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Comment on "He-Laplace variational iteration method for solving the nonlinear equations arising in chemical kinetics and population dynamics"

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Abstract In this Comment we argue that the so-called He-Laplace variational iteration method merely yields the Taylor series of the solution to a partial differential equation. The straightforward application of the textbook power series method is far simpler and more efficient because it gives us closed-form analytic recurrence relations for the expansion coefficients. We also argue that the time series of the solution is unsuitable for the analysis of nonlinear problems in chemical kinetics and population dynamics which require expressions valid for sufficiently large time. Besides, Nadeem and He [J. Math. Chem. (2021) 59:1234-1245] applied the approach to some tailor-made toy problems with known exact solutions that do not appear to exhibit any physical application whatsoever.

In a recent paper published in this journal Nadeem and He [1] (NH from now on) proposed an alternative approach for the study of some partial differential equations of supposed interest in chemical kinetics and population dynamics. The so-called He-Laplace variational iteration method couples the variational iteration method with the Laplace transform. By means of the homotopy perturbation method they obtained results with the aid of the so-called He's polynomials. NH showed results for some chosen Fisher-type equations. The purpose of this Comment is the analysis of the results just mentioned.

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As a first example NH chose

$$\frac{\partial\omega}{\partial t}=\frac{\partial^2\omega}{\partial x^2}+\omega(1-\omega),\ \omega(x,0)=\mu. \tag{1}$$

Since the initial condition is independent of x we try $\omega(x,t)=\omega(t)$ and the problem reduces to a textbook differential equation

$$\frac{\partial\omega}{\partial t}=\omega(1-\omega),\ \omega(0)=\mu. \tag{2}$$

Any student in a first course on Calculus will surely be able to obtain the exact result to this extremely simple differential equation

$$\omega=\frac{\mu e^t}{1+\mu(e^t-1)}. \tag{3}$$

Obviously, this problem is of no physical or chemical interest whatsoever and there is no need of an approximate method because its exact solution is already known. However, NH applied the remarkable He-Laplace method and merely obtained the Taylor series of $\omega(t)$ about $t=0$. NH appeared to be aware of this fact but they never took advantage of it. Instead of resorting to the elaborate homotopy perturbation method and the *celebrated* He's polynomials, in what follows we simply apply the straightforward power-series method and look for a solution of the form

$$\omega(x,t)=\sum_{j=0}^{\infty}\omega_j(x)t^j. \tag{4}$$

If we insert this expansion into the differential equation (1) we easily obtain a recurrence relation for the coefficients ω_j:

$$\omega_{n+1}=\frac{1}{n+1}\left(\frac{\partial^2\omega_n}{\partial x^2}+\omega_n-\sum_{j=0}^{n}\omega_j\omega_{n-j}\right),\ n=0,1,\ldots,\ \omega_0=\mu. \tag{5}$$

This simple recurrence relation, which can be easily programmed in any computer algebra software, yields all the coefficients of the Taylor series (4) for the exact solution (3). By means of the remarkable He-Laplace method NH obtained the first terms up to order $\mathcal{O}\left(t^4\right)$. There is no doubt that the straightforward textbook Taylor-series approach is by far simpler and more efficient.

In order to illustrate the application of equation (5) to a somewhat more realistic toy model, in what follows we assume that $\omega(x,0)=f(x)$ is a differentiable function. One can easily verify that the first terms are

$$\begin{aligned}\omega_0(x) &= f(x),\ \omega_1(x) = f''(x) - f(x)^2 + f(x),\\ \omega_2(x) &= \frac{1}{2}\left\{f^{iv}(x) + 2f''(x)\left[1-2f(x)\right] - 2f'(x)^2\right.,\\ &\left.+f(x)\left[2f(x)^2 - 3f(x) + 1\right]\right\}. \end{aligned} \tag{6}$$

There is no point in a large-order calculation because it is not clear if the model exhibits any physical application and because the expansions of the form (4) are utterly useless.

We want to stress the following fact: by means of the remarkable He-Laplace method NH simply obtained the Taylor series about $t = 0$ of an extremely simple toy problem with no physical interest whatsoever. Besides, this Taylor series is commonly useless in the case of chemical kinetics and population dynamics where one requires the solution for large values of t. For example, the radius of convergence of the Taylor series for the solution (3) of equation (2) is $|t_c|$, $t_c = \ln\left(\frac{\mu-1}{\mu}\right)$. Note that if $\mu \gg 1$ then $t_c = -1/\mu - 1/(2\mu^2) + \ldots$ and the series is valid only for $t < |t_c| \approx 1/\mu \ll 1$.

Before proceeding with the next example let us first address the somewhat trivial problem of expanding ω^α, α real, in t-power series

$$\omega^\alpha = \sum_{j=0}^{\infty} \omega_{\alpha,j} t^j. \tag{7}$$

The coefficients $\omega_{\alpha,j}$ can be easily expressed in terms of the coefficients ω_j by means of the following expression

$$\begin{aligned}\omega_{\alpha,n+1} &= \frac{1}{(n+1)\omega_0}\left[\alpha\sum_{j=0}^{n}(n+1)\omega_{j+1}\omega_{\alpha,n-j} - \sum_{j=0}^{n-1}(j+1)\omega_{\alpha,j+1}\omega_{n-j}\right],\\ n &= 0,1,\ldots,\ \omega_{\alpha,0} = \omega_0^\alpha. \end{aligned} \tag{8}$$

Note that this strategy is far more efficient than the *celebrated* He's polynomial formula [1]

The second example is

$$\frac{\partial\omega}{\partial t} = \frac{\partial^2\omega}{\partial x^2} + \omega(1-\omega^6),\ \omega(x,0) = \left(1 + e^{3x/2}\right)^{-1/3}. \tag{9}$$

In this case the recurrence relation for the coefficients of the Taylor series (4) is given by

$$\omega_{n+1} = \frac{1}{n+1}\left(\frac{\partial^2\omega_n}{\partial x^2} + \omega_n - \omega_{7,n}\right),\ n = 0, 1, \ldots,\ \omega_0 = \left(1 + e^{3x/2}\right)^{-1/3}, \quad (10)$$

where $\omega_{7,n}$ can be calculated by equation (8). This recurrence relation yields all the results produced by the remarkable He-Laplace method in a much simpler and straightforward way. By simple comparison we realize that the NH's term for p^1 is wrong (in fact it corresponds to their example 3).

As in the previous example, the only outcome of the application of the remarkable He-Laplace method are the first few coefficients of the Taylor series about $t = 0$ for an exactly-solvable toy model with no physical utility (at least, the authors did not mention any). As said above, this series is completely useless for most physical purposes that commonly require large values of t. In order to illustrate this fact we consider the exact solution to the differential equation (9)

$$\omega(x,t) = \left\{\frac{1}{2}\tanh\left(-\frac{3}{4}\left[x - \frac{5}{2}t\right]\right) + \frac{1}{2}\right\}^{1/3}, \quad (11)$$

that we will compare with the Taylor-series expansion of fifth order. Figure 1 shows that there is a discrepancy between exact and approximate results when $t = 1$. We have chosen the value of t slightly larger than the values considered by NH ($0 \leq t \leq 0.5$) but not too large. The reader may easily verify that for $t = 2$ the time series yields utterly ridiculous results. NH chose a range of values of t that does reveal the discrepancy and, consequently, their approach appears to be sound. It is not difficult to explain the numerical results shown in Figure 1. In this case the time series converges for $t < |t_c(x)| = (2/5)\sqrt{x^2 + 4\pi^2/9}$ which clearly reveals the failure of the approach in the neighbourhood of $x = 0$ because $|t_c(x)| \geq |t_c(1)| = 4\pi/15 < 1$.

The third example is

$$\frac{\partial\omega}{\partial t} = \frac{\partial^2\omega}{\partial x^2} + \omega(1-\omega)(\omega - a),\ \omega(x,0) = \left(1 + e^{-x/\sqrt{2}}\right)^{-1}, \quad (12)$$

and the recurrence relation for the coefficients of the Taylor series is

$$\omega_{n+1} = \frac{1}{n+1}\left(\frac{\partial^2\omega_n}{\partial x^2} - a\omega_n + (a+1)\sum_{j=0}^{n}\omega_j\omega_{n-j} - \sum_{j=0}^{n}\omega_{n-j}\sum_{k=0}^{j}\omega_k\omega_{j-k}\right),$$
$$n = 0, 1, \ldots, \quad \omega_0 = \omega(x,0). \quad (13)$$

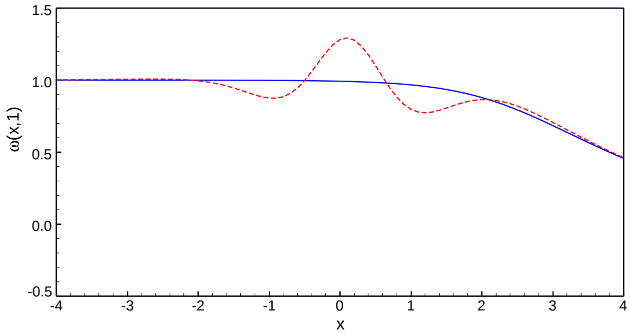

Fig. 1 Exact solution $\omega(x, 1)$ to the differential equation (9) (continuous line) and fifth-order expansion (dashed line)

From this recurrence relation we can obtain as many coefficients as desired because, as explained above, one can easily write a program for that purpose. We again stress the fact that this exactly-solvable toy problem does not appear to exhibit any physical utility and that the expansion of $\omega(x, t)$ about $t = 0$ is unsuitable for the description of the dynamical problem at large t.

Summarizing: by means of the remarkable He-Laplace variational iteration method NH were only able to obtain the Taylor series of $\omega(x, t)$ about $t = 0$. We have shown that the straightforward power-series method is simpler and considerably more efficient because it enables us to derive closed-form recurrence relations for the systematic calculation of the coefficients. The *celebrated* He's polynomials [1] do not appear to be suitable for obtaining such compact expressions. However, it is most important to note that any expansion of the form (4), disregarding the way we derive it, is unsuitable for most applications to chemical kinetics and population dynamics where it is relevant to know what happens at long times.

References

1. M. Nadeem and J.-H. He, J. Math. Chem. **59**, 1234 (2021).