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REMARKS ON ANNIHILATORS PRESERVING CONGRUENCE RELATIONS

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ABSTRACT. In this note we shall give some results on annihilators preserving congruence relations, or AP-congruences, in bounded distributive lattices. We shall give some new characterizations, and a topological interpretation of the notion of annihilator preserving congruences introduced in [JANOWITZ, M. F.: *Annihilator preserving congruence relations of lattices*, Algebra Universalis **5** (1975), 391–394]. As an application of these results, we shall prove that the quotient of a quasicomplemented lattice by means of a AP-congruence is a quasicomplemented lattice. Similarly, we will prove that the quotient of a normal latttice by means of a AP-congruence is also a normal lattice.

1. Introduction and preliminaries

In [7] M. F. Janowitz defined the notion of annihilator preserving congruence relation, or AP-congruence, in a bounded distributive lattice A as a lattice-congruence θ such that for all $a, b \in A$, $a \wedge b \equiv_\theta 0$ implies that there exists $c \in A$ such that $a \wedge c = 0$ and $c \equiv_\theta b$. It is easy to see that if A is a pseudocomplemented bounded distributive lattice, then a lattice-congruence θ is an AP-congruence iff it is a congruence of A. The main aim of this paper is to give some new characterizations of this notion.

The paper is organized in the following fashion. In the rest of this section we shall give some necessary notations and definitions. We will recall the Priestley representation for bounded distributive lattices, and we shall recall some

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properties of annihilators. In Section 2 we will show some new characterizations of AP-congruences in bounded distributive lattices. Particularly, we shall see that the AP-congruences in a bounded distributive lattice A are in bijective correspondence with certains closed subsets of the Priestley space of A. This correspondence is an extension of already known result for congruences in pseudocomplemented distributive lattices (see [6]). In Section 3 we shall give some applications of AP-congruences. We will prove that the quotient of a quasicomplemented lattice (see [11]) by means of an AP-congruence is a quasicomplemented lattice. Similarly, we will prove that the quotient of a normal lattice by means of an AP-congruence is a normal lattice.

The variety of bounded distributive lattices is denoted by DLat. The filter, and the ideal generated by a subset $X \subseteq A$ will be denoted by $F(X)$, and $I(X)$, respectively. When $X = \{a\}$ we will write $F(a)$ or $[a)$, and $I(a)$ or $(a]$, for denote the filter and the ideal generated by $\{a\}$. The lattice of all filters is denoted by $\mathrm{Fi}(A)$. The family of the prime filters of A is denoted by $X(A)$. Given $A \in \mathsf{DLat}$, let $\phi\colon A \to \mathcal{P}(X(A))$ be the Stone map defined by $\phi(a) = \{P \in X(A) \mid a \in P\}$, for each $a \in A$.

Let us consider a poset $\langle X, \leq\rangle$. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*) if for all $x, y \in X$ such that $x \in U$ ($y \in U$) and $x \leq y$, we have $y \in U$ ($x \in U$).

For each $Y \subseteq X$, the *increasing* (*decreasing*) set generated by Y is

$$[Y) = \big\{x \in X \mid (\exists y \in Y)(y \leq x)\big\} \qquad \big((Y] = \big\{x \in X \mid (\exists y \in Y)(x \leq y)\big\}\big).$$

If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. Let $\max X$ denote the maximal elements of X. For each $x \in X$, let

$$\max[x) = \max X \cap [x)\,.$$

A *totally order-disconnected topological space* is a triple $X = \langle X, \leq, \tau_X\rangle$, where $\langle X, \leq\rangle$ is a poset, $\langle X, \tau_X\rangle$ is a topological space and given $x, y \in X$ such that $x \not\leq y$ there is a clopen up-set U such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space. If X is a Priestley space the family of all clopen up-sets of X is denoted by $D(X)$, and it is well known that $D(X) = \langle D(X), \cup, \cap, \emptyset, X\rangle$ is a bounded distributive lattice.

The Priestley space of a bounded distributive lattice A is the triple $X(A) = \langle X(A), \subseteq, \tau_{X(A)}\rangle$, where $\tau_{X(A)}$ is the topology generated by taking as a subbase the family

$$\{\phi(a) \mid a \in A\} \cup \{\phi(a)^c \mid a \in A\}\,,$$

where $\phi(a)^c = X(A) \smallsetminus \phi(a)$ (see [9]).

Let I be an ideal of A. Consider the subset

$$\psi(I) = \big\{P \in X(A) \mid I \cap P \neq \emptyset\big\}.$$

It is easy to see that $\psi(I) = \bigcup \{\phi(a) \mid a \in I\}$, and $\phi(a) = \psi((a])$, for each $a \in A$.

Let $A \in \mathsf{DLat}$. The *annihilator* of $a \in A$ is the set

$$a^\circ = \{c \in A \mid a \wedge c = 0\}.$$

It is easy to see that the annihilator of an element a is an ideal. If a° is a principal ideal, i.e., there exists $x \in A$ such that $a^\circ = (x]$, then x is the pseudocomplement of a. The concept of annihilator is a natural generalization of the notion of pseudocomplement. Annihilators in distributive lattices have been studied by many authors. For instance in [3] W. H. Cornish proved that a lattice A with 0 (resp. with 0 and 1) is a generalized Stone lattice (resp. Stone lattice) if and only if the lattice $\{a^\circ \mid a \in A\}$ is a relatively complemented sublattice (resp. a Boolean subalgebra) of the lattice of ideals of A. For more results on annihilators see [2], [3], or [5].

The following results are known and give a representation of the annihilator of an element $a \in A$ in terms of certain subsets of prime filters. We shall sketch some steps of the proof in order to keep the paper reasonably self-contained.

Lemma 1. *Let $A \in \mathsf{DLat}$. Then for each $a \in A$, and for each $P \in X(A)$,*

(1) *$a^\circ \cap P = \emptyset$ iff there exists $Q \in X(A)$ such that $P \subseteq Q$ and $a \in Q$.*

(2) *$\psi(a^\circ) = (\phi(a)]^c$,*

(3) *$a^\circ \cap P = \emptyset$ if and only if there exists $M \in \max X(A)$ such that $P \subseteq M$ and $a \in M$.*

Proof.

(1) Let $a^\circ \cap P = \emptyset$. Let us consider the filter $F(P \cup \{a\})$. This filter is proper because otherwise there exists $c \in P$ such that $a \wedge c = 0$, i.e., $c \in a^\circ \cap P$, which is a contradiction. Thus, there exists $Q \in X(A)$, such that $P \subseteq Q$ and $a \in Q$. The other direction is immediate.

The item (2) follows from (1).

We prove (3). Suppose that $a^\circ \cap P = \emptyset$. Then there exists $Q \in X(A)$ such that $P \subseteq Q$ and $a \in Q$. Let us consider the family

$$\mathcal{Z} = \{F \in \mathrm{Fi}(A) - \{A\} \mid P \subseteq F \text{ and } a \in F\}.$$

It is clear that $\mathcal{Z} \neq \emptyset$, since $Q \in \mathcal{Z}$. Due to Zorn's lemma there is a maximal element in $\mathcal{Z}$, since every chain of elements of $\mathcal{Z}$, ordered by inclusion, has supremum in $\mathcal{Z}$. Let M be such an element. Clearly, M is proper. We prove that M is a maximal filter. Let $b \notin M$. We prove that there exists $c \in M$ such that $c \wedge b = 0$. If $b^\circ \cap M = \emptyset$, then the filter $F = F(M \cup \{b\})$ is proper and $F \in \mathcal{Z}$, which is a contradiction because M is a maximal element in $\mathcal{Z}$. Therefore there exists $c \in M$ such that $c \wedge b = 0$, i.e., M is a maximal filter. $\square$

Lemma 2. *Let $A \in \mathsf{DLat}$. Then,*

$$P \in \max X(A) \iff (\forall a \in A)(a \notin P \iff a^\circ \cap P \neq \emptyset).$$

P r o o f. Assume that $P \in \max X(A)$. Let $a \notin P$. Since P is a maximal filter, $F(P \cup \{a\}) = A$. So, there exists $p \in P$ such that $p \wedge a = 0$, i.e. $p \in a^\circ \cap P$. If $a^\circ \cap P \neq \emptyset$ and $a \in P$, then there exists $p \in P$ such that $p \wedge a = 0$, which is a contradiction.

Conversely. Let Q be a filter such that $P \subset Q$. Then there exists $a \in Q$ and $a \notin P$. As $a \notin P$, $a^\circ \cap P \neq \emptyset$. So there exists $p \in P$ such that $p \wedge a = 0$. As $P \subset Q$, $p \wedge a = 0 \in Q$. Thus, $Q = A$, and this implies that P is a maximal filter. □

2. Annihilator preserving congruences

We now shall give an apparently different definition of AP-congruence, but in Theorem 6 we shall prove that this definition is equivalent to the one given by M. F. Janowitz in [7].

Let $A \in \mathsf{DLat}$ and let θ be a lattice-congruence of A. We will write $(a, b) \in \theta$ or $a \equiv_\theta b$. The equivalence class of an element $a \in A$ is denoted by $|a|_\theta = \{b \in A \mid a \equiv_\theta b\}$, or directly by $|a|$. The *canonical* or *natural map* with respect to θ is the function $q\colon A \to A/\theta$ defined by $q(a) = |a|_\theta$. For a subset $S \subseteq A$, we will write $|S|_\theta = \{|a|_\theta \mid a \in S\}$.

Definition 3. Let $A \in \mathsf{DLat}$ and let θ be a lattice-congruence of A. We say that θ is a *congruence preserving annihilators*, or *AP-congruence* of A, if

(AP) For each pair $a, b \in A$, $a \equiv_\theta b$ implies that for each $x \in a^\circ$ there exists $y \in b^\circ$ such that $x \equiv_\theta y$.

Let θ be a congruence of a bounded distributive lattice A. To indicate that the pair (a, b) satisfies the condition for each $x \in a^\circ$ there exists $y \in b^\circ$ such that $x \equiv_\theta y$ of the above definition we will use the following notation:

$$(a^\circ, b^\circ) \in \tilde{\theta}, \qquad \text{or} \qquad a^\circ \equiv_{\tilde{\theta}} b^\circ.$$

Thus, a lattice-congruence θ is an AP-congruence if for all $a, b \in A$, $a^\circ \equiv_{\tilde{\theta}} b^\circ$, whenever $a \equiv_\theta b$.

We recall that the complete lattice of the congruences of a bounded ditributive lattice A is dually isomorphic to the complete lattice of the closed subsets of its Priestley space $X(A)$. The isomorphism is given by the map $\theta(\cdot)$ from the closed sets of $X(A)$ to the set of all lattices-congruences Con$A$ of A, and is defined as follows. For every closed set Y of $X(A)$ the set

$$\theta(Y) = \left\{(a, b) \in A^2 \mid \phi(a) \cap Y = \phi(b) \cap Y\right\},$$

is a lattice-congruence of A. Then for each lattice-congruence θ of A there exists a closed subset $Y \subseteq X(A)$ such that $\theta = \theta(Y)$. We also note that for $\theta = \theta(Y)$ and for each $P_\theta \in X(A/\theta)$, $q^{-1}(P_\theta) \in Y$.

Lemma 4. *Let $A \in \mathsf{DLat}$ and let Y be a closed set of $X(A)$. Let $a \equiv_{\theta(Y)} b$. Then following conditions are equivalent,*

(1) $a^\circ \equiv_{\tilde{\theta}(Y)} b^\circ$,
(2) $\psi(a^\circ) \cap Y = \psi(b^\circ) \cap Y$.

Proof. Let us assume (1). We prove that $\psi(a^\circ) \cap Y \subseteq \psi(b^\circ) \cap Y$. Suppose $P \in \psi(a^\circ) \cap Y = \bigcup \{\phi(e) \mid e \in a^\circ\} \cap Y$. Then there exists $e \in a^\circ$ such that $e \in P$. As $(a^\circ, b^\circ) \in \tilde{\theta}(Y)$, there exists $f \in b^\circ$ such that $e \equiv_{\theta(Y)} f$, i.e. $\phi(e) \cap Y = \phi(f) \cap Y$. So, $P \in \phi(e) \cap Y = \phi(f) \cap Y \subseteq \psi(b^\circ) \cap Y$.

Now, assume (2). Then

$$\bigcup \{\phi(e) \mid e \in a^\circ\} \cap Y = \bigcup \{\phi(f) \mid f \in b^\circ\} \cap Y.$$

Let $e \in a^\circ$. We need to prove that there exists $f \in b^\circ$ such that $(e, f) \in \theta$. We note that:

$$\begin{aligned} \phi(e) \cap Y &\subseteq \bigcup \{\phi(x) \mid x \in a^\circ\} \cap Y = \psi(a^\circ) \cap Y = \psi(b^\circ) \cap Y \\ &= \bigcup \{\phi(y) \mid y \in b^\circ\} \cap Y \subseteq \bigcup \{\phi(y) \mid y \in b^\circ\}. \end{aligned}$$

Since $\phi(e) \cap Y$ is closed and the space $X(A)$ is compact, there are $f_1, \dots, f_n \in b^\circ$ such that

$$\phi(e) \cap Y \subseteq \phi(f_1) \cup \cdots \cup \phi(f_n) = \phi(f_1 \vee \cdots \vee f_n) = \phi(f).$$

Therefore $\phi(e) \cap Y = \phi(e \wedge f) \cap Y$. Let $h = e \wedge f$. Consequently $e \equiv_{\theta(Y)} h$. Moreover, since $h \leq f$ and $f \in b^\circ$, we obtain that $h \in b^\circ$. $\square$

Remark 5. Let $A \in \mathsf{DLat}$. It is easy to see that $(a] \cap b^\circ = (a] \cap (b \wedge a)^\circ$, for all $a, b \in A$.

In the following theorem we prove that the definition of AP-congruence given here is equivalent to the definition given by M. F. Janowitz [7].

Theorem 6. *Let $A \in \mathsf{DLat}$. Let θ be a lattice-congruence and let $Y \subseteq X(A)$ be a closed subset such that $\theta = \theta(Y)$. Then the following conditions are equivalent.*

(1) *If $a \equiv_\theta 0$, then $a^\circ \equiv_\theta A$.*
(2) *θ is an AP-congruence.*
(3) *$|a^\circ|_\theta = |a|_\theta^\circ$, for all $a \in A$.*
(4) *For all $a, b \in A$, if $a \wedge b \equiv_\theta 0$, then there exists $c \in A$ such that $a \wedge c = 0$ and $c \equiv_\theta b$.*

Proof.

(1) $\Longrightarrow$ (2). Let $a, b \in A$ such that $a \equiv_\theta b$. By Lemma 4 we need to prove that $\psi(a^\circ) \cap Y = \psi(b^\circ) \cap Y$. Take $P \in X(A)$ such that $a^\circ \cap P \neq \emptyset$ and $P \in Y$. Then there exists $z \in A$ such that $a \wedge z = 0$ and $z \in P$. As $a \equiv_\theta b$, $0 = a \wedge z \equiv_\theta b \wedge z$. Then, $(b \wedge z)^\circ \equiv_\theta A$, i.e.,

$$\psi((b \wedge z)^\circ) \cap Y = \psi(A) \cap Y = X(A) \cap Y = Y. \tag{$*$}$$

On the other hand, as $(z] \cap b^\circ = (z] \cap (b \wedge z)^\circ$, we have

$$\phi(z) \cap \psi(b^\circ) = \phi(z) \cap \psi((b \wedge z)^\circ).$$

Then,

$$\phi(z) \cap \psi(b^\circ) \cap Y = \phi(z) \cap \psi((b \wedge z)^\circ) \cap Y \overset{\text{by } (*)}{=} \phi(z) \cap Y.$$

As $P \in \phi(z) \cap Y$, $b^\circ \cap P \neq \emptyset$. Thus, $\psi(a^\circ) \cap Y \subseteq \psi(b^\circ) \cap Y$. The other inclusion is similar and left to the reader.

(2) $\Longrightarrow$ (3). Let $a \in A$. We prove the inclusion $|a|_\theta^\circ \subseteq |a^\circ|_\theta$. Suppose that there exists $x \in A$ such that $|x|_\theta \in |a|_\theta^\circ$ but $|x|_\theta \notin |a^\circ|_\theta$. Then $x \wedge a \equiv_\theta 0$ and as $|a^\circ|_\theta$ is an ideal, there exists $P_\theta \in X(A/\theta)$ such that $|a^\circ|_\theta \cap P_\theta = \emptyset$ and $|x|_\theta \in P_\theta$. Since the function $q \colon A \to A/\theta$ is surjective, $a^\circ \cap P = \emptyset$, where $P = q^{-1}(P_\theta) \in Y$. From Lemma 1, there exists $M \in \max X(A)$ such that $P \subseteq M$ and $a \in M$. Hence θ is an AP-congruence and $x \wedge a \equiv_\theta 0$, $(x \wedge a)^\circ \equiv_\theta 0^\circ = A$. So,

$$\psi((x \wedge a)^\circ) \cap Y = X(A) \cap Y = Y,$$

i.e., $Y \subseteq \psi((x \wedge a)^\circ)$. Then $(x \wedge a)^\circ \cap P \neq \emptyset$. As $x \in P \subseteq M$, and $a \in M$, we get $x \wedge a \in M$, which is a contradiction. Therefore, $|a|_\theta^\circ \subseteq |a^\circ|_\theta$.

We prove the inclusion $|a^\circ|_\theta \subseteq |a|_\theta^\circ$. Let $|x|_\theta \in |a^\circ|_\theta$. Then there exists $y \in a^\circ$ such that $|x|_\theta = |y|_\theta$. As $y \wedge a = 0$,

$$|y \wedge a|_\theta = |y|_\theta \wedge |a|_\theta = |x|_\theta \wedge |a|_\theta = |0|_\theta .$$

Then $|x|_\theta \in |a|_\theta^\circ$.

(3) $\Longrightarrow$ (4). Let $a, b \in A$ such that $a \wedge b \equiv_\theta 0$. Then $|b|_\theta \in |a|_\theta^\circ = |a^\circ|_\theta$. So there exists $c \in a^\circ$ such that $c \equiv_\theta b$.

(4) $\Longrightarrow$ (1). Let $a, b \in A$ such that $a \equiv_\theta 0$. We need to prove that for every $b \in A$ there exists $c \in a^\circ$ such that $b \equiv_\theta c$. Let $b \in A$. Then $a \wedge b \equiv_\theta 0$. So, there exists $c \in A$ such that $a \wedge c = 0$ and $c \equiv_\theta b$. □

We will now characterize the closed sets of $X(A)$ corresponding to AP-congruences.

Proposition 7. *Let $A \in \mathsf{DLat}$ and let Y be a closed set of $X(A)$. Then $\theta(Y)$ is an AP-congruence of A if and only if* $\max[P) \subseteq Y$*, for each $P \in Y$.*

P r o o f. Suppose that $\theta(Y)$ is an AP-congruence. Let $P \in Y$, $Q \in \max[P)$, and suppose that $Q \notin Y$. Since Y is a closed set, there are $a, b \in A$ such that

$$Q \notin \phi(a)^c \cup \phi(b), \qquad \text{and} \qquad \phi(a) \cap Y = \phi(a) \cap \phi(b) \cap Y.$$

Hence we have $(a, a \wedge b) \in \theta(Y)$, $a \in Q$ and $b \notin Q$. Let us consider the set $P^\circ = \{p \mid p^\circ \cap P \neq \emptyset\}$. It is easy to see that P° is an ideal. As Q is maximal, $F(Q \cup b) \cap P^\circ \neq \emptyset$. Then there exists $c \in Q$ such that $c \wedge b \in P^\circ$, i.e. $(c \wedge b)^\circ \cap P \neq \emptyset$. So there exists $d \in (c \wedge b)^\circ \cap P$. As $\theta(Y)$ is a congruence, $a \wedge c \equiv_{\theta(Y)} a \wedge b \wedge c$, and as it is a $\circ$-congruence,

$$(a \wedge c)^\circ \equiv_{\tilde{\theta}(Y)} (a \wedge b \wedge c)^\circ.$$

Since $d \in (c \wedge b)^\circ \subseteq (a \wedge b \wedge c)^\circ$, there exists $f \in (a \wedge c)^\circ$ such that $f \equiv_{\theta(Y)} d$. Then $P \in \phi(d) \cap Y = \phi(f) \cap Y$. So $f \in P \subseteq Q$, but then $f \wedge a \wedge c = 0 \in Q$, which is an absurd. Thus, $\max[P) \subseteq Y$.

Assume that $\max[P) \subseteq Y$, for each $P \in Y$. Let $a, b \in A$ such that $a \equiv_{\theta(Y)} b$, i.e. $\phi(a) \cap Y = \phi(b) \cap Y$. We prove that $a^\circ \equiv_{\tilde{\theta}(Y)} b^\circ$. By Lemma 4 it is enough to prove that $\psi(a^\circ) \cap Y = \psi(b^\circ) \cap Y$. Let $P \in \psi(a^\circ) \cap Y = \bigcup \{\phi(c) \mid c \in a^\circ\} \cap Y$. Then there exists $c \in a^\circ$ such that $c \in P$, i.e. $a^\circ \cap P \neq \emptyset$. Suppose that $b^\circ \cap P = \emptyset$. By Lemma 2 there exits $Q \in \max X(A)$ such that $P \subseteq Q$ and $b \in Q$. Then $Q \in \phi(b) \cap Y = \phi(a) \cap Y$. So, $a \in Q$. As $c \in P \subseteq Q$, we get $a \wedge c = 0 \in Q$, which is a contradiction. □

Definition 8. Let $A \in \mathsf{DLat}$ and let Y be a closed set of $X(A)$. We shall say that Y is *AP-closed* if $\max[P) \subseteq Y$, for each $P \in Y$.

Theorem 9. *Let $A \in \mathsf{DLat}$. Then the map θ establishes a dual isomorphism between the set of AP-closed subsets of $X(A)$ ordered by inclusion and the set of the AP-congruences of A ordered by inclusion.*

P r o o f. It follows from the Proposition 7 and the dual isomorphism between the lattice of congruences of a bounded distributive lattice and the lattice of closed sets of its dual Priestley space. □

Let $A \in \mathsf{DLat}$. We shall say that A is *AP-subdirectly irreducible* if there exists a minimal non-trivial AP-congruence θ in A. Similarly, we shall say that A is *AP-simple* if A has only two AP-congruences. We note that $\mathrm{Cl}(\max X(A))$ is an AP-closed subset of $X(A)$, because $\max[P) \subseteq \max X(A) \subseteq \mathrm{Cl}(\max X(A))$, for each $P \in \mathrm{Cl}(\max X(A))$. On the other hand, we note that A is a Boolean lattice iff $\max X(A) = X(A)$.

Proposition 10. *Let $A \in \mathsf{DLat}$. Then,*

(1) *A is* AP-simple *iff A is a Boolean lattice.*

(2) *A is AP-subdirectly irreducible but non-AP-simple iff there exists $P \in X(A)$ with $P \notin \mathrm{Cl}(\max X(A))$ such that $\{P\} \cup \mathrm{Cl}(\max X(A)) = X(A)$.*

Proof.

(1) If A is AP-simple, and taking into account that $\mathrm{Cl}(\max X(A))$ is an AP-closed set, $\mathrm{Cl}(\max X(A)) = X(A)$, and clearly this implies that $\max X(A) = X(A)$, because $X(A)$ is closed. So, A is a Boolean lattice. The converse direction is immediate.

(2) Suppose that A is AP-subdirectly irreducible but non-AP-simple. Then from Theorem 9 there exists a greatest proper AP-closed subset Y of $X(A)$. As Y is proper there exists $P \in X(A) - Y$. It is easy to see that $\mathrm{Cl}(\max X(A)) \cup \{P\}$ is an AP-closed subset. Then, $\mathrm{Cl}(\max X(A)) \cup \{P\} \subseteq Y$, and as $P \notin Y$, we conclude that $\mathrm{Cl}(\max X(A)) \cup \{P\} = X(A)$.

Conversely, suppose that there exists $P \in X(A)$ with $P \notin \mathrm{Cl}(\max X(A))$ such that $\mathrm{Cl}(\max X(A)) \cup \{P\} = X(A)$. If Z is a proper AP-closed subset of $X(A)$, then $Z \subseteq \mathrm{Cl}(\max X(A))$. As $\mathrm{Cl}(\max X(A))$ is the greatest proper AP-closed subset of $X(A)$, we get that A is AP-subdirectly irreducible. □

It is known that in a pseudocomplemented bounded distributive lattice A, $\max X(A)$ is a closed subset of the Priestley space $X(A)$, i.e., $\mathrm{Cl}(\max X(A)) = \max X(A)$. From this remark and the previous Proposition we obtain the following known results for pseudocomplemented bounded distributive lattices:

(1) A is simple iff A is a Boolean algebra.

(2) A is subdirectly irreducible but non-simple iff there exists $P \in X(A)$ with $P \notin \max X(A)$ such that $\{P\} \cup \max X(A) = X(A)$, i.e., $A = B \oplus 1$, where B is a Boolean algebra.

3. Some applications

According to Speed [11] (see also [4] or [2, Definition 5.1]) a bounded distributive lattice A with zero 0 is *quasicomplemented* or a *$\circ$-lattice* if for each $a \in A$ there exists $b \in A$ such that $a^{\circ\circ} = b^{\circ}$, where

$$a^{\circ\circ} = \{c \in A \mid (\forall e \in a^{\circ})(c \wedge e = 0)\}.$$

It is clear that this class includes the class of distributive pseudocomplemented lattices. We note that in general the element b is non-unique.

The quotient lattice A/θ of a quasicomplemented lattice A by means of a lattice-congruence θ does not necessarily produce a quasicomplemented lattice.

But, when θ is an AP-congruence, A/θ is a quasicomplemented lattice, as is proved in the following theorem.

Theorem 11. *Let A be a $\circ$-lattice. Let θ be an AP-congruence. Then A/θ is quasicomplemented.*

Proof. We need to prove that for $|a| \in A/\theta$ there exists $|b| \in A/\theta$ such that $|a|^{\circ\circ} = |b|^{\circ}$. Take $a \in A$. As A is quasicomplemented, there exists $b \in A$ such that $a^{\circ\circ} = b^{\circ}$. First we prove that $|a^{\circ\circ}| \subseteq |a|^{\circ\circ}$. Let $|y| \in |a^{\circ\circ}|$. Then there exists $y' \in a^{\circ\circ}$ such that $|y| = |y'|$. Take an element $|k| \in |a|^{\circ}$. As $|a|^{\circ} = |a^{\circ}|$, there exists $k' \in a^{\circ}$ such that $|k| = |k'|$. We note that $y' \wedge k' = 0$, because $y' \in a^{\circ\circ}$ and $k' \in a^{\circ}$. Then $|y| \wedge |k| = |y'| \wedge |k'| = |y' \wedge k'| = |0|$. So, $|y| \in |a|^{\circ\circ}$. Therefore, $|b|^{\circ} = |b^{\circ}| = |a^{\circ\circ}| \subseteq |a|^{\circ\circ}$.

We prove now that $|a|^{\circ\circ} \subseteq |b|^{\circ}$. Let $|x| \in |a|^{\circ\circ}$. As $a^{\circ\circ} = b^{\circ}$, $a^{\circ} = a^{\circ\circ\circ} = b^{\circ\circ}$, and taking into account that $b \in b^{\circ\circ}$, we have that $b \wedge a = 0$. Since θ is a lattice-congruence, $|b \wedge a| = |b| \wedge |a| = |0|$, i.e. $|b| \in |a|^{\circ}$. As $|x| \in |a|^{\circ\circ}$, $|x| \wedge |b| = |0|$, i.e. $|x| \in |b|^{\circ}$. □

Let us recall that a bounded distributive lattice A is *normal* if each prime filter P is contained in a unique maximal filter [2]. While studying normal lattices W. Cornish [2] has given several characterizations of normal lattices. For instance, a bounded distributive lattice A is normal iff $a \wedge b = 0$ implies that $a^{\circ} \vee b^{\circ} = A$, for all $a, b \in A$.

Lemma 12. *Let A be a bounded distributive lattice. Let θ be an AP-congruence. For each $P \in \max X(A/\theta)$, $q^{-1}(P) \in \max X(A)$.*

Proof. Let $P \in \max X(A/\theta)$. It is clear that $q^{-1}(P) \in X(A)$. Now let $a \notin q^{-1}(P)$. Then $q(a) = |a| \notin P$, and as P is maximal, there exists $|b| \in P$ such that $|a| \wedge |b| = |0|$, i.e., $|b| \in |a|^{\circ}$. Since θ is an AP-congruence, $|a|^{\circ} = |a^{\circ}|$. Then there exists $c \in a^{\circ}$ such that $|b| = |c|$. So, $|c| \in P$, i.e., $c \in q^{-1}(P)$ and $c \in a^{\circ}$. Thus, $q^{-1}(P)$ is maximal. □

Theorem 13. *Let A be a normal lattice. Let θ be an AP-congruence. Then A/θ is normal.*

Proof. Let $P \in X(A/\theta)$, and $U_1, U_2 \in \max X(A/\theta)$ such that $P \subseteq U_1$ and $P \subseteq U_2$. From Lemma 12, $q^{-1}(U_1)$, $q^{-1}(U_2) \in \max X(A)$, and as $q^{-1}(P) \subseteq q^{-1}(U_1) \cap q^{-1}(U_2)$, we get $q^{-1}(U_1) = q^{-1}(U_2)$. So, $U_1 = q(q^{-1}(U_1)) = q(q^{-1}(U_2)) = U_2$. Thus A/θ is normal. □

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