ at a time t' after a time t is shown to be based only on the state of the process at time t, if we consider the generalized process defined by the state of the joint probability distribution of $\mathbf{B}$ and its first $N - 1$ derivatives at time t. Therefore, local-in-time noise processes are generally non-Markovian and in our generalized framework appear

as a natural extension of Markovian noise processes, as they maintain many of their properties while being able to model a greater variety of environments. They thus provide a natural model to study non-Markovian environments that do not contain long-time correlations.

1. Markovian process conditions

A stochastic process **B** is Markovian if the probability of the field $\mathbf{B}_f$ at time t_f starting from a value $\mathbf{B}_0$ at t_0 is [79,116–118]

$$P(\mathbf{B}_f, t_f \,|\, \mathbf{B}_0, t_0) = \int d\mathbf{B}_1 P(\mathbf{B}_f, t_f \,|\, \mathbf{B}_1, t_1) P(\mathbf{B}_1, t_1 \,|\, \mathbf{B}_0, t_0) \tag{30}$$

for any $t_f > t_0$, where $P(\mathbf{B}_i, t_i | \mathbf{B}_j, t_j)$ are the conditional probabilities that define the propagator of the process. A Markovian process is said to be memoryless, as the random variable "forgets" its previous state as it evolves [79,116–118]. Therefore, in general, a stochastic process determined by the kernel operator $\mathbb{D}$ as described in Eq. (7) is non-Markovian as it correlates the fluctuating field states at two distant times. Only local-in-time noise processes as defined in Sec. V A may be Markovian. We then only consider this case to evaluate the conditions required for the nonstationary noise process to be Markovian.

If the action $\mathcal{A}[\mathbf{B}]$ of Eq. (24) contains derivatives of order N higher than 1, it implies a differentiability condition for the field paths so that the action does not diverge and they contribute to the propagator (see Appendix D). Therefore, in this case **B** and its first $N - 1$ derivatives must be continuous functions. As Eq. (30) only imposes that the paths are continuous at t_1 to define a finite integral, if the *action* imposes more strict conditions on the possible paths, requiring *continuity* from their *derivatives*, the process of **B** will not be Markovian as the propagator requires information of different previous times. This implies in particular that the derivative of **B** is continuous, and therefore it contains information about the history of **B**. Then Eq. (30) cannot be satisfied as it does not include information about **B** encoded in its derivatives. Therefore, we find that the stochastic process that describes **B** is Markovian if and only if its *kernel operator* is given by the *local-in-time differential operator* of the form

$$\mathbb{D}_{\text{Mark}}(t) = \mathbb{D}_0^H(t) + \overleftarrow{\partial}_t \mathbb{D}_1^H(t)\, \partial_t + \tfrac{1}{2}[\overleftarrow{\partial}_t \mathbb{D}_1^A(t) - \mathbb{D}_1^A(t)\partial_t], \tag{31}$$

where the derivatives of the paths are not continuous (see Appendix F for a full proof). In the frequency basis, the kernel becomes

$$\begin{aligned}
\mathbb{D}_{\text{Mark}}(\omega_1, \omega'_2) &= (\omega_1 | \mathbb{D}_{\text{Mark}} | \omega_2) \\
&= \mathbb{D}_0^H(\omega_1 - \omega_2) + \omega_1 \omega_2 \mathbb{D}_1^H(\omega_1 - \omega_2) \\
&\quad + \tfrac{1}{2} i(\omega_1 + \omega_2) \mathbb{D}_1^A(\omega_1 - \omega_2),
\end{aligned} \tag{32}$$

following Eq. (27). Equations (31) and (32) are definitions of Markovian Gaussian processes within the presented framework that are equivalent to the conventional form of Eq. (30).

In the particular case of stationary noise processes, the most general n-dimensional Markovian noise spectrum is therefore

$$\mathbb{S}_{\text{Mark}}(\omega) = [\mathbb{D}_0^H + i\mathbb{D}_1^A\omega + \mathbb{D}_1^H\omega^2]^{-1}, \tag{33}$$

according to Eq. (29). For the simplest case of one-dimensional stationary noise processes, this means that the process that describes them is Markovian if and only if the noise spectrum is constant, i.e., white noise [$N = 0$ in Eq. (29)] or Lorentzian [$N = 1$ in Eq. (29)]. Figure 2 compares a Markovian field fluctuation for $N = 1$, where the field is not differentiable, and a non-Markovian field fluctuation with a continuous derivative for a one-dimensional stationary noise process.

An important application of the noise spectrum form derived in Eqs. (27) and (33) is that they allow us to determine when a noise process is Markovian or non-Markovian by experimentally measuring its spectrum with dynamical decoupling noise spectroscopy (Sec. IV C).

2. Generalized Markovian process for local-in-time nonstationary, non-Markovian noise processes

If the stochastic process is non-Markovian, but local in time, it can still be described by a generalized Markovian process if all the derivatives of **B** are included in the stochastic process. We define a general conditional probability, or propagator, of the stochastic process based on the ordered set of all the derivatives of **B** of order lower than N,

$$\begin{aligned}
\{\mathbf{B}^{(k)}(t)\} &\equiv \{\mathbf{B}^{(k)}(t)\colon k = 0, \ldots, N-1\} \\
&= \{\mathbf{B}(t), \dot{\mathbf{B}}(t), \ldots, \mathbf{B}^{(N-1)}(t)\}. \tag{34}
\end{aligned}$$

The Markovian condition of Eq. (30) is now satisfied for the stochastic process of $\{\mathbf{B}^{(k)}(t)\}$, as the equation imposes continuity on the first $N - 1$ derivatives of **B**. This means that both the action and the integral over $\{\mathbf{B}_1^{(k)}(t)\}$ impose that $\{\mathbf{B}^{(k)}(t)\}$ is continuous, implying that the *stochastic process* $\{\mathbf{B}^{(k)}(t)\}$ *is Markovian* (see the proof details in Appendix G). Therefore, the family of *fluctuating local-in-time nonstationary noise processes* considered here imply that if **B** is not Markovian, the state of **B** is not only

determined by the *probability distribution* of **B**, but by the *joint probability distribution* of **B** and its derivatives. This means that the information of the initial condition is encoded in the *derivatives* of **B**.

Since local-in-time noise processes can be described by the generalized Markovian process of $\{\mathbf{B}^{(k)}(t)\}$, its *propagator* $P(\{\mathbf{B}_f^{(k)}\}, t_f | \{\mathbf{B}_0^{(k)}\}, t_0)$ contains all the *information* about the stochastic process that describes the field paths **B**. Moreover, we find that the path integral framework allows us to calculate the *propagator* of $\{\mathbf{B}^{(k)}\}$ without actually requiring to perform *path integrals*. Instead, the propagator can be obtained by solving an *ordinary linear differential equation* with different types of *boundary conditions*. The full propagator expression and its derivation are given in Appendix H.

In summary, the local-in-time stochastic process of **B** is Markovian if and only if the differential operator $\mathbb{D}(t)$ does not contain terms with k higher than 1. All other processes are non-Markovian. Yet, the generalized local-in-time process of $\{\mathbf{B}^{(k)}(t)\}$ is always Markovian, and it follows that the information of previous states of **B** is encoded in its derivatives when they are continuous.

3. The long time limit of local-in-time stationary noise processes is indistinguishable from a Markovian process

The spectral density $\mathbb{S}(\omega)$ for local-in-time stationary noise processes of Eq. (29) is infinitely differentiable, as it is the inverse of a nonsingular matrix polynomial, and therefore the integral of its derivatives is finite $\int d\omega \, |(d/d\omega)^k \mathbb{S}_{ij}(\omega)| < \infty$ for all kth derivative orders. Since the corresponding correlation functions are $\mathbb{G}_{ij}(t) = \int (d\omega/2\pi) \mathbb{S}_{ij}(\omega) e^{-i\omega t}$, one can see that

$$\mathbb{G}_{ij}(t) = (it)^{-k} \int \frac{d\omega}{2\pi} \left(\frac{d}{d\omega}\right)^k \mathbb{S}_{ij}(\omega) e^{-i\omega t} \qquad (35)$$

and that they are bounded by the expression [119]

$$|\mathbb{G}_{ij}(t)| \leq \int \frac{d\omega}{2\pi} \left| \left(\frac{d}{d\omega}\right)^k \mathbb{S}_{ij}(\omega) \right| \Big/ |t^k|. \qquad (36)$$

Therefore, we obtain $\lim_{|t| \to \infty} \mathbb{G}(t)/t^k = 0$ for all k, implying that $\mathbb{G}(t)$ decays to zero exponentially (or faster) for long times $t \to \infty$. Since $\mathbb{S}(\omega)$ is the inverse of a matrix polynomial, we know that it is not entire when analytically continued to the complex plane of ω for $N > 0$. Therefore, for $N > 0$, the correlation functions $\mathbb{G}(t)$ cannot decay faster than an exponential in the long time limit, so they must decay exactly as exponentials [120]. Only, for $N = 0$, the correlation function decays faster as its corresponding noise is white noise. Therefore, in the long time limit, every local-in-time stationary noise is indistinguishable by means of noise spectroscopy from a Markovian noise, as its self-correlation functions decay exponentially or are zero

for long times, which coincides with the correlation functions derived from what we showed are Markovian noise processes.

4. Short time limit $t \to 0$ of local-in-time stationary noise processes

We consider an arbitrary differential operator $\mathbb{D}$ for a stationary local-in-time noise. In general, it is not possible to find a closed formula for the correlation functions $\mathbb{G}(t)$, since that would imply finding the roots of a polynomial of an arbitrarily high degree. However, it is possible to get information about the general behavior of $\mathbb{G}(t)$ at short times, by analyzing its representation in the frequency basis, i.e., the noise spectrum $\mathbb{S}(\omega)$.

In the large frequency ω limit, the noise spectrum is $\mathbb{S}(\omega) = \mathbb{D}^{-1}(\omega) \sim \mathbb{D}_N^{-1} \omega^{-2N}$ according to Eq. (29). Since $\mathbb{D}(\omega)$ is positive definite, the spectral density $\mathbb{S}(\omega)$ remains finite for all ω and the integral $\int d\omega |\omega^{2N-2} \mathbb{S}_{ij}(\omega)| < \infty$ is bounded. This implies that $\mathbb{G}(t)$ is continuously differentiable $2N - 2$ times, since its derivatives are

$$\mathbb{G}_{ij}^{(k)}(t) = \int \frac{d\omega}{2\pi} (-i\omega)^k \mathbb{S}_{ij}(\omega) e^{-i\omega t} \qquad (37)$$

and are continuous [119]. This means that, for processes with $N > 1$, the correlation functions decay quadratically or slower around their maxima. Then, as the correlation functions of each component of **B** satisfy

$$\mathbb{G}_{ii}(t) = \langle B_i(t) B_i(0) \rangle = \text{Cov}[B_i(t) B_i(0)] \leq \langle B_i^2(0) \rangle, \quad (38)$$

they have a global maximum at $t = 0$, implying that if $N > 1$, they decay at least quadratically for short times. This agrees well with the expected behavior of correlation functions predicted by quantum mechanics [19,20,121–127].

Therefore, every stationary noise process generated by a quantum mechanical process is not Markovian. Note that the Markovian noise correlation functions for $N \leq 1$ are Dirac delta functions that describe a white-noise spectrum ($N = 0$), and for one-dimensional noise with $N = 1$, we obtain a Lorentzian spectrum that describes an *Ornstein-Uhlenbeck process* where $\mathbb{G}(t) = e^{-|t|/\tau_c}/(2D_0 \tau_c)$, with $\tau_c = \sqrt{D_1/D_0}$. These Markovian noise processes disagree with what quantum mechanics predicts, that, for short times, $\mathbb{G}$ should decay at least quadratically. This is true for all Markovian processes, as Eq. (26) demands the first derivative of $\mathbb{G}(t, t')$ to be discontinuous at $t = t'$ when $N = 1$.

VI. IMPLEMENTATION OF THE FRAMEWORK ON PARADIGMATIC NONSTATIONARY NOISE PROCESSES

In this section, we show how to implement the presented path integral framework to two paradigmatic examples of

nonstationary noise processes. We consider noise determined from a quench on the environment and noise that acts near a point in time. Both examples show some characteristic features that arise from our framework that distinguish nonstationary from stationary noise effects on the qubit-probe dephasing.

A. Quenched diffusion

We here consider a quenched environment described by a diffusion process of the fluctuating field that began at an instant in time. This sets a paradigmatic model of typical environments that can be suddenly quenched to put them out of equilibrium, where excitations start spreading over a large number of degrees of freedom [13,50,56,64,74]. The environment dynamics sensed by the qubit probe can be represented with a generalized diffusion process, and possible examples can be encountered in spin ensembles coupled to single nitrogen-vacancy centers in diamond [94,128,129], macromolecular dynamics [130–133], spin diffusion on environments that becomes out of equilibrium [13,35,86,134,135], dynamics of spin and current fluctuations in a material at the nanoscale probed by magnetic noise sensors [136,137], and molecular diffusion out of equilibrium [63,138–141].

As a simple model, in particular we consider that the fluctuating field is zero and at a given time suddenly starts to fluctuate, driven by a one-dimensional Ornstein-Uhlenbeck diffusion process [47,48,93,94,129,136]. Considering that the quench is at time $t = 0$, the fluctuating field is therefore confined to a point for $t < 0$. For $t > 0$, the noise process is described by a differential operator equal to a stationary one that gives a Lorentzian spectrum

$$S_0(\omega) = \frac{1}{D_0 + D_1\omega^2}. \tag{39}$$

This quenched diffusion noise process is local in time and is modeled by the differential operator

$$\mathbb{D}(t) = D_1\partial_t^2 + D_0(t) \tag{40}$$

with

$$D_0(t) = \begin{cases} D_0, & t > 0, \\ +\infty, & t < 0, \end{cases} \tag{41}$$

where we have considered $\mathbb{D}$ as a scalar kernel operator. The action and the differential operator that describe the possible field paths are equivalent to those derived from a Schrödinger equation that describes an infinite potential wall at position 0. Based on this analogy, one can consider that $D_1\partial_t^2$ is mapped to $-\partial_x^2/(2m)$ and $D_0(t)$ is mapped to the energy potential that describes the wall by replacing

t with the position x. Therefore, the eigenmode basis that diagonalizes the kernel operator $\mathbb{D}$ is

$$|\Omega\rangle = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} dt \, \sin(\Omega t)|t\rangle, \tag{42}$$

where $(\Omega|\Omega') = \delta(\Omega - \Omega')$ and $\Omega \geq 0 \in \mathbb{R}$. Following Eq. (18), the noise spectrum on the eigenmode basis of the nonstationary field fluctuations is (Fig. 3)

$$S(\Omega) = \frac{1}{D_0 + D_1\Omega^2}, \tag{43}$$

and the generalized filter function is defined by

$$F(\Omega) = (\Omega|\mathbf{f}) = \int_0^{+\infty} dt \sqrt{\frac{2}{\pi}} \sin(\Omega t)f(t). \tag{44}$$

Note how the functional form of the nonstationary noise spectrum of Eq. (43) is *equal* to that of a *stationary* case for $\omega \geq 0$ [Eq. (39)] replacing Ω by ω. However, the *eigenmodes* are different to those obtained in the stationary case. In this example, this means that the difference between the stationary and nonstationary cases is encoded in the filter function F. For a stationary Ornstein-Uhlenbeck process, F is obtained by performing the complex Fourier transform of $f(t)$, but in the nonstationary process F is obtained by a real Fourier transform based on the sinusoidal component given by Eq. (44).

1. Manifestation of the quench on the noise correlation functions

The correlation function is the solution of the local-in-time differential equation $\mathbb{D}(t)\mathbb{G}(t, t') = \delta(t - t')$. The correlation function satisfies $\mathbb{G}(t, t') = \mathbb{G}^\dagger(t', t)$, $G(0, t') = 0$ due to the fact that the fluctuating field is 0 at $t = 0$, and it also satisfies $\lim_{t\to\infty} \mathbb{G}(t, t') = 0$ for a fixed t' as the correlation function of the Ornstein-Uhlenbeck process decays exponentially to 0. The solution for the differential equation is thus

$$\mathbb{G}(t, t') = \frac{1}{2D_0\tau_c}[e^{-|t-t'|/\tau_c} - e^{-(t+t')/\tau_c}]\Theta(t)\Theta(t'), \tag{45}$$

where $\tau_c = \sqrt{D_1/D_0}$ is the correlation time and $\Theta(t)$ is the Heaviside function.

For long times after the quench $t/\tau_c, t'/\tau_c \gg 1$, we recover the correlation function for an Ornstein-Uhlenbeck stationary diffusion process

$$\mathbb{G}(t, t') \simeq \frac{1}{2D_0\tau_c}e^{-|t-t'|/\tau_c}. \tag{46}$$

In this case $\mathbb{G}(t, t')$ is given by the Green function of the differential kernel operator $D_1\partial_t^2 + D_0$ acting over time-dependent functions with $t \in \mathbb{R}$ instead of $t > 0$ as in the
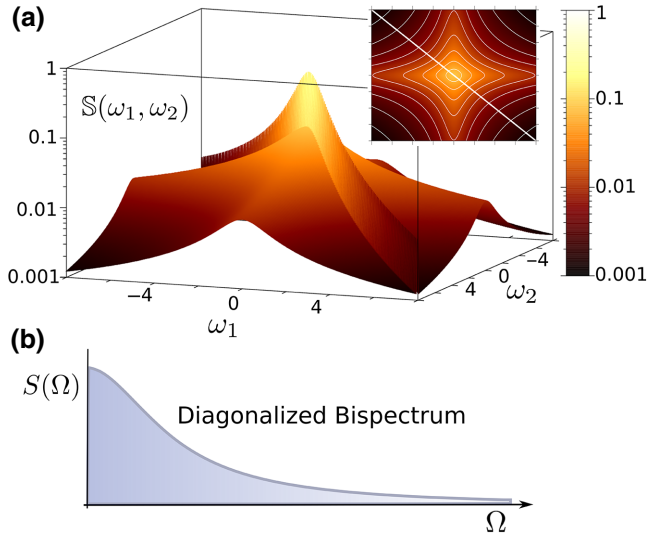
(a)

(b)

FIG. 3.   Nonstationary noise spectrum of a one-dimensional quenched diffusion process of the environmental field fluctuations, driven by a one-dimensional Ornstein-Uhlenbeck stochastic process. (a) The absolute value of the corresponding spectral density $\mathbb{S}(\omega_1, \omega_2)$ in the frequency basis, the bispectrum given by Eq. (47), is shown. Only the diagonal $\omega_1 = \omega_2$ contains the stationary component of the noise spectrum that is dominant at long times after the quench. As the stationary component is $\propto \delta(\omega_1 - \omega_2)$, we plot it by replacing the $\delta(\omega_1 - \omega_2)$ function with a nondivergent Gaussian function for presentation purposes. The inset shows a surface colored map of the bispectrum with contour lines in white. (b) The corresponding generalized, non-stationary noise spectrum $S(\Omega)$, given in the noise eigenmode basis $|\Omega\rangle$ that diagonalizes the noise spectrum.

quenched case. Therefore, for times much longer than τ_c, the environment *forgets* the *boundary condition* imposed by the quench. The nonstationary effects induced by the quench are thus modeled by the second term

$$\frac{-1}{2D_0\tau_c} e^{-(t+t')/\tau_c}$$

of Eq. (45). The quench effects are thus only manifested for *times much lower than* τ_c with respect to the quench time $t = 0$, in contrast with the stationary effects that depend on the time correlation difference $|t - t'|$.

The corresponding spectral density in the frequency basis of Eq. (12) is the bispectrum

$$\mathbb{S}(\omega_1, \omega_2) = \delta(\omega_1 - \omega_2)\frac{1}{D_0}\frac{1}{1 + \omega_1^2 \tau_c^2}$$
$$- \frac{\tau_c}{4\pi D_0}\frac{1}{(1 - i\omega_1 \tau_c)(1 + i\omega_2 \tau_c)}, \quad (47)$$

where the first term is the stationary noise spectrum of Eq. (39), and the other term is due to the quench. Figure 3(a) shows this nonstationary bispectrum, where the stationary

component is only manifested on the diagonal $\omega_1 = \omega_2$. The noise eigenmodes [Eq. (42)] and the nonstationary noise eigenspectrum [Eq. (43)] are obtained by diagonalising this bispectrum. Figure 3(b) shows the corresponding diagonalized nonstationary spectrum $S(\Omega)$ of Eq. (43).

2. Manifestation of the quench on the qubit-probe dephasing

The difference in the qubit-probe decay induced by this quenched diffusion, between the nonstationary effects and its stationary counterpart, is given by

$$\frac{-1}{2D_0\tau_c}\int dt_1 dt_2\, \mathbf{f}(t)e^{-(t+t')/\tau_c}\mathbf{f}(t')$$
$$= \frac{-1}{2D_0\tau_c}\left[\int dt\, \mathbf{f}(t)e^{-t/\tau_c}\right]^2. \quad (48)$$

Therefore, the nonstationary effects for this type of quenched noise always *reduce* the dephasing induced on the qubit probe compared with a stationary process. We consider in particular the control modulation function of the typical continuous-wave irradiation to show this manifestation of the nonstationary effects on the qubit-probe dephasing. We start the control at time t_0 after the quench

$$\mathbf{f}(t) = g\cos[\omega(t - t_0)]\Theta(t - t_0)\Theta(t_0 + T - t), \quad (49)$$

and it acts during a time T. The argument of the dephasing given in Eq. (8) at time $t_0 + T$ is

$$(\mathbf{f}|\mathbb{G}|\mathbf{f}) = (\mathbf{f}|\mathbb{G}_0|\mathbf{f}) - \frac{g^2\tau_c}{2D_0}$$
$$\times\left\{\frac{1 + [\omega\tau_c\sin(\omega T) - \cos(\omega T)]e^{-T/\tau_c}}{1 + \omega^2\tau_c^2}\right\}^2$$
$$\times e^{-2t_0/\tau_c}, \quad (50)$$

where $(\mathbf{f}|\mathbb{G}_0|\mathbf{f})$ is the dephasing obtained for the stationary case, and the second term is due to the quench—nonstationary—effects. This second term provides a dephasing term that oscillates with the control frequency ω as a function of the duration of the control modulation T. This oscillation is attenuated with the exponential decay e^{-T/τ_c}, and thus it disappears at long times after the quench. Then, if the control modulation duration is $T \gg \tau_c$, we still have an extra term due to the quench effects

$$(\mathbf{f}|\mathbb{G}|\mathbf{f}) = (\mathbf{f}|\mathbb{G}_0|\mathbf{f}) - \frac{g^2\tau_c}{2D_0}\frac{e^{-2t_0/\tau_c}}{(1 + \omega^2\tau_c^2)^2}. \quad (51)$$

Note that this last term provides a constant term in the dephasing that contains information about the quench if

$t_0 \lesssim \tau_c$ as it decays exponentially with time t_0 when qubit-probe control started. This prediction thus gives a tool to probe the quench effects by monitoring the dephasing change as a function of t_0, and gives a method to probe the self-correlation times induced by the quench. Moreover, if one monitors the decay rate of the qubit probe as typically done in several noise spectroscopy approaches [35,40,45], one would obtain only the stationary decay rate given by

$$(\mathbf{f}|\mathbb{G}_0|\mathbf{f}) \propto S_0(\omega)T. \qquad (52)$$

Therefore, the effects of the quench are not manifested on the decay rate, but they are evidenced on a *shift* of the decaying signal given by the second term of Eq. (51).

B. Noise produced near a point in time

As one of our results is that nonstationary spectrums can be discrete according to Eq. (18), we exploit an analogy with the Schrödinger equation to describe a paradigmatic example that manifests this discrete nature. Noise that contains these discrete features, for example, acts only near an instant in time, a pulsed noise interaction. This is a paradigmatic model for a quantum probe that interacts with noise during a finite duration of time. Examples of this can be moving charges or particles that pass near a quantum sensor [76,138,139,142–144], forces or interactions detected by a moving cantilever or a tip that contains the sensor [77,129,145], and biomedical applications as the detection of neuronal activity [75,76,78].

All nonstationary noise described by a differential operator of the form

$$\mathbb{D}(t) = -D_1 \partial_t^2 + \mathbb{D}_0(t), \qquad (53)$$

where D_1 is a scalar, can be mapped to a quantum-mechanical problem with the Hamiltonian

$$\mathcal{H} = -\frac{1}{2m}\partial_x^2 + V(x) \qquad (54)$$

by replacing $x \to t$, $m \to 1/(2D_1)$, and $V(x) \to \mathbb{D}_0(t)$. Therefore, for every solvable quantum-mechanical Hamiltonian with positive eigenvalues, we obtain a solution for the noise spectrum and its eigenmodes of a Markovian local-in-time, nonstationary Gaussian noise. The energy levels of the Hamiltonian must be positive so that the differential operator $\mathbb{D}(t)$ is positive definite, but every bounded from below Hamiltonian can be transformed into a positive definite one by adding a large enough constant C to $\mathcal{H} \to \mathcal{H} + C$.

Based on this analogy, we describe the paradigmatic example of pulsed noise with a one-dimensional, nonstationary noise process that is local in time. We consider the noise described by a differential operator $\mathbb{D}$ that is

mappable to the Hamiltonian of a quantum harmonic oscillator

$$\mathcal{H} = -\frac{1}{2m}\partial_x^2 + \frac{1}{2}m\omega_0^2 x^2 + D_0, \qquad (55)$$

where D_0 is an additive constant that does not change the eigenvectors of the Hamiltonian $\mathcal{H}$. The corresponding differential operator is

$$\mathbb{D}(t) = -D_1 \partial_t^2 + D_0 + \alpha t^2, \qquad (56)$$

where the necessary map is $m \to 1/(2D_1)$, $\omega_0 \to \sqrt{4\alpha D_1}$, and $x \to t$. This stochastic process models noise probed by the qubit system that appears near a point in time, where the fluctuating field paths are forced to be 0 for times $|t| \to \infty$, since $D_0(|t| \to \infty) \to \infty$. Therefore, the fluctuating fields are only allowed to deviate from 0 near the local instant of time $t = 0$ (see the blue dashed line in Fig. 4). The noise eigenmodes of $\mathbb{D}(t)$ are

$$|\Omega_n\rangle = \frac{1}{\sqrt{2^n n!}}\left(\sqrt{\frac{\alpha}{D_1\pi^2}}\right)^{1/4}$$
$$\times \int_{-\infty}^{+\infty} dt \, \exp\left\{-\sqrt{\frac{\alpha}{4D_1}}t^2\right\} H_n\left\{\left(\frac{\alpha}{D_1}\right)^{1/4} t\right\}|t\rangle, \qquad (57)$$

where H_n are the Hermite polynomials with $n = 0, 1, 2, \ldots$ in analogy with the eigenfunctions of the quantum harmonic oscillator [Fig. 4(a)]. The nonstationary noise spectrum expressed in its eigenmode basis is then discrete, given by

$$S(\Omega_n) = \frac{1}{\Omega_n + D_0}, \quad \Omega_n = \left(n + \frac{1}{2}\right)\omega_0, \qquad (58)$$

as shown in Fig. 4(a). Since the differential operator $\mathbb{D}(t)$ is positive definite if and only if all its eigenvalues are positive, the model is well defined only for $\omega_0 > -2D_0$.

This noise model sets a paradigmatic example of some of the differences between probing nonstationary and stationary noise spectra with dynamical decoupling noise spectroscopy. In this case, the natural control modulation would be $f_n(t) = \langle t|\Omega_n\rangle$ for $n = 0, 1, 2, \ldots$, as shown in Fig. 4(a). Here the dephasing in the long time limit after applying these modulations will saturate to a constant value that provides the noise eigenvalues

$$(\Omega_n|\mathbb{G}|\Omega_n) = \frac{1}{\Omega_n + D_0}. \qquad (59)$$

This is in contrast to the predicted exponential decay with a constant rate for stationary noise processes typically used for noise spectroscopy [35]. Figure 4(b) shows a schematic
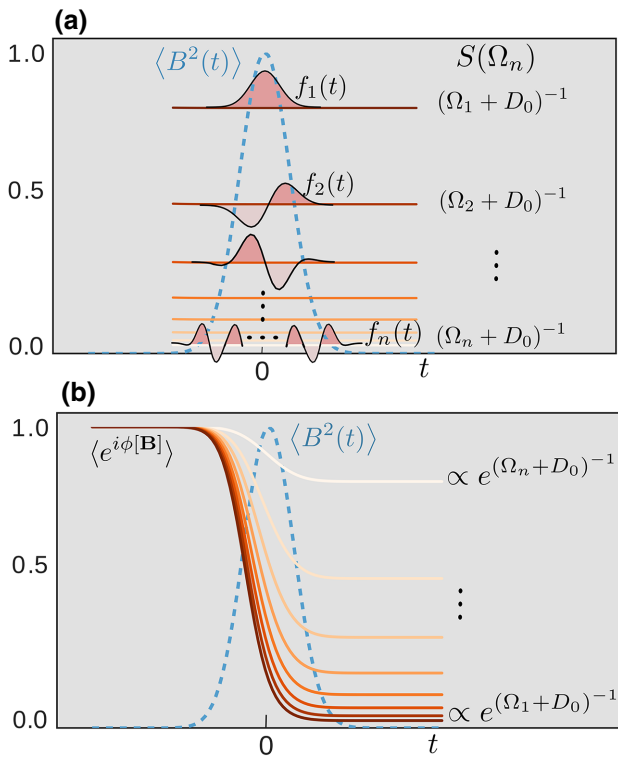
FIG. 4.   Scheme for the characteristic features of a paradigmatic local-in-time, nonstationary noise that acts only near a point in time and therefore produces a discrete, generalized noise spectrum. The variance of one-dimensional noise that appears near a point in time is shown with a blue dashed line. (a) The generalized nonstationary noise spectrum $S(\Omega_n) = 1/(\Omega_n + D_0)$ in the noise eigenmode basis is shown with orange horizontal lines. The corresponding noise eigenmodes $|\Omega_n\rangle$ are shown in the time basis $f_n(t) = (t|\Omega_n)$ for $n = 0, 1, 2, \ldots$ with solid black lines. They also provide the natural control modulations to probe the noise eigenmodes. (b) Schematic representation of the qubit-probe signal decay when it is controlled by the modulation functions $f_n(t)$. They feature a signal saturation at long times given by $\exp\{-(\Omega_n|\mathbb{G}|\Omega_n)\} = \exp\{-S(\Omega_n)\}$.

representation of the qubit-probe signal decay when it is controlled by the modulation functions $f_n(t) = (t|\Omega_n)$. This behavior is universal for all noise processes with a discrete spectrum; therefore, this example shows how noise spectroscopy must be done for such noise processes.

VII. SUMMARY AND CONCLUSIONS

We introduced a path integral framework for determining the dephasing on a quantum probe induced by nonstationary Gaussian noise processes. This type of noise process models a fluctuating qubit-probe interaction with a quantum environment that is out of equilibrium under the so-called weak coupling approximation. We showed that the noise generated by this nonstationary environment is completely determined by a bispectrum defined by the inverse of the kernel operator that describes the probability

density of the field paths. This complements recent results, where similar bispectra functions characterize stationary noise processes that are non-Gaussian [42,67].

The presented framework introduced a generalized noise spectrum for nonstationary environments, defined by the inverse of the eigenvalues of the kernel operator that determines the probability of the noise field paths. The noise eigenmodes define the proper basis to generalize a filter function derived from the control, whose overlap with the noise spectrum determines the qubit-probe dephasing. This results in an extension of the validity of the universal formula for the dephasing of open quantum systems that depends on the overlap between a noise spectral density and a qubit-control filter function. The main result of this generalization is that it allows us to implement two important tools already developed for stationary noise processes to probe spectral properties and to mitigate decoherence effects of nonstationary environments.

We then considered a broad subclass of nonstationary noise processes that we called local in time. We showed they are described by a differential operator based on constraints to the derivatives of the fluctuating field paths. We also showed how these constraints are reflected on the functional behavior of the nonstationary noise spectrums. Such a subclass of the discussed nonstationary noise processes is the only one that can be Markovian if the first derivative of the fluctuating fields is not continuous, e.g., white noise processes and stationary noise processes with Lorentzian noise spectra. Therefore, if the field derivative is continuous or the noise is not local in time, the noise process is non-Markovian. Remarkably, we found that local-in-time, nonstationary Gaussian noise processes that are non-Markovian can still be described by a generalized Markovian noise process that includes the field and a finite number of its derivatives. This approach simplifies the noise description as it is fully determined by a differential equation with the constraints on its derivatives thus reducing the dimensionality for the characterization of the noise spectra. An important application for the derived forms of the noise spectra is that they allow us to determine when a noise process is Markovian or non-Markovian by experimentally measuring its spectrum with dynamical decoupling noise spectroscopy. In the particular case of stationary noise processes that are local in time, we found that the differential equation gives noise spectra given by the inverse of matrix polynomials of the frequency modes. The polynomial coefficients associated with a power of the frequency are given by the constraint to the corresponding order of the fluctuating field derivative. This thus allows the implementation of parametric estimation methods to determine the noise spectra, rather than using nonparametric estimation that is much more complex and requires more available experimental time.

We have shown that in some cases local-in-time, nonstationary noise processes can be mapped to the Schrödinger

equation. Thus, every *solvable one-dimensional quantum mechanical* Hamiltonian H for a single particle in one spatial dimension with eigenvalues bounded from below generates a whole class of solvable noise probability distributions. We have used this map to apply the presented path integral framework to two paradigmatic nonstationary noise processes: noise acting near a point in time—pulsed noise—by analogy to a quantum harmonic oscillator and a quench on the environment that suddenly starts an Ornstein-Uhlenbeck diffusion process. Both of these examples describe out-of-equilibrium environments. The pulsed noise is a paradigmatic model for a quantum probe interacting with noise during a finite duration of time that manifests a discrete nature on the generalized nonstationary noise spectrum. In this case the noise spectrum is obtained by saturation of the qubit-probe dephasing rather than on its decay rate. The quenched diffusion noise sets a paradigmatic model for typical environments that can be suddenly quenched to put them out of equilibrium, where excitations start spreading over a large number of degrees of freedom. In this case we showed some features on the spin dephasing and noise spectra that manifest the quench—nonstationary—effects, evidenced by a bispectrum and a reduced dephasing compared with stationary noise. In particular, we showed how the quench correlation can be probed by monitoring this dephasing.

The results presented here thus set a general framework for a large universal class of nonstationary—out-of-equilibrium—noise sources of decoherence, allowing us to probe and interpret noise spectral properties and time correlations by a quantum probe. Thus, they provide tools and insights for probing and understanding the dynamics of quantum information of out-of-equilibrium complex quantum systems via a quantum sensor [13,49, 50,53,57,61,74,135,137,146]. In particular, they can be useful for quantum sensing the dynamics of single, but complex, large molecules as proteins [133,147–149] and neuronal activity [75,76,78] with potential applications in biology and medicine. At the same time the general framework sets a universal formula to allow finding optimal control for protecting against decoherence generated by more realistic environments, that at atomic scales produces nonstationary—out-of-equilibrium—noise fluctuations. This tool is very important for implementing quantum technologies that can range from memory storage to information processing in quantum devices [1–4,144].

APPENDIX A: PATH INTEGRALS FOR GAUSSIAN NOISE PROBABILITY DISTRIBUTIONS

Path integrals can be calculated exactly when the integrand is determined by Gaussian probability distributions as [107]

$$
\begin{aligned}
&\int \mathcal{D}\mathbf{B} \exp\left[-\frac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}) + (\mathbf{J}|\mathbf{B}) \right] \\
&= \int \mathcal{D}\mathbf{B} \exp\left[-\frac{1}{2}\int dt \int dt' \mathbf{B}^{\dagger}(t)\mathbb{D}(t,t')\mathbf{B}(t') \right. \\
&\quad \left. + \int dt\mathbf{J}^{\dagger}(t)\mathbf{B}(t) \right] \\
&= \mathrm{Det}\left(\frac{\mathbb{D}}{2\pi}\right)^{1/2} \exp\left[\frac{1}{2}\int dt \int dt' \mathbf{J}^{\dagger}(t)\mathbb{G}(t,t')\mathbf{J}(t') \right],
\end{aligned}
\tag{A1}
$$

where $\mathbb{D}$ is any real Hermitian operator, $\mathbf{J}$ is any function over $\mathbb{R}^n$, and $\mathbb{G}$ is the inverse of the kernel operator $\mathbb{D}$. The inverse relation is determined by

$$
\int dt\mathbb{D}(t_1,t)\,\mathbb{G}(t,t_2) = \delta(t_1-t_2), \tag{A2}
$$

where δ is the Dirac delta distribution and the boundary condition $\lim_{|t|\to\infty}\mathbb{G}(t,t')=0$ must be satisfied. The path integral is well defined if and only if $\mathbb{D}$ is a *positive definite* operator. For these operators, functions such that $\lim_{|t|\to\infty}\mathbf{B}(t)\neq 0$ do not contribute to the integral, as $(\mathbf{B}|\mathbb{D}|\mathbf{B})$ diverges.

Since, for any real anti-Hermitian operator $\mathbb{D}^A$, $(\mathbf{B}|\mathbb{D}^A|\mathbf{B})=0$, one can only consider integrals where $\mathbb{D}$ is a Hermitian operator without loss of generality, as in Eq. (A1). This can be demonstrated by considering that an arbitrary $\mathbb{D}$ can be decomposed as

$$
\mathbb{D} = \mathbb{D}^H + \mathbb{D}^A \tag{A3}
$$

with $\mathbb{D}^H = (\mathbb{D}+\mathbb{D}^{\dagger})/2$ and $\mathbb{D}^A = (\mathbb{D}-\mathbb{D}^{\dagger})/2$ its Hermitian and anti-Hermitian parts, respectively. The value of the integral thus depends only on $\mathbb{D}^H$, and therefore we can consider $\mathbb{D}^A = 0$. In this article, we have considered $\mathbf{J} = i\mathbf{f}$ and $\mathbf{B}$ the noise fluctuating fields; therefore,

$$
\begin{aligned}
\langle e^{i\phi[\mathbf{B},\mathbf{f}]}\rangle &= \int \mathcal{D}\mathbf{B}\exp\left[-\frac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}) + i(\mathbf{f}|\mathbf{B}) \right] \\
&= \exp\left[-\frac{1}{2}(\mathbf{f}|\mathbb{G}|\mathbf{f}) \right]. \tag{A4}
\end{aligned}
$$

APPENDIX B: PARAMETRIZATION OF THE NOISE EIGENMODE BASIS

As described in Sec. IV A, Ω is a parameter whose physical meaning in general depends specifically on the nonstationary process, and on the parametrization of the functional behavior $S(\Omega)$, as it is always possible to reparametrize the eigenmodes by a different parameter. Similarly, while the frequency ω seems a natural choice for stationary environments, different parameters can be used to also describe the noise eigenmodes in stationary systems.

Here, we show examples of possible reparametrization of the noise eigenmodes for stationary noise processes. Specifically, rather than using $\Omega = (\omega, m)$ as discussed in Sec. IV B, we can chose $\Omega' = (\lambda, m) = (\omega^{1/3}, m)$. Therefore, the noise spectrum is $S'(\Omega') = S'_m(\lambda) = S_m(\omega = \lambda^3)$ and the noise eigenmode $|\Omega'\rangle = |\lambda, m\rangle = \sqrt{3}|\lambda|\mathbf{b}_m(\omega = \lambda^3)|\omega = \lambda^3\rangle$, where the prefactors are derived from the orthonormal relation $(\lambda, m|\lambda', m') = \delta_{m,m'}\delta(\lambda - \lambda')$, and $F'_m(\lambda) = (\lambda, m|\mathbf{f})$. Applying these the change of variable $\omega = \lambda^3$ to Eq. (21), one obtains

$$\langle e^{i\phi[\mathbf{B},\mathbf{f}]}\rangle = \exp\left[-\frac{1}{2}\sum_m \int d\lambda\, S'_m(\lambda)|F'_m(\lambda)|^2\right],$$

which is how Eq. (20) is expressed with this new parametrization.

Another example is analogous to how the free particle Hamiltonian eigenstates can be indexed either by their momentum or by their energy and parity. In this case, the complex exponentials are replaced by trigonometric functions and the noise eigenmodes are $|E, \pi, m\rangle$. Here

$$(t|E, 1, m) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{E}}\mathbf{b}_m(\omega = \sqrt{E})\cos(\sqrt{E}t)$$

and

$$(t|E, -1, m) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{E}}\mathbf{b}_m(\omega = \sqrt{E})\sin(\sqrt{E}t),$$

where the prefactors are again obtained from the normalization condition $(E, \pi, m|E', \pi', m') = \delta(E - E')\delta_{\pi,\pi'}\delta_{m,m'}$.

APPENDIX C: DERIVATION OF THE LOCAL-IN-TIME DIFFERENTIAL OPERATOR $\mathbb{D}$

The most general action that describes a local-in-time, nonstationary noise process is of the form

$$\mathcal{A}[\mathbf{B}] = \int dt \sum_k \sum_{l \leq k} \mathbf{B}^{\dagger(k)}(t)\mathbb{D}_{k,l}(t)\mathbf{B}^{(l)}(t), \qquad (C1)$$

as in Eq. (23) of the main text. Since the only paths that contribute to the dephasing integral satisfy

$\lim_{|t|\to\infty}\mathbf{B}^{(k)}(t) = 0$ for all $k \leq N$, we integrate by parts each of the action terms and find that

$$\int dt\, \mathbf{B}^{\dagger(k)}(t)\mathbb{D}_{k,l}(t)\mathbf{B}^{(l)}(t) = -\int dt\, \mathbf{B}^{\dagger(k-1)}(t)$$
$$\times [\mathbb{D}_{k,l}(t)\mathbf{B}^{(l+1)}(t) + \dot{\mathbb{D}}_{k,l}(t)\mathbf{B}^{(l)}(t)], \qquad (C2)$$

where $\dot{\mathbb{D}}_{k,l}(t)$ is the derivative of $\mathbb{D}_{k,l}(t)$. Integrating by parts thus transforms a single term containing the derivatives of orders k and l of $\mathbf{B}$ into two terms, one containing the derivatives of orders $k - 1$ and $l + 1$ and another containing the derivatives of orders $k - 1$ and l. Since $l \leq k$, this integration by parts reduces the difference between the orders of the derivatives when applied to terms with $k - l > 1$. By repeating this procedure successively, one can express the action such that it contains only terms with $k = l$ and $k = l + 1$:

$$\mathcal{A}[\mathbf{B}] = \int dt \sum_{k=0}^{N} \mathbf{B}^{\dagger(k)}(t)\tilde{\mathbb{D}}_k^H(t)\mathbf{B}^{(k)}(t)$$
$$+ \sum_{k=1}^{N} \mathbf{B}^{\dagger(k)}(t)\tilde{\mathbb{D}}_k^A(t)\mathbf{B}^{(k-1)}(t) \qquad (C3)$$

with $\tilde{\mathbb{D}}_k^H(t)$ and $\tilde{\mathbb{D}}_k^A(t)$ real matrices.

Considering the facts that the derivative

$$\frac{d}{dt}[\mathbf{B}^{\dagger(k)}(t)\tilde{\mathbb{D}}_{k+1}^A(t)\mathbf{B}^{(k)}(t)]$$
$$= \mathbf{B}^{\dagger(k+1)}(t)\tilde{\mathbb{D}}_{k+1}^A(t)\mathbf{B}^{(k)}(t)$$
$$+ \mathbf{B}^{\dagger(k)}(t)\tilde{\mathbb{D}}_{k+1}^A(t)\mathbf{B}^{(k+1)}(t) + \mathbf{B}^{\dagger(k)}(t)\dot{\tilde{\mathbb{D}}}_{k+1}^A(t)\mathbf{B}^{(k)}(t), \qquad (C4)$$

that

$$\int dt \frac{d}{dt}[\mathbf{B}^{\dagger(k)}(t)\tilde{\mathbb{D}}_k^A(t)\mathbf{B}^{(k)}(t)] = 0, \qquad (C5)$$

and that

$$\mathbf{B}^{\dagger(k)}(t)\tilde{\mathbb{D}}_{k+1}^A(t)\mathbf{B}^{(k+1)}(t) = \mathbf{B}^{\dagger(k+1)}(t)\tilde{\mathbb{D}}_{k+1}^{A\dagger}(t)\mathbf{B}^{(k)}(t), \qquad (C6)$$

the action can be written as

$$\mathcal{A}[\mathbf{B}] = \int dt \sum_{k=0}^{N} \mathbf{B}^{\dagger(k)}(t)\left(\tilde{\mathbb{D}}_k^H(t) - \frac{1}{2}\dot{\tilde{\mathbb{D}}}_{k+1}^A(t)\right)\mathbf{B}^{(k)}(t)$$
$$+ \sum_{k=1}^{N} \frac{1}{2}\mathbf{B}^{\dagger(k)}(t)[\tilde{\mathbb{D}}_k^A(t) - \tilde{\mathbb{D}}_k^{A\dagger}(t)]\mathbf{B}^{(k-1)}(t), \quad (C7)$$

where $\dot{\tilde{\mathbb{D}}}_{N+1}^A(t) = 0$. The terms $[\mathbb{D}_k^A(t) - \mathbb{D}_k^{A\dagger}(t)]$ are real anti-Hermitian matrices and the value of $\mathbf{B}^{\dagger(k)}(t)[\tilde{\mathbb{D}}_k^H(t) -$

$\frac{1}{2}\dot{\tilde{\mathbb{D}}}_{k+1}^A(t)]\mathbf{B}^{(k)}(t)$ depends only on the Hermitian part of $\tilde{\mathbb{D}}_k^H(t) - \frac{1}{2}\dot{\tilde{\mathbb{D}}}_k^A(t)$. Therefore, defining the real Hermitian matrices as

$$\mathbb{D}_k^H = \frac{\tilde{\mathbb{D}}_k^H(t) - \dot{\tilde{\mathbb{D}}}_{k+1}^A(t)/2 + \tilde{\mathbb{D}}_k^{H\dagger}(t) - \dot{\tilde{\mathbb{D}}}_{k+1}^{A\dagger}(t)/2}{2} \quad (C8)$$

and the real anti-Hermitian matrices as

$$\mathbb{D}_k^A = \frac{\mathbb{D}_k^A(t) - \mathbb{D}_k^{A\dagger}(t)}{2}, \quad (C9)$$

the action can be simplified to

$$\mathcal{A}[\mathbf{B}] = \int dt \sum_{k=0}^N \mathbf{B}^{\dagger(k)}(t)\mathbb{D}_k^H(t)\mathbf{B}^{(k)}(t)$$

$$+ \int dt \sum_{k=1}^N \mathbf{B}^{\dagger(k)}(t)\mathbb{D}_k^A(t)\mathbf{B}^{(k-1)}(t). \quad (C10)$$

Defining then the Hermitian differential operator as

$$\mathbb{D} = \sum_{k=0}^N \overset{\leftarrow}{\partial_t^k}\mathbb{D}_k^H(t)\,\partial_t^k$$

$$+ \frac{1}{2}\sum_{k=1}^N \left[\overset{\leftarrow}{\partial_t^k}\mathbb{D}_k^A(t)\partial_t^{k-1} - \overset{\leftarrow}{\partial_t^{k-1}}\mathbb{D}_k^A(t)\partial_t^k\right], \quad (C11)$$

we thus obtain

$$\mathcal{A}[\mathbf{B}] = \tfrac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}), \quad (C12)$$

as defined in the main text in Eq. (24).

The operator $\mathbb{D}$ acts on the field function $\mathbf{B}(t)$ as

$$(t|\mathbb{D}|\mathbf{B}) = \sum_{k=0}^N (-1)^k \partial_t^k [\mathbb{D}_k^H(t)\partial_t^k \mathbf{B}(t)]$$

$$+ \frac{1}{2}\sum_{k=1}^N (-1)^k \{\partial_t^k[\mathbb{D}_k^A(t)\partial_t^{k-1}\mathbf{B}(t)]\}v$$

$$+ \frac{1}{2}\sum_{k=1}^N (-1)^k \{\partial_t^{k-1}[\mathbb{D}_k^A(t)\partial_t^k \mathbf{B}(t)]\}, \quad (C13)$$

where $\overset{\leftarrow}{\partial_t} = -\partial_t$ over the space of functions considered. Integration by parts, we obtain

$$(f|\overset{\leftarrow}{\partial_t}|g) = \int dt \dot{f}(t)g(t) = -\int f(t)\dot{g}(t) = -(f|\partial_t|g). \quad (C14)$$

This implies that, for a stationary process,

$$(\mathbf{B}|\overset{\leftarrow}{\partial_t^k}\mathbb{D}_k^H \partial_t^k|\mathbf{B}) = \int dt \mathbf{B}^{(k)\dagger}(t)\mathbb{D}_k^H \mathbf{B}^{(k)}(t)$$

$$= (\mathbf{B}|(-1)^k \mathbb{D}_k^H \partial_t^{2k}|\mathbf{B}) \quad (C15)$$

and

$$(\mathbf{B}|\overset{\leftarrow}{\partial_t^k}\mathbb{D}_k^A \partial_t^{k-1}|\mathbf{B}) = \int dt \mathbf{B}^{(k)\dagger}(t)\mathbb{D}_k^A \mathbf{B}^{(k-1)}(t)$$

$$= (\mathbf{B}|(-1)^k \mathbb{D}_k^H \partial_t^{2k-1}|\mathbf{B}), \quad (C16)$$

so that the differential operator is

$$\mathbb{D} = \sum_{k=0}^N (-1)^k \mathbb{D}_k^H \partial_t^{2k} + \sum_{k=1}^N (-1)^k \mathbb{D}_k^A \partial_t^{2k-1}. \quad (C17)$$

APPENDIX D: DIFFERENTIABILITY OF THE FIELD PATHS FOR LOCAL-IN-TIME NOISE PROCESSES

In this appendix we demonstrate that the field paths and their first $N-1$ derivatives must be continuous for local-in-time noise processes. We thus study the differentiability of the paths allowed by the action. Considering a particular $\mathbf{B}(t)$, let us assume that at some time $\bar{t}$ some of the first $N-1$ derivatives of $\mathbf{B}(t)$ are not continuous at $t = \bar{t}$. Then, the Nth-order derivative

$$\mathbf{B}^{(N)}(t) = \mathbf{B}_0^{(N)}(t) + \sum_{k=0}^{N-1} \Delta\mathbf{B}^{(k)}\delta^{(N-1-k)}(t-\bar{t}), \quad (D1)$$

where the coefficients

$$\Delta\mathbf{B}^{(k)} = \lim_{t\to\bar{t}^+} \mathbf{B}^{(k)}(t) - \lim_{t\to\bar{t}^-} \mathbf{B}^{(k)}(t) \quad (D2)$$

for functions for which these limits exist and $\mathbf{B}_0^{(N)}(t)$ is a distribution such that $\mathbf{B}_0^{(N)}(\bar{t})$ is finite. Then, terms like the following contribute to the action:

$$\int dt \,\mathbf{B}^{(N)\dagger}(t)\mathbb{D}_N^H(t)\mathbf{B}^{(N)}(t)$$

$$= \int dt \,\mathbf{B}_0^{(N)\dagger}(t)\mathbb{D}_N^H(t)\mathbf{B}_0^{(N)}(t)$$

$$+ \int dt 2\mathbf{B}_0^{(N)\dagger}(t)\mathbb{D}_N^H(t)\sum_{k=0}^{N-1}\Delta\mathbf{B}^{(k)}\delta^{(N-1-k)}(t-\bar{t})$$

$$+ \int dt\left[\sum_{k=0}^{N-1}\Delta\mathbf{B}^{(k)\dagger}\delta^{(N-1-k)}(t-\bar{t})\right]\mathbb{D}_N^H(t)$$

$$\times\left[\sum_{k=0}^{N-1}\Delta\mathbf{B}^{(k)}\delta^{(N-1-k)}(t-\bar{t})\right]. \quad (D3)$$

The last term of this integral gives a positively divergent contribution that depends quadratically on the discontinuity of $\mathbf{B}(t)$, given by $\Delta\mathbf{B}^{(k)}$, and the first $N-1$ derivatives of the field at $t = \bar{t}$. The discontinuities of higher derivatives do not appear in the action. Since the operator $\mathbb{D}(t)$ is definite positive, the total contribution of these discontinuous paths to the action must be infinite. Therefore, the probability of discontinuous paths and paths with discontinuities in their first $N-1$ derivatives is zero, and does not contribute to the path integral. Thus we have concluded that the field paths and their first $N-1$ derivatives must be continuous for local-in-time processes.

APPENDIX E: DISCRETIZATION OF LOCAL-IN-TIME PROCESSES

In this appendix we show an interpretation of the meaning of local-in-time processes by implementing a time discretization. We replace the continuous-time variable t by the discrete sequence of times $t_j = j\,\Delta t$. The action is thus

$$\mathcal{A}[\mathbf{B}] = \sum_j \Delta t \sum_{k=0}^{N} \mathbf{B}^{\dagger(k)}(t_j)\mathbb{D}_k^H(t_j)\mathbf{B}^{(k)}(t_j)$$
$$+ \sum_j \Delta t \sum_{k=1}^{N} \mathbf{B}^{\dagger(k)}(t_j)\mathbb{D}_k^A(t_j)\mathbf{B}^{(k-1)}(t_j),$$

where the field derivatives have been replaced using finite differences $\dot{\mathbf{B}}(t_j) = [\mathbf{B}(t_{j+1}) - \mathbf{B}(t_{j-1})]/2\Delta t$ with N the highest derivative order that contributes to the action. Within this discrete representation, the fluctuating field $\mathbf{B}$ can be defined by a vector $\mathbf{B}_j = \mathbf{B}(t_j)$ and the kernel operator $\mathbb{D}$ can be represented by the matrix $\mathbb{D}_{jj'} = \mathbb{D}(t_j, t_{j'})$. Therefore, the kth derivative of the field $[\mathbf{B}^{(k)}]_j = \sum_{j'=j-k}^{j+k}(c_k/\Delta t^k)\mathbf{B}_{j'}$ is a linear combination of the values that $\mathbf{B}$ takes at the $2k+1$ times closest to j, where the coefficients c_k are independent of Δt. Thus, a kernel operator corresponding to a local-in-time process has coefficients that satisfy $\mathbb{D}_{jj'} = 0$ for $|j - j'| > N$. Within this discrete picture, local-in-time processes are those whose kernel operator is N diagonal, i.e., its matrix representation has nonzero coefficients only in the first N central diagonals. In this representation only, if $N = 0$, the kernel operator is diagonal in the time basis and therefore $\mathbf{B}(t)$ and $\mathbf{B}(t')$ are uncorrelated for $t \neq t'$. However, for $N \geq 1$, the field at the Nth closest times are correlated. Still, local-in-time processes are such that long-time correlations do not exist.

APPENDIX F: NON-MARKOVIAN PROOF FOR $N > 1$

In this appendix we analyze the conditions for nonstationary noise to be non-Markovian. As we stated in the

main text, noise processes that are not local in time cannot be Markovian since the action explicitly relates the value of the fluctuating field $\mathbf{B}$ at different times. A noise with $N = 0$ is the so-called white noise, and therefore it is Markovian. Therefore, in the following demonstration we only consider local-in-time noise processes with $N \geq 1$. In Appendix G we prove that the generalized noise process given by $\{\mathbf{B}^{(k)}(t)\}$ is Markovian. As a particular case, if $N = 1$ then $\{\mathbf{B}^{(k)}(t)\} \equiv \{\mathbf{B}^{(k)}(t): k = 0\} = \{\mathbf{B}(t)\}$, therefore showing that processes with $N = 1$ are Markovian. In Appendix D, we prove that the field paths that contribute to the path integral for local-in-time noise processes are not only continuous, but are $N-1$ times continuously differentiable. This property allows us to demonstrate that noise processes with $N > 1$ are not Markovian.

To do this, we consider the values of the field at two infinitesimally close times t_0 and t_1 to be $\mathbf{B}_0$ and $\mathbf{B}_1$, respectively. According to this assumption, the field derivative $\dot{\mathbf{B}}(t_1) = (\mathbf{B}_1 - \mathbf{B}_0)/(t_1 - t_0)$ must be continuous. If we now consider the field value at a time t_f, infinitesimally close and after t_1, we know again that $\dot{\mathbf{B}}$ will change continuously. Therefore, the mean value of the probability distribution $P(\mathbf{B}_f, t_f | \mathbf{B}_1, t_1; \mathbf{B}_0, t_0)$ is $\mathbf{B}_1 + \dot{\mathbf{B}}(t_1)(t_f - t_1) = \mathbf{B}_1[1 + (t_f - t_1)/(t_1 - t_0)] - \mathbf{B}_0(t_f - t_1)/(t_1 - t_0)$. This means that the state of the field at t_f depends on the states $\mathbf{B}_0$ and $\mathbf{B}_1$ at times t_0 and t_1, respectively. Therefore, the probability distribution of the field depends on the state of the system at least at two previous times; thus,

$$P(\mathbf{B}_f, t_f | \mathbf{B}_1, t_1; \mathbf{B}_0, t_0) \neq P(\mathbf{B}_f, t_f | \mathbf{B}_1, t_1). \tag{F1}$$

A Markovian process must satisfy Eq. (30); therefore, in this case we have

$$P(\mathbf{B}_f, t_f | \mathbf{B}_0, t_0)$$
$$= \int d\mathbf{B}_1 P(\mathbf{B}_f, t_f | \mathbf{B}_1, t_1; \mathbf{B}_0, t_0)P(\mathbf{B}_1, t_1 | \mathbf{B}_0, t_0)$$
$$\neq \int d\mathbf{B}_1 P(\mathbf{B}_f, t_f | \mathbf{B}_1, t_1)P(\mathbf{B}_1, t_1 | \mathbf{B}_0, t_0), \tag{F2}$$

and thus the process is not Markovian. This example evidences the key considerations that show why a local-in-time noise process is not Markovian if the differential operator $\mathbb{D}(t)$ contains derivatives higher than order $N = 1$. In this case of $N > 1$, the state at a given time depends on the information about previous states that is encoded on the continuous derivatives of the field.

APPENDIX G: MARKOVIAN GENERALIZED STOCHASTIC PROCESS

In this appendix, we find a formula for the propagator of the generalized noise process

$$\{\mathbf{B}^{(k)}(t)\} \equiv \{\mathbf{B}^{(k)}(t) \colon k = 0, \ldots, N-1\}$$
$$= \{\mathbf{B}(t), \dot{\mathbf{B}}(t), \ldots, \mathbf{B}^{(N-1)}(t)\}, \qquad \text{(G1)}$$

introduced in Sec. V of the main text, in terms of path integrals. We show that it can be separated into three independent path integrals with boundary conditions. We use this integral decomposition to prove that the process that describes $\{\mathbf{B}^{(k)}(t)\}$ is Markovian.

We consider how the probability distribution of the generalized field fluctuations $\{\mathbf{B}^{(k)}(t)\}$ evolves with time. To demonstrate the Markovian condition of Eq. (30) of the main text, we need to calculate the probability of the state $\{\mathbf{B}_f^{(k)}\}$ at time t_f given that at time t_0 the field and its derivatives are $\{\mathbf{B}_0^{(k)}\}$. This conditional probability is given by

$$P(\{\mathbf{B}_f^{(k)}\}, t_f | \{\mathbf{B}_0^{(k)}\} t_0)$$
$$= \frac{1}{P(\{\mathbf{B}_0^{(k)}\}, t_0)} \int_{\{\mathbf{B} | \{\mathbf{B}^{(k)}(t_{0,f})\} = \{\mathbf{B}_{0,f}^{(k)}\}\}} \mathcal{D}\mathbf{B} \exp$$
$$\times \left[-\frac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}) \right], \qquad \text{(G2)}$$

where the path integral runs over all the paths $\mathbf{B}(t)$ such that $\{\mathbf{B}^{(k)}(t_0)\} = \{\mathbf{B}_0^{(k)}\}$ and $\{\mathbf{B}^{(k)}(t_f)\} = \{\mathbf{B}_f^{(k)}\}$, denoted by $\{\mathbf{B} | \{\mathbf{B}^{(k)}(t_{0,f})\} = \{\mathbf{B}_{0,f}^{(k)}\}\}$ in the integral.

We first determine the probability of the state

$$\{\mathbf{B}_0^{(k)}\} \equiv \{\mathbf{B}_0, \ldots, \mathbf{B}_0^{(N-1)}\} \qquad \text{(G3)}$$

at time t_0, which is given by

$$P(\{\mathbf{B}_0^{(k)}\}, t_0) = \int_{\{\mathbf{B} | \{\mathbf{B}^{(k)}(t_0)\} = \{\mathbf{B}_0^{(k)}\}\}} \mathcal{D}\mathbf{B} \exp\left[-\frac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}) \right],$$
$$\text{(G4)}$$

where the integral runs over all paths that pass through $\{\mathbf{B}_0^{(k)}\}$ at time $t = t_0$. The continuity conditions of $\{\mathbf{B}^{(k)}\}$ demonstrated in Appendix D imply that integrating over all paths such that $\{\mathbf{B}^{(k)}(t_0)\} = \{\mathbf{B}_0^{(k)}\}$ is the same as integrating over all paths that end at t_0 with $\{\mathbf{B}^{(k)}(t_0)\} = \{\mathbf{B}_0^{(k)}\}$, and over all paths that start at t_0 with $\{\mathbf{B}^{(k)}(t_0)\} = \{\mathbf{B}_0^{(k)}\}$, and then multiplying these two integrals. Therefore, the probability

$$P(\{\mathbf{B}_0^{(k)}\}, t_0) = \mathcal{I}_{(0,-\infty)}^{(\{\mathbf{B}_0^{(k)}\}, t_0)} \times \mathcal{I}_{(\{\mathbf{B}_0^{(k)}\}, t_0)}^{(0, +\infty)}, \qquad \text{(G5)}$$

where we have introduced the integrals with boundary conditions for the paths

$$\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\}, t_0)}^{(0, +\infty)} \equiv \int_{(\{\mathbf{B}_0^{(k)}\}, t_0)}^{(0, +\infty)} \mathcal{D}\mathbf{B} \exp$$
$$\times \left[-\frac{1}{2} \int_{t_0}^{+\infty} dt \, \mathbf{B}^\dagger(t) \mathbb{D}(t) \mathbf{B}(t) \right], \qquad \text{(G6)}$$

where the integral runs over all paths that start at time t_0 with value $\{\mathbf{B}_0^{(k)}\}$ and go to $t \to +\infty$ with $\lim_{t \to \infty}\{\mathbf{B}^{(k)}(t)\} = 0$, and

$$\mathcal{I}_{(0,-\infty)}^{(\{\mathbf{B}_0^{(k)}\}, t_0)} \equiv \int_{(0,-\infty)}^{(\{\mathbf{B}_0^{(k)}\}, t_0)} \mathcal{D}\mathbf{B} \exp\left[-\frac{1}{2} \int_{-\infty}^{t_0} dt \, \mathbf{B}^\dagger(t) \mathbb{D}(t) \mathbf{B}(t) \right],$$
$$\text{(G7)}$$

where the integral runs over all paths that come from $t \to -\infty$ with $\lim_{t \to -\infty}\{\mathbf{B}^{(k)}(t)\} = 0$, and end at time t_0 with value $\{\mathbf{B}_0^{(k)}\}$.

We then calculate the integral on the numerator of Eq. (G2), which gives

$$\int_{\{\mathbf{B} | \{\mathbf{B}^{(k)}(t_{0,f})\} = \{\mathbf{B}_{0,f}^{(k)}\}\}} \mathcal{D}\mathbf{B} \exp\left[-\frac{1}{2}(\mathbf{B}|\mathbb{D}|\mathbf{B}) \right]$$
$$= \mathcal{I}_{(0,-\infty)}^{(\{\mathbf{B}_0^{(k)}\}, t_0)} \times \mathcal{I}_{(\{\mathbf{B}_0^{(k)}\}, t_0)}^{(\{\mathbf{B}_f^{(k)}\}, t_f)} \times \mathcal{I}_{(\{\mathbf{B}_f^{(k)}\}, t_f)}^{(0, +\infty)}, \qquad \text{(G8)}$$

where we have introduced the integral with boundary conditions

$$\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\}, t_0)}^{(\{\mathbf{B}_f^{(k)}\}, t_f)} \equiv \int_{(\{\mathbf{B}_0^{(k)}\}, t_0)}^{(\{\mathbf{B}_0^{(k)}\}, t_0)} \mathcal{D}\mathbf{B} \exp\left[-\frac{1}{2} \int_{t_0}^{t_f} dt \, \mathbf{B}^\dagger(t) \mathbb{D}(t) \mathbf{B}(t) \right],$$
$$\text{(G9)}$$

where the integral runs over all the paths that start at time t_0 with value $\{\mathbf{B}_0^{(k)}\}$ and end at time t_f with value $\{\mathbf{B}_f^{(k)}\}$. Again, we have considered the continuity conditions on the field paths $\{\mathbf{B}^{(k)}\}$ demonstrated in Appendix D. The integrals $\mathcal{I}_{(\{\mathbf{B}_f^{(k)}\}, t_f)}^{(0, +\infty)}$ and $\mathcal{I}_{(0,-\infty)}^{(\{\mathbf{B}_0^{(k)}\}, t_0)}$ are defined in Eqs. (G6) and (G7), respectively.

Now, using Eqs. (G5) and (G8), we obtain the conditional probability of the noise process

$$P(\{\mathbf{B}_f^{(k)}\}, t_f \,|\{\mathbf{B}_0^{(k)}\}|t_0) = \frac{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)} \times \mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}}{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}}. \tag{G10}$$

In order to prove that this process is Markovian, we introduce an intermediate time t_1 and calculate the integral

$$\int d\{\mathbf{B}_1^{(k)}\} P(\{\mathbf{B}_f^{(k)}\}, t_f \,|\{\mathbf{B}_1^{(k)}\}, t_1) P(\{\mathbf{B}_1^{(k)}\}, t_1|\{\mathbf{B}_0^{(k)}\}, t_0) = \int d\{\mathbf{B}_1^{(k)}\} \frac{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_1^{(k)}\},t_1)} \times \mathcal{I}_{(\{\mathbf{B}_1^{(k)}\},t_1)}^{(0,+\infty)}}{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}} \frac{\mathcal{I}_{(\{\mathbf{B}_1^{(k)}\},t_1)}^{(\{\mathbf{B}_f^{(k)}\},t_f)} \times \mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}}{\mathcal{I}_{(\{\mathbf{B}_1^{(k)}\},t_1)}^{(0,+\infty)}}$$

$$= \frac{\mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}}{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}} \int d\{\mathbf{B}_1^{(k)}\} \mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_1^{(k)}\},t_1)} \times \mathcal{I}_{(\{\mathbf{B}_1^{(k)}\},t_1)}^{(\{\mathbf{B}_f^{(k)}\},t_f)}, \tag{G11}$$

which appears in the Markovian condition of Eq. (30), where $d\{\mathbf{B}_1^{(k)}\} \equiv d\mathbf{B}_1 d\dot{\mathbf{B}}_1 \cdots d\mathbf{B}^{(N-1)}$.

The continuity conditions for $\{\mathbf{B}^{(k)}\}$ at t_1 imply that $\mathbf{B}^{(k)}(t_1^-) = \mathbf{B}^{(k)}(t_1^+)$ for $0 \leq k \leq N-1$. Then these continuity conditions imply that the integral

$$\int d\{\mathbf{B}_1^{(k)}\} \mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_1^{(k)}\},t_1)} \times \mathcal{I}_{(\{\mathbf{B}_1^{(k)}\},t_1)}^{(\{\mathbf{B}_f^{(k)}\},t_f)} = \mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)}, \tag{G12}$$

and, thus,

$$\int d\{\mathbf{B}_1^{(k)}\} P(\{\mathbf{B}_f^{(k)}\}, t_f \,|\{\mathbf{B}_1^{(k)}\}, t_1) P(\{\mathbf{B}_1^{(k)}\}, t_1|\{\mathbf{B}_0^{(k)}\}, t_0)$$

$$= \frac{\mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}}{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}} \mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)}$$

$$= P(\{\mathbf{B}_f^{(k)}\}, t_f \,|\{\mathbf{B}_0^{(k)}\}, t_0), \tag{G13}$$

demonstrating that the generalized noise process is Markovian.

APPENDIX H: MARKOVIAN PROPAGATOR FOR THE GENERALIZED STOCHASTIC PROCESS

Since local-in-time noise processes can be described by the generalized Markovian process of $\{\mathbf{B}^{(k)}(t)\}$, its *propagator* $P(\{\mathbf{B}_f^{(k)}\}, t_f \,|\{\mathbf{B}_0^{(k)}\}, t_0)$ contains all the *information* about the stochastic process that describes the field paths **B**. The path integral framework allows us to calculate the *propagator* of $\{\mathbf{B}^{(k)}\}$ without actually requiring to perform *path integrals*. Instead, it can be obtained by solving an *ordinary linear differential equation* with three different types of *boundary condition*. The propagator can be expressed as

$$P(\{\mathbf{B}_f^{(k)}\}, t_f \,|\{\mathbf{B}_0^{(k)}\}|t_0) = \frac{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)} \times \mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}}{\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}}, \tag{H1}$$

according to Eq. (G10).

In order to obtain the explicit formula for the propagator for local-in-time nonstationary noise processes, we only need to calculate the integrals $\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)}$, $\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}$, and $\mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}$. To do this, we introduce the classical field $\mathbf{B}_{\text{cl}}$, which is defined as the path that minimizes the action with fixed boundary conditions. Using variational analysis, one can find that $\mathbf{B}_{\text{cl}}$ is the solution of the differential equation

$$\sum_{k=0}^{N} (-1)^k \partial_t^k [\mathbb{D}_k^H(t) \partial_t^k \mathbf{B}_{\text{cl}}(t)]$$

$$+ \frac{1}{2} \sum_{k=1}^{N} (-1)^k \{\partial_t^k [\mathbb{D}_k^A(t) \partial_t^{k-1} \mathbf{B}_{\text{cl}}(t)]\}$$

$$+ \frac{1}{2} \sum_{k=1}^{N} (-1)^k \{\partial_t^{k-1} [\mathbb{D}_k^A(t) \partial_t^k \mathbf{B}_{\text{cl}}(t)]\} = 0 \tag{H2}$$

with the corresponding boundary conditions for each of the integrals. They are

$$\{\mathbf{B}_{\text{cl}}^{(k)}(t_0)\} = \{\mathbf{B}_0^{(k)}\}, \tag{H3}$$

$$\{\mathbf{B}_{\text{cl}}^{(k)}(t_f)\} = \{\mathbf{B}_f^{(k)}\} \tag{H4}$$

for $\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)}$,

$$\{\mathbf{B}_{\mathrm{cl}}^{(k)}(t_0)\} = \{\mathbf{B}_0^{(k)}\}, \tag{H5}$$

$$\lim_{t \to +\infty} \{\mathbf{B}_{\mathrm{cl}}^{(k)}(t)\} = 0 \tag{H6}$$

for $\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}$, and

$$\{\mathbf{B}_{\mathrm{cl}}^{(k)}(t_f)\} = \{\mathbf{B}_f^{(k)}\}, \tag{H7}$$

$$\lim_{t \to +\infty} \{\mathbf{B}_{\mathrm{cl}}^{(k)}(t)\} = 0 \tag{H8}$$

for $\mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}$. Note that Eq. (H2) is a linear differential equation of degree $2N$, and the boundary constraints provide the necessary conditions for the solution to be unique. For the $\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)}$ case, the classical field $\mathbf{B}_{\mathrm{cl}}(t)$ is the path that minimizes the action $\mathcal{A}[\mathbf{B}]$ of all paths between t_0 and t_f with fixed endpoints at the bounds of the integral. The classical field encodes the dependency of the integral on the boundary conditions $(\{\mathbf{B}_0^{(k)}\}, t_0)$ and $(\{\mathbf{B}_f^{(k)}\}, t_f)$. For $\mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}$ and $\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}$, the classical field is the path that minimizes the action over the corresponding time intervals, and encodes the dependency on the corresponding boundary conditions.

By performing the change of variables $\mathbf{B}(t) = \Delta\mathbf{B}(t) + \mathbf{B}_{\mathrm{cl}}(t)$, we can write the integral as

$$\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)} = \int_{(\{\Delta\mathbf{B}^{(k)}\}=0,t_0)}^{(\{\Delta\mathbf{B}^{(k)}\}=0,t_f)} \mathcal{D}\Delta\mathbf{B}\, e^{-\mathcal{A}[\Delta\mathbf{B}+\mathbf{B}_{\mathrm{cl}}]}, \tag{H9}$$

where the integral runs over all paths that start at t_0 and end at t_f with value $\{\Delta\mathbf{B}^{(k)}\} = 0$ and we have considered that the path integral is *invariant under shifts*. Note that the limits of the path integral do not depend on the value of the fields at t_0 and t_f. Similar expressions are obtained for the other two integrals.

The action can thus be written as

$$\mathcal{A}[\mathbf{B}_{\mathrm{cl}} + \Delta\mathbf{B}] = \mathcal{A}[\mathbf{B}_{\mathrm{cl}}] + \mathcal{A}[\Delta\mathbf{B}]$$
$$+ 2\int_{t_0}^{t_f} \Delta\mathbf{B}(t)\mathbb{D}(t)\mathbf{B}_{\mathrm{cl}}(t). \tag{H10}$$

Equation (H2) holds if and only if $\int_{t_0}^{t_f} \Delta\mathbf{B}(t)\mathbb{D}(t)\mathbf{B}_{\mathrm{cl}}(t) = 0$ for any $\Delta\mathbf{B}(t)$ with $\{\Delta\mathbf{B}^{(k)}(t_f)\} = \{\Delta\mathbf{B}^{(k)}(t_0)\} = 0$. Therefore, the last term in Eq. (H10) vanishes and we obtain

$$\mathcal{A}[\mathbf{B}_{\mathrm{cl}} + \Delta\mathbf{B}] = \mathcal{A}[\mathbf{B}_{\mathrm{cl}}] + \mathcal{A}[\Delta\mathbf{B}], \tag{H11}$$

and the integral becomes

$$\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)} = \int_{(\{\Delta\mathbf{B}^{(k)}\}=0,t_0)}^{(\{\Delta\mathbf{B}^{(k)}\}=0,t_f)} \mathcal{D}\Delta\mathbf{B}\, e^{-(\mathcal{A}[\Delta\mathbf{B}]+\mathcal{A}[\mathbf{B}_{\mathrm{cl}}])}. \tag{H12}$$

We are thus able to separate the original path integral into two parts: a path integral with no dependence on the initial and final fields

$$M_{t_0}^{t_f} = \int_{(\{\Delta\mathbf{B}^{(k)}\}=0,t_0)}^{(\{\Delta\mathbf{B}^{(k)}\}=0,t_f)} \mathcal{D}\Delta\mathbf{B}\, e^{-\mathcal{A}[\Delta\mathbf{B}]}, \tag{H13}$$

and a term without path integrals that does depend on the initial and final fields

$$e^{-\mathcal{A}[\mathbf{B}_{\mathrm{cl}}]}. \tag{H14}$$

The path integral $M_{t_0}^{t_f}$ is just a constant that depends on the initial and final times but not on the initial and final fields. Since $M_{t_0}^{t_f}$ does not depend on the values of $\{\mathbf{B}_0^{(k)}\}$ and $\{\mathbf{B}_f^{(k)}\}$, its effect on the propagator is that of a renormalization constant that can be calculated by demanding the normalization of the conditional probability $\int d\{\mathbf{B}_f\}P(\{\mathbf{B}_f\}, t_f\,|\{\mathbf{B}_0\}, t_0) = 1$. Then, if we define the classical action

$$\mathcal{A}_{\mathrm{cl}}(\{\mathbf{B}_f^{(k)}\}, t_f; \{\mathbf{B}_0^{(k)}\}, t_0) \equiv \mathcal{A}[\mathbf{B}_{\mathrm{cl}}], \tag{H15}$$

the integral

$$\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(\{\mathbf{B}_f^{(k)}\},t_f)} = M_{t_0}^{t_f}\, e^{-\mathcal{A}_{\mathrm{cl}}(\{\mathbf{B}_f\},t_f;\{\mathbf{B}_0\},t_0)} \tag{H16}$$

is just a normal integral based on the solution of the classical field for the ordinary differential equation. Analogous results hold for $\mathcal{I}_{(\{\mathbf{B}_0^{(k)}\},t_0)}^{(0,+\infty)}$ and $\mathcal{I}_{(\{\mathbf{B}_f^{(k)}\},t_f)}^{(0,+\infty)}$, where one must solve the same differential equation, but change the boundary conditions.

We have therefore shown here that the problem of calculating the Markovian propagator reduces to solving an ordinary differential equation with three different boundary conditions up to a normalization constant. Its solution is thus given by

$$P(\{\mathbf{B}_f^{(k)}\}, t_f\,|\{\mathbf{B}_0^{(k)}\}|t_0) = \frac{M_{t_0}^{t_f} \times M_{t_f}^{\infty}}{M_{t_0}^{\infty}}$$
$$\times \exp\left(-[\mathcal{A}_{\mathrm{cl}}(\{\mathbf{B}_f\}, t_f; \{\mathbf{B}_0\}, t_0) + \mathcal{A}_{\mathrm{cl}}(0, +\infty; \{\mathbf{B}_f\}, t_f) - \mathcal{A}_{\mathrm{cl}}(0, +\infty; \{\mathbf{B}_0\}, t_0)]\right),$$

where $(M_{t_0}^{t_f} \times M_{t_f}^{\infty})/M_{t_0}^{\infty}$ is a normalization constant.

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