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A Note on the Oscillatory Nature of Lienard Equation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this paper we consider the problem about the conditions on f(x), g(x) and a(t) to ensure that all solutions of (1) are continuable and oscillatory using non usual assumptions.

Keywords: Oscillation; Lienard equation; continuability; asymptotic behavior.

1 Preliminaries

Consider the nonautonomous Liénard equation

$$x''+f(x)x'+a(t)g(x)=0, \tag{1}$$

where a, f and g are continuous in their arguments, satisfying suitable assumptions. In this paper we assume that f, g and a satisfying some properties that guarantees local existence and uniqueness of solutions of (1).

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As it is well known, the problems deal with the qualitative properties of solutions of the above equation or his generalizations are very important in the theory and applications of differential equations. For example, equation (1) is frequently encountered in mathematical model of most dynamics process in electromechanical systems in physics and engineering. However, it should be noted that details of these and other applications will not be given here.

Further details on the qualitative properties of equation (1) can be found in [1].

Various questions on stability, oscillation and periodicity of solutions of (1) have received a considerable amount of attention in the last decades (see [2-4,5,6,7-11] and [12] and references cited therein) under classical assumptions $f(x)>0$ for $x\in\mathbf{R}$. In this paper we study the oscillatory nature of solutions of (1) without makes use of this condition, using a new method in which the usual assumptions on involved functions in (1) are not used.

A solution of (1) is *oscillatory* if there exists a infinite sequence $\{t_n\}$ tending monotonically to ∞ such that $x(t_n)=0$.

The equation (1) is equivalent to the system

$$x'=y-F(x),\qquad (2)$$
$$y'=-a(t)g(x),$$

where $F(x)=\int_0^x f(u)du$, $x\in\mathbf{R}$. Also, we assume that satisfy the following assumptions:

a) $xg(x)>0$ for $x\neq 0$.
b) $\int_0^x g(s)ds=+\infty$.
c) $0<a\leq a(t)\leq A<+\infty$ for $t\in[0,+\infty)$.

The condition a) shows that (0,0) is the only point of equilibrium of system (2) and the condition b) ensures that results obtained are in global sense. From [1] obtain that condition c) is consistent with the common sense.

We consider now the following system equivalent to (1):

$$x'=y,\qquad (3)$$
$$y'=-f(x)y-a(t)g(x).$$

We will show that the solutions of the equation

$$\frac{dy}{dx}=\frac{f(x)y-a(t)g(x)}{y}\qquad (4)$$

do not admit vertical asymptotes, consequently the solutions of (3) (and (1)) do not admit them either. It is enough, to this end, suppose that (4) has a solution $y=y(x)$, $a\leq x\leq b$ such that

$$\lim_{x\to b^-} y(x)=+\infty.\qquad (5)$$

We can assume with no loss of generality, that $0<y(a)\leq y(x)$ for $a\leq x\leq b$, and

d) let K is the max of f on $a\leq x\leq b$ and M the max of g on $a\leq x\leq b$.

It follows that

$$y(x)-y(a)\leq\{K+AM/y(a)\}(b-a),$$

which is a clear contradiction with (5). The other situations can be analyzed in a similar way. The obtained allows assuring the global existence of the solutions of (1).

2 On the Oscillatory Solutions of (1) with a(t)≡1 (I)

For any point $P(x_0,y_0)\in \mathbf{R}^2$ let $\gamma^+(P)$ the positive semi trajectory of (2) passing by P.

Theorem 1. Under conditions a)-d) we suppose in addition that

e) The functions g and $F(x)+\frac{g(x)}{h}$ are no decreasing for $|x|>N>0$ and h>0.
f) $\limsup_{|x|\to+\infty}\frac{F(x)}{-x}\leq 0$.

Then for all $P(x_0,y_0)$ with $x_0\geq 0$, $y_0>F(x_0)$, the positive semi trajectory of (2) passing by P cross the curve y=F(x) in a point (s,F(s)), with $s>x_0$.

Proof. Let $P(x_0,y_0)$ with $x_0>0$, $y_0>F(x_0)$ and let (x(t),y(t)) the solution of (2) with initial conditions (x_0,y_0). We suppose that $x'(t)=y(t)-F(x(t))>0$ for $t\geq 0$. From this we obtain that x(t) is increasing strictly monotonically on $[0,+\infty)$ and $x(t)\to+\infty$ when $t\to+\infty$. If this is no true then there exists a positive number j such that for all $t\geq 0$ we have $0<x(0)\leq x(t)\leq j$ and from c) we obtain $\lim_{t\to+\infty}y(t)=-\infty$ so the curve (x(t),y(t)) cross y=F(x) in some point of $[x_0,j]\mathrm{x}\mathbf{R}$ contradicting the initial supposition.

In the successive we distingue two cases:

Case 1. $\lim_{x\to+\infty}F(x)=-\infty$.

As $\lim_{t\to+\infty}x(t)=+\infty$ there exists a positive number δ such that x(t)>N for t≥δ. Let now t_0, with $t_0>\delta$ so we have a positive number k, k<h such that

$$y(t_0)<F(x(t_0))+g(x(t_0))/k\leq F(x(t))+g(x(t))/k \text{ for all } t\geq t_0.$$

Then we have for all $t\geq t_0$ that $y(t)<y(t_0)<F(x(t))+g(x(t))/k$, i.e. for $t\geq t_0$

$$\frac{-g(x(t))}{y(t)-F(x(t))}\leq -k$$

So the slope of the trajectory of the solution satisfy $\frac{dz}{dx}=\frac{-g(x)}{y-F(x)}\leq -k$ if $x\geq x(t_0)$, where z is a real function definite on **R** such that $z(x(t))=y(t)$ for all $t\geq t_0$. From this we have

$$z(x(t))\leq -k(x(t)-x(t_0))+z(x(t_0)).$$

From e) for all real number b and all positive number a, there exists a positive number η such that if x>η we have $-ax+b\leq F(x)$. As $\lim_{t\to+\infty}x(t)=+\infty$ there exists a positive number $t_1>t_0$ such that

$$y(t)=z(x(t))\leq -kx(t)+[x(t_0))+z(x(t_0)]\leq F(x(t)),\ t\geq t_1.$$

But this contradicts the initial supposition.

Case 2. $\lim_{x\to+\infty}F(x)>-\infty$.

Again we have $\lim_{t\to+\infty} y(t) = -\infty$, and as $\lim_{t\to+\infty} x(t) = +\infty$ and $\lim_{x\to+\infty} g(x) = m > 0$ (g is no decreasing on (M,+∞)). From the first equation of (2) we have that $\lim_{t\to+\infty} F(x(t)) = -\infty$ and so $\lim_{x\to+\infty} F(x) = -\infty$. But this contradicts the supposition of this case.

As the (0,0) is the unique singular point of (2) we can conclude that any positive semi trajectory passing by (x_0,y_0) with $x_0\geq 0$, $y_0>F(x_0)$, cross the curve y=F(x) in a point (s,F(s)) with $s>x_0$.

Remark 1. In a similar way we can prove the existence of a point (u,F(u)) with $u< x_1$ in the case $x_1\leq 0$, $y_1<F(x_1)$.

Remark 2. In the proof of Case 2 the assumptions d) and e) don't used completely.

In the prove of following result we recommend to [4, Theorem 1].

Theorem 2. Under the same assumptions of Theorem 1, we suppose in addition that

g) There exists a positive number ε such that xF(x)<0 if 0<|x|<ε.

Then all solutions of system (2) are oscillatory.

Proof. Using the Theorem 1 we have that any positive semi trajectory passing by (x_0,y_0) with $x_0\geq 0$, $y_0>F(x_0)$, cross the curve y=F(x) in a point (s,F(s)) with $s>x_0$ (analogously we can prove the existence of a point (u,F(u)) with $u< x_1$ in the case $x_1\leq 0$, $y_1<F(x_1)$).

Now we will prove that all nontrivial solutions of (1) are oscillatory it is sufficient to prove that any positive semiorbit of (2) intersects the y axis in a finite time. This will result from the following considerations.

From a) and g) we have that the (0,0) is a fold and this is unique. Also if x>0 then –g(x)<0 so that a nontrivial trajectory which intersects the curve y=F(x) for x>0 can pass only from the part of the semiplane x>0 where y>F(x) to the part where y<F(x). The same remark applies to the semiplane x<0: a nontrivial trajectory which intersects the curve y=F(x) for x<0 can pass only from the region where y<F(x) to the part where y>F(x). We also know that if $P(x_0,y_0)$ with $x_0\geq 0$, $y_0>F(x_0)$ then $\gamma^+(P)$ intersects y=F(x) at a point of the semiplane x>0 whereas if $R(x_0,y_0)$ is such that $x_0<0$, $y_0<F(x_0)$ $\gamma^+(R)$ intersects y=F(x) at a point of the semiplane x<0.

We consider a point $P(x_0,y_0)$ with $x_0\geq 0$, $y_0<F(x_0)$. From the above the positive semi trajectory $\gamma^+(P)$ no cross the curve y=F(x) when x>0 and neither tends to the origin because this is repulsive.

So $\gamma^+(P)$ cross the negative y axis necessarily if y<0. If this is not true we have $0\leq x(t)$, $x'(t)<0$, $y'(t)\leq 0$ if $t\geq 0$. So, $x(t)\in[0,x_0]$ if $t\geq 0$ and there exist positive numbers α and t_1 such that $x'(t)=y(t)-F(x(t))<-\alpha$ fi $t\geq t_1$ and so $\gamma^+(P)$ cross the y axis with y<0 against the supposed.

To obtain the final picture we have to show that if a trajectory is starting from $S(x_0,y_0)$ with $x_0\leq 0$, $y_0<F(x_0)$, then the positive semiorbit $\gamma^+(S)$ cross the y axis with y<0 and that a trajectory starting from $T(x_0,y_0)$ with $x_0\leq 0$ and $y_0>F(x_0)$ crosses the y axis where y>0. Together with the previous remaks, this ensures that the nontrivial trajectories of (2) are clockwise around the origin and leads us to the conclusion that all nontrivial solutions of (2) are oscillatory.

Let $S(x_0,y_0)$ with $x_0\leq 0$, $y_0<F(x_0)$, be fixed. The trajectory $\gamma^+(S)$ cannot cross the curve y=F(x) in the part where x>0 by previous remarks and cannot accumulated to the only stationary point of the system (2) since the origin is repulsive. To show that $\gamma^+(S)$ has to cross the y axis at y<0 we have to prove that it cannot stay forever in the region {(x,y):x≥0, y<F(x)}. If it could stay forever in this region we would have for the unique global solution (x(t),y(t)) with initial data (x_0,y_0) that

$$0\leq x(t),\ x'(t)<0,\ y'(t)\leq 0,\ t\geq 0$$

Thus $x(t)\in[0,x_0]$, $0\le t$ and $\lim_{t\to+\infty} y(t) = -\infty$ which is impossible since the slope of the trajectory is bounded for $t\ge 0$ (see Preliminaries) big enough (we have $\lim_{t\to+\infty} y(t) = -\infty$ and $x(t)\in[0,x_0]$, $0\le t$, implies that $F(x(t))$ and $g(x(t))$ are bounded for $t\ge 0$). This contradiction show that $\gamma^+(S)$ crosses the y axis at $y<0$.

The second case, $T(x_0,y_0)$ with $x_0\le 0$ and $y_0>F(x_0)$, may be treated exactly in the same way.

From this we obtain that all solution of (2) is oscillatory, surrounding the origin clockwise and the proof of the theorem is completed.

The following result generalizes the above.

Theorem 3. In addition to a) and g) we suppose that there exists λ and a real function $\Phi\in C^1(|x|>\lambda)$ verifying $\Phi'(x)<0$ if $|x|>\lambda$, and

h) $\frac{g(x)}{-\Phi'(x)}$ is no decreasing on $(-\infty,-\lambda)\cup(\lambda,+\infty)$.

i) There exists a positive number h such that $F(x)+\frac{g(x)}{h[-\Phi'(x)]}$ is no decreasing on $(-\infty,-\lambda)\cup(\lambda,+\infty)$.

j) For all positive number a and all real number b is $a\Phi(x)+b\le F(x)$ $(x>\lambda)$ and $a\Phi(x)+b\ge F(x)$ $(x<-\lambda)$.

Then, all solutions of (2) are oscillatory.

Proof. Taking $\Phi(x)=-x$ we have the Theorem 2. We just have to follow the proofs of Theorems 1 and 2.

Remark 3. Taking into account the Remark 2 If instead of the conditions e) and f) by $\lim\ \sup_{x\to+\infty} F(x) > -\infty$, $\lim\ \sup_{x\to-\infty} F(x)<+\infty$, $\lim\ \inf_{x\to+\infty} g(x)=m>0$, we obtain a modification of Theorem1 of [4].

Example 1. We consider in (1) $g(x)=x$,
$$f(x)=\begin{cases}-\frac{1}{3}\left(x+1-\frac{1}{\sqrt{27}}\right)^{-\frac{2}{3}}, x\le -1\\ -1, -1<x<1\\ -\frac{1}{3}\left(x-1-\frac{1}{\sqrt{27}}\right)^{-\frac{2}{3}}, x\ge 1\end{cases}.$$

We define the function $V(x,y)=\frac{y^2}{2}+G(x)$ with $G(x)=x^2/2$.

We suppose a solution of (2) definite on $[t_0,T)$ for some $T>t_0$. Then there exists a sequence $\{t_n\}$ on $[t_0,T)$ such that $x_n\to T$ when $n\to\infty$ and $|x(t_n)|+|y(t_n)|\to+\infty$ with $n\to+\infty$.

The derivative of V along this solution gives $V'(x(t),y(t))\le x^2\le 2V(x(t),y(t))$, from this is clear

$$\ln(V(x(t_n),y(t_n)))\le \ln(V(x(t_0),y(t_0)))+2(T-t_0),\ t_0\le t<T$$

but this is false because $\ln(V(x(t_n),y(t_n)))$ is no bounded. So the supposition is no true and the solution is definite for all $t\ge t_0$.

If h=1 it's easy to see that $F+\frac{g}{h}$ is a no decreasing function on $|x|>2$ and f) holds. Then in virtue of Theorem 2 we have that all solutions of system (2) with f and g as above, are oscillatory.

Remark 4. This example is no covered with results of [4].

Example 2. We consider now in (1), $f(x)=-4$ and $g(x)=x$. The assumption f) of Theorem 2 hold taking $h=1/5$, but e) is no fulfilled. However according to theory of linear differential equation the characteristic equation in this case is $r^2-4r+1=0$, with positive roots so we have nonoscillatory solutions ([13, pp.75-86]).

3 On the Oscillatory Solutions of (1) with a(t)≡1 (II)

In this section we will give other results that involve somewhat different conditions on the functions of (1).

Theorem 4. Let F(x) and g(x) such that

a) $xg(x)>0$ for $x\neq0$.
b) There exist a positive number h such that the function $F(x)+\frac{g(x)}{h}$ is bounded below on $[0,+\infty)$ and bounded above on $(-\infty,0]$.
c) $\lim\sup_{x\to+\infty}[F(x)+hx]=+\infty$ (or $\lim\inf_{x\to+\infty}\frac{F(x)}{-hx}<1$), $\lim\inf_{x\to-\infty}[F(x)+hx]=-\infty$ (or $\lim\inf_{x\to-\infty}\frac{F(x)}{-hx}<1$).
d) $\lim\inf_{|x|\to+\infty}|g(x)|=m>0$.

Then for all point $P(x_0,y_0)$ with $x_0\geq0$, $y_0>F(x_0)$ the positive semi trajectory of (2) passing by P cross the curve $y=F(x)$ in the point $(s,F(s))$ with $s>x_0$.

Analogously, the positive semi trajectory of (2) passing by $Q(x_1,y_1)$ with $x_1\leq0$, $y_1<F(x_1)$ cross the curve $y=F(x)$ in the point $(s,F(s))$ with $u<x_1$.

Proof. Let a point $P(x_0,y_0)$ with $x_0>0$, $y_0>F(x_0)$ and let $(x(t),y(t))$ the solution of (2) with initial conditions (x_0,y_0) and suppose that $x'(t)=y(t)-F(x(t))>0$ for all $t\geq0$. From here we have that x(t) is monotone increasing strictly on $[0,+\infty)$ and $\lim_{t\to+\infty}x(t)=+\infty$.

If the assumption were not true would exist a positive number j such that for all $t\geq0$ be $0<x(0)\leq x(t)\leq j$ and from a) we have $\lim_{t\to+\infty}y(t)=-\infty$ and then $(x(t),y(t))$ will intercept to $y=F(x)$ in some point of band $[x(0),j]\text{x}\mathbf{R}$ against the supposition.

In what follows we will consider two cases.

Case 1. $\lim_{x\to+\infty}F(x)=-\infty$.

We suppose that k is above bound of $F(x)+\frac{g(x)}{h}$ on $[0,+\infty)$. From d) we have $\lim_{t\to+\infty}y(t)=-\infty$ so there will be a positive number t_0 such that $y(t)<k$ and

$$\frac{-g(x(t))}{y(t)-F(x(t))}\leq-k \text{ for } t\geq t_0.$$

So the slope of the trajectory satisfy $\frac{dw}{dx}=\frac{-g(x)}{y-F(x)}\leq-k$ if $x\geq x(t_0)$, where w is a real function definite on $\mathbf{R}$ such that $w(x(t))=y(t)$ for all $t\geq t_0$. From this we have

$$w(x(t))\leq-k(x(t)-x(t_0))+w(x(t_0)).$$

From c) for all real number b and all positive number a, there exists a positive number $\eta=\eta(a,b)$ such that if $x>\eta$ we have $-a\eta+b\leq F(\eta)$. As $\lim_{t\to+\infty}x(t)=+\infty$ there exists a positive number $t_1>t_0$ such that

$$y(t_1)=w(x(t_1))\leq-kx(t_1)+[x(t_0))+z(x(t_0)]\leq F(x(t_1)).$$

But this contradicts the initial supposition.

Case 2. $\lim_{x\to+\infty} F(x) > -\infty$.

Again we have $\lim_{t\to+\infty} y(t) = -\infty$, and as $\lim_{t\to+\infty} x(t) = +\infty$ and $\lim_{x\to+\infty} g(x) = m > 0$ (g is no decreasing on (M,+∞)). From the first equation of (2) we have that $\lim_{t\to+\infty} F(x(t)) = -\infty$ and so $\lim_{x\to+\infty} F(x) = -\infty$. But this contradicts the supposition of this case.

As the (0,0) is the unique singular point of (2) we can conclude that any positive semi trajectory passing by (x_0,y_0) with $x_0\geq 0$, $y_0>F(x_0)$, cross the curve y=F(x) in a point (s,F(s)) with $s>x_0$.

Remark 5. In a similar way we can prove the existence of a point (u,F(u)) with $u< x_1$ in the case $x_1\leq 0$, $y_1<F(x_1)$.

Remark 6. In the proof of Case 2, the conditions b) and c) have not been used.

Remark 7. If in this theorem we replace conditions b) and c) by $\lim\ \sup_{x\to+\infty}[F(x)] > -\infty$ and $\lim\ \sup_{x\to-\infty}[F(x)] < +\infty$ we obtain a modification of Theorem 1 of [4].

Theorem 5. Under the same assumptions of Theorem 4, we suppose in addition that

e) There exists a positive number ε such that xF(x)<0 if 0<|x|<ε.

Then all solutions of system (2) are oscillatory.

Proof. It is sufficient to follow the proof of Theorem 1 of [4] and the proof of Theorem 2 of the preceding section.

Theorem 6. We suppose that F(0)=0 and we suppose that assumptions a) and e) are fulfilled if in addition there exists a positive number β and a real function $\Phi\in C^1(|x|>\beta)$ verifying Φ'(x)<0 if |x|>β, and

f) There exists a positive number h such that $F(x) + \frac{g(x)}{h[-\Phi'(x)]}$ is below bounded on [β,+∞) and above bounded on (-∞,β].
g) $\lim\ \sup_{x\to+\infty}[F(x) - h\Phi(x)] = +\infty$ and $\lim\inf_{x\to-\infty}[F(x) - h\Phi(x)] = -\infty$.

The, all solutions of (2) are oscillatory.

Proof. It is similar to the proof of Theorem 3 of the preceding section.

Remark 8. The simple case x''-2x'+x=0; with non-oscillatory solution $x(t) = e^t$, is no contradictory with our results.

Remark 9. Our results are consistent with those obtained in [2,3,5,8,9] and [12].

Remark 10. The results obtained completes those obtained in [8], about the construction of a stability region for the equation (1).

Remark 11. Finally we give examples of functions f(x) which show that our results contains those in [10] and [11] with a(t)≡1.

Example 1.

$$f(x) = \begin{cases} x, |x| \leq 1, \\ x^{-1}, |x| > 1 \end{cases}$$

Example 2.

$$f(x) = \begin{cases} 1, x \geq 1 \\ x, |x| < 1, \\ -1, x \leq -1 \end{cases}$$

These examples do not satisfy the conditions of Repilado and Ruiz, but if are covered by results of this paper.

4 Conclusion

The motivation of this work arises from the following problem:

Under assumptions $f(x) \geq f_0 > 0$ for some positive constant f_0, the class of equation (1) with oscillating solutions is not very large. We can exhibit equations that do not satisfy the above condition and have no oscillating solutions. For example

$$x'' - \left(\frac{1}{4} ln^2 x + 3\right) x' + 2(t^2 + 1)x = 0$$

has the no oscillating $x(t)=e^{2t}$.

The following question then arises naturally **Are there conditions under which we can ensure that equation (1) has all its oscillating solutions?** It is clear that the results obtained outline a first answer to this question.

Competing Interests

Authors have declared that no competing interests exist.

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