Applied Mathematics & Optimization (2023) 87:8
https://doi.org/10.1007/s00245-022-09922-8

A New Carleman Inequality for a Linear Schrödinger Equation on Some Unbounded Domains

Constanza Sánchez F. de la Vega[1] · Luz de Teresa[2] · Pablo Torres[3]

Accepted: 8 September 2022 / Published online: 7 November 2022

Abstract

This article presents a new Carleman inequality for a linear Schrödinger equation which is suitable for both bounded and unbounded domains. We characterize the conditions on the auxiliary function necessary to obtain the global inequality. The novelty of this result is the construction of the auxiliary function on some unbounded domains and for a corresponding valid control region ω. As a consequence, we prove some results on the controllability of a linear Schrödinger equation on unbounded domains.

Keywords Carleman inequalities · Schrödinger equation · Controllability · Unbounded domains

Mathematics Subject Classification 93B05 · 93B07 · 35J10

1 Introduction

Given $T > 0$, $\Omega \subset \mathbb{R}^n$ a connected open set with boundary $\partial\Omega$ at least of class $C^{0,1}$ uniformly. For ω a nonempty open subset we will consider the linear Schrödinger equation

✉ Constanza Sánchez F. de la Vega
csfvega@dm.uba.ar

Luz de Teresa
ldeteresa@im.unam.mx

Pablo Torres
torrespablo@live.com.ar

1 IC (UBA–CONICET) and Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, C1428EGA Buenos Aires, Argentina

2 Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 Mexico City, Mexico, Mexico

3 IMAS (UBA-CONICET), Ciudad Universitaria, Pabellón I, C1428EGA Buenos Aires, Argentina

$$\begin{cases} iw_t &= -\Delta w + h_\omega \quad \text{in} \quad Q = \Omega \times (0,T) \\ w &= 0 \quad \text{in} \quad \Sigma = \partial\Omega \times (0,T) \\ w(x,0) &= w_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{P}$$

with initial datum $w_0(x) \in H^{-1}(\Omega)$ and a control function $h_\omega \in L^2(0,T; H^{-1}(\Omega))$ to be determined. Here h_ω denotes a functional with support in $\omega \times (0,T)$ in the sense of distributions.

In this paper we give sufficient conditions on ω and Ω such that problem (P) is exactly controllable in $H^{-1}(\Omega)$. We recall that (P) is exactly controllable in $H^{-1}(\Omega)$ at time $T > 0$ if for every pair $w_0, w_1 \in H^{-1}(\Omega)$ there exists a control $h_\omega \in L^2(0,T; H^{-1}(\Omega))$ such that the corresponding solution to (P) satisfies $w(T) = w^1$. From the reversibility of Schrödinger equation, exact controllability is reduced to null controllability. We prove the following controllability result

Theorem 1 (Theorem 5 of Sect. 4) *Assume that ω and Ω are open sets such that the existence of a function satisfying (2) is warrantied. Then, given $w_0 \in H^{-1}(\Omega)$, there exists a control $h_\omega \in L^2(0,T; H^{-1}(\Omega))$ with supp $h_\omega \subset \omega \times (0,T)$ such that the corresponding solution to problem (P) satisfies $w(T) = 0$.*

Remark 1 The examples given in Sect. 3 imply the existence of unbounded sets Ω and ω that satisfy the assumptions on the auxiliar function, (2), and in particular, Schrödinger equation is exactly controllable on these unbounded domains, with the control acting on $\omega \times (0,T)$.

In order to obtain the controllability result, we follow the standard controllability–observability duality, that reduces it to prove an observability inequality for the solutions of the adjoint system, given by the homogeneous linear Schrödinger equation

$$\begin{cases} iu_t = -\Delta u \quad \text{in} \quad Q \\ u \quad = 0 \quad \text{in} \quad \Sigma \end{cases} \tag{P^*}$$

with initial datum $u(x,T) = u_T(x)$ in Ω.

Carleman inequalities are a very powerfull tool widely used in the last years to prove null controllability using distributed or boundary controls. Following the ideas of [1], concerning Schrödinger equations, these inequalities have been used with positive results on a coupled kdV-Schrödinger system with internal control in [2] and on a Schrödinger equation with a bounded potential and boundary observations in [3]. In both of theses articles, the domain Ω is assumed to be bounded.

The null controllability of linear and semilinear parabolic problems when Ω is a bounded domain has been analyzed in several recent papers, among others, let us mention [4–6]. When the domain Ω is unbounded, the firsts results were negative (see [7, 8]). For parabolic problems, there are not general positive results of the existence of a Carleman weight in the case of an unbounded domain Ω. Nevertheless, for the heat equation, it was possible to obtain positive results assuming technical considerations on the control region ω. See for instance [9] where they assume $\Omega - \omega$ bounded, [10] where $\Omega = (0, +\infty)$ and ω is an unbounded open set of the form $\omega = \cup_{n\in\mathbb{N}}\omega_n$ with some technical assumptions on ω_n and [11] where they give sufficient conditions on the auxiliar Carleman function, that include the sets in [10] for unbounded domains $\Omega \subset \mathbb{R}^N$.

The main results of this article are a global Carleman inequality given in Sect. 2 and the examples of domains ω, Ω (which can be unbounded) and the auxiliary function $\varphi(x)$, given in Sect. 3, that satisfies the assumptions needed to prove the Carleman inequality

Theorem 2 (Theorem 3 of Sect. 2) *Let Ω be an open connected set with boundary $\partial\Omega$ at least of class $C^{0,1}$ uniformly and ω a nonempty subset. Let $\Psi(x,t) = \varphi(x)\phi(t)$ where ϕ is given by (3) and φ satisfies conditions (2). Then there exist two constants $s_0 = S(\Omega, \omega, T) > 0$ and $C = C(\Omega, \omega) > 0$ such that, for any $s \geq s_0$ the following inequality holds:*

$$
\begin{aligned}
&\int_Q e^{-2s\Psi}\left(|\widetilde{B_1}(u)|^2 + |\widetilde{B_2}(u)|^2\right) dxdt + s^3 \int_Q e^{-2s\Psi} |\phi|^3 |u|^2 dxdt \\
&\quad + s \int_Q e^{-2s\Psi} |\phi| |\nabla u|^2 dxdt \leq C \left(\int_Q e^{-2s\Psi} |iu_t + \Delta u|^2 dxdt \right. \\
&\quad \left. + s^3 \int_{Q_\omega} e^{-2s\Psi} |\phi|^3 |u|^2 dxdt + s \int_{Q_\omega} e^{-2s\Psi} |\phi| |Re \nabla u|^2 dxdt \right)
\end{aligned}
$$

for all $u \in C([0,T], H_0^1(\Omega))$ such that $Bu := iu_t + \Delta u \in L^2(0,T; L^2(\Omega))$, where $\widetilde{B_1}u := B_1(e^{-s\Psi}u)$, $\widetilde{B_2}u := B_2(e^{-s\Psi}u)$ and B_1, B_2 are defined later in (7).

We will use the following well posedness for linear Schrödinger equations that is well known (see e.g. [12]).

Proposition 1 *Let X be either $H_0^1(\Omega)$, $L^2(\Omega)$ or $H^{-1}(\Omega)$. Given $v_0 \in X$ and $f \in L^1(0,T; X)$, there exists a unique solution $v \in C([0,T], X)$ of equation (1)*

$$
\begin{cases} iv_t &= -\Delta v + f \quad in \quad Q \\ v &= 0 \quad in \quad \Sigma \end{cases} \tag{1}
$$

such that $v(x,0) = v_0(x)$ in Ω.

Moreover, from the reversibility of the linear Schrödinger equation, given $v_T \in X$, there exists a unique solution $v \in C([0,T], X))$ of equation (1) such that $v(x,T) = v_T(x)$ in Ω.

This paper is organized as follows: In Sect. 2 we first establish a global Carleman estimate for a linear Schrödinger equation under general assumptions on the domains ω and Ω for which we assume the existence of an auxiliary weight function. In Sect. 3 we provide examples of unbounded domains with its corresponding auxiliary function. Finally, in Sect. 4 we use the Carleman estimate to prove the observability inequality (4) and deduce the controllability result given in Theorem 5.

2 A Global Carleman Inequality

This section is devoted to the proof of an appropriate Carleman estimate which will be useful in Sect. 4 to prove the observability inequality and then the controllability result for the linear Schrödinger equation. Let ω be a nonempty open subset of Ω and define the set $Q_\omega = \omega \times [0, T]$. To begin with, we will assume there exists a real function φ satisfying the following conditions (2a)–(2e):

There exist constants $K = K(\Omega, \omega) > 0$, $\alpha > 0$ and $\beta > 0$ such that

$$\varphi \in C^3(\Omega). \tag{2a}$$

$$0 < \varphi \leq K, \ ||\nabla\varphi|| \leq K, \ ||H\varphi|| \leq K, \ ||\nabla\Delta\varphi|| \leq K, \ \text{for all } x \in \Omega, \tag{2b}$$

$$\frac{\partial\varphi}{\partial\eta}(x) \geq 0 \quad \text{for all} \quad x \in \partial\Omega, \tag{2c}$$

$$||\nabla\varphi(P)||^2 \geq \alpha \quad \text{for all} \quad P \in \Omega \setminus \omega, \tag{2d}$$

$$\mathbf{X}^t \left[-H\varphi(P)\right] \mathbf{X} \geq \beta|\mathbf{X}|^2 \quad \text{for all} \quad P \in \Omega \setminus \omega, \ \mathbf{X} \in \mathbb{C}^n. \tag{2e}$$

Here the expression $H\varphi$ denotes the Hessian matrix of the function φ.

Let $\phi : (0, T) \to \mathbb{R}$ be the function given by

$$\phi(t) = \frac{1}{t(T-t)} \tag{3}$$

which has the properties:

$$|\phi(t)| \leq C(T^2)|\phi(t)|^2 \tag{4a}$$

$$|\phi'(t)| \leq C(T)|\phi(t)|^2 \tag{4b}$$

$$|\phi''(t)| \leq C(T^2)|\phi(t)|^3. \tag{4c}$$

Then we define on $\Omega \times (0, T)$ the function $\Psi(x, t) = \varphi(x)\phi(t)$.

Theorem 3 *Let Ω be an open connected set with boundary $\partial\Omega$ at least of class $C^{0,1}$ uniformly and ω a nonempty subset of Ω. Let $\Psi(x, t) = \varphi(x)\phi(t)$ where ϕ is given by (3) and φ satisfies conditions (2). Then there exist two constants $s_0 = S(\Omega, \omega, T) > 0$ and $C = C(\Omega, \omega) > 0$ such that, for any $s \geq s_0$ the following inequality holds:*

$$\begin{aligned}
&\int_Q e^{-2s\Psi}\left(|\widetilde{B_1}(u)|^2 + |\widetilde{B_2}(u)|^2\right) dxdt + s^3\int_Q e^{-2s\Psi}|\phi|^3|u|^2 dxdt \\
&+ s\int_Q e^{-2s\Psi}|\phi||\nabla u|^2 dxdt \leq C\left(\int_Q e^{-2s\Psi}|iu_t + \Delta u|^2 dxdt\right. \\
&\left.+ s^3\int_{Q_\omega} e^{-2s\Psi}|\phi|^3|u|^2 dxdt + s\int_{Q_\omega} e^{-2s\Psi}|\phi||Re\nabla u|^2 dxdt\right)
\end{aligned} \tag{5}$$

for all $u \in C([0, T], H_0^1(\Omega))$ such that $Bu := iu_t + \Delta u \in L^2(0, T; L^2(\Omega))$, where $\widetilde{B_1}u := B_1(e^{-s\Psi}u)$, $\widetilde{B_2}u := B_2(e^{-s\Psi}u)$ and B_1, B_2 are defined later in (7).

Proof In what follows, $C(\Omega, \omega)$ will denote a generic constant depending only on Ω and ω that may change from line to line. The classic first step in this kind of inequality is to change the variable of the function u by its multiplication with an appropriate weight function.

With that in mind we take $u \in C^2(\overline{Q})$ with $u = 0$ in Σ, set $v = e^{-s\Psi}u$ and compute:

$$B_\Psi v := e^{-s\Psi} B(e^{s\Psi} v) = is\Psi_t v + iv_t + \Delta v + s^2||\nabla\Psi||^2 v + sv\Delta\Psi + 2s\nabla\Psi \cdot \nabla v$$

A "classical" approach of this problem consist in representing this operator as the sum of an adjoint and a skew-adjoint operators B_1 and B_2:

$$B_\Psi v = B_1 v + B_2 v \tag{6}$$

where

$$B_1 v = 2s\nabla\Psi \cdot \nabla v + sv\Delta\Psi + is\Psi_t v \tag{7a}$$

$$B_2 v = iv_t + \Delta v + s^2||\nabla\Psi||^2 v. \tag{7b}$$

□

To simplify the notation, we will denote by $B_j^i v$ $(1 \leq j \leq 2,\ 1 \leq i \leq 3)$ the ith term in the expression of $B_j v$ given in (7) and $\diamondsuit_{ij}$ the L^2 real inner product in Q between the i-th term of $B_1 v$ and the conjugate j-th term of $B_2 v$. Using this notation we get from (6)

$$\int_Q |B_1 v|^2 dxdt + \int_Q |B_2 v|^2 dxdt + 2Re\int_Q B_1 v\overline{B_2 v}\, dxdt = \int_Q |B_\Psi v|^2 dxdt. \tag{8}$$

Remark 2 The term $B_1^3 v = is\Psi_t v$ does not contribute with something significant in the development of the inequality and can be put substracting in the left hand side to obtain another decomposition somehow easier to compute:

$$B_\Psi v - is\Psi_t v = B_1 v + B_2 v.$$

Nevertheless we will keep it on the right hand side in order to follow the classical approach.

The goal is to find lower bounds of the term:

$$Re\int_Q B_1 v\overline{B_2 v}\, dxdt = \sum_{i,j=1,2,3} \diamondsuit_{ij}. \tag{9}$$

We multiply each term of $B_1 v = is\Psi_t v$ by each term of $\overline{B_2 v}$ and develop the nine terms appearing in $B_1 v\overline{B_2 v}$. For this, we will integrate by parts with respect to the time and space variables and make use of the properties of v.

First, we have

$$\begin{aligned}\lozenge_{11} &= Re \int_Q 2s(\nabla\Psi \cdot \nabla v)\overline{i v_t}\, dxdt = 2sIm \int_Q (\nabla\Psi \cdot \nabla v)\overline{v_t}\, dxdt \\ &= -2sIm \int_Q (\nabla\Psi_t \cdot \nabla v)\overline{v}\, dxdt - 2sIm \int_Q (\nabla\Psi \cdot \nabla v_t)\overline{v}\, dxdt \\ &:= A_1 + A_2.\end{aligned}$$

We also have

$$\begin{aligned}\lozenge_{21} &= Re \int_Q s\Delta\Psi v\overline{i v_t}\, dxdt = sIm \int_Q \Delta\Psi v\overline{v_t}\, dxdt \\ &= -sIm \int_Q \nabla\Psi \cdot \nabla(v\overline{v_t})\, dxdt \\ &= -sIm \int_Q (\nabla\Psi \cdot \nabla v)\overline{v_t}\, dxdt - sIm \int_Q (\nabla\Psi \cdot \nabla\overline{v_t})v\, dxdt.\end{aligned}$$

Integrating by parts in time in the first term and using the identity $-Im(z) = Im(\overline{z})$ for $z \in \mathbb{C}$, we get

$$\begin{aligned}\lozenge_{21} &= sIm \int_Q (\nabla\Psi_t \cdot \nabla v)\overline{v}\, dxdt + 2sIm \int_Q (\nabla\Psi \cdot \nabla v_t)\overline{v}\, dxdt \\ &= -\frac{A_1}{2} - A_2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\lozenge_{32} &= Re \int_Q is\Psi_t v\Delta\overline{v}\, dxdt = -sIm \int_Q \Psi_t v\Delta\overline{v}\, dxdt \\ &= -sIm \int_Q \nabla(\Psi_t v)\nabla\overline{v}\, dxdt \\ &= -sIm \int_Q (\nabla\Psi_t \cdot \nabla\overline{v})v\, dxdt - sIm \int_Q \Psi_t|\nabla v|^2\, dxdt.\end{aligned}$$

Since Ψ is a real valued function we have $sIm \int_Q \Psi_t|\nabla v|^2\, dxdt = 0$ and consequently:

$$\lozenge_{32} = \frac{A_1}{2}.$$

Adding up this first three terms, we get

$$\lozenge_{11} + \lozenge_{21} + \lozenge_{32} = A_1.$$

Now, using the conditions (2b) and (4b) we obtain

$$|\nabla\Psi_t| \le C|\phi|^2 \text{ for all } (x,t) \in \Omega \times (0,T)$$

and therefore using Young's inequality we can estimate the term A_1 in the following way

$$\begin{aligned} A_1 &= -2sIm \int_Q (\nabla\Psi_t \cdot \nabla v)\overline{v}\, dxdt \\ &\le 2s \int_Q |\nabla\Psi_t \cdot \nabla v||\overline{v}|\, dxdt \\ &\le C \int_Q \left(|\phi|^{1/2}|\nabla v|\right)\left(s|\phi|^{3/2}|v|\right) dxdt \\ &\le C \int_Q |\phi||\nabla v|^2\, dxdt + Cs^2 \int_Q |\phi|^3|v|^2\, dxdt. \end{aligned}$$

In conclusion:

$$\Diamond_{11} + \Diamond_{21} + \Diamond_{32} \ge -C \int_Q |\phi||\nabla v|^2\, dxdt - Cs^2 \int_Q |\phi|^3|v|^2\, dxdt. \tag{10}$$

For the next inequality we notice that if $v = 0$ in $\partial\Omega \times (0,T)$ then:

$$\nabla v\Big|_{\partial\Omega\times(0,T)} = \frac{\partial v}{\partial\eta}\eta$$

with η denoting the outward unit vector. Therefore

$$\begin{aligned} \Diamond_{12} &= Re \int_Q 2s(\nabla\Psi \cdot \nabla v)\Delta\overline{v}\, dxdt \\ &= 2sRe \int_0^T \int_{\partial\Omega} \frac{\partial\Psi}{\partial\eta}\left|\frac{\partial v}{\partial\eta}\right|^2 dSdt - 2sRe \int_Q \nabla\left(\nabla\Psi \cdot \nabla v\right) \cdot \nabla\overline{v}\, dxdt \\ &= D_1 + D_2. \end{aligned}$$

Note that from condition (2c), it follows that $D_1 \ge 0$. Now, we expand the term D_2

$$\begin{aligned} D_2 &= -2sRe \int_Q \nabla\left(\nabla\Psi \cdot \nabla v\right) \cdot \nabla\overline{v}\, dxdt \\ &= -2s \int_Q \nabla v^t \cdot H\Psi \cdot \nabla\overline{v}\, dxdt - 2sRe \int_Q \nabla\Psi^t \cdot Hv \cdot \nabla\overline{v}\, dxdt. \end{aligned}$$

here $H\Psi$ denotes the Hessian matrix , with respect to the space variables, of Ψ. We use Green formula on the second term

$$\begin{aligned}
&- s\int_Q \nabla\Psi^t \cdot 2Re\,(Hv\cdot\nabla\overline{v})\;dxdt\\
&= -s\int_Q \nabla\Psi\cdot\nabla|\nabla v|^2\\
&= -sRe\int_0^T\int_{\partial\Omega}\frac{\partial\Psi}{\partial\eta}\left|\frac{\partial v}{\partial\eta}\right|^2\,dSdt + s\int_Q \Delta\Psi|\nabla v|^2\,dxdt\\
&= -\frac{D_1}{2} + s\int_Q \Delta\Psi|\nabla v|^2\,dxdt.
\end{aligned}$$

Thus, we get

$$\diamondsuit_{12} = \frac{D_1}{2} - 2s\int_Q \nabla v^t\cdot H\Psi\cdot\nabla\overline{v}\,dxdt + s\int_Q \Delta\Psi|\nabla v|^2\,dxdt. \tag{11}$$

Moreover,

$$\begin{aligned}
\diamondsuit_{22} &= sRe\int_Q \Delta\Psi v\Delta\overline{v}\,dxdt\\
&= -sRe\int_Q (\nabla[\Delta\Psi]\cdot\nabla v)\overline{v}\,dxdt - s\int_Q \Delta\Psi|\nabla v|^2\,dxdt.
\end{aligned}$$

The last equality shows why we need $\varphi\in C^3(\Omega)$. Adding up the last two terms that we developed we obtain

$$\diamondsuit_{12} + \diamondsuit_{22} = \frac{D_1}{2} + 2s\int_Q \nabla v^t\cdot[-H\Psi]\cdot\nabla\overline{v}\,dxdt - sRe\int_Q (\nabla[\Delta\Psi]\cdot\nabla v)\overline{v}\,dxdt. \tag{12}$$

Using conditions (2b) and (4a) and Young's inequality we can estimate the last term appearing in the right hand side of (12) as follows

$$\begin{aligned}
\left|-sRe\left[\int_Q (\nabla[\Delta\Psi]\cdot\nabla v)\overline{v}\right]\,dxdt\right| &\le C\int_Q |\phi|^{1/2}|(\nabla[\Delta\varphi]\cdot\nabla v)||\phi|^{3/2}s|\overline{v}|\,dxdt\\
&\le C\int_Q |\phi||\nabla v|^2\,dxdt + Cs^2\int_Q |\phi|^3|v|^2\,dxdt.
\end{aligned}$$

We obtain

$$
\begin{aligned}
&-sRe\left[\int_Q (\nabla[\Delta\Psi]\cdot\nabla v)\overline{v}\right]\,dxdt \\
&\quad \geq -C\int_Q |\phi||\nabla v|^2\,dxdt - Cs^2\int_Q |\phi|^3|v|^2\,dxdt. \qquad (13)
\end{aligned}
$$

To finish the estimations of this terms we need to provide lower bounds for the term $2s\int_Q \nabla v^t\cdot[-H\Psi]\cdot\nabla\overline{v}\,dxdt$. Using conditions (2b) and (2e) we have:

$$
\begin{aligned}
&2s\int_Q \nabla v^t\cdot[-H\Psi]\cdot\nabla\overline{v}\,dxdt \\
&\geq 2s\int_{Q_\omega} \nabla v^t\cdot[-H\Psi]\cdot\nabla\overline{v}\,dxdt + C(\beta)s\int_{Q\setminus Q_\omega} |\phi||\nabla v|^2\,dxdt \\
&\geq -Cs\int_{Q_\omega} |\phi||\nabla v|^2\,dxdt + C(\beta)s\int_{Q\setminus Q_\omega} |\phi||\nabla v|^2\,dxdt. \qquad (14)
\end{aligned}
$$

Therefore, from (12), using that $D_1 \geq 0$, (13) and (14) we have

$$
\begin{aligned}
&\Diamond_{12} + \Diamond_{22} \\
&\quad \geq -C\int_Q |\phi||\nabla v|^2\,dxdt - Cs^2\int_Q |\phi|^3|v|^2\,dxdt - Cs\int_{Q_\omega} |\phi||\nabla v|^2\,dxdt \\
&\quad\quad + C(\beta)s\int_{Q\setminus Q_\omega} |\phi||\nabla v|^2\,dxdt. \qquad (15)
\end{aligned}
$$

The term

$$
C(\beta)s\int_{Q\setminus Q_\omega} |\phi||\nabla v|^2\,dxdt
$$

plays an important role in the development of the inequality since it is positive and has order 1 in s. In order to obtain estimates on Q and absorb the terms with less order on s, from (10) and (15) we get that

$$
\begin{aligned}
&\Diamond_{11} + \Diamond_{21} + \Diamond_{32} + \Diamond_{12} + \Diamond_{22} + C(\beta)s\int_{Q_\omega} |\phi||\nabla v|^2\,dxdt \\
&\quad \geq -C\int_Q |\phi||\nabla v|^2\,dxdt - Cs^2\int_Q |\phi|^3|v|^2\,dxdt + C(\beta)s\int_Q |\phi||\nabla v|^2\,dxdt. \qquad (16)
\end{aligned}
$$

Moving forward to the last part of the estimations, we have

$$
\Diamond_{13} = 2s^3 Re\int_Q \nabla\Psi\cdot\nabla v\,||\nabla\Psi||^2\overline{v}\,dxdt = s^3\int_Q ||\nabla\Psi||^2[\nabla\Psi\cdot 2Re(\overline{v}\nabla v)]\,dxdt
$$

$$= s^3 \int_Q (\nabla\Psi^t \cdot [-H\Psi] \cdot \nabla\Psi)|v|^2 \, dxdt - s^3 \int_Q ||\nabla\Psi||^2 \Delta\Psi |v|^2 \, dxdt.$$

Also

$$\Diamond_{23} = s^3 \int_Q ||\nabla\Psi||^2 \Delta\Psi |v|^2 \, dxdt$$

and

$$\begin{aligned}\Diamond_{31} = Re \int_Q is\Psi_t v \overline{i v_t} \, dxdt &= \int_Q s\Psi_t Re(v\overline{v_t}) \, dxdt. \\ &= -\frac{1}{2} s \int_Q \Psi_{tt} |v|^2 \, dxdt.\end{aligned}$$

Finally, again since Ψ is real valued we get

$$\Diamond_{33} = -s^3 Im \int_Q \Psi_t ||\nabla\Psi|||v|^2 \, dxdt = 0.$$

Adding up the last four terms that we have developed

$$\begin{aligned}\Diamond_{13} + \Diamond_{23} + \Diamond_{31} + \Diamond_{33} = s^3 &\int_Q (\nabla\Psi^t \cdot [-H\Psi] \cdot \nabla\Psi)|v|^2 \, dxdt \\ &- \frac{1}{2} s \int_Q \Psi_{tt} |v|^2 \, dxdt. \end{aligned} \tag{17}$$

Using (2b), (4a), (4c), the second term appearing in the right hand side of (17) can be estimated as follows

$$\left| -s \int_Q \Psi_{tt} |v|^2 \, dxdt \right| \le Cs \int_Q |\phi|^3 |v|^2 \, dxdt$$

to get

$$-\frac{1}{2} s \int_Q \Psi_{tt} |v|^2 \, dxdt \ge -Cs \int_Q |\phi|^3 |v|^2 \, dxdt. \tag{18}$$

For the first term in the right hand side of (17) we use hypothesis (2d), (2e) and (2b) to obtain

$$\begin{aligned}&s^3 \int_Q (\nabla\Psi^t \cdot [-H\Psi] \cdot \nabla\Psi)|v|^2 \, dxdt \\ &\qquad \ge -Cs^3 \int_{Q_\omega} |\phi|^3 |v|^2 \, dxdt + C(\alpha, \beta) s^3 \int_{Q \setminus Q_\omega} |\phi|^3 |v|^2 \, dxdt.\end{aligned} \tag{19}$$

Then, from (17), using (18) and (19)

$$\begin{aligned}\Diamond_{13}+\Diamond_{23}+\Diamond_{31}+\Diamond_{33}\geq &-Cs\int_{Q}|\phi|^3|v|^2\,dxdt-Cs^3\int_{Q_\omega}|\phi|^3|v|^2\,dxdt\\&+C(\alpha,\beta)s^3\int_{Q\setminus Q_\omega}|\phi|^3|v|^2\,dxdt.\end{aligned}\tag{20}$$

In this case, the positive term

$$C(\alpha,\beta)s^3\int_{Q\setminus Q_\omega}|\phi|^3|v|^2\,dxdt$$

plays an important role since it can be used to absorb the terms with less order in s. In the same way as before, in order to obtain estimates on Q and absorb the terms with less order on s. We get

$$\begin{aligned}&\Diamond_{13}+\Diamond_{23}+\Diamond_{31}+\Diamond_{33}+C(\alpha,\beta)s^3\int_{Q_\omega}|\phi|^3|v|^2\,dxdt\\&\geq -Cs\int_{Q}|\phi|^3|v|^2\,dxdt+C(\alpha,\beta)s^3\int_{Q}|\phi|^3|v|^2\,dxdt.\end{aligned}\tag{21}$$

Altogether, the two positive terms

$$C(\beta)s\int_{Q}|\phi||\nabla v|^2\,dxdt\quad\text{and}\quad C(\alpha,\beta)s^3\int_{Q}|\phi|^3|v|^2\,dxdt$$

are the dominant ones since the negative terms on the right hand sides of equations (16) and (21) have lower order in s and thus can be absorbed. Therefore, from (8) and (9), we conclude that there exist constants $D=D(\alpha,\beta,T)>0, C=C(\alpha,\beta,T)>0$ and $s_0=s_0(\alpha,\beta,T)$ such that for all $s\geq s_0$:

$$\begin{aligned}&\int_{Q}|B_\Psi v|^2\,dxdt+Ds^3\int_{Q_\omega}|\phi|^3|v|^2dxdt+Ds\int_{Q_\omega}|\phi||\nabla v|^2\,dxdt\geq\int_{Q}|B_1v|^2dxdt\\&+\int_{Q}|B_2v|^2dxdt+Cs\int_{Q}|\phi||\nabla v|^2\,dxdt+Cs^3\int_{Q}|\phi|^3|v|^2\,dxdt.\end{aligned}\tag{22}$$

To obtain the desired inequality in u since $v=e^{-s\Psi}u$ we have the identities

$$|v|^2=e^{-2s\Psi}|u|^2$$

and

$$e^{-s\Psi}|\phi|^{1/2}s^{1/2}\nabla u=s^{3/2}\nabla\varphi|\phi|^{3/2}v+s^{1/2}|\phi|^{1/2}\nabla v$$

from where, together with the properties of Ψ, we get the estimations

$$Cs\int_{Q_\omega} e^{-2s\Psi}|\phi||\nabla u|^2 + Cs^3\int_{Q_\omega} e^{-2s\Psi}|\phi|^3|u|^2 \geq s\int_{Q_\omega}|\phi||\nabla v|^2 \tag{23}$$

and

$$s^3\int_Q e^{-2s\Psi}|\phi|^3|u|^2 + \int_Q s|\phi|^2|\nabla v|^2 \geq sC\int_Q e^{-2s\Psi}|\phi||\nabla u|^2. \tag{24}$$

Using both estimations (23), (24) and the fact that $B_\Psi(v) = e^{-s\Psi}(iu_t + \Delta u)$ we arrive to an inequality similar to (5) with the difference that the last term in the right hand side depends on $|\nabla u|^2$ and not on $|Re\nabla u|^2$. To get the sharper inequality we can follow the technique developed in [2] and conclude the proof of (5). □

3 Examples

This section is devoted to give some examples of unbounded sets Ω and ω for which we can exhibit an auxiliary weight function φ satisfying conditions (2). We begin with an example in $\mathbb{R}$ and then we prove a general lemma that will let us built a family of examples in $\mathbb{R}^n$.

Example 1 In this first example we follow the ideas of [11]. For $\Omega = [0, \infty)$ we take a family of disjoint intervals in the following way

$$\omega = \bigcup_{n\in\mathbb{N}} \omega_n, \quad \omega_n = [a_n, b_n]$$

with some technical assumptions on the intervals $\omega_n = [a_n, b_n]$. More precisely we need the existence of constants $m > 0$ and $L > 0$ and the existence of $n_0 \in \mathbb{N}$ such

$$\text{For all } \ n \geq n_0 \ \text{ we have } \ b_n - a_n \geq m \tag{25}$$

$$\text{For all } \ n \geq n_0 \ \text{ we have } \ a_{n+1} - b_n \leq L. \tag{26}$$

Without lost of generality we can assume that conditions (25) and (26) are satisfied for all $n \in \mathbb{N}$ since if this is not the case then the same function φ_0 that we define bellow works in the interval $[0, a_{n_0}]$. We will show that there exists a function $\widehat{\varphi}$ defined on Ω satisfying conditions (2). We define (see Figure 1)

$$\widehat{\varphi}(x) = \begin{cases} \varphi_0(x) \text{ in } [0, c_1 - m/4] \\ L_n(x) \text{ in } [c_n - m/4, c_n - m/8]\ n \geq 1 \\ \varphi_n(x) \text{ in } [c_n - m/8, c_{n+1} - m/4]\ n \geq 1 \end{cases}.$$

Here the centers $\{c_n\}_{n\in\mathbb{N}}$ are defined as $c_n = b_n - m/8$ and the functions $\{\varphi_n\}_{n\geq 0}$ as follows:

$$\varphi_0(x) = 1 - e^{-k(x-c_1)^2}; \quad \varphi_n(x) = 1 - e^{-k(x-c_n)^2}, n \geq 1 \tag{27}$$

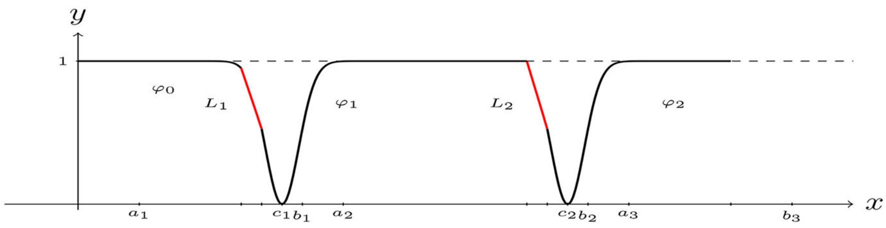

Fig. 1 Graphic of the function $\widehat{\varphi}$

where k is a constant that satisfies

$$k > \frac{32}{m^2}. \tag{28}$$

The functions $L_n(x)$ are lines that connects the points P_n and Q_n given by $P_n = (c_n - m/4, \varphi_{n-1}(c_n - m/4))$ and $Q_n = (c_n - m/8, \varphi_n(c_n - m/8))$. With these definitions we find that $0 \leq \widehat{\varphi} \leq 1$, $\widehat{\varphi}$ is continuous in $[0, +\infty)$ and C^3 in $[0, \infty) - \{c_n - m/4; c_n - m/8\}_{n \in \mathbb{N}}$. Finally we take a cutoff function $\rho(x) \in C^3(\mathbb{R})$ that satisfies

$$\rho(x) = \begin{cases} 0 \text{ in } (-\infty, -m/8] \\ 1 \text{ in } [0, c_1 - 3m/8] \\ 0 \text{ in } [c_n - m/4, c_n - m/8], \; n \geq 1 \\ 1 \text{ in } [b_n, c_{n+1} - 3m/8], \; n \geq 1 \end{cases}$$

and such that $|\rho'| \leq C(m)$, $|\rho''| \leq C(m)$, $|\rho'''| \leq C(m)$ where $C(m)$ is a constant that depends only of m.

Then the weight function

$$\Psi(x, t) = \varphi(x)\phi(t) \text{ with } \varphi(x) = \widehat{\varphi}(x)\rho(x) \tag{29}$$

is $C^3(\Omega)$ and satisfies properties 2b–2e. To see this we use the fact that for $x > 0$ there is a constant N such that $e^{-x}x^j \leq N$ for $j = 0, 1, 2, 3$; and we obtain that for $x \in Dom(\varphi_n)$ and for all $n \geq 0$

$$|\varphi_n'| \leq C(L, m); \quad |\varphi_n''| \leq C(L, m) \quad |\varphi_n'''| \leq C(L, m).$$

Moreover, if we define the sets $K_n = Dom(\varphi_n) \cap (\Omega - \omega) = [b_n, a_{n+1}]$ for $n \geq 1$; $K_0 = [0, a_1]$, then for $x \in K_n$

$$|x - c_n| \geq \frac{m}{8} = C(m); \tag{30}$$

$$|x - c_n| \leq |a_{n+1} - c_n| \leq L + \frac{m}{8} = C(L, m) \tag{31}$$

which together with (28) implies that for $x \in K_n$ and for all $n \geq 0$

$$|\varphi_n'| \geq D(L, m); \quad -\varphi_n'' \geq D(L, m)$$

where all the constants appearing here depends only on L and m.

Analogously, if we consider now

$$\omega = \bigcup_{n\in\mathbb{N}} [a_n, b_n] \cup \bigcup_{n\in\mathbb{N}} [a_{-n}, b_{-n}]$$

with $a_{-n} > b_{-n-1}$ and the same assumptions as (25)–(26), we can set up an example in $\mathbb{R}$.

In order to produce new examples in $\mathbb{R}^N$, we will use the following lemma.

Lemma 1 *Assume that we have* $\Omega = A_1 \times A_2 \times \cdots \times A_n$*,*

$$\omega = \bigcup_{j=1\ldots n} A_1 \times \cdots \times \omega_{A_j} \times \cdots \times A_n$$

with $\omega_{A_j} \subset A_j$*,* $Q_\omega = \omega \times [0, T]$ *;* $\mathbf{X_j} \in A_j \subset \mathbb{R}^{n_j}$ *where* A_j *for* $j = 1, \ldots, n$ *are* C^1*. If for all* j *there is a function* $f_j \in C^3(A_j) \cap W^{3,\infty}(A_j)$ *satisfying the following conditions*

1. *For all* $j = 1 \ldots n$*,*

$$\left.\frac{\partial f_j}{\partial \eta}\right|_{\partial A_j} \geq 0\,.$$

2. *There are constants* $C_j > 0$ *and* $E > 0$ *such that*

$$\mathbf{Z_j}^t\left[-Hf(P_j)\right]\mathbf{Z_j} \geq C_j|\mathbf{Z_j}|^2$$

   for all $\mathbf{Z_j} \in \mathbb{C}^{n_j}$ *and* $P_j \in A_j \setminus \omega_{A_j}$*.*
3. *Additionally for some* j_0 *there is a constant* $D_{j_0} > 0$ *such that for all* $P_{j_0} \in A_{j_0} \setminus \omega_{A_{j_0}}$ *we have*

$$||\nabla f_{j_0}(P_{j_0})||^2 \geq D_{j_0}.$$

Then the function

$$\varphi(\mathbf{X_1}, \ldots, \mathbf{X_n}) = \sum_{j=1}^{n} f_j(\mathbf{X_j})$$

is a weight function for the Carleman estimate 5 (satisfies conditions (2)).

Proof We begin by noting that

$$\Omega - \omega = A_1 \setminus \omega_{A_1} \times A_2 \setminus \omega_{A_2} \cdots \times A_n \setminus \omega_{A_n}.$$

From the definition, it is clear that $\varphi \in C^3(\Omega) \cap W^{3,\infty}(\Omega)$ and therefore conditions (2a)-(2b) are satisfied. To finish the proof we check the rest of the properties. For (2c) since

$$\partial\Omega = \bigcup_j A_1 \times \cdots \times \partial A_j \cdots \times A_n$$

we find that

$$\left.\frac{\partial \varphi}{\partial \eta}\right|_{A_1\times\cdots\times\partial A_j\times\cdots\times A_n} = \frac{\partial f_j}{\partial \eta_{A_j}} \geq 0 \ .$$

For (2e) we compute the Hessian for $P = (P_1, \ldots, P_n, Q)$:

$$-H\varphi(P) = \begin{pmatrix} -Hf_1(P_1) & 0 & \ldots & \ldots & 0 \\ 0 & \ddots & \ldots & \ldots & \vdots \\ \vdots & \vdots & -Hf_j(P_j) & \ldots & \vdots \\ \vdots & \ldots & \ldots & \ddots & \vdots \\ \vdots & \ldots & \ldots & \ldots & -Hf_n(P_n) \end{pmatrix}$$

and therefore for $\mathbf{Z} = (\mathbf{Z_1}, \ldots, \mathbf{Z_n})$ we obtain

$$\begin{aligned} \mathbf{Z}^t[-H\varphi]\mathbf{Z} &= \sum_j (0, \ldots \mathbf{Z_j} \ldots 0)^t[-Hf_j(P_j)](0, \ldots \mathbf{Z_j} \ldots 0) \\ &\geq \sum_j C_j ||(0, ..\mathbf{Z_j}...0)||^2 \geq \left(\min_j C_j\right) ||\mathbf{Z}||^2. \end{aligned}$$

Finally, for (2d) if we take $P = (P_1, \ldots, P_n)$ we see that

$$||\nabla\varphi(P)||^2 = \sum_j ||\nabla f_j(P_j)||^2 \geq ||\nabla f_{j_0}(P)||^2 \geq D_{j_0}.$$

□

We now exhibit a family of examples in $\mathbb{R}^N$.

Example 2 Using the first example in $\mathbb{R}$ and the lemma below, we can show the existence of a weight function φ satisfying conditions (2) for $\Omega = [0, \infty)^j$ and $\Omega = \mathbb{R}^j$ where

$$\omega = \bigcup_{n\in\mathbb{N}} [a_n^1, b_n^1] \times \cdots \times \bigcup_{n\in\mathbb{N}} [a_n^j, b_n^j]$$

and

$$\omega = \bigcup_{n\in\mathbb{N}} [a_n^1, b_n^1] \cup \bigcup_{n\in\mathbb{N}} [a_{-n}^1, b_{-n}^1] \times \cdots \times \bigcup_{n\in\mathbb{N}} [a_n^j, b_n^j] \cup \bigcup_{n\in\mathbb{N}} [a_{-n}^j, b_{-n}^j]$$

repectively, with $\varphi = \sum_{i=1}^{j} \varphi_i$.

Example 3 We can combine the first example with any bounded set that have an auxiliary Carleman function. For instance for any C^1 bounded set with control region around the whole boundary we can take an auxiliary function of the form $\varphi(x) = \rho(x) f(x)$ with f an strict concave function without critical points in Ω and ρ a suitable cut-off function of a neighbourhood on the boundary. For example for Ω a disk we can obtain examples in infinite cylinders and so forth.

4 Observability and Controllability

We are now in conditions to prove the controllability inequality for domains ω, Ω for which there exists an auxiliary function φ satisfying conditions (2), (for instance, the domains given in Sect. 3).

Theorem 4 *Let ω and Ω be such that there exists an auxiliary function satisfying conditions (2). Given $u_0 \in H_0^1(\Omega)$, let $Q_\omega = \omega \times (0, T)$ and $u \in C([0, T], H_0^1(\Omega))$ be the solution of the dual problem (P^*) given by Proposition 1, then*

$$\|u(\cdot, 0)\|^2_{H_0^1(\Omega)} \leq C \left(\int_{Q_\omega} |u(x,t)|^2 + |\nabla u(x,t)|^2 \, dxdt \right). \tag{32}$$

Proof We define

$$E(t) = \int_\Omega |u(x,t)|^2 + |\nabla u(x,t)|^2 \, dx. \tag{33}$$

Using the estimations:

$$\begin{aligned} m_1 \leq s|\phi| e^{-2s\Psi}; \;\; m_2 \leq s^3 |\phi|^3 e^{-2s\Psi} \;\; &\text{for all } \;(x,t) \in \Omega \times (T/4, 3T/4) \\ s|\phi| e^{-2s\Psi} \leq M_1; \;\; s^3|\phi|^3 e^{-2s\Psi} \leq M_2 \;\; &\text{for all } \;(x,t) \in \Omega \times (0, T) \end{aligned}$$

and Carleman inequality (5), we find that there exists a constant $C(m_1, m_2, M_1, M_2, s_0)$ such that:

$$\begin{aligned} E(t) &\leq C \left(\int_{T/4}^{3T/4} \int_\Omega s^3 e^{-2s\Psi} |\phi|^3 |u|^2 + s e^{-2s\Psi} |\phi| |\nabla u|^2 \, dxdt \right) \\ &\leq C \left(\int_{\omega \times (0,T)} s^3 e^{-2s\Psi} |\phi|^3 |u|^2 + s e^{-2s\Psi} |\phi| |Re(\nabla u)|^2 \, dxdt \right) \\ &\leq C \left(\int_{\omega \times (0,T)} |u|^2 + |\nabla u|^2 \, dxdt \right). \end{aligned}$$

Moreover, since $E'(t) = 0$ we obtain the observability inequality in $H_0^1(\Omega)$:

$$\|u(\cdot, 0)\|^2_{H_0^1(\Omega)} \leq C \left(\int_{Q_\omega} |u(x,t)|^2 + |\nabla u(x,t)|^2 \, dxdt \right). \tag{34}$$

□

In order to prove null controllability, we begin by characterizing a control that drives the system to zero at time T. We recall that $h \in L^2(0, T; H^{-1}(\Omega))$ has support in $\omega \times (0, T)$ if for almost every $t \in (0, T)$ and every $\theta \in H_0^1(\Omega)$ such that $\theta|_\omega = 0$ we have that $\langle h(t), \theta \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = 0$. Note that every $h \in L^2(0, T; (H^1(\omega))')$ can be identified with $h \in L^2(0, T; H^{-1}(\Omega))$ with support in $\omega \times (0, T)$. In fact, let $\theta \in H_0^1(\Omega)$ such that $\theta|_\omega = 0$, then

$$\langle h, \theta \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} = \langle h, \theta|_\omega \rangle_{L^2(0,T;(H^1(\omega))') \times L^2(0,T;H^1(\omega))} = 0. \tag{35}$$

Lemma 2 *A control $h_\omega \in L^2(0,T;H^{-1}(\Omega))$ drives system (P) from $w(0)=w_0$ to $w(T)=0$ if and only if*

$$\langle -iw_0, u(0)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \langle h_\omega, u\rangle_{L^2(0,T;H^{-1}(\Omega))\times L^2(0,T;H_0^1(\Omega)))} \tag{36}$$

for all u solution of (P^*).

Proof The proof follows using integration by parts for regular solutions and using usual density arguments. □

Theorem 5 *Assume that ω and Ω are open sets such that the existence of a function satisfying (2) is warrantied. Then, given $w_0 \in H^{-1}(\Omega)$, there exists a control $h_\omega \in L^2(0,T;H^{-1}(\Omega))$ with supp $h_\omega \subset \omega\times(0,T)$ such that the corresponding solution to problem (P) satisfies $w(T)=0$.*

Proof We define $\Lambda : H^1(\omega) \to (H^1(\omega))'$ the usual isomorphism given by Riesz's Theorem and the application (HUM)

$$\Gamma : H^1(\Omega) \to H^{-1}(\Omega), \quad \Gamma(u_0) = -iw(0) \tag{37}$$

defined as follows: given u_0, we consider u the solution of system (P^*) with initial data $u(0)=u_0$, then w is the solution backwards in time of equation (P) with $w(T)=0$ and $h_\omega(t)=\Lambda(u(t)|_\omega)$ where $h_\omega \in L^2(0,T;H^{-1}(\Omega))$ has support in $\omega\times(0,T)$ from (35). From lemma 2 we have that

$$\begin{aligned}\langle \Gamma(u_0), u_0\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} &= \langle -iw(0), u_0\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\ &= \langle \Lambda(u|_\omega), u|_\omega\rangle_{L^2(0,T;(H^1(\omega))')\times L^2(0,T;H^1(\omega)))}\\ &= \|u\|^2_{L^2(0,T;,H^1(\omega))} \geq C\|u_0\|_{H^1(\Omega)}\end{aligned}$$

where we have used the observability inequality (34). Thus, from Lax Milgram's Theorem we deduce that Γ is an isomorphism and therefore, we obtain the existence of the sought control h_ω. □

Acknowledgements The research was supported by the project *Differential Equations, Dynamical Systems and Control* from the Union Iberoamericana de Universidades and project UNAM-PAPIIT IN109522.

Declarations

Competing Interests The authors have not disclosed any competing interests.

References

1. Imanuvilov, O., Yamamoto, M.: Global Lipschitz stability in an inverse hyperbolic problem by interior observations. Inverse Probl. **17**(4), 717–728 (2001). https://doi.org/10.1088/0266-5611/17/4/310
2. Araruna, F., Cerpa, E., Mercado, A., Santos, M.: Internal null controllability of a linear Schrödinger-KdV system on a bounded interval. J. Differ. Equ. **260**(1), 653–687 (2016). https://doi.org/10.1016/j.jde.2015.09.009
3. Baudouin, L., Puel, J.: Uniqueness and stability in an inverse problem for the Schrödinger equation. Inverse Probl. **18**(6), 1537–1554 (2002). https://doi.org/10.1088/0266-5611/18/6/307

4. Aniţa, S., Barbu, V.: Null controllability of nonlinear convective heat equations. ESAIM Control Optim. Calc. Var. **5**, 157–173 (2000). https://doi.org/10.1051/cocv:2000105
5. Doubova, A., Fernández-Cara, E., González-Burgos, M., Zuazua, E.: On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. SIAM J. Control. Optim. **41**(3), 798–819 (2002). https://doi.org/10.1137/S0363012901386465
6. Fernández-Cara, E., Zuazua, E.: Null and approximate controllability for weakly blowing up semilinear heat equations. Ann. Inst. Henri Poincaré **17**(5), 583–616 (2000). https://doi.org/10.1016/S0294-1449(00)00117-7
7. Micu, S., Zuazua, E.: On the lack of null-controllability of the heat equation on the half space. Port. Math. (N.S.) **58**(1), 1–24 (2001)
8. Micu, S., Zuazua, E.: On the lack of null-controllability of the heat equation on the half-line. Trans. Am. Math. Soc. **353**(4), 1635–1659 (2001). https://doi.org/10.1090/S0002-9947-00-02665-9
9. Cabanillas, V., De Menezes, S.S.B., Zuazua, E.E.: Null controllability in unbounded domains for the semilinear heat equation with nonlinearities involving gradient terms. J. Optim. Theory Appl. **110**(2), 245–264 (2001). https://doi.org/10.1023/A:1017515027783
10. Cannarsa, P., Martinez, P., Vancostenoble, J.: Null controllability of the heat equation in unbounded domains by a finite measure control region. ESAIM Control Optim. Calc. Var. **10**, 381–408 (2004). https://doi.org/10.1051/cocv:2004010
11. González-Burgos, M., Luz de Teresa, L.: Some results on controllability for linear and nonlinear heat equations in unbounded domains. Adv. Differ. Equ. **12**(11), 1201–1240 (2007)
12. Cazenave, T., Haraux, A.: An Introduction to Semilinear Evolution Equations. Trans. by Yvan Martel., vol. 13. Clarendon Press, Oxford (1998)