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Comment on: "Spin-0 scalar particle interacts with scalar potential in the presence of magnetic field and quantum flux under the effects of KKT in 5D cosmic string spacetime"

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We show that the meaning of the eigenvalues of a radial equation derived from a power-series method was misunderstood. The roots stemming from the truncation of the power series through a three-term recurrence relation are not the energies of the quantum-mechanical model under study but isolated particular eigenvalues of different models. The supposed dependence of the intensity of the magnetic field on the quantum numbers is just an artifact of the truncation method.

Keywords: Radial eigenvalue equation; power-series method; three-term recurrence relation; truncation condition; conditionally-solvable model.

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In a recent paper, Ahmed[1] studied the relativistic quantum dynamics of a spin-0 scalar particle that interacts with a scalar potential in the presence of a uniform magnetic field and a quantum flux in the background of the Kaluza–Klein theory. He solved the Klein–Gordon equation and analyzed the relativistic analogue of the Aharonov–Bohm effect for bound states. He showed that the energy levels depend on the global parameters characterizing the spacetime, scalar potential and the magnetic field which break their degeneracy. Ahmed considered four cases where the relevant dynamical equations are separable in cylindrical coordinates. In order to solve the resulting radial equation, Ahmed resorted to the Frobenius power-series method and obtained exact analytical expressions for the eigenvalues for some restricted values of the model parameters. This restriction comes from the truncation of the power series so that the radial part of the eigenfunction reduces to a polynomial times and exponential factor. In this way, Ahmed predicted a dependence of the intensity of the magnetic field on the quantum numbers. The purpose of this Comment is the analysis of the effect of the truncation of the power series on the physical conclusions derived by the author.

In what follows, we briefly discuss the examples studied by Ahmed. In the first one (Case A), he derived the eigenvalue equation

$$\left[\frac{d^2}{d\rho^2}+\tfrac{1}{\rho}\tfrac{d}{d\rho}+\zeta-\frac{j^2}{\rho^2}-\rho^2-\frac{\eta}{\rho}-\theta\rho\right]\psi(\rho),$$

$$\zeta=\frac{\lambda}{\Omega},\quad \eta=\frac{a}{\sqrt{\Omega}},\quad \theta=\frac{b}{\Omega^{3/2}},$$

$$\lambda=E^2-k^2-q^2-m^2-2\eta_c\eta_L-2m\omega\frac{l-\frac{q\Phi}{2\pi}}{\alpha},\tag{1}$$

$$\Omega=\sqrt{m^2\omega^2+\eta_L^2},\quad j=\sqrt{\frac{\left(l-\frac{q\Phi}{2\pi}\right)^2}{\alpha^2}+\eta_L^2},\quad \omega=\frac{qB_0}{2m},$$

$$a=2m\eta_c,\quad b=2m\eta_L,$$

where m is the mass of the particle, E is the energy, B_0 is the intensity of the magnetic field, Φ is the Aharonov–Bohm flux, α is a wedge parameter, η_c and η_L are the parameters in the Cornell-type potential $S(r)=\eta_c/r+\eta_L r$, $l=0\pm1,\pm2$ the rotational quantum number and q and k constants introduced during the separation of variables.

Ahmed solved the eigenvalue equation (1) by means of a suitable ansatz and a power-series method that leads to a three-term recurrence relation for the coefficients. We will analyze this approach later on but for the time being we just outline his results and conclusions. From a truncation condition based on two equations, Ahmed derived an analytical expression for the energies that he denoted $E_{n,l}$, where n appears to be a radial quantum number (which it is not as shown below). In particular, for $n=1$, Ahmed obtained $E_{1,l}$ and called it the ground-state energy. However, the ground-state energy is supposed to be the lowest eigenvalue and should not depend on any quantum number (unless there is degeneracy which is not the case here). In addition to it, the polynomial solution for $n=1$ appears only for some particular values $\Omega_{1,l}$ that are roots of a cubic equation. The author states that "The magnetic field $B_0^{1,l}$ is so adjusted that Eq. (22) can be satisfied and we have simplified by labeling as ..." and shows expressions for $\omega_{1,l}$ and $B_0^{1,l}$. Unfortunately, Ahmed did not indulge in the analysis of the physical meaning of these particular values of $B_0^{1,l}$. In the conclusions he only stated that "We have observed a quantum effect due to the dependence of the magnetic field on the quantum numbers of the system which we determined by a relation for the different radial modes $n=1,2,\ldots$" It is clear that $E_{1,l}$ and $E_{1,l'}$, $l\neq l'$, are eigenvalues of two different models with potential parameters determined by $\Omega_{1,l}$ and $\Omega_{1,l'}$. With the same reasoning, we conclude that the eigenvalues $E_{n,l}$ do not provide the spectrum of a given quantum-mechanical model but are eigenvalues of different problems.

A simple analysis of Eq. (1) reveals that the behavior of $\psi(\rho)$ at origin and infinity is determined by the terms j^2/ρ^2 and ρ^2, respectively. Therefore, Eq. (1) exhibits bound states (square-integrable solutions $\psi(\rho)$) for all real values of η and θ and, consequently, for all positive values of Ω. This fact makes the existence of particular values of Ω and B_0 most intriguing. The only possible explanation is that Ahmed argued that the factor $H(\rho)$ in the ansatz for $\psi(\rho)$, "is the Heun polynomials" because it is a solution to a biconfluent Heun's equation. However, the simple analysis just carried above shows that the polynomial solutions are merely some particular square-integrable solutions that appear for particular values of the model parameters. Do they have any physical meaning or are they just a mathematical curiosity? We will try to answer this question later on.

Ahmed derived the equations for Case B in detail; however, as far as we understand Case B is merely Case A with $\eta_L = 0$. For this reason, we do not deem it necessary to abound in details and simply discuss the author's results and conclusions. The author repeated the whole calculation procedure unnecessarily and derived an analytical expression for the energies $E_{n,l}$. In particular, for $n = 1$, he obtained $E_{1,l}$ for some particular values of $\omega_{1,l}$ or $B_0^{1,l}$. In this case, the author states that "Equation (37) corresponds to the allowed values of the energy level for the radial mode $n = 1$ of the system in the context of Kaluza–Klein theory." If this sentence suggests that there are no other eigenvalues beyond those given by the truncation condition, then it is wrong as argued above and as shown later on in this Comment.

The author also went through a detailed calculation for Case C, which is, in our opinion, unnecessary because it is a particular case of Case A with $\eta_c = 0$. Therefore, we will not outline all the equations and simply comment on Ahmed's results and conclusions. As before, he derived an analytical expression for the energies $E_{n,l}$ and, in particular, for $E_{1,l}$ that is valid for particular values of the model parameters $\Omega_{1,l}$, $\omega_{1,l}$ and $B_0^{1,l}$. In this case, he states that "Equation (57) corresponds to the allowed values of relativistic energy level for the radial mode $n = 1$ of the system subject to linear confining potential in the background of Kaluza–Klein theory." However, as argued above, such particular values of the field intensity do not make sense.

As a special case of Case C, Ahmed considered the limit of vanishing magnetic field $B_0 \to 0$ and went through the calculation procedure once more (unnecessarily in our opinion). In this case, he obtained $E_{n,l}$ and, in particular, $E_{1,l}$ for particular values of η_L.

In order to analyze Ahmed's results and conclusions with more detail and somewhat more rigorously, we consider the general eigenvalue equation

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{\gamma^2}{\rho^2} - \frac{a}{\rho} - b\rho - \rho^2 + W\right]R(\rho) = 0 \tag{2}$$

and only assume that γ, a and b are arbitrary real parameters. We are interested in those solutions $R(\rho)$ that are square integrable:

$$\int_0^\infty |R(\rho)|^2 \rho \, d\rho < \infty, \tag{3}$$

which only take place for particular values of $W = W_{\nu,|\gamma|}(a,b)$, $\nu = 0, 1, \ldots$. It is convenient, for present purposes, to label the eigenvalues with the value of $|\gamma|$ instead of the actual quantum number l on which γ may depend. Since the behavior of $R(\rho)$ at origin and at infinity is determined by the terms $\gamma^2\rho^{-2}$ and ρ^2, respectively, it is clear that there are square-integrable solutions for all real values of a and b. More precisely, the eigenvalues $W_{\nu,|\gamma|}(a,b)$ are continuous functions of a and b that satisfy the Hellmann–Feynman theorem[2,3]

$$\frac{\partial W}{\partial a} = \left\langle \frac{1}{\rho} \right\rangle > 0, \quad \frac{\partial W}{\partial b} = \langle \rho \rangle > 0. \tag{4}$$

In order to obtain exact solutions to Eq. (2), we apply the Frobenius method by means of the ansatz

$$R(\rho) = \rho^s \exp\left(-\frac{b}{2}\rho - \frac{\rho^2}{2}\right) P(\rho), \quad P(\rho) = \sum_{j=0}^{\infty} c_j \rho^j, \quad s = |\gamma|. \tag{5}$$

The expansion coefficients c_j satisfy the three-term recurrence relation

$$\begin{gathered} c_{j+2} = A_j(a,b) c_{j+1} + B_j(W,b) c_j, \quad j = -1, 0, 1, 2, \ldots, \; c_{-1} = 0, \; c_0 = 1, \\ A_j(a,b) = \frac{2a + b(2j + 2s + 3)}{2(j+2)[j + 2(s+1)]}, \quad B_j(W,b) = \frac{4(2j + 2s - W + 2) - b^2}{4(j+2)[j + 2(s+1)]}. \end{gathered} \tag{6}$$

If the truncation condition $c_{n+1} = c_{n+2} = 0$, $c_n \neq 0$, $n = 0, 1, \ldots$, has physically acceptable solutions for a, b and W then we obtain exact eigenfunctions because $c_j = 0$ for all $j > n$. This truncation condition is equivalent to $B_n = 0$, $c_{n+1} = 0$ or

$$W_s^{(n)} = 2(n + s + 1) - \frac{b^2}{4}, \quad c_{n+1}(a,b) = 0, \tag{7}$$

where the second equation determines a relationship between the parameters a and b. On setting $W = W_s^{(n)}$ the coefficient B_j takes a simpler form

$$B_j(W_s^{(n)}, b) = \frac{2(j-n)}{(j+2)[j + 2(s+1)]}. \tag{8}$$

It is clear that the truncation condition (7) cannot provide all the bound-state solutions to the eigenvalue equation (2) because it forces a relationship between the model parameters a and b. As stated above, there are bound states for all $-\infty < a, b < \infty$ and those coming from the truncation condition are valid in a considerably more restricted domain of these model parameters. More precisely, there are bound states in the whole $a - b$ plane and polynomial solutions only for some curves $c_{n+1}(a,b)=0$ in this plane. For this reason, this kind of models is commonly called quasi-exactly solvable or conditionally solvable (see Refs. 4 and 5 and references therein).

Since $B_j(W_s^{(n)},b)$ is independent of a and b and $A_j(-a,-b)=-A(a,b)$ we conclude that $c_j(-a,-b)=(-1)^j c_j(a,b)$. The coefficient $c_j(a,b)$ is a polynomial function of order j in each of the variables a and b; therefore, the condition $c_{n+1}(a,b)=0$ has solutions of the form $a_s^{(n,i)}(b)$ or $b_s^{(n,i)}(a)$, $i=1,2,\ldots,n+1$, and it can be proved that all the roots are real.[6,7] The exact solutions to the radial eigenvalue equation (2), given by the truncation method, are of the form

$$R_s^{(n,i)}(\rho)=\rho^s\exp\left(-\frac{b}{2}\rho-\frac{\rho^2}{2}\right)P^{(n,i)}(\rho),\quad P^{(n,i)}(\rho)=\sum_{j=0}^{n}c_{j,s}^{(n,i)}\rho^j. \tag{9}$$

These solutions already satisfy Eqs. (2) and (3) but, as stated above, they are not the only allowed solutions to the radial eigenvalue equation.

For a given value of b all the roots $W_s^{(n,i)}=W_s^{(n)}=2(n+s+1)-\frac{b^2}{4}$, $i=1,2,\ldots,n+1$, have the same value; on the other hand, for a given value of a the roots $W_s^{(n,i)}$, $i=1,2,\ldots,n+1$, are points on the inverted parabola $W_s^{(n,i)}=2(n+s+1)-\frac{[b_s^{(n,i)}]^2}{4}$.

As an illustrative example, we choose $b=0$ that is equivalent to Ahmed's Case B. In this case, we have $W_s^{(n)}=2(n+s+1)$ and arrange the roots so that $a_s^{(n,i)}>a_s^{(n,i+1)}$, $i=1,2,\ldots,n$. Since $c_j(-a)=(-1)^j c_j(a)$ then the roots of $c_{n+1}=0$ satisfy $a_s^{(n,i)}=-a_s^{(n,n+2-i)}$, $i=1,2,\ldots,\frac{n+1}{2}$ for n odd and $a_s^{(n,i)}=-a_s^{(n,n+2-i)}$, $i=1,2,\ldots,\frac{n}{2}$, $a_s^{(n,j)}=0$, $j=\frac{n}{2}+1$, for n even. In other words, the roots $a_s^{(n,i)}$ are symmetrically distributed with respect to the W-axis in the $a-W$ plane. Ahmed failed to realize the existence of such a multiplicity of roots, a fact of fundamental importance as discussed in what follows.

It follows from the Hellmann–Feynman theorem (4) and the chosen arrangement of roots that $(a_s^{(n,i)},W_s^{(n)})$ is a point on the curve $W_{i-1,s}(a)=W_{i-1,s}(a,0)$. In order to verify this fact, we need the actual eigenvalues $W_{\nu,s}$ that should be obtained by means of a suitable approximate method because the eigenvalue equation (2) is not exactly solvable.[4,5,7] Here, we resort to the well-known Rayleigh-Ritz variational method that is known to yield upper bounds to all the eigenvalues[8,9] and, for simplicity, choose the non-orthogonal basis set of Gaussian functions $\left\{\varphi_{j,s}(\rho)=\rho^{s+j}\exp\left(-\frac{\rho^2}{2}\right),\ j=0,1,\ldots\right\}$. It is worth noticing that the chosen basis set takes into account the correct behavior of the bound states at origin and infinity. Besides, it is complete because the eigenfunctions of the dimensionless two-dimensional harmonic oscillator with potential $V(\rho)=\frac{\gamma^2}{\rho^2}+\rho^2$ are linear combinations of these Gaussian functions. Figure 1 shows the first eigenvalues $W_{\nu,0}(a)$ calculated in this way (blue, continuous lines) and the roots $W_0^{(n)}$ given by the truncation condition (red points). There is no doubt that the former connects the latter exactly as stated above. This figure makes it clear that the roots $W_s^{(n)}$ given by the truncation condition are, by themselves, meaningless if one does not arrange and

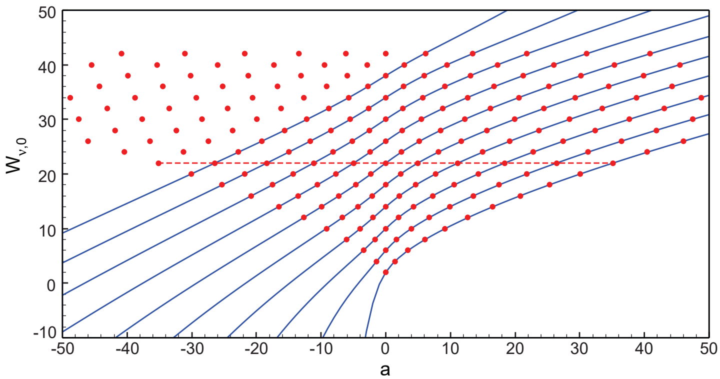

Fig. 1. (Color online) Eigenvalues $W_0^{(n)}$ ($b = 0$) from the truncation condition (red points) and $W_{\nu,0}(a)$ (blue, continuous lines) obtained by means of the variational method.

connect them properly. Figure 1 also shows an horizontal line at $W = W_0^{(10)}$ (red, dashed) that connects all the roots $a_0^{(10,i)}$, $i = 1, 2, \ldots, 11$. It should be clear, from the discussion above, that the exact eigenvalue $W_s^{(n)}$ is shared by $n+1$ different quantum-mechanical problems given by model parameters $a_s^{(n,i)}$, $i = 1, 2, \ldots, n+1$. This fact is also revealed, from a different angle, by the application of supersymmetric quantum mechanics[4] and other suitable algebraic approaches.[5]

From the results above, we draw the following conclusions (for simplicity we restrict ourselves to the case $b = 0$): first, the actual eigenvalues of the radial equation (2), $W_{\nu,s}(a)$, $\nu = 0, 1, \ldots$, are continuous functions of $-\infty < a < \infty$. They are associated to square-integrable functions $R_{\nu,s}(\rho)$. Clearly, there are no allowed values of a. Second, the truncation of the Frobenius expansion for $R(\rho)$ leads to polynomial solutions $R_s^{(n,i)}(\rho)$ that occur for some values of $W_s^{(n)}$, $n = 1, 2, \ldots$ and $a_s^{(n,i)}$, $i = 1, 2, \ldots, n+1$. The integer n is not a quantum number but merely an artifact of the arbitrary truncation condition $c_n \neq 0$, $c_{n+1} = c_{n+2} = 0$. Note that from this truncation procedure stems another relevant integer, i, that Ahmed overlooked completely. The roots $W_s^{(n)}$ and $W_s^{(n')}$ are not part of the spectrum of a given quantum-mechanical model but solutions to different problems with parameters $a_s^{(n,i)}$, $i = 1, 2, \ldots, n+1$ and $a_s^{(n',i')}$, $i' = 1, 2, \ldots, n'+1$. Neither $W_s^{(n)}$ nor $a_s^{(n,i)}$ have any physical meaning. From a mathematical point of view, they are particular values of the eigenvalue and model parameter, respectively, for which the eigenvalue equation admits polynomial solutions. They are useless artifacts of the truncation method unless one is able to organize and connect them properly as we have done in Fig. 1. Third, the allowed values of B_0 and η_L conjectured by Ahmed have no physical meaning because the radial eigenvalue equation exhibits square-integrable solutions for any pair of values of such model parameters. The allowed values of the energy are determined by the requirement of square-integrability and not for the desire that the radial part of the wavefunction exhibits a polynomial factor.

Ahmed, based on his work on several earlier papers, applied the same approach basically to the same equation and drew similar conclusions. For this reason, it is worth revealing the origin of the misunderstanding. Several years ago, Verçin[10] derived a radial eigenvalue equation similar to (2) with $b = 0$, applied the power-series method and argued that there are bound states if and only if the series terminates. He concluded that there are bound states only for certain discrete values of the magnetic-field intensity. Later, Myrheim *et al.*[11] analyzed Verçin's results more rigorously finding that there are square-integrable solutions for all values of the magnetic field. Therefore, the existence of allowed cyclotron frequencies or allowed magnetic field intensities was proved to be an artifact of the truncation method. Unfortunately, Myrheim *et al.*[11] did not stress this point with sufficient clarity and left room for one of the greatest misunderstandings in the field of mathematical physics. It is worth mentioning that Fig. 2 in their paper shows that the energy is a continuous function of the Coulomb coupling, somewhat similar to what we do in present Fig. 1. Furtado *et al.*[12] managed to derive the same radial eigenvalue equation for another problem. Although they were aware of the results derived by both Verçin[10] and Myrheim *et al.*,[11] they surprisingly overlooked (or did not understand or who knows what) the latter more rigorous analysis and, based on the former, concluded that the cyclotron frequency and the magnetic field should depend on the quantum numbers. This mistake gave rise to a series of papers in which the authors conjectured that cyclotron frequencies, oscillator frequencies, field intensities and other physical quantities should have some particular discrete values in order to have bound states.[13–32]

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