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Some New Results on Nonconformable Fractional Calculus

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Abstract

In this paper, we present some results for a local fractional derivative, not conformable, defined by the authors in a previous work, and which are closely related to some of the classic calculus.

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1 Preliminaries

The asymptotic behaviors of functions have been analyzed by velocities or rates of change in functions, while there are very small changes occur in the independent variables. The concept of rate of change in any function versus change in the independent variables was defined as derivative,first of an integer order, and this concepts attracted many scientists and mathematicians such as Newton, L'Hospital, Leibniz, Abel, Euler, Riemann, etc. Later, several types of fractional derivatives have been introduced to

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date Euler, Riemann–Liouville, Abel, Fourier, Caputo, Hadamard, Grunwald–Letnikov, Miller–Ross, Riesz among others, extended the derivative concept to fractional order derivative (see [10, 13, 14]). Most of these derivatives are defined on the basis of the corresponding fractional integral in the Riemann–Liouville sense. Among the inconsistencies of the above fractional derivatives D^α are:

1. Most of the fractional derivatives except Caputo-type, do not satisfy $D^\alpha(1) = 0$, if α is not a natural number.

2. All fractional derivatives do not satisfy the familiar product rule for two functions $D^\alpha(fg) = gD^\alpha(f) + fD^\alpha(g)$.

3. All fractional derivatives do not satisfy the familiar quotient rule for two functions $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$ with $g \neq 0$.

4. All fractional derivatives do not satisfy the chain rule for composite functions $D^\alpha(f \circ g)(t) = D^\alpha(f(g))D^\alpha g(t)$.

5. The fractional derivatives do not have a corresponding "calculus".

6. All fractional derivatives do not satisfy the indices rule $D^\alpha D^\beta(f) = D^{\alpha+\beta}(f)$.

The fractional calculus attracted many researches in the last and present centuries. The impact of this fractional calculus in both pure and applied branches of science and engineering (cf. [2, 9, 12]) started to increase substantially during the last two decades, in particular this meant that these notions of "global" fractional derivatives have been extended to the local sense (see, for example [1,3–8,15]). As we pointed out before, one of the drawbacks that exists is the absence of a theoretical body relative to these local fractional derivatives. In this direction, this paper should be understood as a continuation of [3] and we will present some of the most important theorems of fractional calculus for the derivative $N_1^\alpha f(t)$, a new fractional derivative of local type defined therein.

2 Main Results

First, let's remember the definition of $N_1^\alpha f(t)$, a nonconformable fractional derivative of a function in a point t defined in [3] and that is the basis of our results, that are close resemblance of those found in classical calculus.

Definition 2.1. Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by $N_1^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon e^{t^{-\alpha}}) - f(t)}{\varepsilon}$ for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, and $\lim_{t \to 0^+} N_1^{(\alpha)} f(t)$ exists, then define $N_1^{(\alpha)} f(0) = \lim_{t \to 0^+} N_1^{(\alpha)} f(t)$.

Following the same procedure of the ordinary calculus, we can prove the following result.

Theorem 2.2. *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*

i) f is continuous on $[a, b]$,

ii) f is N-differentiable for some $\alpha \in (0, 1)$.

Then, we have that if $N_1^\alpha f(t) \geq 0$ (≤ 0), then f is a nondecreasing (increasing) function.

Analogously we have the following result.

Theorem 2.3 (Racetrack Type Principle)**.** *Let $a > 0$ and $f, g : [a, b] \to \mathbb{R}$ be given functions satisfying*

i) f and g are continuous on $[a, b]$,

ii) f and g are N-differentiable for some $\alpha \in (0, 1)$,

iii) $N_1^\alpha f(t) \geq N_1^\alpha g(t)$ for all $t \in (a, b)$.

Then, we have that following:

I) If $f(a) = g(a)$, then $f(t) \geq g(t)$ for all $t \in (a, b)$.

II) If $f(b) = g(b)$, then $f(t) \leq g(t)$ for all $t \in (a, b)$.

Proof. Consider the auxiliary function $h(t) = f(t) - g(t)$. Then h is continuous on $[a, b]$ and N-differentiable for some $\alpha \in (0, 1)$. From here, we obtain that $N_1^\alpha h(t) \geq 0$ for all $t \in (a, b)$, so by Theorem 2.2, h is a nonincreasing function. Hence, for any $t \in [a, b]$, we have that $h(a) \leq h(t)$ and since $h(a) = f(a) - g(a) = 0$ by assumption, the result follows. In a similar way, the second part is proved. This concludes the proof. □

We will discuss the occurrence of local maxima and local minima of a function. In fact, these points are crucial to many questions related to application problems.

Definition 2.4. A function f is said to have a local maximum at c iff there exists an interval I around c such that $f(c) \geq f(x)$ for all $x \in I$. Analogously, f is said to have a local minimum at c iff there exists an interval I around c such that $f(c) \leq f(x)$ for all $x \in I$. A local extremum is a local maximum or a local minimum.

Remark 2.5. As in the classic calculus, if the function f is N-differentiable at a point c where it reaches an extreme, then $N_1^\alpha f(c) = 0$.

Theorem 2.6 (Rolle's Theorem)**.** *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies* *i)* $f \in C[a, b]$,

ii) f *is* N*-differentiable on* (a, b) *for some* $\alpha \in [0, 1]$,

iii) $f(a) = f(b)$.

Then, there exists $c \in (a, b)$ *such that* $N_1^\alpha f(c) = 0$.

Proof. We prove this using contradiction. From assumptions, since f is continuous in $[a, b]$ and $f(a) = f(b)$, there is $c \in (a, b)$, at least one, which is a point of local extreme. On the other hand, as f is N-differentiable in (a, b) for some α, we have

$$\begin{aligned} N_1^\alpha f(c) &= N_1^\alpha f(c^+) = \lim_{h \to 0^+} \frac{f(c + \ he^{c^{-\alpha}}) - f(c)}{h} \\ &= N_1^\alpha f(c^-) = \lim_{h \to 0^-} \frac{f(c + he^{c^{-\alpha}}) - f(c)}{h} \end{aligned}$$

but $N_1^\alpha f(c^+)$ and $N_1^\alpha f(c^-)$ have opposite signs. Hence $N_1^\alpha f(c) = 0$. If $N_1^\alpha f(c^+)$ and $N_1^\alpha f(c^-)$ they have the same sign then as $f(a) = f(b)$, we have that f is constant and the result is trivially followed. This concludes the proof. □

Theorem 2.7 (Mean Value Theorem)**.** *Let* $a > 0$ *and* $f : [a, b] \to \mathbb{R}$ *be a function that satisfies*

i) f *is continuous in* $[a, b]$,

ii) f *is* N*-differentiable on* (a, b), *for some* $\alpha \in (0, 1]$.

Then, exists $c \in (a, b)$ *such that*

$$N_1^\alpha f(c) = \left[\frac{f(b) - f(a)}{b - a}\right] e^{c^{-\alpha}}.$$

Proof. Consider the function

$$g(t) = f(t) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](t - a).$$

The auxiliary function g satisfies all conditions of Theorem 2.6, and, therefore, there exists $c \in (a, b)$ such that $N_1^\alpha g(c) = 0$. Then, we have

$$N_1^\alpha g(t) = N_1^\alpha \left(f(t) - f(a)\right) - \frac{f(b) - f(a)}{b - a} N_1^\alpha (t - a),$$

and from here it follows that

$$N_1^\alpha g(c) = N_1^\alpha f(c) - \frac{f(b) - f(a)}{b - a} e^{c^{-\alpha}} = 0,$$

from where

$$N_\alpha[f(c)] = \frac{f(b) - f(a)}{b - a} e^{c^{-\alpha}}.$$

This concludes the proof. □

Theorem 2.8. *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*

i) *f is continuous on $[a, b]$,*

ii) *f is N-differentiable for some $\alpha \in (0, 1)$.*

If $N_1^\alpha f(t) = 0$ for all $t \in (a, b)$, then f is a constant on $[a, b]$.

Proof. It is sufficient to apply Theorem 2.7 to the function f over any nondegenerate interval contained in $[a, b]$. □

As a consequence of the previous theorem, we have the following.

Corollary 2.9. *Let $a > 0$ and $F, G : [a, b] \to \mathbb{R}$ be functions such that for all $\alpha \in (0, 1)$, $N_1^\alpha F(t) = N_1^\alpha G(t)$ for all $t \in (a, b)$. Then there exists a constant C such that $F(t) = G(t) + C$.*

Along the same lines of classic calculus, one can use the previous results to prove the following result.

Theorem 2.10. *Let $f : [a, b] \to \mathbb{R}$ be N-differentiable for some $\alpha \in (0, 1)$. If*

i) *$N_1^\alpha f(t)$ is bounded on $[a, b]$ with $a > 0$, then f is uniformly continuous on $[a, b]$, and hence f is bounded.*

ii) *$N_1^\alpha f(t)$ is bounded on $[a, b]$ and continuous at a with $a > 0$, then f is uniformly continuous on $[a, b]$, and hence f is bounded.*

Theorem 2.11 (Extended Mean Value Theorem)**.** *Let $\alpha \in (0, 1]$ and $a > 0$. If $f, g : [a, b] \to \mathbb{R}$ are functions that satisfy*

i) *f, g are continuous in $[a, b]$,*

ii) *f, g are N-differentiable on (a, b), for some $\alpha \in (0, 1]$,*

iii) *$N_1^\alpha g(t) \neq 0$ for all $t \in (a, b)$.*

Then, there exists $c \in (a, b)$ such that

$$\frac{N_1^\alpha f(c)}{N_1^\alpha g(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Remark 2.12. If $g(t) = t$, then this is just the statement of Theorem 2.7.

Proof of Theorem 2.11. Let us now define a new function as

$$F(t) = f(t) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(t) - g(a)).$$

Then the auxiliary function F satisfies the assumptions of Theorem 2.6. Thus, there exists $c \in (a, b)$ such that $N_1^\alpha F(c) = 0$ for some $\alpha \in (0, 1)$. From here, we have

$$N_1^\alpha F(c) = N_1^\alpha f(c) - \frac{f(b) - f(a)}{g(b) - g(a)} N_1^\alpha g(c) = 0.$$

Now the desired result is obtained. □

Our last result is a generalization of one of the most important, and oldest, theorems of mathematical analysis, the Taylor series, which establishes under what conditions a function f can be approximated in a neighborhood of a point $t = a$, by a linear combination of polynomials.

Theorem 2.13. *Let $f : [v, w] \to \mathbb{R}$ be n times continuously N-differentiable and $n + 1$ times N-differentiable in (v, w) and let $a \in (v, w)$. Then, for each $t \in (v, w)$ with $t \neq a$, there exists a point $\xi \in (a, t)$, respectively (t, a), such that*

$$f(t) = \sum_{j=0}^{n} \frac{f_j(a)}{j!}(t - a) + R_{n+1}(f) \tag{2.1}$$

with

$$f_{j+1}(\cdot) = N_1^\alpha f(\cdot) e^{-(\cdot)^{-\alpha}}, f_0(\cdot) = f(\cdot) \tag{2.2}$$

holds, whereby the remainder term can be written as

$$R_{n+1}(f) = \frac{f_{n+1}(\xi)}{(n + 1)!}(t - a)^{n+1}.$$

Proof. Define

$$g(y) = f(t) - f(y) - f_1(y)(t - y) - f_2(y)\frac{(t - y)^2}{2!} - \ldots - f_n(y)\frac{(t - y)^n}{n!} - \frac{M}{(n + 1)!}(t - y)^{n+1}, \tag{2.3}$$

where M is chosen so that $g(a) = 0$ is fulfilled. Using $g(a) = g(t) = 0$, Theorem 2.6 allows us to affirm that there exist $\xi \in (a, t)$, respectively (t, a), with $N_1^\alpha g(\xi) = 0$. Since

$$N_1^\alpha g(y) = -\frac{N_1^\alpha f_n(y)}{n!}(t - y)^n - \frac{M}{n!}(t - y)^n(-1)e^{y^{-\alpha}},$$

it follows that

$$0 = N_1^\alpha g(\xi) = -N_1^\alpha f_n(\xi) + M e^{\xi^{-\alpha}}$$

and thus $M = f_{n+1}(\xi)$. Setting $y = a$ in (2.3) we obtain the desired result. □

3 Concluding Remarks

In the present paper, we obtained some new results concerning a new local fractional derivative obtained in [3], give some analogues results of classical theorems (Rolle, mean value, etc.) those who come to complete the necessary theoretical body of this new calculus. As in ordinary calculus, the function $g(t)$ used to prove the mean value theorem from Rolle's theorem is not the only one. In fact, following [16] we can apply Rolle's theorem to the function

$$h(t) = [f(t) - f(a)](t - b) + [f(t) - f(b)](t - a)$$

and get the next result. If $f(x)$ is continuous on $[a, b]$ and N-differentiable on (a, b), $0 < \alpha < 1$, then there is a c in (a, b) such that $(c \neq \dfrac{a+b}{2})$

$$f'(c) = \frac{f(c) - \frac{f(a)+f(b)}{2}}{c - \frac{a+b}{2}}.$$

To prove this result, it is enough to verify that the function $h(x)$ satisfies Rolle's theorem and then

$$N_1^{\alpha} h(t) = e^{t^{-\alpha}} \left[f'(t)(2t - a - b) + 2f(t) - f(a) - f(b)\right] = 0$$

and from here the desired result is derived. The search for new auxiliary functions that comply with Rolle's theorem, would allow obtaining new results similar to those of the ordinary calculation, we recommend [10] and [17] for works in this direction in the ordinary case. As we pointed before, in the proof of Rolle's theorem (and of Lagrange, of course) we look for a function that satisfies the hypotheses of this theorem and in this way guarantee the existence of (at least) a point c inside the interval, where the derivative of said function becomes zero. We wanted to present an application of this idea, to the resolution of equations, where the derivative N_1^{α} plays a prominent role. *Problem.* Let f be a continuous function on $[a, b]$ and differentiable on (a, b), solve the equation

$$\alpha t^{-\alpha-1} f(t) + f'(t) = 0,$$

subject to the condition $f(a) = f(b) = 0$ with $0 < a < b$ and $0 < \alpha < 1$. From [3], we know that $N_1^{\alpha}(e^{-t^{-\alpha}}) = \alpha t^{-\alpha-1}$, so we can write the above equation of the form $N_1^{\alpha}\left[e^{-t^{-\alpha}} f(t)\right] = 0$, taking $h(t) = e^{-t^{-\alpha}} f(t)$, we observe that all conditions of Rolle's theorem are satisfied for the function $h(t)$ and therefore, there exists at least one point in (a, b) where its derivative is canceled, which is the solution sought from the original equation. We can exemplify this with a concrete application. Show that the equation

$$t^{-\frac{3}{2}} \sin(t^2 - 3t + 2) + 2(2t - 3)\cos(t^2 - 3t + 2) = 0$$

has a solution between 1 and 2. This equation can be written in the form

$$\frac{1}{2}t^{-\frac{3}{2}}\sin((t-1)(t-2)) + (2t-3)\cos((t-1)(t-2)) = 0,$$

from here we have $h(t) = e^{-t^{-\alpha}}\sin((t-1)(t-2))$, $\alpha = \frac{1}{2}$, $a = 1$ and $b = 2$. Applying Rolle's theorem, we have that there is at least one $c \in (1,2)$ such that $h'(c) = 0$. Moreover, from the mean value theorem we can obtain that this local fractional derivative provides us the coefficient A in the approximation of $f(x)$ by the function

$$f(x) = f(a) + A(x-a),\ A = \frac{N_1^{\alpha} f(c)}{e^{c^{-\alpha}}},\ a < c < x,\ 0 < \alpha < 1,$$

in the neighborhood of x. This generalizes the geometric interpretation of derivatives in terms of "tangents", because if there is the limit of the secant of the points $(t, f(t))$ and $((t+\varepsilon), f(t+\varepsilon))$, then the limit of the points will exist $(t, f(t))$ and $((t+\varepsilon), f(t+\varepsilon exp(t^{-\alpha})))$, since the latter is contained in the former, since $t + \varepsilon exp(t^{-\alpha}) < t + \varepsilon$. All these results are consistent with the Taylor Series obtained later. It is clear that the results obtained in this work, extend the various obtained for compliant fractional local derivatives, because the introduction of the term e in the case of differentiable functions in the classical sense, opens a totally new picture in the study of functions.

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