Comment on: "Effects of rotation on the Landau levels in an elastic medium with a spiral dislocation"

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A R T I C L E   I N F O

A B S T R A C T

In this Comment we analyse a model proposed recently with the purpose of studying the effects of rotation on the interaction of a point charge with a uniform magnetic field in an elastic medium with a spiral dislocation. In particular we focus on the approximation proposed by the authors that consists of changing the left boundary condition in order to obtain analytical results. We show that this approximation leads to quantitative and qualitative errors, the most relevant one being a wrong prediction of the level spacing.

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1. Introduction

In a recent paper Maia and Bakke [1] analysed the effects of rotation on the interaction of a point charge with a uniform magnetic field in an elastic medium with a spiral dislocation. In order to obtain analytical solutions to the Schrödinger equation in a rotating frame with a constant angular velocity Maia and Bakke [1] changed the boundary conditions of the model. Based on this approximation they concluded that both the topology of the defect and rotation modify the degeneracy of the Landau levels.

The purpose of this paper is to analyse the effect of the change of the left boundary condition on the results because Maia and Bakke [1] did not discuss this point in detail and their weak argument leaves much to be desired. In Section 2 we derive the analytical results by means of the Frobenius

method because it is clearer than the approach based on the confluent hypergeometric function followed by Maia and Bakke [1]. In Section 3 we solve the eigenvalue equation with the correct left boundary condition and compare the numerical results thus obtained with the analytical ones. In Section 4 we briefly analyse two papers where the same authors attempted to take into account the left cut-off in the differential equation. Finally, in Section 5 we summarize the main results and draw conclusions.

2. Analytical results

In this section we outline some of the results derived by Maia and Bakke [1]. Our starting point is the radial differential equation

$$
\left(1 + \frac{\beta^2}{r^2}\right) h'' + \left(\frac{1}{r} - \frac{\beta^2}{r^3}\right) h' - \frac{l^2 + m\omega l\beta^2}{r^2 + \beta^2} h - \frac{m^2 \delta^2 r^4}{4\left(r^2 + \beta^2\right)} h
$$
$$
- \frac{m^2 \beta^2 \left(\delta^2 - \omega^2\right) r^2}{4\left(r^2 + \beta^2\right)} h + \left[2m\left(E + \Omega l\right) - k^2 + m\omega l\right] h = 0, \tag{1}
$$

derived by those authors. By means of the change of variables $y = m\delta\left(r^2 + \beta^2\right)/2$ they obtained the simpler equation

$$
y^2 g'' + y g' - \frac{\gamma^2}{4} g - \frac{y^2}{4} g + \tau y g = 0,
$$
$$
\gamma = l + \frac{m\omega\beta^2}{2},
$$
$$
\tau = \frac{1}{2m\delta}\left[2m\left(E + \Omega l\right) - k^2 + m\omega l + \frac{m^2\left(\delta^2 + \omega^2\right)\beta^2}{4}\right]. \tag{2}
$$

Maia and Bakke [1] resorted to the same function h for Eqs. (1) and (2). However, we decided to use a different name because $h(z)$ and $g(z)$ are obviously different because $h(r) = (g \circ y)(r) = g(y(r))$. The authors stated that "Henceforth, let us impose that $h(y) \to 0$ when $y \to \infty$ and $y \to 0$. Note that, since $0 < \beta < 1$, then, we can assume that $\beta^2 \ll 1$. Thus, when $r \to 0$, we can consider $y \to 0$ without loss of generality [13]". In what follows we estimate the effect of such drastic change of the left boundary condition.

Maia and Bakke [1] derived the allowed values of τ by writing $g(y)$ in terms of the confluent hypergeometric function. Here, we resort to the Frobenius method and write

$$
g(y) = y^{|\gamma|/2} e^{-y/2} \sum_{j=0}^{\infty} c_j y^j, \tag{3}
$$

that leads to the following recurrence relation for the coefficients:

$$
c_{j+1} = A_j c_j, \quad A_j = \frac{|\gamma| + 2j - 2\tau + 1}{2(j+1)(|\gamma| + j + 1)}. \tag{4}
$$

In order to obtain solutions with the correct behaviour when $y \to \infty$ we have to choose τ so that the infinite series in Eq. (3) terminates [2]. The requirement $c_n \neq 0$ and $c_{n+1} = 0$, $n = 0, 1, \ldots,$ leads to $c_j = c_{jn} = 0$ for all $j > n$ and the exact solutions

$$
g_n(y) = y^{|\gamma|/2} e^{-y/2} \sum_{j=0}^{n} c_{jn} y^j, \tag{5}
$$

for

$$
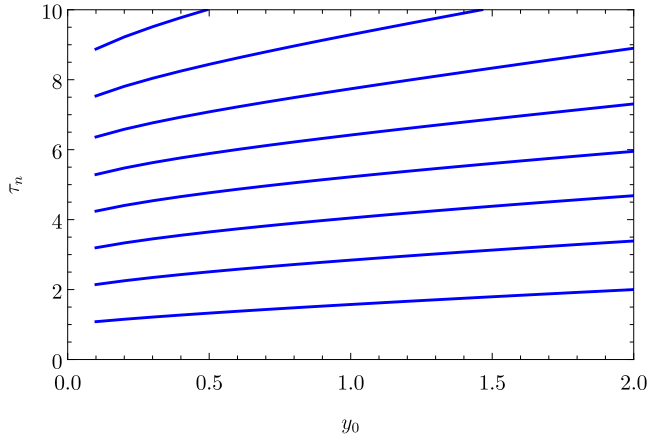\tau = \tau_n = \frac{1}{2}\left(2n + 1 + |\gamma|\right), \tag{6}
$$

Fig. 1. Numerical eigenvalues τ_n vs. y_0 for $\gamma = 1$.

that agrees with the one derived by Maia and Bakke [1]. The factor A_j takes the simpler form

$$A_j = \frac{j - n}{(j + 1)\,(|\gamma| + j + 1)}. \tag{7}$$

3. Exact boundary conditions

When $r = 0$ then $y = y_0 = m\delta\beta^2/2$ and the correct boundary condition for the differential equation (2) is $g(y_0) = 0$. The choice $y_0 = 0$ is rather unphysical because it leads to $r^2 = -\beta^2$; however, Maia and Bakke [1] stated that they could resort to this approximation *without loss of generality* because $\beta^2 \ll 1$ as mentioned above. In what follows we analyse to which extent this approximation is reasonable. To begin with, note that the behaviour at origin of the solution to the differential equation (1) is $h \sim r^2$ while, on the other hand, the behaviour at origin of the solution to (2) chosen by Maia and Bakke [1] is $g \sim y^{|\gamma|/2}$. Under the approximation $\beta \sim 0$ it leads to $g \sim r^{|\gamma|}$ that is consistent with the correct asymptotic behaviour at origin only when $|\gamma| = 2$. Besides, $g'(y)$ diverges at $y = 0$ when $0 < |\gamma| < 2$.

In order to solve the differential equation (2) with the proper boundary condition we resort to the shooting method. The choice of suitable values for the model parameters is rather difficult because the equations reported by Maia and Bakke [1] do not exhibit unit consistency. They did not indicate the chosen units explicitly but we assume that they followed early papers in which they stated that "we shall use the units $\hbar = 1$, $c = 1$" [3]. Unfortunately, this expression does not mean anything because we do not know what are actually the units of length, energy, etc., (see [4] for a clear pedagogical discussion of the subject and an earlier criticism of this undesirable practice [5]). The fact is that all their equations lack of unit consistency. Consider, for example, the expression for γ in Eq. (2) where we appreciate that l is an integer and $m\omega\beta^2/2$ exhibits units of energy. It is not clear how the authors converted the latter term into a dimensionless one because they did not explain it. As a result the expression for the energy $E_{n,l,k}$ shown by Maia and Bakke [1] is extremely inconsistent. For example, the term ωl should be multiplied by $\hbar$ in order to have units of energy; on the other hand $k^2/(2\,m)$ should be multiplied by $\hbar^2$ in order to have the same units. What should we do with the term $m\omega\Omega\beta^2/2$ that exhibits units of energy?. Since it is not possible to estimate physically reasonable values of y_0 and γ we choose them arbitrarily.

Fig. 1 shows that the eigenvalues τ_n increase monotonously with y_0 and that this behaviour increases with the radial quantum number. This effect has not been taken into account by Maia and Bakke [1] because they only considered $y_0 = 0$.
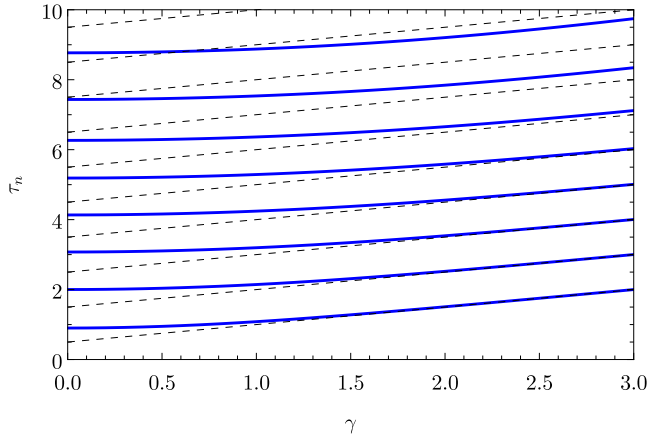
Fig. 2. Eigenvalues τ_n vs. γ for $y_0 = 0.1$. The continuous and dashed lines indicate numerical and analytical ($y_0 = 0$) results, respectively.
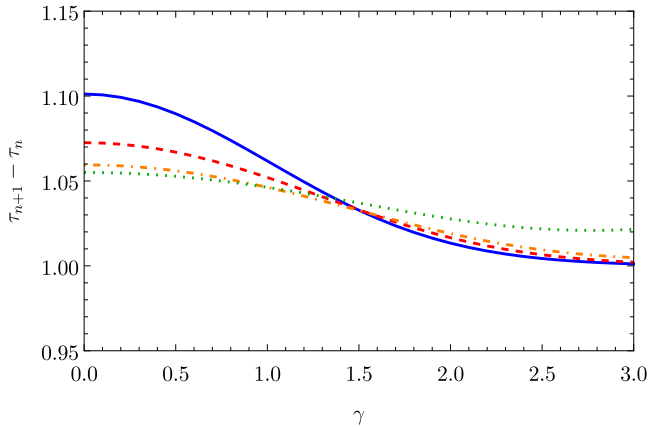

Fig. 3. $\tau_{n+1} - \tau_n$ vs. γ for $n = 0$ (blue, continuous line), $n = 1$ (red, dashed line), $n = 2$ (orange, dash-point line), $n = 3$ (green, point line) and $y_0 = 0.1$.

Fig. 2 shows that the eigenvalues τ_n increase monotonously with γ, exactly as the analytical expressions (6). The discrepancy produced by the boundary condition at y_0 seems to be less relevant as γ increases. We will discuss this point in more detail below.

The difference between two analytical eigenvalues (6) is independent of γ: $\tau_n - \tau_j = n - j$. Fig. 3 shows that if we choose the correct left boundary condition $\tau_{n+1} - \tau_n$, $n = 0, 1, 2, 3$, change with γ. The unphysical approximation proposed by Maia and Bakke [1] does not take into account this fact. Note that the purpose of their paper was to determine the effect of the dislocation given by the model parameter β on the Landau levels and they failed to do it properly. The variation of $\Delta\tau$ with γ may appear to be weak at first sight but it should be taken into account that we have chosen a relatively small value of $y_0 = 0.1$. We are unable to estimate reasonable physical values of $y_0 = m\delta\beta^2/2$ because Maia and Bakke [1] did not show suitable values for the model parameters and also because their equations are inconsistent with respect to units.

Fig. 4 shows $g_0(y)$ for $\gamma = 0.5$, $\gamma = 1$ and $\gamma = 2$ with the boundary conditions $g(0) = 0$ and $g(0.1) = 0$. We appreciate that the agreement is better when $\gamma = 2$ as argued above (only in this case the maxima are reasonably close). As γ increases both kind of eigenfunctions become
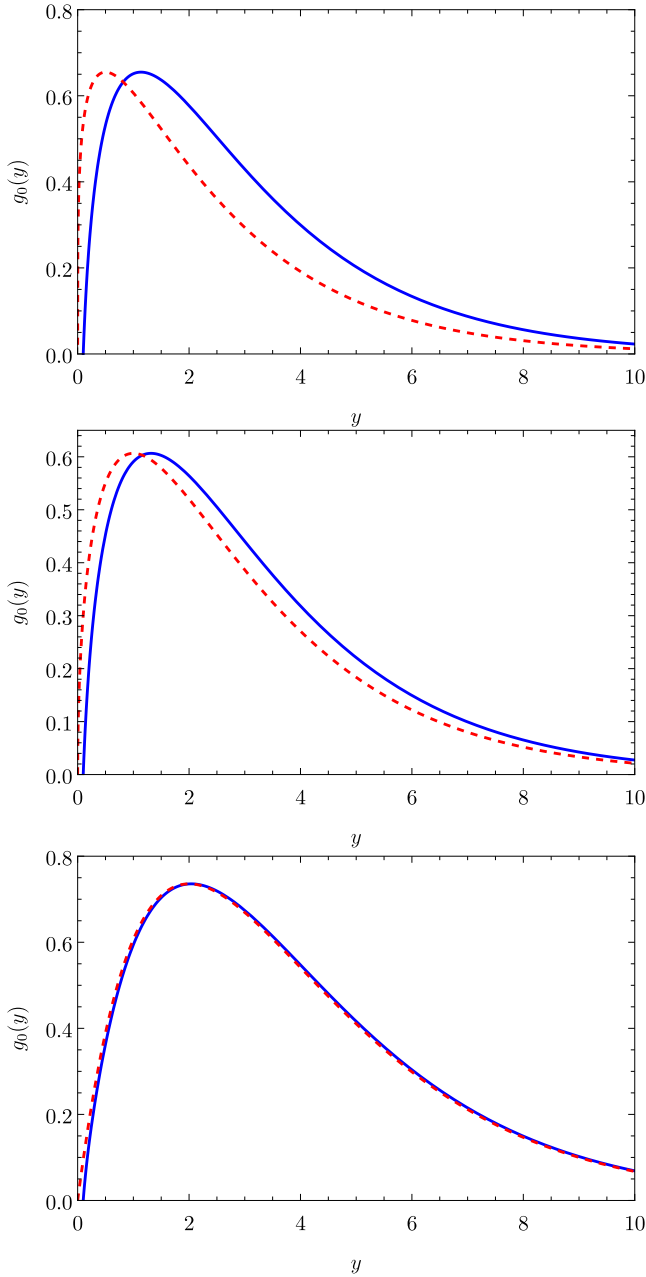
Fig. 4. Analytical (red dashed line, $y_0 = 0$) and numerical (blue continuous line, $y_0 = 0.1$) values of $g_0(y)$ for $\gamma = 0.5$, $\gamma = 1$ and $\gamma = 2$ (top to bottom).

increasingly small in a neighbourhood of the origin; consequently, if y_0 is small enough then the effect of the Dirichlet boundary condition at y_0 is expected to be less noticeable. Obviously, the discrepancy between both kinds of solutions increases as y_0 increases.

4. Failed attempts to introduce the left boundary condition

In two recent papers Maia and Bakke [6,7] attempted to introduce the left cut-off into two almost identical models: the harmonic oscillator and the electric field produced by a uniform distribution of electric charges inside a long nonconductor cylinder. The interaction potentials are $V(r) = \frac{1}{2} m \omega^2 r^2$ and $V(r) = \frac{|q|\lambda}{4} r^2$ in the former and latter case, respectively, where $q = -|q|$ is the electric charge and $\lambda > 0$ is a constant associated with the uniform distribution of electric charges. It is obvious that we can treat both problems simultaneously by simply setting $\omega^2 = \frac{|q|\lambda}{2m}$ in the latter.

From the properties of the confluent hypergeometric function Maia and Bakke [7] derived the approximate analytical spectrum

$$E_n = -\omega \left(2n + \frac{1}{2} + \frac{m\omega\beta^2}{2} \right) + \frac{4m\omega^2\beta^2}{\pi^2} \left[1 \pm \sqrt{1 - \frac{\pi^2(4n+1)}{4m\omega\beta^2}} \right] + \frac{k^2}{2m}, \tag{8}$$

where $n = 0, 1, \ldots$ is the radial quantum number. The units in this expression are clearly inconsistent; for example: $2n + \frac{1}{2}$ is a rational number and $\frac{m\omega\beta^2}{2}$ has units of energy. Since their energies do not depend on the angular momentum quantum number Maia and Bakke concluded that they are infinitely degenerate. This fact deserves further analysis because the results shown in this Comment suggest otherwise. A most intriguing fact is that Maia and Bakke [6,7] did not indulge in explaining the meaning of the $\pm$ sign in Eq. (8); does it mean the existence of two different spectra? However, the most striking fact about this expression is that the eigenvalues are real provided that $m\omega\beta^2 \geq \pi^2(4n+1)/4$ which is inconsistent with the condition $0 < \beta < 1$ considered in earlier papers by the same authors.

A thorough analysis of these results is beyond the scope of this Comment; however these models give us the opportunity to discuss suitable natural units. Consider, for example, the harmonic oscillator. In order to obtain an equation that is dimensionally consistent we choose the natural unit of length $L = (\hbar/m\omega)^{1/2}$ (see [4] for details) and define the dimensionless variables $\tilde{r} = r/L$, $\tilde{z} = z/L$ and model parameter $\tilde{\beta} = \beta/L$ so that $d\tilde{s}^2 = d\tilde{r}^2 + 2\tilde{\beta} d\tilde{r} d\phi + (\tilde{\beta}^2 + \tilde{r}^2) d\phi^2 + d\tilde{z}^2$, where $d\tilde{s} = ds/L$ is dimensionless. Thus, the dimensionless Hamiltonian reads

$$\tilde{H} = \frac{H}{\hbar\omega} = -\frac{1}{2}\tilde{\nabla}^2 + \frac{1}{2}\tilde{r}^2, \tag{9}$$

where $\tilde{\nabla}^2 = L^2 \nabla^2$. Note that all the results of Maia and Bakke [8] can be written in a consistent way by just setting $m = 1$, and $\omega = 1$ (in addition to $\hbar = 1$) and then multiplying the energies by $\hbar\omega$ [5]. More precisely, if $\tilde{E}_{n,l}\left(\tilde{\beta}, \tilde{k}\right)$, where $\tilde{k} = kL$, is an eigenvalue of $\tilde{H}$, then $E_{n,l} = \hbar\omega\tilde{E}_{n,l}\left(\tilde{\beta}, \tilde{k}\right)$ is the corresponding eigenvalue of H. An advantage of this dimensional analysis is that $\tilde{\beta}$ in those equations is dimensionless and, consequently, it makes sense to state that $0 < \tilde{\beta} < 1$ (instead of $0 < \beta < 1$ as done by Maia and Bakke [1,3,8]).

5. Conclusions

The purpose of this Comment is to analyse the effect of changing the left boundary condition in the model proposed by Maia and Bakke [1]. Our results clearly show that the errors are not only quantitative but also qualitative. If y_0 is small enough the discrepancy between the solutions with Dirichlet boundary conditions at $y = 0$ and $y = y_0 > 0$ decreases as γ increases. The reason is that the larger the value of γ the smaller the magnitude of the eigenfunctions in a neighbourhood of the origin. In all our calculations we have chosen y_0 small because it is the condition under which the approach of Maia and Bakke [1] should work better. However, the main drawback of their approximation is a wrong prediction of the level spacing. Under the unphysical boundary condition at $y = 0$ the level spacing does not depend on γ while the use of the correct boundary condition at $y_0 > 0$ predicts that the level spacing depends on that model parameter. Another relevant point is that the equations derived by Maia and Bakke [1] lack unit consistency. For this reason they are completely useless for any physical application. These authors carried out a similar mistake in an earlier paper [8] but in that case it was easier to derive the equations properly [5].

CRediT authorship contribution statement

Paolo Amore: Software, Formal analysis, Resources, Writing – review & editing, Visualization. **Francisco M. Fernández:** Conceptualization, Software, Formal analysis, Writing – original draft, Writing – review & editing, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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