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On edge-sets of bicliques in graphs

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ABSTRACT

A *biclique* is a maximal induced complete bipartite subgraph of a graph. We investigate the intersection structure of edge-sets of bicliques in a graph. Specifically, we study the associated *edge-biclique hypergraph* whose hyperedges are precisely the edge-sets of all bicliques. We characterize graphs whose edge-biclique hypergraph is *conformal* (i.e., it is the clique hypergraph of its 2-section) by means of a single forbidden induced obstruction, the triangular prism. Using this result, we characterize graphs whose edge-biclique hypergraph is *Helly* and provide a polynomial time recognition algorithm. We further study a hereditary version of this property and show that it also admits polynomial time recognition, and, in fact, is characterized by a finite set of forbidden induced subgraphs. We conclude by describing some interesting properties of the 2-section graph of the edge-biclique hypergraph.

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1. Introduction

The intersection graph of a collection of sets is defined as follows. The vertices correspond to the sets, and two vertices are adjacent just if the corresponding sets intersect. Intersection graphs are a central theme in algorithmic graph theory because they naturally occur in many applications. Moreover, they often exhibit elegant structure which allows efficient solution of many algorithmic problems. Of course, to obtain a meaningful notion, one has to restrict the type of sets in the collection. In fact, [19], every graph can be obtained as the intersection graph of some collection of sets. By considering intersections of intervals of the real line, subtrees of a tree, or arcs on a circle, one obtains interval, chordal, or circular-arc graphs, respectively. For these classes, a maximum clique or a maximum independent set can be found in polynomial time [9]. We note that one can alternatively define an interval graph as an intersection graph of connected subgraphs of a path; similarly intersection graphs of connected subgraphs of a tree produce chordal graphs, and intersection graphs of connected subgraphs of a cycle produce circular-arc graphs. More generally, one can consider intersections of particular subgraphs of arbitrary graphs. This naturally leads to intersections of edges, cliques, or bicliques of graphs which correspond to line graphs, clique graphs, and biclique graphs, respectively.

We focus on edge intersections of subgraphs. The edge intersection graph of a collection of subgraphs is defined in the obvious way, as the intersection graph of their edge-sets. In hypergraph terminology, this can be defined as the line graph of the hypergraph whose hyperedges are the edge-sets of the subgraphs. We say that subgraphs are edge intersecting if they share at least one edge of the graph. For instance, the EPT graphs from [10] are exactly the edge intersection graphs of paths in trees. For another example, consider the double stars of a graph G , i.e., the subgraphs formed by the sets of edges

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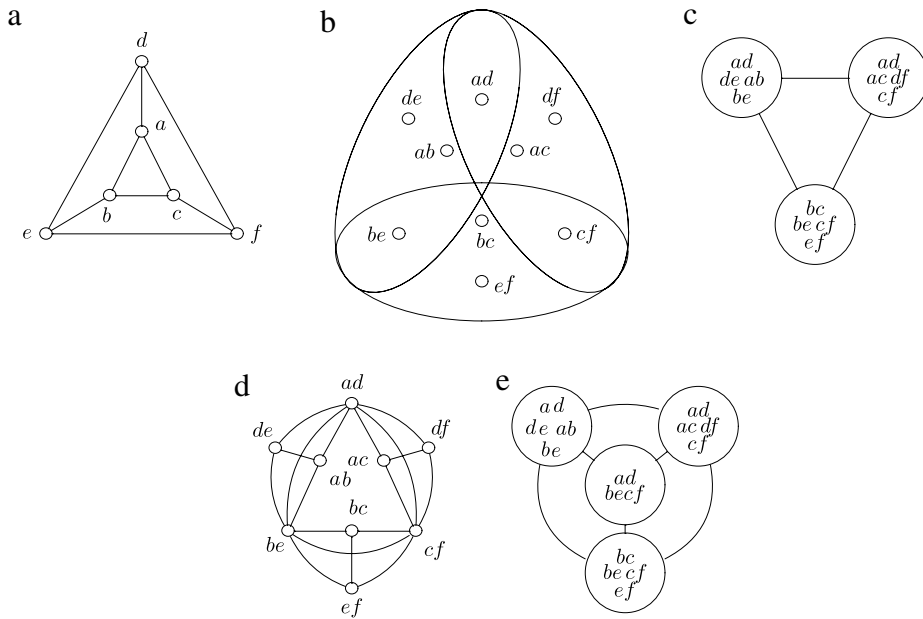


Fig. 1. (a) G , (b) $\mathcal{EB}(G)$, (c) the line graph of $\mathcal{EB}(G)$, (d) $L_G =$ the 2-section of $\mathcal{EB}(G)$, (e) the clique graph of L_G .

graph with vertex set E where two edges of E are adjacent if they belong to a common biclique of G . Note that L_G is the 2-section of $\mathcal{EB}(G)$.

For the reader's convenience, we summarize these notions in the following two tables.

$\mathcal{H} = (V, \mathcal{E})$	notation	vertices	(hyper)edges
dual	\mathcal{H}^*	hyperedges	hyperedges sharing a common vertex
line graph	$L(\mathcal{H})$	hyperedges	two intersecting hyperedges
2-section	$(\mathcal{H})_2$	vertices	two vertices in a common hyperedge

$G = (V, E)$	notation	vertices	(hyper)edges
line graph	$L(G)$	edges	two edges sharing a vertex
biclique line graph	L_G	edges	two edges in a common biclique
clique graph	$K(G)$	cliques	two intersecting cliques
clique hypergraph	$\mathcal{K}(G)$	vertices	cliques
edge-biclique hypergraph	$\mathcal{EB}(G)$	edges	edge-sets of bicliques

We also refer the reader to Fig. 1 for an illustration of these concepts.

We say that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *reduced* if there are no hyperedges $X, X' \in \mathcal{E}$ with $X \subsetneq X'$. In other words, a hypergraph \mathcal{H} is reduced if every hyperedge of \mathcal{H} is inclusion-wise maximal among the hyperedges of \mathcal{H} . If \mathcal{H} is not reduced, then the *reduction* of \mathcal{H} is the partial hypergraph of \mathcal{H} containing only the inclusion-wise maximal hyperedges of \mathcal{E} . Note that the reduction of \mathcal{H} is always a reduced hypergraph. Also, observe that $\mathcal{K}(G)$ and $\mathcal{EB}(G)$ are reduced hypergraphs by definition.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *Helly* if for every subcollection $\mathcal{E}' \subseteq \mathcal{E}$ satisfying $X \cap X' \neq \emptyset$ for all $X, X' \in \mathcal{E}'$, we have $\bigcap_{X \in \mathcal{E}'} X \neq \emptyset$. A hypergraph \mathcal{H} is *conformal* if every clique of the 2-section of \mathcal{H} is contained in a hyperedge of \mathcal{H} . In particular, if \mathcal{H} is reduced, then \mathcal{H} is conformal if and only if it is the clique hypergraph of its 2-section. Alternatively [3], \mathcal{H} is conformal if and only if the dual of \mathcal{H} is Helly.

We say that H is a line graph, or a clique graph, or a biclique line graph if, respectively, $H = L(G)$, or $H = K(G)$, or $H = L_G$, for some G . Note that where appropriate we shall refer to the vertices of L_G and $K(G)$ as edges and cliques, respectively, and refer to the hyperedges of $\mathcal{EB}(G)$ as bicliques.

As usual, we shall denote by $V(G)$ and $E(G)$ the vertex set respectively the edge set of a graph G . For hypergraphs, we shall not use special notation for vertices and hyperedges for simplicity.

We emphasize that, in this paper, cliques and bicliques are always maximal, and they are usually viewed as vertex sets, rather than subgraphs. For any further terminology, please consult [3,20].

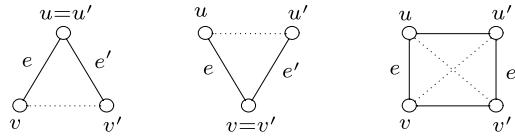


Fig. 2. Adjacent edges in biclique line graphs.

3. The conformal property

In this section, we characterize graphs G whose edge-biclique hypergraph $\mathcal{EB}(G)$ is conformal. We do this by studying the 2-section of $\mathcal{EB}(G)$. Recall that we use L_G to denote the 2-section of $\mathcal{EB}(G)$ and call this graph the biclique line graph of G . We start with some useful observations about L_G . The following is a restatement of the definition.

Proposition 1. *If $e = uv$ and $e' = u'v'$ are edges of G , then e and e' are adjacent in L_G if and only if either $u = u'$ and $vv' \notin E(G)$, or $v = v'$ and $uu' \notin E(G)$, or u, v, u', v' induces a four-cycle in G (see Fig. 2). □*

In the next lemma and subsequent statements, \bar{P}_3 denotes the complement of the path on 3 vertices.

Lemma 2. *If $e_1 = ab$ and $e_2 = cd$ are edges of G such that $G[a, b, c, d]$ contains a triangle or an induced \bar{P}_3 , then e_1e_2 is not an edge of L_G .*

Proof. Let ab, cd be such edges, and let $H = G[a, b, c, d]$. First, suppose that H contains a triangle. Without loss of generality, let a, b, c be a triangle of H . If $\{a, b\} \cap \{c, d\} \neq \emptyset$, then H itself is a triangle, and hence, by Proposition 1, the edges $e_1 = ab$ and $e_2 = cd$ are not adjacent in L_G . So $\{a, b\} \cap \{c, d\} = \emptyset$, but then H is not a four-cycle implying again that $e_1e_2 \notin E(L_G)$.

Now, suppose that H contains an induced \bar{P}_3 . Without loss of generality, let a, b, c induce a \bar{P}_3 in H with $ac, bc \notin E(G)$. This yields $\{a, b\} \cap \{c, d\} = \emptyset$. Hence, if $e_1e_2 \in E(L_G)$, it follows from Proposition 1 that this can only be if a, b, c, d induces a four-cycle. But this contradicts $ac, bc \notin E(G)$. □

Next, observe that the edge sets of bicliques of G are complete subgraphs of L_G . In the following, we show that they are, in fact, cliques of L_G .

Lemma 3. *The edge-biclique hypergraph of G is a partial hypergraph of the clique hypergraph of L_G .*

Proof. For the proof, we shall show that for every biclique of G , its edge set is a clique in L_G . Consider a biclique B of G , and let C denote the edges of $G[B]$. We shall show that C is a clique of L_G .

Since all edges in the set C belong to a complete bipartite subgraph of G , the set C induces a complete subgraph of L_G , as observed above the claim, by the definition of L_G . Suppose that C is not a clique of L_G , that is, there exists an edge $uv = e \notin C$ such that $C \cup \{e\}$ is a complete subgraph of L_G . We show that $G[B \cup \{u, v\}]$ is a complete bipartite graph, which will contradict our assumption that B is a biclique of G . If $G[B \cup \{u, v\}]$ is not a complete bipartite graph, then it contains a triangle or an induced \bar{P}_3 whose at least one vertex is u or v . In particular, if u, v, a induces in G a triangle or a \bar{P}_3 for some $a \in B$, we let b be any vertex of B adjacent to a (possibly $b = u$ or $b = v$), and conclude that ab and uv are edges in $C \cup \{e\}$. This, however, contradicts Lemma 2, since then $G[a, b, u, v]$ contains a triangle or an induced \bar{P}_3 . If u, a, b or v, a, b is a triangle or an induced \bar{P}_3 in G for $a, b \in B$ where $ab \in E(G)$, we again have edges ab, uv in $C \cup \{e\}$ contradicting Lemma 2. So, we let a, b be non-adjacent vertices of B , and let c be any vertex of B adjacent to a (and hence to b). In particular, ac and bc are edges in C , and u, v are not both in $\{a, b, c\}$, since $a, b, c \in B$ and $e \notin C$. If exactly one of u, v is in $\{a, b, c\}$, then we conclude that neither u, a, b nor v, a, b induces a \bar{P}_3 in G , since otherwise we contradict Lemma 2 for the edges ac, uv or bc, uv . So, $\{u, v\} \cap \{a, b, c\} = \emptyset$, and we conclude, by Proposition 1, that both a, c, u, v and b, c, u, v induce a four-cycle in G . In other words, $vc \in E(G)$ if and only if $va, vb \notin E(G)$ if and only if $ua, ub \in E(G)$. Hence, both u, a, b and v, a, b do not induce a \bar{P}_3 in G . Consequently, $G[B \cup \{u, v\}]$ is a complete bipartite graph, a contradiction. □

Now, assuming that G contains no induced subgraph isomorphic to the triangular prism (the graph in Fig. 1(a)) we show that there are no other cliques in L_G than the ones arising from bicliques of G .

Lemma 4. *If G contains no induced subgraph isomorphic to the triangular prism, then the edge-biclique hypergraph of G is equal to the clique hypergraph of L_G .*

Proof. Assume that G contains no induced subgraph isomorphic to the triangular prism. By Lemma 3 it remains to prove that every clique of L_G is the set of edges of some biclique of G . Consider a clique C of L_G , and let B denote the vertices of G incident to the edges in the set C . We show that B is a biclique of G , and C is precisely the set of edges of $G[B]$ which will prove the claim.

First, we show that the set of edges of $G[B]$ is precisely C . Suppose otherwise, and let $e = uv$ be an edge of $G[B]$ that is not in C . Since $u, v \in B$, we have, by the definition of B , edges $au = e^* \in C$ and $bv = e^{**} \in C$. Clearly, $a \neq v$ and $b \neq u$, since $e \notin C$. Also, $a \neq b$, because otherwise $G[a, b, u, v]$ contains a triangle contradicting Lemma 2 for e^* and e^{**} which are adjacent in L_G . Hence, we conclude that the vertices a, b, u, v induce a four-cycle. Now, recall that C is a clique of L_G , that is,

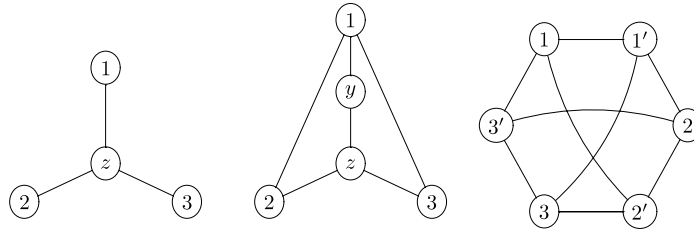


Fig. 3. Graph B of a B -template.

we observe that u is adjacent to exactly one of $\{a, b\}$, since otherwise $G[B_1]$ contains a triangle or an induced \bar{P}_3 , and thus B_1 is not a biclique. Without loss of generality, suppose that $ua \in E(G)$ and $ub \notin E(G)$. This implies that $vb \in E(G)$ and $va \notin E(G)$. Therefore, $vc \notin E(G)$, since otherwise $G[B_2]$ contains a triangle. Thus the vertices v, a, c induce a \bar{P}_3 in $G[B_3]$, a contradiction. \square

Theorem 8. *The edge-biclique hypergraph of G is Helly if and only if the clique hypergraph of L_G is Helly.*

Proof. By Lemma 3, the edge sets of bicliques of G are the cliques of L_G . Hence, if the cliques of L_G satisfy the Helly property, then the edge sets of bicliques of G must satisfy the Helly property. Conversely, if $\mathcal{EB}(G)$ is Helly, we conclude, by Lemma 7, that G contains no induced triangular prism. Hence, by Lemma 4, the cliques of L_G are the edge sets of bicliques of G . So, if the edge sets of bicliques of G satisfy the Helly property, then the cliques of L_G must satisfy the Helly property. \square

Corollary 9. *There is a polynomial time algorithm for the recognition of graphs whose edge-biclique hypergraph is Helly.*

Proof. Clearly, the graph L_G can be constructed in polynomial time. By [15], graphs whose clique hypergraph is Helly can be recognized in polynomial time. This with Theorem 8 implies the claim. \square

To be more precise, the complexity of the algorithm is $O(|E(G)|^4)$. This follows from $O(|E(G)|^2)$ complexity [15] of recognizing graphs whose clique hypergraph is Helly. Since we apply this to the graph L_G , the total complexity is $O(|E(L_G)|^2) = O(|E(G)|^4)$. For this note that L_G can have $O(|E(G)|^2)$ edges, and this is tight, for example, if G is a complete bipartite graph. Finally, the construction of the biclique line graph L_G from G can be realized in time $O(|E(G)|^2)$ by a straightforward implementation.

We remark that Berge described in [3] a polynomial time condition for a family of sets to be Helly. However, we cannot apply this condition directly, as a graph can have exponentially many bicliques.

5. The hereditary Helly property

In this section, we look at a hereditary version of the Helly property for edge-biclique hypergraphs. This is in a direct analogy with similar classes of graphs based on the Helly property (e.g., clique-Helly, disk-Helly) whose corresponding hereditary classes have been considered in the literature (cf. [12]).

We say that a hypergraph \mathcal{H} is *hereditary Helly* if the reduction of every induced subhypergraph of \mathcal{H} is Helly. We require only reductions of induced subhypergraphs to be Helly so that we obtain a more general notion also suitable for derived hypergraphs (see below).

We study graphs G for which the edge-biclique hypergraph $\mathcal{EB}(G)$ is hereditary Helly. It can be seen from the definition that $\mathcal{EB}(G)$ is hereditary Helly if and only if for every induced subgraph H of G , the hypergraph $\mathcal{EB}(H)$ is Helly. Using this, we describe (in Theorem 11) a finite forbidden induced subgraph characterization of graphs whose edge-biclique hypergraph is hereditary Helly.

A *B-template* is a graph H that consists of a complete bipartite graph B and three additional vertices x_1, x_2, x_3 satisfying one of the following:

1. $V(B) = \{1, 2, 3, z\}$ and $E(B) = \{1z, 2z, 3z\}$, or $V(B) = \{1, 2, 3, y, z\}$ and $E(B) = \{1y, 1z, 2y, 2z, 3y, 3z\}$ (see Fig. 3(a) and (b)) and for each $i \in \{1, 2, 3\}$
 - (a) $H[B \setminus \{i\} \cup \{x_i\}]$ is a complete bipartite graph,
 - (b) $H[B \cup \{x_i\}]$ is not a complete bipartite graph,
2. $V(B) = \{1, 1', 2, 2', 3, 3'\}$ and $E(B) = \{11', 12', 13', 21', 22', 23', 31', 32', 33'\}$ (see Fig. 3(c)) and for each $i \in \{1, 2, 3\}$
 - (a) $H[B \setminus \{i, i'\} \cup \{x_i\}]$ is a complete bipartite graph,
 - (b) $H[B \setminus \{i\} \cup \{x_i\}]$ and $H[B \setminus \{i'\} \cup \{x_i\}]$ are not complete bipartite graphs.

All possible B -templates are illustrated in Fig. 4. For the proof of our characterization, we shall need the following useful lemma.

Lemma 10. *Let G be a graph with a vertex x and sets of vertices $B_1 \subseteq V(G), B_2 \subseteq V(G)$ such that*

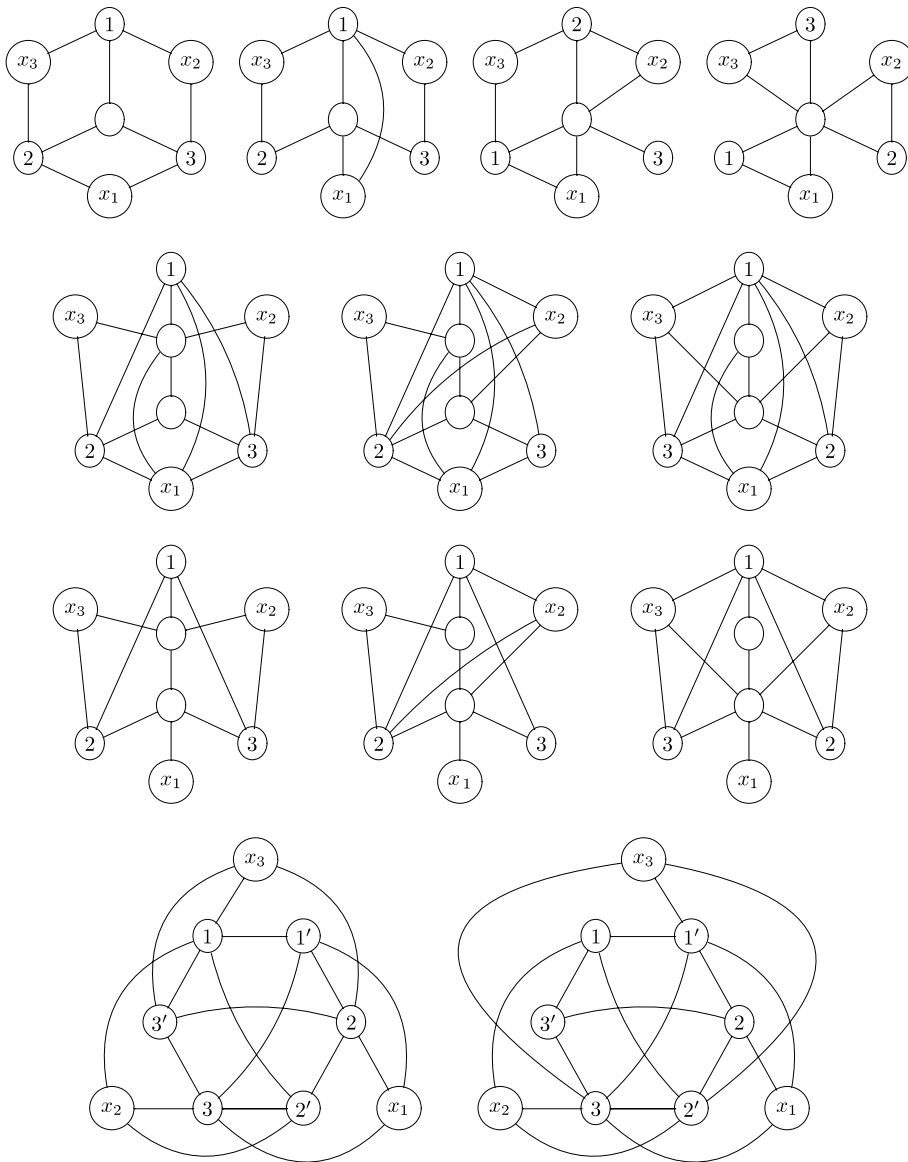


Fig. 4. List of all B -templates (excluding the edges between x_1, x_2, x_3).

- (i) $B_1 \cap B_2 \neq \emptyset$,
- (ii) $B_1 \cup B_2$ induces in G a complete bipartite graph, and
- (iii) $B_1 \cup \{x\}$ and $B_2 \cup \{x\}$ induce in G complete bipartite graphs.

Then $B_1 \cup B_2 \cup \{x\}$ induces in G a complete bipartite graph.

Proof. Suppose that $G[B_1 \cup B_2 \cup \{x\}]$ is not a complete bipartite graph. It follows that there must be vertices $a \in B_1 \setminus B_2$ and $b \in B_2 \setminus B_1$ such that x, a, b induce in G either a triangle or a \bar{P}_3 .

Let z be any vertex of $B_1 \cap B_2$. First, suppose that x, a, b induce a triangle in G . Since $G[B_1 \cup B_2]$ is a complete bipartite graph, the vertices a, b, z induce neither a triangle nor a \bar{P}_3 , and hence, up to symmetry, we must have $az \notin E(G)$ and $bz \in E(G)$. It follows that $xz \in E(G)$, since otherwise x, a, z induce a \bar{P}_3 contradicting the fact that $G[B_1 \cup \{x\}]$ is a complete bipartite graph. Thus x, b, z induce a triangle contradicting the fact that $G[B_2 \cup \{x\}]$ is a complete bipartite graph.

Hence, we conclude that x, a, b induce a \bar{P}_3 . If $ab \in E(G)$, we may again assume $az \notin E(G)$ and $bx \in E(G)$. This yields $xz \in E(G)$, since otherwise x, b, z induce a \bar{P}_3 . Thus x, a, z induce a \bar{P}_3 , a contradiction. Therefore, $ab \notin E(G)$, and up to symmetry, we may assume $ax \in E(G)$, and $bx \notin E(G)$. Yet again, we conclude $xz \in E(G)$, since otherwise $bz \notin E(G)$ which implies $az \notin E(G)$ and x, a, z induce a \bar{P}_3 . Consequently, we have $bz \in E(G)$, since otherwise x, b, z induce a \bar{P}_3 . This implies $az \in E(G)$, since otherwise a, b, z induce a \bar{P}_3 . But now x, a, z induce a triangle, a contradiction. \square

Theorem 11. For every graph G , the edge-biclique hypergraph of G is hereditary Helly if and only if G contains no triangular prism and no B -template as an induced subgraph.

Proof. For the forward direction, it suffices to verify that the edge-biclique hypergraph of neither the triangular prism nor any B -template is Helly. This is left for the reader as an exercise.

For the converse, let G be a graph such that $\mathcal{EB}(G)$ is not Helly, and let $\mathcal{B} = \{B_1, B_2, \dots, B_k\}$ be a minimal family of pairwise edge intersecting bicliques of G without a common edge. Define $\mathcal{B}_i = \mathcal{B} \setminus \{B_i\}$ for $i \in \{1, 2, 3\}$. Since the family \mathcal{B} is minimal, the bicliques in \mathcal{B}_i have a common edge e_i for each $i \in \{1, 2, 3\}$. In addition, e_i is not an edge of $G[B_i]$, since the bicliques in \mathcal{B} have no common edge. In particular, for each $i \in \{1, 2, 3\}$, we have that e_i is an edge of $G[B_j]$ iff $j \neq i$.

There are only three possible cases: the edges e_1, e_2, e_3 have a common vertex, or two of the edges, say e_2, e_3 have a common vertex not in e_1 , or the three edges share no vertices.

Case 1: the edges e_1, e_2, e_3 have a common vertex z . It follows that the edges induce a complete bipartite graph with vertices $\{z, 1, 2, 3\}$ where $e_1 = (1, z)$, $e_2 = (2, z)$, and $e_3 = (3, z)$ as depicted in Fig. 3(a). By definition, we have $\{z, 2, 3\} \subseteq B_1$, $\{z, 1, 3\} \subseteq B_2$, $\{z, 1, 2\} \subseteq B_3$, and $1 \notin B_1, 2 \notin B_2, 3 \notin B_3$. However, $\{z, 1, 2, 3\}$ induces a complete bipartite graph, and therefore there must exist vertices $x_1 \in B_1, x_2 \in B_2$, and $x_3 \in B_3$ such that none of $\{x_1, z, 1, 2, 3\}$, $\{x_2, z, 1, 2, 3\}$ and $\{x_3, z, 1, 2, 3\}$ induces a complete bipartite graph. In fact, the three vertices x_1, x_2, x_3 must be different. Suppose otherwise, and say $x_1 = x_2$. Then $\{x_2, z, 1, 3\} = \{x_1, z, 1, 3\}$, and hence, $\{x_1, z, 1, 3\}$, $\{x_1, z, 2, 3\}$, and $\{z, 1, 2, 3\}$ induce complete bipartite graphs whereas their union $\{x_1, z, 1, 2, 3\}$ does not. This contradicts Lemma 10 when applied to $\{z, 1, 3\}$, $\{z, 2, 3\}$ and x_1 . Hence, the vertices x_1, x_2, x_3 are all distinct yielding a B -template $\{x_1, x_2, x_3, z, 1, 2, 3\}$ induced in G .

Case 2: the edges e_2, e_3 share a common vertex z not in e_1 . It follows that the edges induce a complete bipartite graph with vertices $\{y, z, 1, 2, 3\}$ where $e_1 = (1, y)$, $e_2 = (2, z)$, and $e_3 = (3, z)$ as depicted in Fig. 3(b). In particular, we have $\{z, 2, 3\} \subseteq B_1$, $\{y, z, 1, 3\} \subseteq B_2$, and $\{y, z, 1, 2\} \subseteq B_3$. Also, $2 \notin B_2$ and $3 \notin B_3$. For B_1 , we have two possibilities: $y \notin B_1$ or $1 \notin B_1$. If $y \notin B_1$, we can replace the edge e_1 with $e'_1 = (y, z)$ to obtain edges e'_1, e_2, e_3 satisfying the conditions of Case 1. Hence, we may assume $y \in B_1$ and $1 \notin B_1$. Now, since $\{y, z, 1, 2, 3\}$ induces a complete bipartite graph, we again have vertices $x_1 \in B_1, x_2 \in B_2$, and $x_3 \in B_3$ such that none of $\{x_1, y, z, 1, 2, 3\}$, $\{x_2, y, z, 1, 2, 3\}$, $\{x_3, y, z, 1, 2, 3\}$ induces a complete bipartite graph. We also conclude that the vertices x_1, x_2, x_3 are distinct. Indeed, if say $x_1 = x_2$, we contradict Lemma 10 for $\{y, z, 1, 3\}$, $\{y, z, 2, 3\}$ and x_1 . Hence, we obtain a B -template $\{x_1, x_2, x_3, y, z, 1, 2, 3\}$ induced in G .

Case 3: the edges e_1, e_2, e_3 share no vertices. It is not difficult to verify that, unless G contains the triangular prism as an induced subgraph, the edges e_1, e_2, e_3 induce a complete bipartite graph with vertices $\{1, 1', 2, 2', 3, 3'\}$ where $e_1 = (1, 1')$, $e_2 = (2, 2')$, and $e_3 = (3, 3')$ as depicted in Fig. 3(c). Namely, $\{2, 2', 3, 3'\} \subseteq B_1$, $\{1, 1', 3, 3'\} \subseteq B_2$, and $\{1, 1', 2, 2'\} \subseteq B_3$.

We show that we may also assume $1, 1' \notin B_1, 2, 2' \notin B_2$, and $3, 3' \notin B_3$. Suppose otherwise, say $1 \in B_1$. Then $1' \notin B_1$, since e_1 is not an edge of $G[B_1]$. If $2 \in B_2$, then we can replace e_1 with $e'_1 = (1', 2)$ to obtain edges e'_1, e_2, e_3 satisfying Case 2. Hence, $2 \notin B_2$. Moreover, $3' \notin B_3$, since otherwise we can replace e_2 with $e'_2 = (2, 3')$ to obtain edges e_1, e'_2, e_3 satisfying Case 2. However, now we can replace e_3 with $e'_3 = (1, 3')$ to obtain edges e_1, e_2, e'_3 satisfying Case 2. Therefore, we may conclude $1 \notin B_1$, and by symmetry, we have $1, 1' \notin B_1, 2, 2' \notin B_2$, and $3, 3' \notin B_3$.

Now, since $\{1, 1', 2, 2', 3, 3'\}$ induces a complete bipartite graph, there are again $x_1 \in B_1, x_2 \in B_2, x_3 \in B_3$ such that none of $X_1 = \{x_1, 1, 1', 2, 2', 3, 3'\}$, $X_2 = \{x_1, 1, 1', 2, 2', 3, 3'\}$, $X_3 = \{x_3, 1, 1', 2, 2', 3, 3'\}$ induces a complete bipartite graph. In fact, if $X_1 \setminus \{1\}$ induces a complete bipartite graph, we may replace B_1 with a biclique B'_1 containing $X_1 \setminus \{1\}$ to obtain bicliques B'_1, B_2, B_3 satisfying Case 3 for edges e_1, e_2, e_3 . However, $1' \in B'_1$ implies that the argument from the above paragraph reduces this situation again to Case 2. Therefore, we may assume that $X_1 \setminus \{1\}$ does induce not a complete bipartite graph, and by symmetry, none of $X_1 \setminus \{1\}, X_1 \setminus \{1'\}, X_2 \setminus \{2\}, X_2 \setminus \{2'\}, X_3 \setminus \{3\}, X_3 \setminus \{3'\}$ induces a complete bipartite graph.

It remains to observe that the three vertices x_1, x_2, x_3 are all distinct. Indeed, if say $x_1 = x_2$, we again contradict Lemma 10 for $\{1, 1', 3, 3'\}$, $\{2, 2', 3, 3'\}$ and x_1 . Hence, $\{x_1, x_2, x_3, 1, 1', 2, 2', 3, 3'\}$ yields a B -template induced in G , and that concludes the proof. \square

Since all forbidden induced subgraphs in the above theorem have at most 9 vertices, we immediately obtain the following consequence.

Corollary 12. There is a polynomial time algorithm for the recognition of graphs whose edge-biclique hypergraph is hereditary Helly.

6. Biclique line graphs

Finally, we discuss some additional interesting properties of biclique line graphs. A word on notation used in this section. By K_ℓ and \overline{K}_ℓ we denote the complete graph on ℓ vertices and its complement, respectively, and C_ℓ denotes the cycle on ℓ vertices. Other special graphs we use are shown in Fig. 6.

First, we have the following property directly from the definition of L_G .

Lemma 13. If G has no triangle and no induced C_4 , then $L_G = L(G)$. \square

Lemma 14. If L_G has no induced \overline{K}_3 and no K_4 , then $L_G = L(G)$.

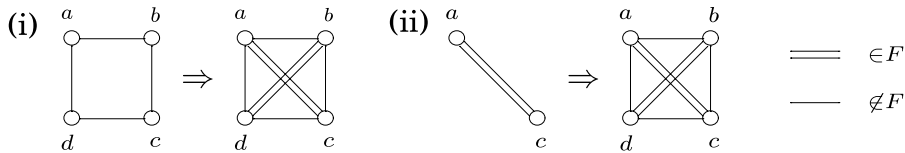


Fig. 5. Conditions (i) and (ii) of Theorem 15.

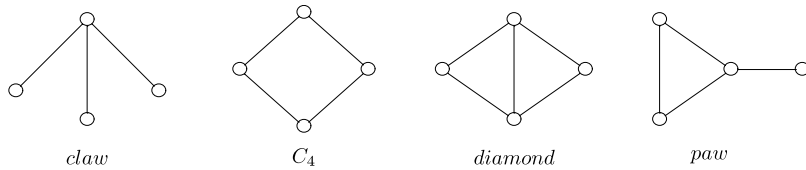


Fig. 6. The graphs claw, C_4 , diamond, and paw.

Proof. Clearly, if L_G contains no K_4 , then G contains no induced $\overline{C_4}$, since the edges of any induced C_4 in G are always pairwise adjacent in L_G . Also, if G contains a triangle, then L_G contains a $\overline{K_3}$, that is, a triple of pairwise non-adjacent vertices, which correspond to the three edges of the triangle. Consequently, if L_G contains no K_4 and no induced $\overline{K_3}$, then G has no induced C_4 and no triangle. Hence, $L_G = L(G)$ by Lemma 13. \square

If we only disallow triangles in G , then $L(G)$ becomes a subgraph of L_G , and moreover, we obtain the following characterization.

Theorem 15. Let H be a graph. Then $H = L_G$ where G is a triangle-free graph if and only if there exists a set $F \subseteq E(H)$ such that $H - F = L(G)$ and

- (i) if $H - F$ contains an induced four-cycle with vertices a, b, c, d and edges ab, bc, cd, ad , then $ac, bd \in F$,
- (ii) if $ac \in F$, then there exist vertices b, d with $bd \in F$ such that a, b, c, d induces a four-cycle in $H - F$ (see Fig. 5).

Proof. Suppose that $H = L_G$ where G is a triangle-free graph. Since G is triangle-free, we have $E(H) \supseteq E(L(G))$. Thus, we choose F to be the set $F = E(H) \setminus E(L(G))$. Clearly, we have $H - F = L(G)$.

For the condition (i), let a, b, c, d be an induced four-cycle of $H - F$ with edges ab, bc, cd, ad . Since $H - F$ is the line graph of G , it is easy to observe that G contains a four-cycle whose edges are a, b, c, d . Moreover, since G is triangle-free, this cycle is induced. Thus a, b, c, d induce a complete subgraph in H , and therefore, $ac, bd \in F$. For the condition (ii), if $ac \in F$, then G contains an induced four-cycle such that a, c are two opposite edges of this cycle. Thus, if b, d are the other two edge of this cycle, we have that a, b, c, d induce a complete subgraph in H , and hence, $bd \in F$.

For the other direction, let F be a set of edges of H satisfying the conditions (i), (ii), and such that $H - F = L(G)$ for some triangle-free graph G .

We show that $H = L_G$. Suppose that there is an edge $ac \in E(H)$ such that $ac \notin E(L_G)$. Since G is triangle-free, we conclude $ac \notin E(L(G))$. Hence, $ac \in F$, and by (ii), there exist b, d such that a, b, c, d induce a four-cycle in $H - F$ and $bd \in F$. Since $H - F$ is a line graph, we again observe that G contains an induced four-cycle whose edges are a, b, c, d . Thus $ac \in E(L_G)$, a contradiction. Conversely, suppose that there is an edge $ac \in E(L_G)$ with $ac \notin E(H)$. Since $H - F = L(G)$, we have $ac \notin E(L(G))$. Hence, G contains an induced four-cycle whose two opposite edges are a, c . If b, d are the other two edges of this cycle, we have that a, b, c, d induce a four-cycle in $L(G)$, and therefore, also in $H - F$. Thus, by (i), we have $ac, bd \in F$, and hence, $ac \in E(H)$, a contradiction. \square

Note that the above characterization does not directly imply a polynomial time algorithm for recognizing biclique line graphs of triangle-free graphs, nor it rules out such possibility. It also does not provide any idea about the complexity of recognizing biclique line graphs of arbitrary graphs. We remark that the corresponding problem for line graphs can be solved in polynomial time as follows from the characterization of [14] and from a more general result of [2]. In these results, polynomial time algorithms are a consequence of a finite forbidden induced subgraph characterization of line graphs. This is possible, in particular, because line graphs are closed under vertex removal. In other words, every induced subgraph of a line graph is again a line graph. Unfortunately, this is not so for biclique line graphs. In fact, biclique line graphs are not closed under any of the standard graph operations (edge, vertex removal, contraction), and hence, it is harder to properly characterize their structure. Furthermore, any arbitrary graph can be made to be an induced subgraph of a biclique line graph as shown in the following claim.

Proposition 16. For every graph G , there exists a graph G' such that G is an induced subgraph of $L_{G'}$.

Proof. We present two constructions. For the first construction, we let G' denote the graph we obtain by adding to the complement \overline{G} of G a new vertex v which we make adjacent to all vertices of \overline{G} . We note that $xy \in E(G)$ if and only if $xy \notin E(\overline{G})$ if and only if the vertices corresponding to the edges xv, yv are adjacent in $L_{G'}$. In other words, the vertices of $L_{G'}$ corresponding to the edges incident to v induce in $L_{G'}$ precisely the graph G .

For the second construction, we let G_1 and G_2 denote two disjoint copies of G , and for every vertex u of G , we let u_1 and u_2 denote the copies of u in G_1 and G_2 , respectively. We construct the graph G' by taking the disjoint union of G_1 and G_2 , and adding the edge u_1u_2 for each vertex u of G . The graph G' in Fig. 1(a) illustrates this construction for $G = K_3$. Now, we let e_u denote the vertex of $L_{G'}$ corresponding to the edge u_1u_2 of G' . By Proposition 1, $e_u e_v \in E(L_{G'})$ implies that u_1, u_2, v_1, v_2 is an induced four-cycle of G' . This implies $u_1v_1 \in E(G')$, and hence, $uv \in E(G)$. On the other hand, if $uv \in E(G)$, then $u_1v_1, u_2v_2 \in E(G')$, and hence, $e_u e_v \in E(L_{G'})$, because u_1, u_2, v_1, v_2 induce a four-cycle in G' . Consequently, the subgraph of $L_{G'}$ induced on $\{e_u \mid u \in V(G)\}$ is precisely the graph G . \square

In order to show that biclique line graphs are not closed under standard operations, we describe some graphs that are not biclique line graphs.

Proposition 17. C_4 , diamond, and claw are not biclique line graphs.

Proof. Clearly, C_4 contains no K_4 and no triple of pairwise non-adjacent vertices. Therefore, if $C_4 = L_G$ for some graph G , we have $L_G = L(G)$ by Lemma 14, and G contains no triangle and no induced C_4 . However, we must conclude $G = C_4$, since C_4 is the only graph whose line graph is C_4 , and hence, G contains an induced C_4 , a contradiction.

Similarly, if $H = \text{diamond}$ and G is a graph with $H = L_G$, then $L_G = L(G)$ by Lemma 14, since H contains no K_4 and no \overline{K}_3 . We must conclude that $G = \text{paw}$ (see Fig. 6), which is the only simple graph whose line graph is H . Thus G contains a triangle, a contradiction.

Finally, let $H = \text{claw}$ and G be a graph such that $H = L_G$. Since H is not a line graph, we conclude, by Lemma 14, that G contains a triangle or an induced C_4 . In fact, G contains a triangle, since H has no K_4 , and the edges of this triangle form a \overline{K}_3 in H . Since there is only one \overline{K}_3 in H , we conclude that G consists of a triangle and an edge that shares a vertex with every edge of the triangle. However, this is not possible. \square

Now, we see that biclique line graphs are not closed under edge removal, since C_4 is a subgraph of $K_4 \cong L_{C_4}$. Similarly, they are not closed under edge contraction, since $C_5 \cong L_{C_5}$ contracts to C_4 . Moreover, they are not closed under vertex removal, since, by Proposition 16, there exists a graph G such that L_G contains C_4 as an induced subgraph.

Finally, we conclude with the following result. A graph H is a *hereditary biclique line graph*, if every induced subgraph of H is a biclique line graph.

Theorem 18. A graph H is a hereditary biclique line graph if and only if H contains no induced claw, diamond, or C_4 .

Proof. Clearly, claw, diamond, and C_4 are not biclique line graphs by Proposition 17. Hence, it follows that if H is a hereditary biclique line graph, then H contains no induced claw, diamond, or C_4 .

Conversely, let H be a graph with no induced claw, diamond, or C_4 , and suppose that H is not a hereditary biclique line graph. This implies that H contains an induced subgraph H' that is not a biclique line graph. Clearly, H' also contains no induced claw, diamond, or C_4 . In [14], it is shown that if H' does not contain these induced subgraphs, then it must be the line graph of some triangle-free graph G . Therefore, since H' contains no induced C_4 , we can apply Theorem 15 to H' with $F = \emptyset$ to conclude that H' is also the biclique line graph of G , a contradiction. \square

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