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# A Study of Continuous Dependence and Symmetric Properties of Double Diffusive Convection: Forchheimer Model

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**Abstract:** In this recent work, the continuous dependence of double diffusive convection was studied theoretically in a porous medium of the Forchheimer model along with a variable viscosity. The analysis depicts that the density of saturating fluid under consideration shows a linear relationship with its concentration and a cubic dependence on the temperature. In this model, the equations for convection fluid motion were examined when viscosity changed with temperature linearly. This problem allowed the possibility of resonance between internal layers in thermal convection. Furthermore, we investigated the continuous dependence of this solution based on the changes in viscosity. Throughout the paper, we found an "a priori estimate" with coefficients that relied only on initial values, boundary data, and the geometry of the problem that demonstrated the continuous dependence of the solution on changes in the viscosity, which also helped us to state the relationship between the continuous dependence of the solution and the changes in viscosity. Moreover, we deduced a convergence result based on the Forchheimer model at the stage when the variable viscosity trends toward a constant value by assuming a couple of solutions to the boundary-initial-value problems and defining a difference solution of variables that satisfy a given boundary-initial-value problem.

**Keywords:** Forchheimer model; double diffusive; salinization; stability; variable viscosity; convergence

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## 1. Introduction

The structural stability of porous media or the study of the continuous dependence of a model itself is considered a prime and challenging problem. As a result, many authors have started intensively studying the subject of double diffusive convection in many different aspects, such as the horizontal layer of porous material saturated with a fluid whose density or volume does not change with pressure, see [1]. Generally, in the domain of partial differential equations, the structural stability and continuum mechanics are noticeable in many areas, see [2]. The reliance of the elasticity field on modeling began with a major publication by Knops and Payne [3,4]. An extensive research of Payene [5–7] in the area of structural stability enhanced it, and, since then, numerous papers have been published in this direction. Resources of this direction can be obtained in an intensive study that was first conducted by Straughan [1]. In addition, new elaborations of this study were conducted by Aulisa et al. [8], Ciarletta et al. [9], Hoang and Ibragimov [10], Liu [11], Liu et al. [12,13], and Ghazi and Ali [14–17]. Our current paper is an advancement of the work conducted by Straughan [18] and Gentile and Straughan [19] who studied the continuous dependence and the structural stability of the supplier of heat in a penetrating convection type and in the porous medium of Forchheimer, such that the density relies quadratically or cubically on the temperature field. In most of the applications, cubic dependence is obligatory, while quadratic dependence has been observed to be insufficient, see McKay and Straughan [20] and Straughan ([21], pp. 143–144). In case where the fluid flow is not small, we can introduce Forchheimer coefficients in the Darcy Equations (see [22,23]) because the pressure gradient is no longer proportional to velocity, see Straughan ([1], p. 12) and Néel [24]. In our present paper, we are dealing with the Forchheimer model of quadratic degree, where we first took into consideration the changes on the supplier of the heat, so we had the ability to obtain the solution of its continuous dependence, and then we analyzed it on the coefficients of the presented Forchheimer model. In fact, each of the parameters were analyzed separately because the bounds that resulted in every stage were different. Usually, in case of liquids, viscosity varies significantly in comparison with other thermo-physical properties. Most of the real liquids are highly viscous, and, consequently, they can be affected functionally by temperature. The variability of viscosity can be easily observed from the tabulated data included in [25–27]. Rossby ([26], p. 334) created a table with the viscosity values of water as well as a 20 centistoke silicone oil at temperatures ranging from 20 to 25 °C. While doing so, the authors observed that, in case of water, the kinematic viscosity varies from 0.01008 $cm^2/s$ at 20 °C to 0.00896 $cm^2/s$ at 25 °C, which is almost a 10% change. On the other hand, its thermal conductivity varies by only around 1.5% over the same range of temperatures. Moreover, in the case of 20 cSt silicone oil, the kinematic viscosity decreases from 0.2137 $cm^2/s$ to 0.1904 $cm^2/s$ from 20 degrees to 25 degrees, which is almost a 20% variation, while the thermal conductivity remains constant over the same interval of temperatures. Over a wide range of temperatures, the viscosity variation may be extremely large. For instance, Weast [27] stated that the viscosity of olive oil varies from 138.0 cP at 10 °C to 12.4 cP at 70 °C. Tippelskirch [28], Palm [29], and Palm et al. [30] started the study of thermal convection in temperature-based viscosity fluids. Generally, there is a non-linear relationship between viscosity and temperature in numerous convection problems, but the linear relationship was adopted in [30]. In another expression, Palm stated that when $\mu(T)$ is the kinetic viscosity, we have

$$\mu(T) = \mu_0(1 - \gamma_0(T - T_0)), \tag{1}$$

where $\mu_0$ and $\gamma_0$ are constants ($\mu_0, \gamma_0 > 0$).

The phenomenon of double-diffusive convection, in which scalar fields are involved, such as the heat and concentration of a solute, affects the density distribution in a fluid-saturated porous medium and has a wide range of applications, including processes arising in chemical engineering, energy technology, geophysics, and oceanography. In particular, some applications include groundwater systems in “karst aquifers”, chemical processing, the convective flow of carbon nanotubes, the propagation of biological fluids, and the simulation of bacterial bio-convection and thermohaline circulation problems; see [31–35].

In this regard, the goal of the present paper was to develop and analyze the double diffusive convection problem in a porous medium layer using a Forchheimer model where viscosity depends linearly on temperature. Furthermore, we took into consideration the density of fluid in such a manner that it depended linearly on the concentration and cubically on the temperature. A priori estimates were derived for a solution to the governing partial differential equations, and these were employed in an analysis of continuous dependence and of convergence. We also proved a convergence result by demonstrating that the solution converges appropriately as the coupling coefficient vanishes.

The structure of this work is organized as follows. In Section 2, we present the basic governing equations. In Section 3, we develop a priori estimates by introducing some given functions as solutions to a group of boundary value problems and continue the process with the aid of the mathematical analysis. In Section 4, we analyze the continuous dependence of a solution to our presented model and check the effective viscosity coefficient, $\gamma$. Finally, we deal with the convergence of the solution to the same presented model and check how its system affects the viscosity coefficient.

## 2. Governing Equations

In this section, similar to [18,36], we study double-diffusive convection by considering a porous medium that fills the three-dimensional region and is governed by the following Forchheimer equations:

$$\begin{aligned} au_i|\mathbf{u}| + bu_i|\mathbf{u}|^2 + (1+\gamma T)u_i &= -p_{,i} + g_i T + h_i T^2 + \mathcal{I}_i T^3 + \mathcal{L}_i C, \\ u_{i,i} &= 0, \\ T_{,t} + u_i T_{,i} &= \Delta T, \\ C_{,t} + u_i C_{,i} &= \Delta C. \end{aligned} \tag{2}$$

For an explanation of the nomenclature, see Table 1.

**Table 1.** Nomenclature used in this study.

| Symbol | Definition |
|---|---|
| $u_i$ | Velocity |
| $T$ | Temperature |
| $p$ | Pressure |
| $C$ | Concentration |
| $\mu$ | A constant |
| $\gamma$ | A positive constant (viscosity coefficient) |
| $r$ | A positive integer |
| $a$ and $b$ | Forchheimer coefficients |
| $g_i, h_i, \mathcal{I}_i$, and $\mathcal{L}_i$ | Vectors for incorporating the gravity field |
| $\mathbf{n}$ | A unit outward vector |
| $h$, $k$, $T_0$, and $C_0$ | Maps |
| $V$ | A bounded domain in $R^3$ |
| $\mu(T)$ | A kinetic viscosity |
| $\mu_0$ and $\gamma_0$ | Positive constants |
| $\Gamma$ | A boundary of the domain $V$ |
| $q$ | A non-zero eigenvalue |
| $\varphi$ | A solution to the Neumann problem with data $f$ |
| $\mathcal{F}$ | A space of admissible functions |
| $v$ | The volume of the domain $V$ |
| $\nabla$ | The gradient |
| $\nabla_s$ | The surface gradient over the boundary $\Gamma$ |
| $\Delta$ | Laplace operator |
| $m(\Gamma)$ | The surface measure of $\Gamma$ |
| $\\|\cdot\\|_n$ | $L^n(\Omega)$ norm |
| $\\|\cdot\\|$ | $L^2(\Omega)$ norm |
| $(\cdot,\cdot)$ | Inner product |
| $_{,i}$ | $\partial/\partial x_i$ |
| $_{,t}$ | $\partial/\partial t$ |
| $u_{i,i}=0$ | The balance of mass equation |

We assume, without loss of generality for the present discussion, that

$$|\mathbf{g}| \le 1, \quad |\mathbf{h}| \le 1, \quad |\mathcal{I}| \le 1 \quad \text{and} \quad |\mathcal{L}| \le 1.$$

Let $V$ be a bounded domain in $\mathbb{R}^3$ with the boundary $\Gamma$. Then, system (2) is defined on $V \times (0, \mathcal{T})$, for $\mathcal{T} < \infty$, the boundary, and the initial conditions

$$u_i n_i = f(x,t), \quad \text{on} \quad \Gamma \times (0, \mathcal{T}), \tag{3}$$

$$T(x,t) = h(x,t), \;\; C(x,t) = k(x,t), \;\; x \;\; \text{on} \;\; \Gamma, \;\; t \in (0, \mathcal{T}), \tag{4}$$

and

$$T(x,0) = T_0(x), \;\; C(x,0) = C_0(x), \;\; x \in V, \tag{5}$$

where $\mathbf{n}$ is defined as a unit outward vector that is perpendicular to $\Gamma$, and $h$, $k$, $T_0$, and $C_0$ are given maps. A collection of many different models with their presentations can be obtained from [1].

## 3. A Priori Estimates

To drive the a priori estimates for $T$ and $C$, we introduce the functions $G(x,t)$, $K(x,t)$, $I(x,t)$, $F(x,t)$, $H(x,t)$, and $M(x,t)$ as a means of resolving the following boundary value problems:

$$\begin{cases} \Delta G(x,t) = 0, & \text{in} \;\; V, \\ G(x,t) = h(x,t), & \text{on} \;\; \Gamma, \end{cases} \tag{6}$$

$$\begin{cases} \Delta K(x,t) = 0, & \text{in} \;\; V, \\ K(x,t) = k(x,t), & \text{on} \;\; \Gamma, \end{cases} \tag{7}$$

$$\begin{cases} \Delta I(x,t) = 0, & \text{in} \;\; V, \\ I(x,t) = h^3(x,t), & \text{on} \;\; \Gamma, \end{cases} \tag{8}$$

$$\begin{cases} \Delta F(x,t) = 0, & \text{in} \;\; V, \\ F(x,t) = h^5(x,t), & \text{on} \;\; \Gamma, \end{cases} \tag{9}$$

$$\begin{cases} \Delta H(x,t) = 0, & \text{in} \;\; V, \\ H(x,t) = h^{2r-1}(x,t), & \text{on} \;\; \Gamma, \end{cases} \tag{10}$$

$$\begin{cases} \Delta M(x,t) = 0, & \text{in} \;\; V, \\ M(x,t) = k^{2r-1}(x,t), & \text{on} \;\; \Gamma, \end{cases} \tag{11}$$

where $r$ is a positive integer to be determined later. Multiplying Equation (2) by $u_i$ and integrating over $V$, we obtain

$$\begin{aligned} & a\|\mathbf{u}\|_3^3 + b\|\mathbf{u}\|_4^4 + \int_V (1+\gamma T)|\mathbf{u}|^2 d\mathbf{x} \\ &= -\oint_\Gamma p_{,i} f dA + g_i(T, u_i) + h_i(T^2, u_i) + \mathcal{I}_i(T^3, u_i) + \mathcal{L}_i(C, u_i) \\ &\le -\oint_\Gamma p f dA + \frac{1}{4}\|\mathbf{u}\|^2 + 4\Big( \|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2 \Big). \end{aligned} \tag{12}$$

To treat the pressure term in (12), we assume that $\varphi$ is the solution to the Neumann problem with data $f$, i.e.,

$$\begin{aligned} \Delta\varphi = 0, & \quad \text{in} \;\; V, \\ \frac{\partial\varphi}{\partial n} = f, & \quad \text{on} \;\; \Gamma, \\ \oint_\Gamma \varphi dA = 0. & \end{aligned} \tag{13}$$

Employing (13) in a pressure term, we obtain

$$
\begin{aligned}
&-\oint_\Gamma pf dA = -\oint_\Gamma p\frac{\partial\varphi}{\partial n}dA = -\int_V p_{,i}\varphi_{,i}d\mathbf{x} \\
&= \int_V \varphi_{,i}\Big[au_i|\mathbf{u}| + bu_i|\mathbf{u}|^2 + (1+\gamma T)u_i - g_iT - h_iT^2 - \mathcal{I}_iT^3 - \mathcal{L}_iC\Big]d\mathbf{x} \\
&\leq a\Big(\int_V |\nabla\varphi|^3 d\mathbf{x}\Big)^{1/3}\Big(\int_V |\mathbf{u}|^3 d\mathbf{x}\Big)^{2/3} + b\Big(\int_V |\nabla\varphi|^4 d\mathbf{x}\Big)^{1/4}\Big(\int_V |\mathbf{u}|^4 d\mathbf{x}\Big)^{3/4} \\
&+\Big(\int_V (1+\gamma T)u_iu_i d\mathbf{x}\Big)^{1/2}\Big(\int_V (1+\gamma T)\varphi_{,i}\varphi_{,i}d\mathbf{x}\Big)^{1/2} + \|\nabla\varphi\|\Big(\|T\| + \|T\|^2 + \|T\|_3^3 + \|C\|\Big),
\end{aligned}
\tag{14}
$$

where the Cauchy–Schwarz and Hölder's inequalities are employed. From the Stekloff inequality, we obtain

$$
\oint_\Gamma \varphi^2 dA \leq \frac{1}{q}\|\nabla\varphi\|^2, \tag{15}
$$

where $q$ is the first non-zero eigenvalue, such that

$$
q = \min_{\xi\in\mathcal{F}} \frac{\|\nabla\xi\|^2}{\oint_\Gamma \xi^2 dA}. \tag{16}
$$

Here, $\mathcal{F}$ is the space of admissible functions. Since $\Delta\varphi = 0$ in $V$, it follows that

$$
\|\nabla\varphi\|^2 = \oint_\Gamma \varphi\frac{\partial\varphi}{\partial n}dA.
$$

Adopting (15) and using the Cauchy–Schwarz inequality, we have

$$
\|\nabla\varphi\|^2 \leq \frac{1}{q}\oint_\Gamma \Big(\frac{\partial\varphi}{\partial n}\Big)^2 dA. \tag{17}
$$

Then, inserting (13) and (17) in (3), with the further use of the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
-\oint_\Gamma pf dA \quad &\leq \quad av^{1/6}\|\nabla\varphi\|_6\|\mathbf{u}\|_3^2 + bv^{1/8}\|\nabla\varphi\|_8\|\mathbf{u}\|_4^3 \\
&+ \quad \Big(\int_V (1+\gamma T)u_iu_i d\mathbf{x}\Big)^{1/2}\Big(\|\nabla\varphi\|^2 + \gamma\|T\|\sqrt{\int_V |\nabla\varphi|^4 d\mathbf{x}}\Big)^{1/2} \\
&+ \quad \frac{1}{\sqrt{q}}\Big(\oint_\Gamma f^2 dA\Big)^{1/2}\Big(\|T\| + \|T\|^2 + \|T\|_3^3 + \|C\|\Big),
\end{aligned}
\tag{18}
$$

where $v$ is the volume of $V$. We next use the Sobolev inequalities

$$
\|\psi\|_6 \leq \mathcal{C}\|\psi\|_{H^1(V)}, \quad \|\psi\|_4^2 \leq \tilde{\mathcal{C}}\|\psi\|_{H^1(V)}^2, \quad \|\psi\|_8 \leq \mathcal{C}\|\psi\|_{H^1(V)},
$$

with (17), to obtain

$$
\begin{aligned}
-\oint_{\Gamma} pf\,dA \;\le\;& av^{1/6}\mathcal{C}\|\mathbf{u}\|_3^2\left(\int_V\left[|\nabla\varphi|^2+\varphi_{,ij}\varphi_{,ij}\right]d\mathbf{x}\right)^{1/2}\\
+\;& bv^{1/8}\mathcal{C}\|\mathbf{u}\|_4^3\left(\int_V\left[|\nabla\varphi|^2+\varphi_{,ij}\varphi_{,ij}\right]d\mathbf{x}\right)^{1/2}\\
+\;& \left(\int_V(1+\gamma T)u_iu_i\,d\mathbf{x}\right)^{1/2}\left\{\frac{1}{q}\oint_{\Gamma} f^2dA+\gamma\tilde{\mathcal{C}}\|T\|\left(\int_V\left[|\nabla\varphi|^2+\varphi_{,ij}\varphi_{,ij}\right]d\mathbf{x}\right)\right\}^{1/2}\\
+\;& \frac{1}{\sqrt{q}}\left(\oint_{\Gamma} f^2dA\right)^{1/2}\left(\|T\|+\|T\|^2+\|T\|_3^3+\|C\|\right).
\end{aligned}
\tag{19}
$$

Now, let $d_1, d_2$, and $d_3$ be the data terms, see [37],

$$
\begin{aligned}
\int_V \varphi_{,ij}\varphi_{,ij}\,d\mathbf{x} &= d_1(t)+d_2(t),\\
\frac{1}{q}\oint_{\Gamma} f^2dA &= d_3(t).
\end{aligned}
\tag{20}
$$

Then, we have

$$
\|\nabla\varphi\|^2 \le d_3(t). \tag{21}
$$

Next, by (19), (20), and (21), we obtain

$$
\begin{aligned}
-\oint_{\Gamma} pf\,dA \;\le\;& \sqrt{d_4}\|\mathbf{u}\|_3^2+\sqrt{d_5}\|\mathbf{u}\|_4^3+(d_3+d_6\|T\|)^{1/2}\left(\int_V(1+\gamma T)u_iu_i\,d\mathbf{x}\right)^{1/2}\\
&+\sqrt{d_3}\left(\|T\|+\|T\|^2+\|T\|_3^3+\|C\|\right),
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
d_4 &= av^{1/6}\mathcal{C}(d_1+d_2+d_3),\\
d_5 &= bv^{1/8}\mathcal{C}(d_1+d_2+d_3),\\
d_6 &= \gamma\tilde{\mathcal{C}}(d_1+d_2+d_3).
\end{aligned}
$$

Hence, using (22) in (12), together with Young's inequality, we obtain

$$
\begin{aligned}
& a\|\mathbf{u}\|_3^3+b\|\mathbf{u}\|_4^4+\int_V(1+\gamma T)|\mathbf{u}|^2d\mathbf{x}\\
&\le \frac{a}{2}\|\mathbf{u}\|_3^3+\frac{64}{81a^2}d_4^{3/2}+\frac{b}{2}\|\mathbf{u}\|_4^4+\frac{27}{32b^3}d_5^2+\frac{1}{4}\|\mathbf{u}\|^2\\
&+4\left(\|T\|^2+\|T\|_4^4+\|T\|_6^6+\|C\|^2\right)+d_6\|T\|\\
&+\frac{1}{4}\int_V(1+\gamma T)|\mathbf{u}|^2d\mathbf{x}+2d_3+\|T\|^2+\|T\|_4^4+\|T\|_6^6+\|C\|^2\\
&\le \frac{a}{2}\|\mathbf{u}\|_3^3+\frac{64}{81a^2}d_4^{3/2}+\frac{b}{2}\|\mathbf{u}\|_4^4+\frac{27}{32b^3}d_5^2+\frac{1}{4}d_6^2+2d_3\\
&+\frac{1}{2}\int_V(1+\gamma T)|\mathbf{u}|^2d\mathbf{x}+6\|T\|^2+5\left(\|T\|_4^4+\|T\|_6^6+\|C\|^2\right),
\end{aligned}
\tag{23}
$$

and so

$$a\|\mathbf{u}\|_3^3+b\|\mathbf{u}\|_4^4+\int_V(1+\gamma T)|\mathbf{u}|^2d\mathbf{x}\le d_7+10\Big(\frac{6}{5}\|T\|^2+\|T\|_4^4+\|T\|_6^6+\|C\|^2\Big), \tag{24}$$

where the data term $d_7$ is given by

$$d_7=\frac{128}{81a^2}d_4^{3/2}+\frac{27}{16b^3}d_5^2+\frac{1}{2}d_6^2+4d_3.$$

Due to (24), we obtain

$$\begin{aligned}\|\mathbf{u}\|_3^3&\le\frac{1}{a}\Big[d_7+10\Big(\frac{6}{5}\|T\|^2+\|T\|_4^4+\|T\|_6^6+\|C\|^2\Big)\Big],\\ \|\mathbf{u}\|_4^4&\le\frac{1}{b}\Big[d_7+10\Big(\frac{6}{5}\|T\|^2+\|T\|_4^4+\|T\|_6^6+\|C\|^2\Big)\Big],\\ \int_V(1+\gamma T)|\mathbf{u}|^2d\mathbf{x}&\le d_7+10\Big(\frac{6}{5}\|T\|^2+\|T\|_4^4+\|T\|_6^6+\|C\|^2\Big).\end{aligned} \tag{25}$$

Now, we form the following expressions:

$$\int_0^t\int_V(T-G)(T_{,s}+u_iT_{,i}-\Delta T)d\mathbf{x}ds=0, \tag{26}$$

$$\int_0^t\int_V(C-K)(C_{,s}+u_iC_{,i}-\Delta C)d\mathbf{x}ds=0, \tag{27}$$

$$\int_0^t\int_V(T^3-I)(T_{,s}+u_iT_{,i}-\Delta T)d\mathbf{x}ds=0, \tag{28}$$

$$\int_0^t\int_V(T^5-F)(T_{,s}+u_iT_{,i}-\Delta T)d\mathbf{x}ds=0, \tag{29}$$

where $t$ is a number, such that $0\le t\le\mathcal{T}$.

Next, integrating by part in (29) and employing the boundary condition (6), we obtain

$$\begin{aligned}\|T\|^2+2\int_0^t\|\nabla T\|^2ds\le{}&\|T_0\|^2+2(G,T)+2\Big|(G_0,T_0)\Big|+2\Big|\int_0^t(G_{,s},T)ds\Big|\\ &+2\int_0^t\int_V Gu_iT_{,i}d\mathbf{x}ds+2\int_0^t\oint_\Gamma h\Big(\frac{\partial G}{\partial n}\Big)dAds+\int_0^t\oint_\Gamma|f|h^2dAds.\end{aligned}$$

Using the Cauchy–Schwarz and arithmetic-geometric mean inequalities on the right, we obtain

$$\begin{aligned}\frac{1}{2}\|T\|^2+\int_0^t\|\nabla T\|^2ds\ \le\ {}&2\|T_0\|^2+2\|G\|^2+\|G_0\|^2+\int_0^t\|G_{,s}\|^2ds\\ &+G_m^2\int_0^t\|\mathbf{u}\|^2ds+\int_0^t\|T\|^2ds+\int_0^t\oint_\Gamma h^2dAds\\ &+\int_0^t\oint_\Gamma\Big(\frac{\partial G}{\partial n}\Big)^2dAds+\int_0^t\oint_\Gamma|f|h^2dAds.\end{aligned} \tag{30}$$

By $(25)_3$ and (30), we have

$$\begin{aligned} \|T\|^2 &+ 2\int_0^t \|\nabla T\|^2 ds \le 4\|T_0\|^2 + 4\|G\|^2 + 2\|G_0\|^2 + 2\int_0^t \|G_{,s}\|^2 ds \\ &+ 2(1+12G_m^2)\int_0^t \|T\|^2 ds + 2G_m^2 \int_0^t \left[d_7 + 10\Big(\|T\|_4^4 + \|T\|_6^6 + \|C\|^2\Big)\right] ds \\ &+ 2\int_0^t \oint_\Gamma \left(\frac{\partial G}{\partial n}\right)^2 dAds + 2\int_0^t \oint_\Gamma (1+|f|)h^2 dAds. \end{aligned} \tag{31}$$

Now, by (27) and by performing an integration by parts along with using the inequalities of the Cauchy–Schwarz and arithmetic-geometric mean, we obtain

$$\begin{aligned} &\frac{1}{4}\|C\|^2 + \frac{3}{4}\int_0^t \|\nabla C\|^2 ds \le \|C_0\|^2 + \frac{1}{2}\|K_0\|^2 + \|K\|^2 + \frac{1}{2}\int_0^t \|C\|^2 ds \\ &+K_m^2 \int_0^t \|\mathbf{u}\|^2 ds + \frac{1}{2}\int_0^t \oint_\Gamma (1+|f|)k^2 dAds \frac{1}{2}\int_0^t \|K_{,s}\|^2 ds + \frac{1}{2}\int_0^t \oint_\Gamma \left(\frac{\partial K}{\partial n}\right)^2 dAds. \end{aligned} \tag{32}$$

Using $(25)_3$, we obtain

$$\begin{aligned} \|C\|^2 &+ 3\int_0^t \|\nabla C\|^2 ds \le 4\|C_0\|^2 + 2\|K_0\|^2 + 4\|K\|^2 + 2(1+20K_m^2)\int_0^t \|C\|^2 ds \\ &+ 4K_m^2 \int_0^t \left[d_7 + 10\Big(\frac{6}{5}\|T\|^2 + \|T\|_4^4 + \|T\|_6^6\Big)\right] ds + 2\int_0^t \oint_\Gamma (1+|f|)k^2 dAds \\ &+2\int_0^t \|K_{,s}\|^2 ds + 2\int_0^t \oint_\Gamma \left(\frac{\partial K}{\partial n}\right)^2 dAds. \end{aligned} \tag{33}$$

Furthermore, employing the same previous procedure on (28) and (29), respectively, we obtain

$$\begin{aligned} \|T\|_4^4 &+ 3\int_0^t \|\nabla T^2\|^2 ds \le \|T_0\|_4^4 + 2\|I_0\|^2 + 2\|T_0\|^2 + 16\|I\|^2 + \frac{1}{4}\|T\|^2 \\ &+ 2\int_0^t \|I_{,s}\|^2 ds + \int_0^t \oint_\Gamma |f|h^4 dAds + 2(1+24I_m^2)\int_0^t \|T\|^2 ds \\ &+ \int_0^t \|\nabla T\|^2 ds + 4I_m^2 \int_0^t \left[d_7 + 10\Big(\|T\|_4^4 + \|T\|_6^6 + \|C\|^2\Big)\right] ds \\ &+ 2\int_0^t \oint_\Gamma h^2 dAds + 2\int_0^t \oint_\Gamma \left(\frac{\partial I}{\partial n}\right)^2 dAds, \end{aligned} \tag{34}$$

$$\begin{aligned} \|T\|_6^6 &+ \frac{5}{3}\int_0^t \|\nabla T^3\|^2 ds \le \|T_0\|_6^6 + 3\|F_0\|^2 + 3\|T_0\|^2 + 36\|F\|^2 + \frac{1}{4}\|T\|^2 \\ &+ 3\int_0^t \|F_{,s}\|^2 ds + \int_0^t \oint_\Gamma |f|h^6 dAds + 3(1+72F_m^2)\int_0^t \|T\|^2 ds \\ &+ \frac{1}{2}\int_0^t \|\nabla T\|^2 ds + 18F_m^2 \int_0^t \left[d_7 + 10\Big(\|T\|_4^4 + \|T\|_6^6 + \|C\|^2\Big)\right] ds \\ &+ 3\int_0^t \oint_\Gamma h^2 dAds + 3\int_0^t \oint_\Gamma \left(\frac{\partial F}{\partial n}\right)^2 dAds, \end{aligned} \tag{35}$$

where $G_m$, $K_m$, $I_m$, and $F_m$ are the maximum values of $G$, $K$, $I$, and $F$, respectively, on $\Gamma \times (0, \mathcal{T})$.

Next, by (31)–(35), we obtain

$$
\begin{aligned}
\frac{1}{2}\|T\|^2 &+ \|T\|_4^4 + \|T\|_6^6 + \|C\|^2 + \frac{1}{2}\int_0^t \|\nabla T\|^2 ds + 3\int_0^t \|\nabla C\|^2 ds + 3\int_0^t \|\nabla T^2\|^2 ds \\
&+ \frac{5}{3}\int_0^t \|\nabla T^3\|^2 ds \le \left(7 + 24G_m^2 + 48K_m^2 + 48I_m^2 + 216F_m^2\right)\int_0^t \|T\|^2 ds \\
&+\left(2 + 20G_m^2 + 40K_m^2 + 40I_m^2 + 180F_m^2\right)\int_0^t \|C\|^2 ds \\
&+\left(20G_m^2 + 40K_m^2 + 40I_m^2 + 180F_m^2\right)\int_0^t \|T\|_4^4 ds \\
&+\left(20G_m^2 + 40K_m^2 + 40I_m^2 + 180F_m^2\right)\int_0^t \|T\|_6^6 ds + \mathcal{E}(t) \\
&\le \mathcal{B}\int_0^t \left(\frac{1}{2}\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2\right) ds + \mathcal{E}(t),
\end{aligned} \tag{36}
$$

where

$$
\mathcal{B} = 2\left(7 + 24G_m^2 + 48K_m^2 + 48I_m^2 + 216F_m^2\right)
$$

and $\mathcal{E}(t)$ is a term that is bounded by data and is given by

$$
\begin{aligned}
\mathcal{E}(t) &= 9\|T_0\|^2 + \|T_0\|_4^4 + \|T_0\|_6^6 + 2\|G_0\|^2 + 4\|C_0\|^2 + 2\|K_0\|^2 + 2\|I_0\|^2 + 3\|F_0\|^2 \\
&+ 4\|G\|^2 + 4\|K\|^2 + 16\|I\|^2 + 36\|F\|^2 + \left(2G_m^2 + 4K_m^2 + 4I_m^2 + 18F_m^2\right)\int_0^t d_7 ds \\
&+2\int_0^t \|G_{,s}\|^2 ds + 2\int_0^t \|K_{,s}\|^2 ds + 2\int_0^t \|I_{,s}\|^2 ds + 3\int_0^t \|F_{,s}\|^2 ds \\
&+\int_0^t \oint_\Gamma (7 + 2|f|)h^2 dAds + 2\int_0^t \oint_\Gamma (1 + |f|)k^2 dAds + \int_0^t \oint_\Gamma |f|h^4 dAds \\
&+\int_0^t \oint_\Gamma |f|h^6 dAds + 2\int_0^t \oint_\Gamma \left(\frac{\partial G}{\partial n}\right)^2 dAds + 2\int_0^t \oint_\Gamma \left(\frac{\partial K}{\partial n}\right)^2 dAds \\
&+2\int_0^t \oint_\Gamma \left(\frac{\partial I}{\partial n}\right)^2 dAds + 3\int_0^t \oint_\Gamma \left(\frac{\partial F}{\partial n}\right)^2 dAds.
\end{aligned} \tag{37}
$$

Straughan and Payne proved in [38] that for a given function $\phi$ that satisfies

$$
\begin{aligned}
&\Delta\phi = 0, \ \text{in } V, \\
&\phi = M, \ \text{on } \Gamma,
\end{aligned} \tag{38}
$$

one may use a Rellich identity [39] to determine $c_1$ and $c_2$, such that

$$
\|\nabla\phi\|^2 + c_1 \oint_\Gamma \left(\frac{\partial\phi}{\partial n}\right)^2 dA \le c_2 \oint_\Gamma |\nabla_s M|^2 dA, \tag{39}
$$

where $\nabla_s$ represents the surface gradient over the boundary $\Gamma$. Furthermore, they showed that

$$
2(\psi\nabla\phi, \nabla\phi) + \|\phi\|^2 \le \psi_1 \oint_\Gamma M^2 dA, \tag{40}
$$

where

$$
\psi_1 = \max_\Gamma \left|\frac{\partial\psi}{\partial n}\right|
$$

with the boundary value condition

$$\Delta\psi = -1, \text{ in } V,$$
$$\psi = 0, \text{ on } \Gamma. \tag{41}$$

Thus, (39) and (40) result in bounds for $\mathcal{E}(t)$ represented by data. We now can directly state that

$$\mathcal{E}(t) \leq \mathcal{D}(t) \tag{42}$$

such that

$$\begin{aligned}
\mathcal{D}(t) \;=\;& 9\|T_0\|^2 + \|T_0\|_4^4 + \|T_0\|_6^6 + 4\|C_0\|^2 + \left(2G_m^2 + 4K_m^2 + 4I_m^2 + 18F_m^2\right)\int_0^t d_7 ds \\
&+2\psi_1 \oint_\Gamma h_0^2 dA + 2\psi_1 \oint_\Gamma k_0^2 dA + 2\psi_1 \oint_\Gamma h_0^6 dA + 3\psi_1 \oint_\Gamma h_0^{10} dA + 4\psi_1 \oint_\Gamma h^2 dA \\
&+4\psi_1 \oint_\Gamma k^2 dA + 16\psi_1 \oint_\Gamma h^6 dA + 36\psi_1 \oint_\Gamma h^{10} dA + 2\psi_1 \int_0^t \oint_\Gamma h_{,\eta}^2 dA d\eta \\
&+2\psi_1 \int_0^t \oint_\Gamma k_{,\eta}^2 dA d\eta + 2\psi_1 \int_0^t \oint_\Gamma h^4 h_{,\eta}^2 dA d\eta + 3\psi_1 \int_0^t \oint_\Gamma h^8 h_{,\eta}^2 dA d\eta \\
&+ \int_0^t \oint_\Gamma (7 + 2|f|) h^2 dA d\eta + 2\int_0^t \oint_\Gamma (1 + |f|) k^2 dA d\eta + \int_0^t \oint_\Gamma |f| h^4 dA d\eta \\
&+ \int_0^t \oint_\Gamma |f| h^6 dA d\eta + \frac{2c_2}{c_1} \int_0^t \oint_\Gamma |\nabla_s h|^2 dA d\eta + \frac{2c_2}{c_1} \int_0^t \oint_\Gamma |\nabla_s k|^2 dA d\eta \\
&+ \frac{2c_2}{c_1} \int_0^t \oint_\Gamma |\nabla_s h^3|^2 dA d\eta + \frac{3c_2}{c_1} \int_0^t \oint_\Gamma |\nabla_s h^5|^2 dA d\eta
\end{aligned} \tag{43}$$

and then, from (36), we may have

$$U' - \mathcal{B}U \leq \mathcal{D}(t), \tag{44}$$

where $U$ is a function defined by

$$U(t) = \int_0^t \left(\frac{1}{2}\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2\right) ds.$$

Next, setting

$$\mathcal{D}_1(t) = \int_0^t \mathcal{D}(s) \exp(\mathcal{B}[t-s]) ds, \tag{45}$$

and integrating (44) to show

$$U(t) \leq \mathcal{D}_1(t). \tag{46}$$

Furthermore, taking $\mathcal{D}_2 = \mathcal{B}\mathcal{D}_1 + \mathcal{D}$ and using (44), we have

$$\frac{1}{2}\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2 \leq \mathcal{D}_2(t). \tag{47}$$

By (36), (46), and (47), we obtain

$$\begin{aligned}
&\int_0^t \|T\|^2 ds \leq 2\mathcal{D}_1, \int_0^t \|T\|_4^4 ds \leq \mathcal{D}_1, \int_0^t \|T\|_6^6 ds \leq \mathcal{D}_1, \int_0^t \|C\|^2 ds \leq \mathcal{D}_1(t), \\
&\|T\|^2 \leq 2\mathcal{D}_2, \|T\|_4^4 \leq \mathcal{D}_2, \|T\|_6^6 \leq \mathcal{D}_2, \|C\|^2 \leq \mathcal{D}_2(t), \\
&\int_0^t \|\nabla T\|^2 ds \leq 2\mathcal{D}_2, \int_0^t \|\nabla C\|^2 ds \leq \frac{1}{3}\mathcal{D}_2, \int_0^t \|\nabla T^2\|^2 ds \leq \frac{1}{3}\mathcal{D}_2, \int_0^t \|\nabla T^3\|^2 ds \leq \frac{3}{5}\mathcal{D}_2.
\end{aligned} \tag{48}$$

We could now start the way of deriving a bound for $\sup_{V\times[0,T]}|T|$, such that

$$\int_0^t\int_V (T^{2r-1}-H)(T_{,s}+u_iT_{,i}-\Delta T)d\mathbf{x}ds=0. \tag{49}$$

By integration by parts, we obtain

$$\begin{aligned}&\int_V T^{2r}d\mathbf{x}+\frac{2(2r-1)}{r}\int_0^t\int_V \nabla T^r\nabla T^r d\mathbf{x}ds=\int_V T_0^{2r}d\mathbf{x}+2r(T,H)-2r(T_0,H_0)\\&-2r\int_0^t\int_V TH_{,s}d\mathbf{x}ds+2r\int_0^t\int_V T_{,i}Hu_id\mathbf{x}ds+2r\int_0^t\oint_\Gamma h\frac{\partial H}{\partial n}dAds-\int_0^t\oint_\Gamma fT^{2r}dAds\\&\leq\int_V T_0^{2r}d\mathbf{x}+2r(\|T\|\|H\|+\|T_0\|\|H_0\|)+2r\left(\int_0^t\|H_{,s}\|^2ds\int_0^t\|T\|^2ds\right)^{1/2}\\&+2rh_m^{2r-1}\left(\int_0^t\left[d_7+12\|T\|^2+10\left(\|T\|_4^4+\|T\|_6^6+\|C\|^2\right)\right]ds\int_0^t\|\nabla T\|^2ds\right)^{1/2}\\&+2r\left(\int_0^t\oint_\Gamma h^2dAds\int_0^t\oint_\Gamma\left(\frac{\partial H}{\partial n}\right)^2dAds\right)^{1/2}+\int_0^t\oint_\Gamma|f|h^{2r}dAds.\end{aligned} \tag{50}$$

Hence, using the arithmetic-geometric mean inequality and the inequalities (39) and (40) together with (48), we obtain

$$\begin{aligned}\int_V T^{2r}d\mathbf{x}\ \leq\ &\int_V T_0^{2r}d\mathbf{x}+2r(\sqrt{2\mathcal{D}_2}+\|T_0\|)\left(\psi_1\oint_\Gamma h^{4r-2}dA\right)^{1/2}\\+\ &2r\left(2\mathcal{D}_1\psi_1\int_0^t\oint_\Gamma\left[h_{,\eta}^{2r-1}\right]^2dAd\eta\right)^{1/2}+rh_m^{2r-1}\left(\int_0^t d_7d\eta+54\mathcal{D}_1+2\mathcal{D}_2\right)\\&+2r\left(\frac{c_2}{c_1}\int_0^t\oint_\Gamma\left[\nabla_\eta h\right]^2dAd\eta\int_0^t\oint_\Gamma h^2dAd\eta\right)^{1/2}+\int_0^t\oint_\Gamma|f|h^{2r}dAd\eta.\end{aligned} \tag{51}$$

Using the Cauchy–Schwarz inequality again, we obtain

$$\left(\oint_\Gamma h^{4r-2}dA\right)^{1/2}\leq h_m^{2r-1}\left(\oint_\Gamma dA\right)^{1/2}=\frac{h_m^{2r}}{h_m}\sqrt{[m(\Gamma)]}, \tag{52}$$

$$\left(\int_0^t\oint_\Gamma h^{4r-4}h_{,\eta}^2dAd\eta\right)^{1/2}\leq\frac{h_m^{2r}}{h_m^2}\left(\int_0^t\oint_\Gamma h_{,\eta}^2dAd\eta\right)^{1/2}, \tag{53}$$

$$\left(\int_0^t\oint_\Gamma h^{4r-4}\left[\nabla_\eta h\right]^2dAd\eta\right)^{1/2}\leq\frac{h_m^{2r}}{h_m^2}\left(\int_0^t\oint_\Gamma\left[\nabla_\eta h\right]^2dAd\eta\right)^{1/2}, \tag{54}$$

where $m(\Gamma)$ is the surface measure of $\Gamma$.

Employing (53) and (54) in (51), we obtain

$$\begin{aligned}\int_V T^{2r}d\mathbf{x}\ \leq\ &\int_V T_0^{2r}d\mathbf{x}+\frac{2rh_m^{2r}}{h_m}(\sqrt{2\mathcal{D}_2}+\|T_0\|)\sqrt{\psi_1[m(\Gamma)]}\\&+\frac{2r(2r-1)h_m^{2r}}{h_m}\left(2\mathcal{D}_1\psi_1\int_0^t\oint_\Gamma h_{,\eta}^2dAd\eta\right)^{1/2}\\&+\frac{rh_m^{2r}}{h_m}\left(\int_0^t d_7d\eta+54\mathcal{D}_1+2\mathcal{D}_2\right)+h_m^{2r}\int_0^t\oint_\Gamma|f|dAd\eta\\&+\frac{2r(2r-1)h_m^{2r}}{h_m^2}\left(\frac{c_2}{c_1}\int_0^t\oint_\Gamma\left[\nabla_\eta h\right]^2dAd\eta\int_0^t\oint_\Gamma h^2dAd\eta\right)^{1/2}.\end{aligned} \tag{55}$$

Taking the power $1/2r$ of (55), we obtain

$$\|T\|_{2r} \leq \left( \|T_0\|_{2r}^{2r} + h_m^{2r} \sum_{i=1}^{5} \alpha_i \right)^{1/2r}, \tag{56}$$

where $\alpha_i$ can be obtained from (55) and

$$h_m = \max_{\Gamma\times[0,\mathcal{T}]} |h|.$$

Taking the limit as $r \to \infty$, we obtain the a priori bound

$$\sup_{V\times[0,\mathcal{T}]} |T| \leq \max\{|T_0|_m, \sup_{[0,\mathcal{T}]} h_m\}, \tag{57}$$

where

$$|T_0|_m = \max_V |T_0|.$$

Finally, we have to find a bound for $\sup_{V\times[0,\mathcal{T}]} |C|$. Now, we form the expression

$$\int_0^t \int_V (C^{2r-1} - M)(C_{,s} + u_i C_{,i} - \Delta C) d\mathbf{x} ds = 0. \tag{58}$$

Following the same procedure in (49)–(57), we have that

$$\sup_{V\times[0,\mathcal{T}]} |C| \leq \max\{|C_0|_m, \sup_{[0,\mathcal{T}]} k_m\}, \tag{59}$$

where

$$|C_0|_m = \max_V |C_0|.$$

## 4. Continuous Dependence on $\gamma$

In order to study the continuous dependence on the coefficient $\gamma$ of the viscosity in the Forchheimer equations that govern the fluid flow defined in (2), we first assume that $(u_i, T, C_1$, and $p)$ and $(v_i, S, C_2$, and $q)$ are solutions of (2)–(5) for the identical data functions $f$, $h$, and $T_0$ but for not the same coefficients of the viscosity, $\gamma_1$ and $\gamma_2$. Next, we introduce the definition of the difference solution $(w_i, \theta, \phi$, and $\pi)$ as

$$w_i = u_i - v_i, \quad \theta = T - S, \quad \phi = C_1 - C_2, \quad \pi = p - q, \quad \gamma = \gamma_1 - \gamma_2. \tag{60}$$

Moreover, by using (2)–(5), it is obvious that this solution holds for the following boundary-initial-value problem

$$\begin{gathered} a[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + b[|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] + w_i + \gamma T u_i + \gamma_2 \theta u_i + \gamma_2 S w_i \\ = -\pi_{,i} + g_i\theta + h_i\theta(T+S) + \mathcal{I}_i\theta(T^2 + TS + S^2) + \mathcal{L}_i\phi, \\ w_{i,i} = 0, \\ \theta_{,t} + w_i S_{,i} + u_i\theta_{,i} = \Delta\theta, \\ \phi_{,t} + w_i C_{2,i} + u_i\phi_{,i} = \Delta\phi, \\ w_i n_i = \theta = \phi = 0 \ \text{ on } \ \Gamma \times (0, \mathcal{T}), \\ \theta(x,0) = \phi(x,0) = 0, \quad x \in V. \end{gathered} \tag{61}$$

Obviously, $(61)_1$ can be rearranged as

$$\begin{gathered} a[|\mathbf{u}|u_i - |\mathbf{v}|v_i] + b[|\mathbf{u}|^2 u_i - |\mathbf{v}|^2 v_i] + w_i + \gamma T v_i + \gamma_2 \theta v_i + \gamma_1 T w_i \\ = -\pi_{,i} + g_i\theta + h_i\theta(T+S) + \mathcal{I}_i\theta(T^2 + TS + S^2) + \mathcal{L}_i\phi. \end{gathered} \tag{62}$$

To achieve this, multiplying (61)$_1$ by $w_i$ and integrating over $V$, we obtain

$$\begin{aligned} a\int_V[|\mathbf{u}|u_i-|\mathbf{v}|v_i]w_i d\mathbf{x}+b\int_V[|\mathbf{u}|^2u_i-|\mathbf{v}|^2v_i]w_i d\mathbf{x}+\int_V(1+\gamma_2 S)w_iw_i d\mathbf{x} \\ \leq -\gamma\int_V Tu_iw_i d\mathbf{x}-\gamma_2\int_V \theta u_iw_i d\mathbf{x}+g_i(\theta,w_i)+h_i[T_m+S_m](\theta,w_i) \\ +\mathcal{I}_i[T_m^2+T_mS_m+S_m^2](\theta,w_i)+\mathcal{L}_i(\phi,w_i), \end{aligned} \tag{63}$$

where $T_m$ and $S_m$ are the maximum values of $T$ and $S$, respectively.

Now, we observe, with the aid of the triangle inequality, that

$$\begin{aligned} a\int_\Omega[|\mathbf{u}|u_i-|\mathbf{v}|v_i]w_i dx &= \frac{a}{2}\int_\Omega[|\mathbf{u}|+|\mathbf{v}|]w_iw_i dx+\frac{a}{2}\int_\Omega[|\mathbf{u}|-|\mathbf{v}|]^2(|\mathbf{u}|+|\mathbf{v}|)dx \\ &\geq \frac{a}{2}\int_\Omega[|\mathbf{u}|+|\mathbf{v}|]w_iw_i dx. \end{aligned} \tag{64}$$

Similarly,

$$\begin{aligned} b\int_\Omega[|\mathbf{u}|^2u_i-|\mathbf{v}|^2v_i]w_i dx &= \frac{b}{2}\int_\Omega[|\mathbf{u}|^2+|\mathbf{v}|^2]w_iw_i dx+\frac{b}{2}\int_\Omega[|\mathbf{u}|^2-|\mathbf{v}|^2]^2(|\mathbf{u}|+|\mathbf{v}|)dx \\ &\geq \frac{b}{2}\int_\Omega[|\mathbf{u}|^2+|\mathbf{v}|^2]w_iw_i dx \geq \frac{b}{4}\int_\Omega[|\mathbf{u}|+|\mathbf{v}|]^2 w_iw_i dx \\ &\geq \frac{b}{4}\int_\Omega |u_i-v_i|^2 w_iw_i dx=\frac{b}{4}\|\mathbf{w}\|_4^4. \end{aligned} \tag{65}$$

Substituting (64) and (65) in (63) and applying the arithmetic-geometric mean and the Hölder's and Cauchy–Schwarz inequalities, we have

$$\begin{aligned} &\frac{a}{2}\int_V[(|\mathbf{u}|+|\mathbf{v}|)w_iw_i d\mathbf{x}+\frac{b}{4}\|\mathbf{w}\|_4^4+\int_V(1+\gamma_2 S)w_iw_i d\mathbf{x} \\ &\leq \frac{1}{3}\|\mathbf{w}\|^2+\frac{3\gamma^2T_m^2}{2}\|\mathbf{u}\|^2+a\int_V|\mathbf{u}|w_iw_i d\mathbf{x}+\frac{\gamma_2^2}{4a}\int_V|\mathbf{u}|\theta^2 d\mathbf{x}+\mathcal{R}\|\theta\|\|\mathbf{w}\|+\frac{3}{2}\|\phi\|^2 \\ &\leq \frac{1}{2}\|\mathbf{w}\|^2+\frac{3\gamma^2T_m^2}{2}\|\mathbf{u}\|^2+a\int_V|\mathbf{u}|w_iw_i d\mathbf{x}+\frac{\gamma_2^2}{4a}\left(\int_V|u|^3 d\mathbf{x}\right)^{1/3}\left(\int_V|\theta|^3 d\mathbf{x}\right)^{2/3} \\ &\quad+\frac{3\mathcal{R}^2}{2}\|\theta\|^2+\frac{3}{2}\|\phi\|^2, \end{aligned} \tag{66}$$

where $\mathcal{R}=1+T_m+S_m+T_m^2+T_mS_m+S_m^2$. Using the Sobolev inequality together with the Cauchy–Schwarz inequality in (66), we obtain

$$\begin{aligned} &\frac{a}{2}\int_V[(|\mathbf{u}|+|\mathbf{v}|)w_iw_i d\mathbf{x}+\frac{b}{4}\|\mathbf{w}\|_4^4+\int_V(\frac{1}{2}+\gamma_2 S)w_iw_i d\mathbf{x} \\ &\leq \frac{3\gamma^2T_m^2}{2}\|\mathbf{u}\|^2+a\int_V|\mathbf{u}|w_iw_i d\mathbf{x}+\frac{\gamma_2^2C^{2/3}}{4a}\|\mathbf{u}\|_3\|\theta\|\|\nabla\theta\|+\frac{3\mathcal{R}^2}{2}(\|\theta\|^2+\|\phi\|^2). \end{aligned} \tag{67}$$

Repeating the same argument starting from (63) to obtain

$$\begin{aligned} &\frac{a}{2}\int_V[(|\mathbf{u}|+|\mathbf{v}|)w_iw_i d\mathbf{x}+\frac{b}{4}\|\mathbf{w}\|_4^4+\int_V(\frac{1}{2}+\gamma_1 T)w_iw_i d\mathbf{x} \\ &\leq \frac{3\gamma^2T_m^2}{2}\|\mathbf{v}\|^2+a\int_V|\mathbf{v}|w_iw_i d\mathbf{x}+\frac{\gamma_2^2C^{2/3}}{4a}\|\mathbf{v}\|_3\|\theta\|\|\nabla\theta\|+\frac{3\mathcal{R}^2}{2}(\|\theta\|^2+\|\phi\|^2). \end{aligned} \tag{68}$$

Combining (67) and (68), we have

$$\frac{b}{2}\|\mathbf{w}\|_4^4 + \int_V (1+\gamma_2 S + \gamma_1 T) w_i w_i d\mathbf{x} \le \frac{3\gamma^2 T_m^2}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) + \frac{\gamma_2^2 \mathcal{C}^{2/3}}{4a}(\|\mathbf{u}\|_3 + \|\mathbf{v}\|_3)\|\theta\|\|\nabla\theta\| + 3\mathcal{R}^2(\|\theta\|^2 + \|\phi\|^2). \tag{69}$$

For $\lambda > 0$, dropping the first terms on the left hand side, we conclude

$$\begin{aligned}\int_V (1+\gamma_2 S + \gamma_1 T) w_i w_i d\mathbf{x} &\le \frac{3\gamma^2 T_m^2}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) + \frac{\gamma_2^4 \mathcal{C}^{4/3}}{64\lambda a^2}(\|\mathbf{u}\|_3 + \|\mathbf{v}\|_3)^2\|\theta\|^2 + \lambda\|\nabla\theta\|^2 + 3\mathcal{R}^2(\|\theta\|^2 + \|\phi\|^2) \\ &\le 3\gamma^2 T_m^2(d_7 + 54\mathcal{D}_2) + \frac{\gamma_2^4 \mathcal{C}^{4/3}}{16\lambda a^{8/3}}(d_7 + 54\mathcal{D}_2)^{2/3}\|\theta\|^2 + \lambda\|\nabla\theta\|^2 + 3\mathcal{R}^2(\|\theta\|^2 + \|\phi\|^2) \\ &\le A\gamma^2 + B(\|\theta\|^2 + \|\phi\|^2) + \lambda\|\nabla\theta\|^2,\end{aligned} \tag{70}$$

where $A = 3T_m^2(d_{7_{\max}} + 54\mathcal{D}_2)$ and $B = a^{-8/3}\gamma_2^4\mathcal{C}^{4/3}(d_{7_{\max}} + 54\mathcal{D}_2)^{2/3}/16\lambda + 3\mathcal{R}^2$.

Furthermore, multiplying $(61)_3$ and $(61)_4$ by $\theta$ and $\phi$, respectively, and integrating over $V$, we obtain

$$\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 \le S_m^2\|\mathbf{w}\|^2, \tag{71}$$

$$\frac{d}{dt}\|\phi\|^2 + \|\nabla\phi\|^2 \le C_{2\,m}^2\|\mathbf{w}\|^2. \tag{72}$$

Collecting (71) and (72) and integrating the result, we have

$$\|\theta\|^2 + \|\phi\|^2 + \int_0^t (\|\nabla\theta\|^2 + \|\nabla\phi\|^2) ds \le (S_m^2 + C_{2\,m}^2)\int_0^t \|\mathbf{w}\|^2 ds, \tag{73}$$

which implies

$$\|\nabla\theta\|^2 \le (S_m^2 + C_{2\,m}^2)\|\mathbf{w}\|^2, \tag{74}$$

where $C_{2\,m}$ is the maximum value of $C_2$.

Next, we assume that $\mathcal{K} = 2B(S_m^2 + C_{2\,m}^2)$, $\mathcal{A} = 2A(S_m^2 + C_{2\,m}^2)$, and $\lambda = 1/2(S_m^2 + C_{2\,m}^2)$. Then, from (70) and (73), we obtain

$$\|\mathbf{w}\|^2 \le \mathcal{A}\gamma^2 + \mathcal{K}\int_0^t \|\mathbf{w}\|^2 ds. \tag{75}$$

By (75), we have

$$\int_0^t \|\mathbf{w}\|^2 ds \le \mathcal{A}\gamma^2 t + \mathcal{K}\int_0^t (t-s)\|\mathbf{w}\|^2 ds \tag{76}$$

and, thus, from (76), we obtain

$$\int_0^t (t-s)\|\mathbf{w}\|^2 ds \le \mathcal{K}_2(t)\gamma^2 \tag{77}$$

and so

$$\int_0^t \|\mathbf{w}\|^2 ds \le \mathcal{K}_3(t)\gamma^2, \tag{78}$$

where

$$\mathcal{K}_2(t) = \int_0^t \mathcal{K}_1(s)\exp(\mathcal{K}[t-s])ds, \;\; \mathcal{K}_1(t) = \mathcal{A}t \;\text{ and }\; \mathcal{K}_3(t) = \mathcal{K}_1 + \mathcal{K}\mathcal{K}_2.$$

Moreover, employing (78) in (73) we can have

$$\|\theta\|^2+\|\phi\|^2+\int_0^t(\|\nabla\theta\|^2+\|\nabla\phi\|^2)ds\le\gamma^2\mathcal{K}_3(S_m^2+C_{2\,m}^2). \tag{79}$$

The continuous dependence on the coefficient $\gamma$ of the viscosity is shown by (79), and it is clearly an a priori bound, such that $\gamma^2$ only relies on the initial data and boundary data.

## 5. Convergence to the Constant Viscosity Solution

Let $(u_i, T, C_1,$ and $p)$ and $(v_i, S, C_2,$ and $q)$ be solutions satisfying the following boundary-initial-value problems:

$$\left.\begin{array}{c} au_i|\mathbf{u}|+bu_i|\mathbf{u}|^2+(1+\gamma T)u_i=-p_{,i}+g_iT+h_iT^2+\mathcal{I}_iT^3+\mathcal{L}_iC_1,\\ u_{i,i}=0,\\ T_{,t}+u_iT_{,i}=\Delta T,\\ C_{1,t}+u_iC_{1,i}=\Delta C_1, \end{array}\right\}\text{ in } V\times(0,\mathcal{T}) \tag{80}$$

and

$$\begin{array}{c} u_in_i=f(x,t),\quad\text{on }\ \Gamma\times(0,\mathcal{T}),\\ T(x,t)=h(x,t),\ \ C_1(x,t)=k(x,t),\ \ x\ \text{ on }\ \Gamma,\ \ t\in(0,\mathcal{T}), \end{array} \tag{81}$$

$$\left.\begin{array}{c} av_i|\mathbf{v}|+bv_i|\mathbf{v}|^2+v_i=-q_{,i}+g_iS+h_iS^2+\mathcal{I}_iS^3+\mathcal{L}_iC_2,\\ v_{i,i}=0,\\ S_{,t}+v_iS_{,i}=\Delta S,\\ C_{2,t}+v_iC_{2,i}=\Delta C_2, \end{array}\right\}\text{ in } V\times(0,\mathcal{T}) \tag{82}$$

and

$$\begin{array}{c} v_in_i=f(x,t),\quad\text{on }\ \Gamma\times(0,\mathcal{T}),\\ S(x,t)=h(x,t),\ \ C_2(x,t)=k(x,t),\ \ x\ \text{ on }\ \Gamma,\ \ t\in(0,\mathcal{T}). \end{array} \tag{83}$$

The variables $w_i$, $\theta$, $\phi$, and $\pi$ are introduced in (60) and satisfy the boundary-initial-value problem:

$$\begin{array}{c} a[|\mathbf{u}|u_i-|\mathbf{v}|v_i]+b[|\mathbf{u}|^2u_i-|\mathbf{v}|^2v_i]+w_i+\gamma Tu_i=-\pi_{,i}+g_i\theta\\ +h_i\theta(T+S)+\mathcal{I}_i\theta(T^2+TS+S^2)+\mathcal{L}_i\phi,\\ w_{i,i}=0,\\ \theta_{,t}+w_iS_{,i}+u_i\theta_{,i}=\Delta\theta,\\ \phi_{,t}+w_iC_{2,i}+u_i\phi_{,i}=\Delta\phi,\\ w_in_i=\theta=\phi=0\ \text{ on }\ \Gamma\times(0,\mathcal{T}),\\ \theta(x,0)=\phi(x,0)=0,\quad x\in V. \end{array} \tag{84}$$

The proof of the maximum principle (57) and (59) for $T$ and $C$ may be shown to hold here. Multiplying $(84)_1$ by $w_i$ and integrating over $V$, and employing the Cauchy–Schwarz and arithmetic-geometric-mean inequalities, we obtain

$$\begin{aligned} a\|\mathbf{w}\|_3^3+b\|\mathbf{w}\|_4^4+\|\mathbf{w}\|^2 &\le \gamma T_m\|\mathbf{w}\|\|\mathbf{u}\|+\mathcal{R}\|\theta\|\|\mathbf{w}\|+\|\phi\|\|\mathbf{w}\| \\ &\le \frac{1}{2}\|\mathbf{w}\|^2+\frac{3}{2}\gamma^2T_m^2\|\mathbf{u}\|^2+\frac{3}{2}\mathcal{R}^2\|\theta\|^2+\frac{3}{2}\|\phi\|^2. \end{aligned} \tag{85}$$

We may drop the non-negative first and second terms in the left to obtain

$$\|\mathbf{w}\|^2\le 3\gamma^2T_m^2\|\mathbf{u}\|^2+3\mathcal{R}^2(\|\theta\|^2+\|\phi\|^2). \tag{86}$$

Next, multiplying (80) by $u_i$, and vs. integrating over $V$, we obtain

$$\begin{aligned} & a\|\mathbf{u}\|_3^3 + b\|\mathbf{u}\|_4^4 + \int_V (1+\gamma T)|\mathbf{u}|^2 d\mathbf{x} \\ & = -\oint_\Gamma p_{,i} f dA + g_i(T, u_i) + h_i(T^2, u_i) + \mathcal{I}_i(T^3, u_i) + \mathcal{L}_i(C, u_i) \\ & \leq -\oint_\Gamma p_{,i} f dA + \frac{1}{4}\|\mathbf{u}\|^2 + 4\Big(\|T\|^2 + \|T\|_4^4 + \|T\|_6^6 + \|C\|^2\Big) \\ & \leq -\oint_\Gamma p_{,i} f dA + \frac{1}{4}\int_V (1+\gamma T)|\mathbf{u}|^2 d\mathbf{x} + 20\mathcal{D}_2. \end{aligned} \tag{87}$$

Now, one could use the Cauchy–Schwarz and Hölder's inequalities along with Young's inequality to obtain

$$\begin{aligned} & -\oint_\Gamma p_{,i} f dA = \int_V \varphi_{,i}\Big[au_i|\mathbf{u}| + bu_i|\mathbf{u}|^2 + (1+\gamma T)u_i - g_i T - h_i T^2 - \mathcal{I}_i T^3 - \mathcal{L}_i C\Big] d\mathbf{x} \\ & \leq a\Big(\int_V |\nabla\varphi|^3 d\mathbf{x}\Big)^{1/3}\Big(\int_V |\mathbf{u}|^3 d\mathbf{x}\Big)^{2/3} + b\Big(\int_V |\nabla\varphi|^4 d\mathbf{x}\Big)^{1/4}\Big(\int_V |\mathbf{u}|^4 d\mathbf{x}\Big)^{3/4} \\ & + \Big(\int_V (1+\gamma T)u_i u_i d\mathbf{x}\Big)^{1/2}\Big(\int_V (1+\gamma T)\varphi_{,i}\varphi_{,i} d\mathbf{x}\Big)^{1/2} + \|\nabla\varphi\|\Big(\|T\| + \|T\|^2 + \|T\|_3^3 + \|C\|\Big) \\ & \leq a\|\mathbf{u}\|_3^3 + \frac{4a}{27}\int_V |\nabla\varphi|^3 d\mathbf{x} + b\|\mathbf{u}\|_4^4 + \frac{27b}{256}\int_V |\nabla\varphi|^4 d\mathbf{x} \\ & + \frac{1}{4}\int_V (1+\gamma T_m)|\mathbf{u}|^2 d\mathbf{x} + \int_V (1+\gamma T_m)|\nabla\varphi|^2 d\mathbf{x} + \frac{1}{4}\|\nabla\varphi\|^2 + 20\mathcal{D}_2. \end{aligned} \tag{88}$$

By (87) and (88), we obtain

$$\|\mathbf{u}\|^2 \leq \mathcal{M}^2, \tag{89}$$

where $\mathcal{M}^2$ is the data term, such that

$$\mathcal{M}^2 = \frac{8a}{27}\int_V |\nabla\varphi|^3 d\mathbf{x} + \frac{27b}{128}\int_V |\nabla\varphi|^4 d\mathbf{x} + 2\int_V (1+\gamma T_m)|\nabla\varphi|^2 d\mathbf{x} + \frac{1}{2}\|\nabla\varphi\|^2 + 80\mathcal{D}_2.$$

Next, multiplying of $(84)_3$ and $(84)_4$ by $\theta$ and $\phi$, respectively, we also obtain

$$\|\theta\|^2 + \|\phi\|^2 \leq \frac{1}{2}(S_m^2 + C_{2m}^2)\int_0^t \|\mathbf{w}\|^2 ds. \tag{90}$$

Thus, by (86) and (90), we have

$$\|\mathbf{w}\|^2 \leq 3\gamma^2 T_m^2 \mathcal{M}^2 + \frac{3}{2}\mathcal{R}^2(S_m^2 + C_{2m}^2)\int_0^t \|\mathbf{w}\|^2 ds. \tag{91}$$

By (91), we obtain

$$\int_0^t \|\mathbf{w}\|^2 ds \leq \frac{2\gamma^2 T_m^2 \mathcal{M}^2 \exp\Big(\frac{3}{2}\mathcal{R}^2(S_m^2 + C_{2m}^2)t\Big)}{\mathcal{R}^2(S_m^2 + C_{2m}^2)}. \tag{92}$$

The above inequality (92) shows that $u_i$ is convergent to $v_i$ as $\gamma \to 0$. Adding (91) and (92), one can clearly see the convergence of $w_i$ in the $L^2(\Omega)$ norm, and from (90), one can obtain the convergence of $\theta$ and $\phi$ in the $L^2(\Omega)$ norm.

## 6. Conclusions

The continuous dependence and structural stability in the problem of double-diffusive convection in a porous medium of the Forchheimer model was investigated throughout this study when the fluid density and viscosity had cubic and linear temperature dependences, respectively. The a priori bound was derived and obtained for both of the temperature and the concentration using coefficients that were dependent only on the boundary and initial data. With the aid of these a priori bounds, we were able to demonstrate the continuous dependence of the solutions on some coefficients. We further showed that the solution depends continuously on a change in the viscosity coefficients. In addition, we studied the continuous dependence on the coefficient $\gamma$ of the viscosity in the Forchheimer equations that govern fluid flow by assuming a couple of solutions to our boundary-initial-value problems and defining a difference solution to the same problem. Then, with the aid of the Cauchy–Schwarz inequality and the integration, we used these two assumed solutions $(u_i, T, C_1, \text{ and } p)$ and $(v_i, S, C_2, \text{ and } q)$, and we proved that $u_i$ from the first solution converges to $v_i$ from the second solution using the difference solution $(w_i, \theta, \phi, \text{ and } \pi)$, which indicates the difference between the two assumed solutions. Consequently, the first solution converges to the second solution as the difference solution converges to zero. Finally, we reached to an inequality that ensures the convergence on the Forchheimer system when the variable viscosity coefficients trend toward a constant viscosity.

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