CLASSIFICATION OF FOUR QUBIT STATES AND THEIR STABILISERS UNDER SLOCC OPERATIONS

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Abstract. We classify four qubit states under SLOCC operations, that is, we classify the orbits of the group $\mathrm{SL}(2,\mathbb{C})^4$ on the Hilbert space $\mathcal{H}_4 = (\mathbb{C}^2)^{\otimes 4}$. We approach the classification by realising this representation as a symmetric space of maximal rank. We first describe general methods for classifying the orbits of such a space. We then apply these methods to obtain the orbits in our special case, resulting in a complete and irredundant classification of $\mathrm{SL}(2,\mathbb{C})^4$-orbits on $\mathcal{H}_4$. It follows that an element of $(\mathbb{C}^2)^{\otimes 4}$ is conjugate to an element of precisely 87 classes of elements. Each of these classes either consists of one element or of a parametrised family of elements, and the elements in the same class all have equal stabiliser in $\mathrm{SL}(2,\mathbb{C})^4$. We also present a complete and irredundant classification of elements and stabilisers up to the action of $\mathrm{Sym}_4 \ltimes \mathrm{SL}(2,\mathbb{C})^4$ where Sym_4 permutes the four tensor factors of $(\mathbb{C}^2)^{\otimes 4}$.

1. Introduction

Entanglement is a fundamental notion in Quantum Information Theory (QIT). The beginning of the XXI[st] century has witnessed many efforts and advances in understanding the nature of entanglement (see the review papers [1,2]). Since entangled states lie at the core of quantum-enhanced applications it is crucially important to know which of these states are *equivalent*, in the sense that they are capable of performing the same QIT tasks almost equally well. Therefore, the classification of the entanglement of pure multipartite quantum states under the group of reversible Stochastic Local Quantum Operations assisted by Classical Communication (SLOCC) is nowadays one of the most prominent challenges in QIT (see [2]).

Since entanglement is deeply related to the non-local properties of a state, its intrinsic nature cannot be affected by local quantum operations, implemented by the SLOCC group [3, 4], which provides the most general local operations that can be implemented without deteriorating the quantum correlations shared by spatially separated physical systems. As mentioned above, two states belonging to the same entanglement class would be able to perform the same tasks, because one should be obtained with nonzero probability from the other using local invertible operations. Group theoretically, SLOCC equivalence classes on n-qubit states are $\mathrm{SL}(2,\mathbb{C})^n$-orbits in the space $\mathcal{H}_n = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (n factors $\mathbb{C}^2$).

SLOCC classifications for $n = 2$ and $n = 3$ are easily determined, yielding two and six SLOCC orbits for 2- and 3-qubit states, respectively. In particular, for what concerns the case of an entangled pure state of two qubits ($n = 2$), it is well-known that it can be converted to the singlet state by SLOCC operations [5]. For what concerns three entangled qubits ($n = 3$), it was proved in a series of works [4,6,7] that any state can be converted by SLOCC operations either to the GHZ-state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, or to the W-state $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |001\rangle)$, thus yielding to two inequivalent ways of entangling three qubits. In general, the GHZ (Greenberger-Horne-Zeilinger)-state is considered as the state with the genuine tripartite entanglement, whereas the W-state enjoys the peculiar property of having the maximal expected amount of twopartite entanglement if one party is traced out [4].

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For n qubits with $n \geqslant 4$ uncountably many SLOCC classes arise [4]. The case of four qubits ($n = 4$) has been the subject of a number of studies; without claim to completeness we mention [7–16]. Here we cannot review all these publications; we just mention the following.

• Verstraete et al. [7] considered the orbits of $\mathrm{SL}(2,\mathbb{C})^4$ on $\mathcal{H}_4$ and also allowed for permutations of the qubits, that is, they considered the action of $\mathcal{S} = \mathrm{Sym}_4 \ltimes \mathrm{SL}(2,\mathbb{C})^4$. Their main result is a list of nine classes such that each $\mathcal{S}$-orbit has a point in exactly one of the classes; it may happen that different elements of the same class are $\mathcal{S}$-conjugate. In [8] this classification was again derived and corrected.

• Wallach [16] also considered the $\mathrm{SL}(2,\mathbb{C})^4$-orbits in $\mathcal{H}_4$. His methods are based on results of Kostant & Rallis [17]. One of the main results is the statement that there are 90 "types" of orbits.

In this paper we present a classification of the orbits of $\mathrm{SL}(2,\mathbb{C})^4$ on $\mathcal{H}_4$. The method that we use is similar to the one employed by Wallach [16]. However, we also use concepts and methods introduced by Vinberg [18], and employ a similar scheme as used by Vinberg & Elashvili [19]. The main idea is to realise the representation of $\mathrm{SL}(2,\mathbb{C})^4$ on $\mathcal{H}_4$ using a symmetric pair of maximal rank corresponding to the simple Lie algebra of type D_4. This yields a Jordan decomposition of the elements of $\mathcal{H}_4$, and allows partitioning its elements and $\mathrm{SL}(2,\mathbb{C})^4$-orbits into three classes: semisimple, nilpotent and mixed. The nilpotent orbits can be classified by general methods such as the ones described in [20,21]. The semisimple orbits are classified by exhibiting a Cartan subspace and studying the action of the (finite) Weyl group on this space. The mixed orbits are classified by listing the nilpotent orbits in the centraliser of a semisimple element. More specifically we have the following:

• There are 31 nilpotent orbits with representatives given in Table 7. This has been proved in [10] by the Kostant-Sekiguchi correspondence; it can also be derived using Vinberg's method of carrier algebras [20]. In the remainder of this paper we will therefore not discuss the nilpotent case further.

• There are 10 parametrised classes of nonzero semisimple elements, as given in Table 2. Every semisimple element is $\mathrm{SL}(2,\mathbb{C})^4$-conjugate to an element in precisely one of these classes. For each class we explicitly determine a finite group Γ with the property that two elements in the class are $\mathrm{SL}(2,\mathbb{C})^4$-conjugate if and only if they are Γ-conjugate. Elements of different classes are not $\mathrm{SL}(2,\mathbb{C})^4$-conjugate. Furthermore, the elements of a class all have the *same* stabiliser in $\mathrm{SL}(2,\mathbb{C})^4$.

• For each semisimple class we explicitly list representatives of the orbits of mixed type whose semisimple part comes from the given class, see Theorem 3.7. This amounts to listing the possible nilpotent parts up to the action of the centraliser of the semisimple part.

This yields the following theorem.

Theorem 1.1. *There are 87 classes of elements of $\mathcal{H}_4$: 31 classes consist of a single nilpotent element, 10 classes consist of semisimple elements, and 46 classes consist of mixed elements. Each element of $\mathcal{H}_4$ is $\mathrm{SL}(2,\mathbb{C})^4$-conjugate to an element of precisely one class; elements of a class all have the same stabiliser in $\mathrm{SL}(2,\mathbb{C})^4$.*

In particular, our results yield the first complete and irredundant classification of the $\mathrm{SL}(2,\mathbb{C})^4$-orbits on $\mathcal{H}_4$; this follows from Theorem 3.2 (semisimple), Theorem 3.7 (mixed), Table 7 (nilpotent), together with Remarks 3.1 and 3.3. We also compare our classifications with those of Verstraete et al. [7] and Chterental & Djoković [8], and we present the first complete and irredundant classification of $(\mathrm{Sym}_4 \ltimes \mathrm{SL}(2,\mathbb{C})^4)$-orbits in $\mathcal{H}_4$, together with their stabilisers.

In [22] it is argued that in many contexts it is important to determine the stabiliser (also called *group of local symmetries*) of a given element in $\mathcal{H}_n$. As a corollary to Theorem 1.1, it follows that the stabiliser of any element of $\mathcal{H}_4$ is conjugate in $\mathrm{SL}(2,\mathbb{C})^4$ to one of 87 stabilisers. In [22] the orbits of $\psi, \phi \in \mathcal{H}_4$ are defined to have the same *type* if the stabilisers of ψ and ϕ in $\mathrm{SL}(2,\mathbb{C})^4$ are conjugate in that group. Hence we conclude that there are at most 87 types of orbits.

When we consider the action of $\mathrm{Sym}_4 \ltimes \mathrm{SL}(2,\mathbb{C})^4$ then every element in $\mathcal{H}_4$ is conjugate to an element of precisely one of 27 classes. As mentioned above we explicitly determine the stabilisers in $\mathrm{SL}(2,\mathbb{C})^4$ for the elements in our classification of $(\mathrm{Sym}_4 \ltimes \mathrm{SL}(2,\mathbb{C})^4)$-orbits, see Tables 3, 6, and 8. From Table 3, Row 1, it is seen that the stabiliser of a generic element is a finite group of order 32. For n-qubits with $n \geqslant 5$ the situation is completely different, as in those cases the stabiliser of a generic element is trivial [22].

1.1. **Structure of this paper.** In Section 2 we describe our general approach to classifying the orbits in symmetric spaces of maximal rank. These are certain representations of reductive algebraic groups that arise from a $\mathbb{Z}/2\mathbb{Z}$-grading of a semisimple Lie algebra. In Section 3.1 we show how the representation of $\mathrm{SL}(2,\mathbb{C})^4$ arises in this way. We then apply our methods to derive a classification of the semisimple orbits, see Theorem 3.2. The orbits of mixed elements are determined in Section 3.2, see Theorem 3.7. In Section 4 we compare our classifications with those of Verstraete et al. [7] and Chterental & Djokovič [8], and we present a complete classification of $(\mathrm{Sym}_4 \ltimes \mathrm{SL}(2,\mathbb{C})^4)$-orbits in $\mathcal{H}_4$. In Section 5 we investigate the ring of invariants for our main example. All explicit calculations have been done in GAP [23] using the GAP packages SLA and Singular; the latter provides an interface to the algebra software Singular [24].

2. Orbits in symmetric spaces of maximal rank

We let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and suppose that $\mathfrak{g}$ is furnished with a $\mathbb{Z}/2\mathbb{Z}$-grading, that is, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \leqslant \mathfrak{g}_{i+j \bmod 2}$ for all i, j; in particular, $\mathfrak{g}_0$ is a subalgebra that acts on $\mathfrak{g}_1$. It is also said that $(\mathfrak{g}, \mathfrak{g}_0)$ is a *symmetric pair*. Associated with this grading is an automorphism $\theta\colon \mathfrak{g} \to \mathfrak{g}$ of order 2 such that each $\mathfrak{g}_i$ is the $(-1)^i$-eigenspace of θ.

Let G be the adjoint group of $\mathfrak{g}$, that is, the identity component of the automorphism group of $\mathfrak{g}$; the Lie algebra of G is $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g} \cong \mathfrak{g}$. Let G_0 be the connected algebraic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}_0$; note that $G_0 \leqslant G^\theta = \{g \in G : \theta g = g\theta\}$. The group G_0 acts naturally on $\mathfrak{g}_1$ and we are interested in listing the orbits of G_0 in $\mathfrak{g}_1$; the study of these orbits was initiated by Kostant & Rallis [17]. The group G_0 with its action on $\mathfrak{g}_1$ is a special case of a θ-group, a concept introduced by Vinberg [18, 20], who also studied the orbits of G_0 on $\mathfrak{g}_1$. Part of Vinberg's theory is covered by a recent book by Wallach [25]; in the sequel we will mainly refer to this book, although all cited results can also be found in the papers by Kostant & Rallis and Vinberg.

A *Cartan subspace* of the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is a subspace of $\mathfrak{g}_1$ maximal with respect to the property that its elements are commuting semisimple elements. By [25, Corollary 3.55] any two Cartan subspaces are G_0-conjugate. In particular, they have the same dimension, which is called the *rank* of $(\mathfrak{g}, \mathfrak{g}_0)$. Here we assume that a Cartan subspace of $\mathfrak{g}_1$ is also a Cartan subalgebra of $\mathfrak{g}$, that is, we assume that the rank of $(\mathfrak{g}, \mathfrak{g}_0)$ is equal to the rank of the root system of $\mathfrak{g}$. Up to conjugacy, there exists a unique symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ of maximal rank, which can be constructed as follows: The split real form $\mathfrak{g}_{\mathbb{R}}$ with complexification $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$, and letting $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be the complexifications of $\mathfrak{k}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$, respectively, we obtain the symmetric pair of maximal rank. In Table 1 we list the symmetric spaces of maximal rank corresponding to the simple complex Lie algebras.

Recall that $x \in \mathfrak{g}$ is semisimple (nilpotent) if the adjoint map $\mathrm{ad}x\colon \mathfrak{g} \to \mathfrak{g}$ is a semisimple (nilpotent) endomorphism. The study of the G_0-orbits in $\mathfrak{g}_1$ starts with the following well-known lemma on the Jordan decomposition; we refer to [17, Proposition 3] for a proof.

Lemma 2.1. *If $x \in \mathfrak{g}_1$, then $x = s + n$ for unique semisimple $s \in \mathfrak{g}_1$ and nilpotent $n \in \mathfrak{g}_1$ with $[n, s] = 0$.*

Accordingly, the G_0-orbits in $\mathfrak{g}_1$ split into three classes: the nilpotent orbits (that consist entirely of nilpotent elements), the semisimple orbits (that consist of semisimple elements) and the mixed orbits (consisting of elements that are neither semisimple nor nilpotent). Methods for listing the nilpotent orbits have been developed by Vinberg [20] and de Graaf [27], so we will not comment on that here. Instead, we will describe how to list the semisimple and mixed orbits. Our methods for that are based on (and very similar to) methods for an analogous problem developed by Vinberg & Èlašvili [19]. We note that in [30] and [31] the authors study the orbits in the symmetric spaces of maximal rank of types E_7 and E_8, respectively, using methods that are also based on the approach in [19].

We continue with a subsection that contains some results on Weyl groups. Subsequently, we discuss semisimple orbits and mixed orbits. Throughout, we use standard notation for Lie algebras and their related combinatorial data (Cartan subalgebras, root systems, Weyl groups, etc), and we refer to the books of Erdmann & Wildon [28] or Humphreys [29] for more details and background information.

type	$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	degrees
A_{n-1}	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})$	$S_0^2\mathbf{n}$	$2,3,4,\ldots,n$
B_n	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})\oplus\mathfrak{so}(n+1,\mathbb{C})$	$\mathbf{n}\otimes(\mathbf{n+1})$	$2,4,6,\ldots,2n$
C_n	$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})$	$S^2\mathbf{n}\oplus\overline{S^2\mathbf{n}}$	$2,4,6,\ldots,2n$
D_n	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})\oplus\mathfrak{so}(n,\mathbb{C})$	$\mathbf{n}\otimes\mathbf{n}$	$n,2,4,6,\ldots,2n-2$
E_6	$\mathrm{E}_6(\mathbb{C})$	$\mathfrak{sp}(8,\mathbb{C})$	$\wedge_0^4\mathbf{8}$	$2,5,6,8,9,12$
E_7	$\mathrm{E}_7(\mathbb{C})$	$\mathfrak{sl}(8,\mathbb{C})$	$\wedge^4\mathbf{8}$	$2,6,8,10,12,14,18$
E_8	$\mathrm{E}_8(\mathbb{C})$	$\mathfrak{so}(16,\mathbb{C})$	$\mathbf{128}$ semispinor	$2,8,12,14,18,20,24,30$
F_4	$\mathrm{F}_4(\mathbb{C})$	$\mathfrak{sp}(6,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$(\wedge_0^3\mathbf{6})\otimes\mathbf{2}$	$2,6,8,12$
G_2	$\mathrm{G}_2(\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$S^3\mathbf{2}\otimes\mathbf{2}$	$2,6$

TABLE 1. Symmetric spaces of maximal rank in simple Lie algebras defined over $\mathbb{C}$. The fourth column displays the structure of $\mathfrak{g}_1$ as $\mathfrak{g}_0$-module; here we denote an irreducible module by its dimension, and a notation like $\wedge_0^4\mathbf{8}$ indicates the quotient of $\wedge^4\mathbf{8}$ by the trivial 1-dimensional module. The last column has the degrees of the homogeneous invariant polynomials that generate the invariant ring, cf. [26, Table 1, p. 59]

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2.1. **Root subsystems.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$ and corresponding root system Φ and Weyl group W. We write $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$ for the corresponding root space decomposition. A subset $\Pi\subseteq\Phi$ is a *root subsystem* if for $\alpha,\beta\in\Pi$ we have $-\alpha\in\Pi$ and, if $\alpha+\beta\in\Phi$, then $\alpha+\beta\in\Pi$. For $\alpha\in\Phi$ let $s_\alpha\in W$ be the corresponding reflection. The group W acts on $\mathfrak{h}$ by $s_\alpha(h)=h-\alpha(h)h_\alpha$, where h_α is the unique element of $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]\leqslant\mathfrak{h}$ with $\alpha(h_\alpha)=2$ (see [21, Remark 2.9.9]). If $w\in W$, $h\in\mathfrak{h}$, and $\alpha\in\Phi$, then we sometimes abbreviate $wh=w(h)$ and $w\alpha=w(\alpha)$. We have the following property

$$\alpha(s_\beta(h))=s_\beta(\alpha)(h)\qquad\text{for all }\alpha,\beta\in\Phi\text{ and }h\in\mathfrak{h},\tag{2.1}$$

which implies that

$$w(\alpha)(h)=\alpha(w^{-1}(h))\qquad\text{for all }\alpha\in\Phi, w\in W\text{ and }h\in\mathfrak{h}.\tag{2.2}$$

For $p\in\mathfrak{h}$ we define Φ_p to be the annihilator of p in Φ, that is,

$$\Phi_p=\{\alpha\in\Phi:\alpha(p)=0\}.$$

It is clear that Φ_p is a root subsystem of Φ, but not all root subsystems arise in this way. The next lemma gives a criterion to decide whether a root subsystem is of the form Φ_p for some $p\in\mathfrak{h}$; recall that a root subsystem $\Psi\subseteq\Phi$ is *complete* if it is not properly contained in a root subsystem of Φ of the same rank.

Lemma 2.2. *Let $\Psi\subseteq\Phi$ be a root subsystem. There exists $p\in\mathfrak{h}$ with $\Psi=\Phi_p$ if and only if Ψ is complete.*

Proof. We first show that a root subsystem $\Pi\subseteq\Phi$ is complete if and only if $V_\Pi\cap\Phi=\Pi$, where V_Π is the $\mathbb{Q}$-space spanned by Π: If Π is complete, then $V_\Pi\cap\Phi=\Pi$ since $V_\Pi\cap\Phi$ is a root subsystem of Φ containing Π of the same rank as Π. For the converse suppose that $V_\Pi\cap\Phi=\Pi$. If $\Pi'\subseteq\Phi$ is a root subsystem containing Π and of the same rank as Π, then $V_{\Pi'}$ contains V_Π and both spaces are of the same dimension, hence they are equal. Thus, Π' is contained in $V_{\Pi'}\cap\Phi=V_\Pi\cap\Phi=\Pi$, so $\Pi'=\Pi$.

Suppose that $\Psi=\Phi_p$. If $\beta\in V_\Psi\cap\Phi$ then β is a linear combination of elements of Φ_p, hence $\beta(p)=0$ and $\beta\in\Phi_p=\Psi$. It follows that Ψ is complete. For the converse, suppose that Ψ is complete. If Ψ and Φ have equal rank, then $\Psi=\Phi$ and $\Psi=\Phi_0$. Now suppose that the rank s of Ψ is less then the rank of Φ. Define $\mathfrak{u}=\{p\in\mathfrak{h}:\alpha(p)=0\text{ for all }\alpha\in\Psi\}$ and $\mathfrak{u}^\circ=\{p\in\mathfrak{u}:\beta(p)\neq0\text{ for all }\beta\in\Phi\setminus\Psi\}$; note that $\dim\mathfrak{u}=\dim\mathfrak{h}-s>0$. If $\beta\in\Phi\setminus\Psi$, then β is not contained in V_Ψ by our claim above. Thus, the space spanned by β and Ψ has dimension $s+1$ and so $\{u\in\mathfrak{u}:\beta(u)=0\}$ has dimension $\dim\mathfrak{h}-s-1$. This shows that β is nonzero on $\mathfrak{u}$, hence the kernel of every $\beta\in\Phi\setminus\Psi$ on $\mathfrak{u}$ has codimension 1. Since any finite union of codimension 1 subspaces of $\mathfrak{u}$ does not cover $\mathfrak{u}$ it follows that there is some $p \in \mathfrak{u}$ with $\beta(p) \neq 0$ for all $\beta \in \Phi \setminus \Psi$, thus $\mathfrak{u}^\circ \neq \emptyset$. Furthermore, for any $p \in \mathfrak{u}^\circ$ we have $\Psi = \Phi_p$. $\square$

For a root subsystem $\Psi \subseteq \Phi$ we define

$$\mathfrak{h}_\Psi^\circ = \{p \in \mathfrak{h} : \Phi_p = \Psi\};$$

note that this is the set of all $p \in \mathfrak{h}$ such that $\alpha(p) = 0$ for every $\alpha \in \Psi$ and $\beta(p) \neq 0$ for every $\beta \in \Phi \setminus \Psi$.

Lemma 2.3. *Let $\Psi, \Pi \subseteq \Phi$ be root subsystems and $u \in W$. Then $\Pi = u\Psi$ if and only if $\mathfrak{h}_\Pi^\circ = u\mathfrak{h}_\Psi^\circ$.*

Proof. First assume that $\Pi = u\Psi$, and note that $u(\Phi \setminus \Psi) = \Phi \setminus \Pi$. Let $h \in \mathfrak{h}_\Psi^\circ$. If $\beta \in \Pi$, then $\beta = u\alpha$ for some $\alpha \in \Psi$, and (2.2) shows that $\beta(uh) = (u\alpha)(uh) = \alpha(h) = 0$. If $\beta \in \Phi \setminus \Pi$, then $\beta = u\alpha$ for some $\alpha \in \Phi \setminus \Psi$, and (2.2) yields $\beta(uh) = \alpha(h) \neq 0$. This shows that $uh \in \mathfrak{h}_\Pi^\circ$. Conversely, let $h \in \mathfrak{h}_\Pi^\circ$. Since $\Psi = u^{-1}\Pi$, the previous argument shows that $u^{-1}h \in \mathfrak{h}_\Psi^\circ$, so $h \in u\mathfrak{h}_\Psi^\circ$. Thus, $\mathfrak{h}_\Pi^\circ = u\mathfrak{h}_\Psi^\circ$, as claimed.

Now suppose that $\mathfrak{h}_\Pi^\circ = u\mathfrak{h}_\Psi^\circ$. The first part of the proof shows that $u\mathfrak{h}_\Psi^\circ = h_{u\Psi}^\circ$, so our assumption is that $\mathfrak{h}_\Pi^\circ = \mathfrak{h}_{u\Psi}^\circ$. The definition of $\mathfrak{h}_\Pi^\circ$ immediately implies that $u\Psi = \Pi$, as claimed. $\square$

By definition, each $p \in \mathfrak{h}$ lies in $\mathfrak{h}_{\Phi_p}^\circ$. Thus, if $\Phi_1, \ldots, \Phi_r$ are, up to W-conjugacy, all the complete root subsystems of Φ (including $\emptyset$ and Φ itself), then every W-orbit in $\mathfrak{h}$ has a point in a unique set $\mathfrak{h}_{\Phi_i}^\circ$.

For a fixed complete root subsystem $\Psi \subseteq \Phi$ we now characterise the W-conjugacy of elements in $\mathfrak{h}_\Psi^\circ$. The proof of the next lemma follows well-known ideas (see, for example, [26, §1.12]), however, we could not find an exact reference in the literature.

Lemma 2.4. *Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a basis of simple roots of Φ.*

a) *Every $p \in \mathfrak{h}$ is W-conjugate to an element in $C = \{h \in \mathfrak{h} : \alpha_i(h) \geqslant 0 \text{ for all } i \in \{1, \ldots, \ell\}\}$ where we write $z > 0$ for a complex number $z = x + \imath y$ with $x, y \in \mathbb{R}$ if either $x > 0$, or $x = 0$ and $y > 0$.*
b) *If $p \in \mathfrak{h}$, then the stabiliser $W_p = \{w \in W : w(p) = p\}$ is generated by $\{s_\alpha : \alpha \in \Phi_p\}$.*

Proof. Throughout the proof we abbreviate $h_i = h_{\alpha_i}$.

a) This is standard: We construct a sequence $k_1 = p, k_2, k_3 \ldots$ of W-conjugate elements until we find some $k_m \in C$. If k_n is defined, but $k_n \notin C$, then $\alpha_i(k_n) < 0$ for some i, and we set $k_{n+1} = s_i(k_n) = k_n + c_i h_i$ with $c_i = -\alpha_i(k_n) > 0$. Thus, by construction, all elements in the sequence $k_1, k_2, \ldots$ are distinct; since the W-orbit of p is finite, we will eventually construct an element $k_m \in C$.

b) We first show that if $p \in C$, then W_p is generated by the s_{α_i} such that $\alpha_i(p) = 0$. By definition of C, we have $\alpha(p) \geqslant 0$ for every positive root and $\alpha(p) \leqslant 0$ for every negative one. Now let $w = s_{i_1} \cdots s_{i_t} \in W_p$ be a reduced expression; the claim follows if each $s_{i_k} \in W_p$. Write $p_{t+1} = p$ and $p_j = s_{i_j} \cdots s_{i_t}(p)$ for $j \in \{1, \ldots, t\}$. It follows from (2.1) that

$$(*) \quad \alpha_{i_{j-1}}(p_j) = s_{i_t} \cdots s_{i_j}(\alpha_{i_{j-1}})(p) \quad \text{ for all } j \in \{2, \ldots, t+1\}.$$

By [29, Corollary 10.2], each $s_{i_t} \ldots s_{i_{j-1}}(\alpha_{i_{j-1}})$ is a negative root, so $s_{i_t} \ldots s_{i_{j-1}}(\alpha_{i_{j-1}})(p) \leqslant 0$ for $p \in C$; since $s_{i_{j-1}}(\alpha_{i_{j-1}}) = -\alpha_{i_{j-1}}$, we deduce from $(*)$ that $c_{j-1} = \alpha_{i_{j-1}}(p_j)$ satisfies $c_{j-1} \geqslant 0$. This shows that $p_{j-1} = s_{i_{j-1}}(p_j) = p_j - c_{j-1}h_{i_{j-1}}$ with $c_{j-1} \geqslant 0$. Since this holds for every j, the equality $w(p) = p$ implies that all $c_{j-1} = 0$, showing that each $p_j = p$, hence $\alpha_{i_{j-1}}(p) = 0$, and so each $s_{i_{j-1}} \in W_p$.

Now let $p \in \mathfrak{h}$ and choose $w \in W$ with $w(p) \in C$. The above shows that $W_p = w^{-1}W_{w(p)}w$ is generated by $\{s_{w^{-1}(\alpha)} : \alpha \in \Phi_{w(p)}\}$; here we use $w^{-1}s_\alpha w = s_{w^{-1}(\alpha)}$, see [29, Lemma 9.2]. On the other hand, $\Phi_{w(p)} = w(\Phi_p)$ by (2.2), so W_p is generated by $\{s_{w^{-1}(\alpha)} : \alpha \in w(\Phi_p)\} = \{s_\alpha : \alpha \in \Phi_p\}$. $\square$

For a root subsystem $\Psi \subseteq \Phi$ we define

$$W_\Psi = \langle s_\alpha : \alpha \in \Psi \rangle \quad \text{and} \quad \Gamma_\Psi = N_W(W_\Psi)/W_\Psi.$$

Let $\Psi \subseteq \Phi$ be a complete subsystem. The previous lemma shows that the stabiliser W_p of $p \in \mathfrak{h}_\Psi^\circ$ in W is generated by all s_α with $\alpha \in \Phi_p$. By definition, $\Phi_p = \Psi$, and so $W_p = W_\Psi$. In particular, for every

$w \in N_W(W_\Psi)$ we have $wW_pw^{-1} = W_p$, and if $q = wp$, then $W_q = W_p$, or $W_{\Phi_q} = W_{\Phi_p}$, which implies that $\Phi_q = \Phi_p = \Psi$, and so $q \in \mathfrak{h}_\Psi^\circ$. We conclude that Γ_Ψ acts naturally on $\mathfrak{h}_\Psi^\circ$ and we have the following:

Proposition 2.5. *Let $\Psi \subseteq \Phi$ be a complete subsystem. Two elements $p, q \in \mathfrak{h}_\Psi^\circ$ are W-conjugate if and only if they are Γ_Ψ-conjugate.*

Proof. If $q = wp$ with $w \in W$, then $W_q = wW_pw^{-1}$; since $W_q = W_p = W_\Psi$, we have $w \in N_W(W_\Psi)$, so p and q are Γ_Ψ-conjugate. The converse is obvious. □

2.2. **Semisimple orbits.** We revert back to the set up from the start of the section, that is we consider a symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ of maximal rank. We are interested in listing the semisimple G_0-orbits in $\mathfrak{g}_1$. Recall that a Cartan subspace of $\mathfrak{g}$ is a maximal subspace of $\mathfrak{g}_1$ consisting of commuting semisimple elements, and any two Cartan subspaces are G_0-conjugate, see [25, Corollary 3.55]. Thus, every semisimple G_0-orbit in $\mathfrak{g}_1$ intersects any given Cartan subspace nontrivially. For a Cartan subspace $\mathfrak{h} \leqslant \mathfrak{g}_1$ let

$$W_\mathfrak{h} = N_{G_0}(\mathfrak{h})/Z_{G_0}(\mathfrak{h})$$

be the *little Weyl group*, also called the *Weyl group of the graded Lie algebra* $\mathfrak{g}$. This group was studied in detail by Vinberg [18], who proved (among other things) that two elements of $\mathfrak{h}$ are G_0-conjugate if and only if they are $W_\mathfrak{h}$-conjugate, see [18, Theorem 2] or [25, Proposition 3.61].

From now on we fix a Cartan subspace $\mathfrak{h}$ in $\mathfrak{g}_1$. By the remarks above, the classification of the semisimple G_0-orbits in $\mathfrak{g}_1$ is reduced to the classification of the $W_\mathfrak{h}$-orbits in $\mathfrak{h}$. Let Φ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let W be the Weyl group of Φ. By definition, $W_\mathfrak{h}$ is naturally a subgroup of $N_G(\mathfrak{h})/Z_G(\mathfrak{h})$; the next result shows that we actually have equality. This will allow us to identify W and $W_\mathfrak{h}$.

Lemma 2.6. *We have $W_\mathfrak{h} = N_G(\mathfrak{h})/Z_G(\mathfrak{h}) \cong W$.*

Proof. It is well-known that $W \cong N_G(\mathfrak{h})/Z_G(\mathfrak{h})$, see [21, Lemma 5.2.22]. To prove $W_\mathfrak{h} = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ we fix $\alpha \in \Phi$ and show that $W_\mathfrak{h}$ contains an element that acts as s_α on $\mathfrak{h}$. Let $x \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$; applying θ (the automorphism of $\mathfrak{g}$ defining the grading) to the equality $[h, x] = \alpha(h)x$, we see that $\theta(x) \in \mathfrak{g}_{-\alpha}$. Let $\mathfrak{u}(\alpha)$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, so $\mathfrak{u}(\alpha) \cong \mathfrak{sl}(2, \mathbb{C})$ is stable under θ. Let $U(\alpha)$ denote the connected subgroup of G with Lie algebra $\mathrm{ad}_\mathfrak{g}\mathfrak{u}$, that is, $U(\alpha)$ is generated by $\exp(\mathrm{ad}_\mathfrak{g} tx_\alpha)$ and $\exp(\mathrm{ad}_\mathfrak{g} tx_{-\alpha})$ with $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ and $t \in \mathbb{C}$. The automorphism group of $\mathfrak{sl}(2, \mathbb{C})$ is the adjoint group $\mathrm{PSL}(2, \mathbb{C})$. By general theory of semisimple algebraic groups, see [21, p. 182], there is a surjective morphism of algebraic groups $U(\alpha) \to \mathrm{PSL}(2, \mathbb{C})$. Let $g_\alpha \in U(\alpha)$ denote an inverse image under this morphism of the restriction $\theta|_{\mathfrak{u}(\alpha)} \in \mathrm{PSL}(2, \mathbb{C})$. If $g \in U(\alpha)$, then $g_\alpha g g_\alpha^{-1} = \theta g \theta^{-1}$: this is easy to check for generators $\exp(\mathrm{ad}_\mathfrak{g} tx_{\pm\alpha})$, the general case follows from that. If we take $g = g_\alpha$, then we get $\theta g_\alpha \theta^{-1} = g_\alpha$, which proves that $g_\alpha \in G_0$. Note that $\mathfrak{h} = \langle h_\alpha \rangle \oplus \hat{\mathfrak{h}}_\alpha$ where $\hat{\mathfrak{h}}_\alpha = \{x \in \mathfrak{h} : \alpha(x) = 0\}$, so all elements of $U(\alpha)$ act as the identity on $\hat{\mathfrak{h}}_\alpha$. Furthermore, $h_\alpha \in \mathfrak{u}(\alpha)$, so $g_\alpha(h_\alpha) = \theta(h_\alpha) = -h_\alpha$. It follows that g_α acts as s_α on $\mathfrak{h}$. □

Now our procedure for obtaining a classification of the semisimple G_0-orbits in $\mathfrak{g}_1$ is as follows. Dynkin devised an algorithm to find the root subsystems of Φ up to W-conjugacy (see [32], and also [21, pp. 221]). We use this algorithm to compute all subsystems up to W-conjugacy and discard those that are not complete. We also add the empty set to the list. Let $\Pi_1, \ldots, \Pi_r$ denote the obtained subsystems. Then for each Π_i we compute the subspace

$$\mathfrak{h}_{\Pi_i} = \{p \in \mathfrak{h} : \alpha(p) = 0 \text{ for all } \alpha \in \Pi_i\}.$$

We call the r sets $\mathfrak{h}_{\Pi_i}^\circ$ the *canonical semisimple sets*. Note that $p \in \mathfrak{h}_{\Pi_i}$ lies in $\mathfrak{h}_{\Pi_i}^\circ$ if and only if $\beta(p) \neq 0$ for all $\beta \in \Phi \setminus \Pi_i$; this leads to a finite number of linear conditions for β. Multiplying them, we obtain a polynomial function F_i on $\mathfrak{h}_{\Pi_i}$ such that $p \in \mathfrak{h}_{\Pi_i}$ lies in $\mathfrak{h}_{\Pi_i}^\circ$ if and only if $F_i(p) \neq 0$. We determine the groups Γ_{Π_i} by computing the normalisers $N_W(W_{\Pi_i})$. (In our main example discussed below we construct the quotient Γ_{Π_i} as a complement to W_{Π_i} in $N_W(W_{\Pi_i})$.) A set Σ of semisimple G_0-orbit representatives in $\mathfrak{g}_1$ can now be obtained by taking the union of the sets of Γ_{Π_i}-representatives in $\mathfrak{h}_{\Pi_i}^\circ$ for all i.

2.3. **Mixed orbits.** We now consider elements of *mixed type*, that is, $x \in \mathfrak{g}_1$ that is neither nilpotent nor semisimple. The investigation of such elements is based on their Jordan decomposition (see Lemma 2.1).

Let Σ be the set of semisimple orbit representatives from the previous section; we also use the sets $\mathfrak{h}^\circ_{\Pi_i}$ defined above. If $x = s + n$ is a mixed element in $\mathfrak{g}_1$, then there exists a unique $s' \in \Sigma$ such that $g(s) = s'$ for some $g \in G_0$, hence $g(x) = s' + g(n)$ with $g(n)$ nilpotent and $[s', g(n)] = 0$. Together with the uniqueness of the Jordan decomposition, this shows the following.

Lemma 2.7. *Every element of mixed type in $\mathfrak{g}_1$ is G_0-conjugate to an element in*

$$\mathcal{M} = \{s + n : s \in \Sigma \text{ and nonzero nilpotent } n \in \mathfrak{z}_{\mathfrak{g}_1}(s)\},$$

and $s + n, s' + n' \in \mathcal{M}$ are G_0-conjugate if and only if $s = s'$ and $n' = g(n)$ for some $g \in Z_{G_0}(s)$.

Because of this lemma we are interested in determining the centraliser $Z_{G_0}(s)$ of semisimple s. This requires the following preliminary lemma, which seems to be well-known; we include a proof because we could not find a precise reference in the literature.

Lemma 2.8. *Let $K \leqslant \mathrm{GL}(n, \mathbb{C})$ be a connected reductive algebraic group with semisimple Lie algebra $\mathfrak{k} \leqslant \mathfrak{gl}(n, \mathbb{C})$. If $x \in \mathfrak{k}$ is semisimple, then $Z_K(x) = \{g \in K : gxg^{-1} = x\}$ is connected.*

Proof. Let U be the smallest algebraic subgroup of K whose Lie algebra $\mathfrak{u}$ contains x; such a subgroup always exists and it is unique and connected, see [21, Theorem 4.1.5]. It follows from [21, Lemma 4.7.3] that $Z_K(U) = \{g \in K : gh = hg \text{ for all } u \in U\}$ equals $Z_K(\mathfrak{u}) = \{g \in K : gu = ug \text{ for all } u \in \mathfrak{u}\}$. Let A be the associative matrix algebra with identity generated by x. It follows from [21, Example 3.6.9] that the unit group A^* of A is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$ with Lie algebra A where the Lie bracket is given by the commutator. By [21, Theorem 4.1.5] we have $U \leqslant A^*$. Since elements in A are linear combinations of powers of x, it follows that every $w \in Z_K(x)$ also centralises A^*, in particular, $w \in Z_K(U)$, thus $Z_K(x) \leqslant Z_K(U) = Z_G(\mathfrak{u})$. Since $x \in \mathfrak{u}$, we have $Z_K(\mathfrak{u}) \leqslant Z_K(x)$, hence $Z_K(x) = Z_K(U)$. Since $U \leqslant A^*$ consists of commuting semisimple elements, $U \leqslant K$ is a subtorus; now it follows from [33, Corollary 8.13a)] that $Z_K(U)$ is connected. □

The next lemma shows that the determination of the centralisers for the infinitely many $p \in \Sigma$ can be reduced to a finite calculation: it suffices to consider one explicit element in each $\mathfrak{h}^\circ_{\Pi_i}$. For $x \in \mathfrak{g}_1$ we denote its centraliser in $\mathfrak{g}$ by $\mathfrak{z}_g(x)$.

Lemma 2.9. *If $x, y \in \mathfrak{h}^\circ_{\Pi_i}$, then $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(y)$ and $Z_{G_0}(x) = Z_{G_0}(y)$.*

Proof. Note that $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(y)$ as both equal $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Pi_i} \mathfrak{g}_\alpha$. Intersecting with $\mathfrak{g}_0$ yields $\mathfrak{z}_{\mathfrak{g}_0}(x) = \mathfrak{z}_{\mathfrak{g}_0}(y)$. As in the proof of Lemma 2.8, let $U_x, U_y \leqslant G$ be the minimal subtori whose Lie algebras $\mathfrak{u}_x$ and $\mathfrak{u}_y$ contain x and y, respectively; the proof also showed that $Z_G(x) = Z_G(U_x) = Z_G(\mathfrak{u}_x)$ and $Z_G(y) = Z_G(U_y) = Z_G(\mathfrak{u}_y)$ are both connected. Moreover, both groups have the same Lie algebra $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(y)$: consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ with differential $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. It follows from [21, Corollary 4.2.8] that $Z_G(x) = \{g \in G : \mathrm{Ad}(g)(x) = x\}$ has Lie algebra $\{y \in \mathfrak{g} : \mathrm{ad}y(x) = 0\} = \mathfrak{z}_{\mathfrak{g}}(x)$. Now [21, Theorem 4.2.2] shows that $Z_G(x) = Z_G(y)$, and so $Z_{G_0}(x) = Z_{G_0}(y)$. □

Remark 2.10. Now let $q_1, q_2 \in \mathfrak{h}^\circ_{\Pi_i}$, so $\mathfrak{z}_{\mathfrak{g}}(q_1) = \mathfrak{z}_{\mathfrak{g}}(q_2)$ and $Z_{G_0}(q_1) = Z_{G_0}(q_2)$. Write $\mathfrak{a} = \mathfrak{z}_{\mathfrak{g}}(q_1)$ and $A_0 = Z_{G_0}(q_1)$. We note that $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ is graded with each $\mathfrak{a}_i = \mathfrak{g}_i \cap \mathfrak{a}$, the Lie algebra of A_0 is $\mathfrak{a}_0$, and A_0 acts on $\mathfrak{a}_1$. Let $n_1, \ldots, n_s$ be representatives of the nilpotent A_0-orbits in $\mathfrak{a}_1$. Then the G_0-orbits of mixed elements with semisimple parts q_1 and q_2 have representatives $q_1 + n_i$ and $q_2 + n_i$, respectively, for $i \in \{1, \ldots, s\}$. In particular, the nilpotent parts of mixed elements with semisimple part in $\mathfrak{h}^\circ_{\Pi_i}$ do not depend on the choice of the particular semisimple element. In order to determine the nilpotent elements n_i, we first determine the nilpotent orbits of the identity component A_0° acting on $\mathfrak{a}_1$. This can be done by an algorithm that only works with the Lie algebra $\mathfrak{a}$, see [21, Section 8.4]. The fusion of these A_0°-orbits in A_0 can be decided by computing a set of representatives in A_0 of the the component group A_0/A_0°.

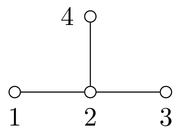
3. The orbits of $\mathrm{SL}(2,\mathbb{C})^4$ on $\mathcal{H}_4$

In this section we classify the orbits of the group

$$\widehat{G} = \mathrm{SL}(2,\mathbb{C})^4$$

acting on the space $\mathcal{H}_4 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. First, we show how this action comes from a symmetric pair as studied in the previous section.

Let $\mathfrak{g}$ be the simple Lie algebra of type D_4 defined over the complex numbers. Let Ψ denote its root system with respect to a fixed Cartan subalgebra $\mathfrak{t}$. Let $\gamma_1, \ldots, \gamma_4$ be a fixed choice of simple roots such that the Dynkin diagram of Ψ is labelled as follows



We now construct a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$: let $\mathfrak{g}_0$ be spanned by $\mathfrak{t}$ along with the root spaces $\mathfrak{g}_\gamma$, where $\gamma = \sum_i k_i\gamma_i$ has k_2 even, and let $\mathfrak{g}_1$ be spanned by those $\mathfrak{g}_\gamma$ where $\gamma = \sum_i k_i\gamma_i$ has k_2 odd. Let $\gamma_0 = \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4$ be the highest root of Ψ. The root system of $\mathfrak{g}_0$ is $\{\pm\gamma_0, \pm\gamma_1, \pm\gamma_3, \pm\gamma_4\}$, hence

$$\mathfrak{g}_0 \cong \mathfrak{sl}(2,\mathbb{C})^4 = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}).$$

Taking $-\gamma_0, \gamma_1, \gamma_3, \gamma_4$ as basis of simple roots of $\mathfrak{g}_0$ we have that $-\gamma_2$ is the highest weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$, which therefore is isomorphic to $\mathcal{H}_4 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We fix a basis $\{e_0, e_1\}$ of $\mathbb{C}^2$ and denote the basis elements of $\mathcal{H}_4$ by

$$|i_1 i_2 i_3 i_4\rangle = e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}.$$

Mapping any nonzero root vector in $\mathfrak{g}_{-\gamma_2}$ to $|0000\rangle$ extends uniquely to an isomorphism $\mathfrak{g}_1 \to \mathcal{H}_4$ of $\mathfrak{sl}(2,\mathbb{C})^4$-modules. We denote by G the adjoint group of $\mathfrak{g}$, and we write G_0 for the connected algebraic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}_0 \cong \mathfrak{sl}(2,\mathbb{C})^4$. The isomorphism $\mathfrak{sl}(2,\mathbb{C})^4 \to \mathfrak{g}_0$ lifts to a surjective morphism $\pi\colon \widehat{G} \to G_0$ of algebraic groups, which makes $\mathfrak{g}_1$ into a $\widehat{G}$-module isomorphic to $\mathcal{H}_4$.

It is well-known (cf. [7]), and we have verified by computer, that

$$u_1 = |0000\rangle + |1111\rangle, \quad u_2 = |0110\rangle + |1001\rangle, \quad u_3 = |0101\rangle + |1010\rangle, \quad u_4 = |0011\rangle + |1100\rangle$$

span a Cartan subspace $\mathfrak{h}$ of $\mathfrak{g}_1$. This shows that the symmetric pair $(\mathfrak{g}_0, \mathfrak{g}_1)$ is of maximal rank.

Let Φ be the root system corresponding to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. (Note that $\mathfrak{h}$ is necessarily different from $\mathfrak{t}$ as $\mathfrak{h} \subset \mathfrak{g}_1$ and $\mathfrak{t} \subset \mathfrak{g}_0$.) By W we denote its Weyl group. Representing a root α by the 4-tuple $(\alpha(u_1), \ldots, \alpha(u_4))$, a choice of simple roots is $\Delta = \{\alpha_1, \ldots, \alpha_4\}$, where

$$\alpha_1 = (0,-2,0,0), \quad \alpha_2 = (1,1,1,1), \quad \alpha_3 = (0,0,-2,0), \quad \alpha_4 = (0,0,0,-2);$$

here we use the same enumeration as in the above Dynkin diagram.

Remark 3.1. Consider the group Sym_4 of all permutations of $\{1,2,3,4\}$. For $\sigma \in \mathrm{Sym}_4$ we define the linear map $\pi_\sigma\colon \mathfrak{g}_1 \to \mathfrak{g}_1$ that maps each $|i_1 i_2 i_3 i_4\rangle$ to $|i_{1^\sigma} i_{2^\sigma} i_{3^\sigma} i_{4^\sigma}\rangle$. Since $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra, there is at most one way in which π_σ extends to an automorphism of $\mathfrak{g}$. We have checked by computer that indeed for all $\sigma \in \mathrm{Sym}_4$ this yields an automorphism of $\mathfrak{g}$. The group generated by all these π_σ fixes u_1 and permutes $\{u_2, u_3, u_4\}$ as Sym_3. Specifically, $\pi_{(2,3)}$ swaps u_3 and u_4, and $\pi_{(2,4)}$ swaps u_2 and u_4. Since $\widehat{G}$ has no elements acting as a π_σ, the space $\mathcal{H}_4$ is acted upon by the split product

$$\mathcal{S} = \mathrm{Sym}_4 \ltimes \mathrm{SL}(2,4)^4.$$

Classifications: We use the techniques described above to classify the $\widehat{G}$-orbits on $\mathcal{H}_4$, which are exactly the G_0-orbits in $\mathfrak{g}_1$. Our classification of semisimple elements is given in Theorem 3.2 and Table 2; our classification of mixed elements is given in Theorem 3.7. We then consider the group $\mathcal{S}$: A classification of semisimple, nilpotent, and mixed elements up to $\mathcal{S}$-conjugacy is described in Theorem 4.2.

3.1. **The orbits of semisimple elements.** We use Dynkin's algorithm to compute all root subsystems of Φ. This yields 11 subsystems, up to W-conjugacy. One of these subsystems is of type $4\mathrm{A}_1$ and therefore not complete. The others are complete and listed in the third column of Table 2. By adding the empty set we obtain 11 subsystems $\Pi_1, \dots, \Pi_{11}$. Table 2 also contains the data that we computed starting from the complete root subsystems. The fourth column has the description of $\mathfrak{h}_{\Pi_i}$ and the fifth column gives the polynomial conditions that an element of $\mathfrak{h}_{\Pi_i}$ has to satisfy to belong to $\mathfrak{h}_{\Pi_i}^\circ$; see Remark 3.4 below for more details. The sixth column describes the groups Γ_{Π_i}, see also the statement of Theorem 3.2 for more details. Here we write $I_4 = \mathrm{diag}(1,1,1,1)$ for the 4×4 identity matrix. Finally, the last column displays the semisimple part of the reductive centraliser $\mathfrak{z}_\mathfrak{g}(p_i)$, where p_i is some element in $\mathfrak{h}_{\Pi_i}^\circ$, cf. Lemma 2.9.

Theorem 3.2. *Up to G_0-conjugacy, the semisimple elements of $\mathfrak{g}_1$ are the Γ_{Π_i}-classes of elements in $\mathfrak{h}_{\Pi_i}^\circ$ for $i = 1, \dots, 11$, as given in Table 2; each Γ_{Π_i} is realised as a complement subgroup to W_{Π_i} in $N_W(W_{\Pi_i})$: the group $\Gamma_{\Pi_2} \cong (\mathbb{Z}/2\mathbb{Z})^3$ is generated by all 4×4 diagonal matrices that have two 1s and two -1s on the diagonal; the groups $\Gamma_{\Pi_4}, \Gamma_{\Pi_5}, \Gamma_{\Pi_6} \cong \mathrm{Dih}_4$ are isomorphic to the dihedral group of order 8 and defined as*

$$\Gamma_{\Pi_4} = \langle \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix} 0&0&0&-1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} \rangle,\ \Gamma_{\Pi_5} = \langle \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix} \rangle,\ \Gamma_{\Pi_6} = \langle \begin{pmatrix} -1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix} \rangle.$$

In the following we denote by Σ a set of G_0-orbit representatives of semisimple elements in $\mathfrak{g}_1$.

i	type of Π_i	roots of Π_i	elements of $\mathfrak{h}_{\Pi_i}$	condition for being in $\mathfrak{h}_{\Pi_i}^\circ$	Γ_{Π_i}	$\mathfrak{z}_\mathfrak{g}(p_i)'$
1	$\emptyset$		$\lambda_1 u_1 + \dots + \lambda_4 u_4$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3 \pm \lambda_4\}$	W	0
2	A_1	α_4	$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$\mathfrak{sl}(2,\mathbb{C})$
3	A_2	α_2, α_4	$\lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3)$	$\lambda_u \neq 0$ and $\lambda_1 \neq -\lambda_2$	$\langle -I_4\rangle$	$\mathfrak{sl}(3,\mathbb{C})$
4	$2\mathrm{A}_1$	α_1, α_3	$\lambda_1 u_1 + \lambda_2 u_4$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2\}$	Dih_4	$\mathfrak{sl}(2,\mathbb{C})^2$
5	$2\mathrm{A}_1$	α_1, α_4	$\lambda_1 u_1 + \lambda_2 u_3$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2\}$	Dih_4	$\mathfrak{sl}(2,\mathbb{C})^2$
6	$2\mathrm{A}_1$	α_3, α_4	$\lambda_1 u_1 + \lambda_2 u_2$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2\}$	Dih_4	$\mathfrak{sl}(2,\mathbb{C})^2$
7	A_3	$\alpha_1, \alpha_2, \alpha_3$	$\lambda_1(u_1 - u_4)$	$\lambda_1 \neq 0$	$\langle -I_4\rangle$	$\mathfrak{sl}(4,\mathbb{C})$
8	A_3	$\alpha_1, \alpha_2, \alpha_4$	$\lambda_1(u_1 - u_3)$	$\lambda_1 \neq 0$	$\langle -I_4\rangle$	$\mathfrak{sl}(4,\mathbb{C})$
9	A_3	$\alpha_2, \alpha_3, \alpha_4$	$\lambda_1(u_1 - u_2)$	$\lambda_1 \neq 0$	$\langle -I_4\rangle$	$\mathfrak{sl}(4,\mathbb{C})$
10	$3\mathrm{A}_1$	$\alpha_1, \alpha_3, \alpha_4$	$\lambda_1 u_1$	$\lambda_1 \neq 0$	$\langle -I_4\rangle$	$\mathfrak{sl}(2,\mathbb{C})^3$
11	D_4	$\alpha_1, \dots, \alpha_4$	0	0	1	$\mathfrak{so}(4,\mathbb{C})$

TABLE 2. Complete root subsystems Π_i of Φ, corresponding sets $\mathfrak{h}_{\Pi_i}$ and $\mathfrak{h}_{\Pi_i}^\circ$ with parameters $\lambda_1, \dots, \lambda_4 \in \mathbb{C}$, and groups Γ_{Π_i}; the last column displays the derived algebra of the centraliser $\mathfrak{z}_\mathfrak{g}(p_i)$ for $p_i \in \mathfrak{h}_{\Pi_i}^\circ$.

Remark 3.3. To determine Σ explicitly, one still has to consider the Γ_{Π_i}-classes of elements in $\mathfrak{h}_{\Pi_i}^\circ$ for every $i = 1, \dots, 10$; note that the case $i = 11$ only contributes the zero element. In view of Remark 3.1, the cases $i = 4, 5, 6$ are all symmetric and classifications for one of these i can be translated to classifications for the other two cases by applying one of the automorphisms $\pi_{(2,3)}$ or $\pi_{(2,4)}$ of $\mathfrak{g}$; similarly, the cases $i = 7, 8, 9$ are symmetric. If $i \in \{3, 7, 8, 9, 10\}$, then the elements listed under $\mathfrak{h}_{\Pi_i}$ in column four of Table 2 need to be reduced modulo multiplication by -1. If $i = 2$, then we can change two signs of u_1, u_2, u_3, u_4 at once; since u_4 is not involved in elements of $\mathfrak{h}_{\Pi_2}$, for fixed $\lambda_1, \lambda_2, \lambda_3$, all the elements $e_1\lambda_1 u_1 + e_2\lambda_2 u_2 + e_3\lambda_3 u_3 \in \mathfrak{h}_{\Pi_2}^\circ$ with $e_1, e_2, e_3 \in \{\pm 1\}$ are Γ_{Π_2}-conjugate. If $i = 4$, then we can swap u_1 and u_4, or change their signs; cases $i \in \{5, 6\}$ are analogous. For $i = 1$ we act with W; a direct calculation shows that every element in W acts as PQ^i for some $i \in \{0, 1, 2\}$ where

$$Q = \frac{1}{2}\begin{pmatrix} 1&-1&-1&1\\ 1&-1&1&-1\\ -1&-1&1&1\\ 1&1&1&1\end{pmatrix}$$

and P is any signed permutation matrix induced by an element in $\langle (1,2)(3,4), (1,3)(2,4)\rangle$.

Remark 3.4. We comment on the construction of $\mathfrak{h}_{\Pi_i}^\circ$ as listed in Table 2. Recall that $p \in \mathfrak{h}_{\Pi_i}$ lies in $\mathfrak{h}_{\Pi_i}^\circ$ if and only if $\beta(p) \neq 0$ for all $\beta \in \Phi \setminus \Pi_i$. Representing a positive root by the 4-tuple of its values on

$u_1,\dots,u_4$, we have $\alpha_1=(0,-2,0,0)$, $\alpha_2=(1,1,1,1)$, $\alpha_3=(0,0,-2,0)$, $\alpha_4=(0,0,0,-2)$, and

$$\begin{aligned}
\alpha_1+\alpha_2 &= (1,-1,1,1), & \alpha_2+\alpha_3 &= (1,1,-1,1)\\
\alpha_2+\alpha_4 &= (1,1,1,-1), & \alpha_1+\alpha_2+\alpha_3 &= (1,-1,-1,1),\\
\alpha_1+\alpha_2+\alpha_4 &= (1,-1,1,-1), & \alpha_2+\alpha_3+\alpha_4 &= (1,1,-1,-1),\\
\alpha_1+\alpha_2+\alpha_3+\alpha_4 &= (1,-1,-1,-1), & \alpha_1+2\alpha_2+\alpha_3+\alpha_4 &= (2,0,0,0).
\end{aligned}$$

Consider the third line of Table 2, so $p=\lambda_1(u_1-u_2)+\lambda_2(u_1-u_3)$; the coordinate vector with respect to the chosen basis of $\mathfrak{h}$ is $(\lambda_1+\lambda_2,-\lambda_1,-\lambda_2,0)$. The positive roots in Π_3 are $\alpha_2,\alpha_4,\alpha_2+\alpha_4$. For p to be in $\mathfrak{h}_{\Pi_3}^\circ$, the inner product of $(\lambda_1+\lambda_2,-\lambda_1,-\lambda_2,0)$ with all positive roots other than $\{\alpha_2,\alpha_4,\alpha_2+\alpha_4\}$ has to be nonzero; it is straightforward to see that this reduces to the condition $\lambda_1\lambda_2(\lambda_1+\lambda_2)\neq 0$. In a similar way, one can determine the conditions for the other sets $\mathfrak{h}_{\Pi_i}^\circ$.

3.2. **The orbits of mixed elements.** As before, $\widehat{G}=\mathrm{SL}(2,\mathbb{C})^4$; recall that there is a surjective morphism $\pi\colon\widehat{G}\to G_0$ of algebraic groups and $\widehat{G}$ acts as G_0 on $\mathfrak{g}_1$. We start with an observation.

Lemma 3.5. *If $x,y\in\mathfrak{h}_{\Pi_i}^\circ$ for some i, then $Z_{\widehat{G}}(x)=Z_{\widehat{G}}(y)$.*

Proof. By Lemma 2.9 we have $Z_{G_0}(x)=Z_{G_0}(y)$. Note that $Z_{\widehat{G}}(x)$ is the preimage of $Z_{G_0}(x)$ in $\widehat{G}$ under π, and similarly for $Z_{\widehat{G}}(y)$; therefore $Z_{\widehat{G}}(x)=Z_{\widehat{G}}(y)$. □

Proposition 3.6. *If $s\in\Sigma$ lies in $\mathfrak{h}_{\Pi_i}^\circ$ as in Table 2, then $Z_{\widehat{G}}(s)$ is given in Row i of Table 3.*

Proof. The group $\widehat{G}$ acts naturally on $\mathfrak{g}_1\cong\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, and we can write down the equations for $g(s)=s$ where $g=(A,B,C,D)\in\mathrm{SL}(2,\mathbb{C})^4$ is a general element with 16 indeterminates $a_{ij},b_{i,j},c_{ij},d_{ij}$ defining the matrices A,B,C,D. We use Gröbner basis techniques to obtain a useful description of $Z_{\widehat{G}}(s)$. Table 3 summarises our results, where for $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ and $u,v\in\mathbb{C}$ with $u\neq 0$ we write

$$\begin{aligned}
&A^{\#}=\begin{pmatrix} d & c\\ b & a\end{pmatrix}, \quad D(u,v)=\begin{pmatrix} u & 0\\ v & u^{-1}\end{pmatrix}, \quad D(u)=\begin{pmatrix} u & 0\\ 0 & u^{-1}\end{pmatrix}, \quad L(v)=D(1,v), \quad L=D(\imath)\\
&M(a,b)=\begin{pmatrix} a & b\\ b & a\end{pmatrix}, \qquad I=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \qquad J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \qquad K=\begin{pmatrix} 0 & \imath\\ \imath & 0\end{pmatrix}.
\end{aligned}\tag{3.1}$$
□

In Table 3 we often have $(J,J,J,J)^2=-(I,I,I,I)\in Z_{\widehat{G}}(s)^\circ$, in which case $Z_{\widehat{G}}(s)^\circ$ does not split in $Z_{\widehat{G}}(s)$, that is, the generators listed in the right column of Table 3 generate a group that is larger than the component group $Z_{\widehat{G}}(s)/Z_{\widehat{G}}(s)^\circ$.

By Lemma 2.7, up to G_0-conjugacy, every mixed element has the form $p+e$ where $p\in\Sigma$ is semisimple (as in Table 2) and $e\in\mathfrak{z}_{\mathfrak{g}}(p)'\cap\mathfrak{g}_1$ is nilpotent; recall that $\mathfrak{z}_{\mathfrak{g}}(p)$ is reductive and its center consists of semisimple elements, so e lies in the semisimple part $\mathfrak{z}_{\mathfrak{g}}(p)'$. Moreover, $p+e$ and $p+e'$ are G_0-conjugate if and only if e and e' are $Z_{G_0}(p)$-conjugate. Writing $\mathfrak{a}=\mathfrak{z}_{\mathfrak{g}}(p)'$ and $\mathfrak{a}_i=\mathfrak{a}\cap\mathfrak{g}_i$ for $i=0,1$, we need to classify the nilpotent $Z_{G_0}(p)$-orbits in $\mathfrak{a}_1$.

Lemma 2.9 shows that any element in the same row of Table 2 has the same centraliser. Moreover, the proof of Lemma 2.9 shows that $Z_G(p)$ is connected with Lie algebra $\mathfrak{z}_{\mathfrak{g}}(p)$. It follows from $p\in\mathfrak{g}_1$ that $\mathfrak{z}_{\mathfrak{g}}(p)=\mathfrak{z}_{\mathfrak{g}_0}(p)\oplus\mathfrak{z}_{\mathfrak{g}_1}(p)$; thus, $\mathfrak{z}_{\mathfrak{g}}(p)$ is a reductive graded Lie algebra with adjoint group $Z_G(p)$, and $Z_{G_0}(p)^\circ\leqslant Z_G(p)$ is the connected algebraic subgroup with Lie algebra $\mathfrak{z}_{\mathfrak{g}_0}(p)$. We can now use standard methods, such as described in [21, Chapter 8.3.2], to classify the (finitely many) nilpotent $Z_{G_0}(p)^\circ$-orbits in $\mathfrak{a}_1$; we have done this in GAP [23] using the GAP package SLA. It remains to reduce the obtained list up to conjugacy under the component group of $Z_{G_0}(p)$.

According to Table 2, we have 11 cases, namely $p\in\Sigma\cap\mathfrak{h}_{\Pi_i}^\circ$ with $i\in\{1,\dots,11\}$. For $i=1$ we have $\mathfrak{a}=0$, so there are no nonzero nilpotent elements, and $p=0$ for $i=11$; in both cases there are no mixed elements. Thus, it remains to consider $i\in\{2,\dots,10\}$; we report on the outcome of our computations:

i	**identity component $Z_{\widehat{G}}(s)^\circ$**	**preimages of generators of $Z_{\widehat{G}}(s)/Z_{\widehat{G}}(s)^\circ$**
1	1	$(J,J,J,J),(-I,-I,I,I),(-I,I,-I,I),(K,K,K,K)$
2	$\{(D(a)^{-1},D(a)^{-1},D(a),D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I),(J,J,J,J)$
3	$\{(A^\#,A^\#,A,A)\ :\ A\in\mathrm{SL}(2,\mathbb{C})\}$	$(-I,-I,I,I),(-I,I,-I,I)$
4	$\{(D(a)^{-1},D(a),D(b)^{-1},D(b))\ :\ a,b\in\mathbb{C}^\times\}$	$(-I,I,-I,I),(J,J,J,J)$
5	$\{(D(a)^{-1},D(b)^{-1},D(a),D(b))\ :\ a,b\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(J,J,J,J)$
6	$\{(D(a)^{-1},D(b),D(b)^{-1},D(a))\ :\ a,b\in\mathbb{C}^\times\}$	$(-I,I,-I,I),(J,J,J,J)$
7	$\{(A^\#,A,B^\#,B):A,B\in\mathrm{SL}(2,\mathbb{C})\}$	$(-I,I,-I,I)$
8	$\{(A^\#,B^\#,A,B):A,B\in\mathrm{SL}(2,\mathbb{C})\}$	$(-I,-I,I,I)$
9	$\{(A^\#,B,B^\#,A):A,B\in\mathrm{SL}(2,\mathbb{C})\}$	$(-I,I,-I,I)$
10	$\{(D(abc)^{-1},D(a),D(b),D(c))\ :\ a,b,c\in\mathbb{C}^\times\}$	(J,J,J,J)

TABLE 3. The groups $Z_{\widehat{G}}(s)$: the entry i is the label of the canonical semisimple set $\mathfrak{h}^\circ_{\Pi_i}$ that contains s, as in Table 2; the notation $A^\#$, $D(a)$, I, J, and K is as explained in (3.1).

Case $i=2$. Here $\mathfrak{a}=\mathfrak{sl}(2,\mathbb{C})$ and there are two nilpotent $Z_{\widehat{G}}(p)^\circ$-orbits in $\mathfrak{a}_1$; these are interchanged by the component group. In conclusion, one nilpotent orbit remains with representative

$$n_{2,1}=|0011\rangle.$$

Case $i=3$. Here $\mathfrak{a}=\mathfrak{sl}(3,\mathbb{C})$ and its grading is induced by an outer automorphism of $\mathfrak{a}$. There are two nilpotent $Z_{\widehat{G}}(p)^\circ$-orbits in $\mathfrak{a}_1$ and the component group acts trivially. In conclusion, there are two nilpotent orbits with representatives

$$n_{3,1}=|0011\rangle \quad\text{and}\quad n_{3,2}=|0111\rangle+|1011\rangle+|0010\rangle+|0001\rangle.$$

Case $i=4,5,6$. First let $i=4$. Here $\mathfrak{a}=\mathfrak{sl}(2,\mathbb{C})^2$ and there are eight nilpotent $Z_{\widehat{G}}(p)^\circ$-orbits in $\mathfrak{a}_1$. Up to the action of the component group, four of them remain, with representatives

$$n_{4,1}=|0110\rangle+|1010\rangle,\qquad n_{4,2}=|0110\rangle+|0101\rangle,\qquad n_{4,3}=|0110\rangle,\qquad n_{4,4}=|0101\rangle.$$

For $i=5,6$, Remark 3.1 yields $n_{5,1}=|0110\rangle+|1100\rangle$, $n_{5,2}=|0110\rangle+|0011\rangle$, $n_{5,3}=|0110\rangle$, $n_{5,4}=|0011\rangle$ and $n_{6,1}=|0011\rangle+|1010\rangle$, $n_{6,2}=|0011\rangle+|0101\rangle$, $n_{6,3}=|0011\rangle$, $n_{6,4}=|0101\rangle$, respectively.

Case $i=7,8,9$. First let $i=7$. Here $\mathfrak{a}=\mathfrak{sl}(4,\mathbb{C})$ and its grading is induced by an outer automorphism of $\mathfrak{a}$; we have that $\mathfrak{a}_0=\mathfrak{sl}(2,\mathbb{C})^2$ and there are six nilpotent $Z_{\widehat{G}}(p)^\circ$-orbits in $\mathfrak{a}_1$. The component group acts trivially, and representatives of nilpotent orbits are

$$\begin{aligned} n_{7,1}&=|1101\rangle+|1011\rangle+|1000\rangle+|0001\rangle, & n_{7,2}&=|1101\rangle+|1010\rangle+|0001\rangle,\\ n_{7,3}&=|1011\rangle+|1000\rangle+|0101\rangle, & n_{7,4}&=|1011\rangle+|1000\rangle,\\ n_{7,5}&=|1101\rangle+|0001\rangle, & n_{7,6}&=|1001\rangle.\end{aligned}$$

Remark 3.1 yields $n_{8,1}=|1011\rangle+|1101\rangle+|1000\rangle+|0001\rangle$, $n_{8,2}=|1011\rangle+|1100\rangle+|0001\rangle$, $n_{8,3}=|1101\rangle+|1000\rangle+|0011\rangle$, $n_{8,4}=|1101\rangle+|1000\rangle$, $n_{8,5}=|1011\rangle+|0001\rangle$, $n_{8,6}=|1001\rangle$ for the case $i=8$, and $n_{9,1}=|1101\rangle+|1110\rangle+|1000\rangle+|0100\rangle$, $n_{9,2}=|1101\rangle+|1010\rangle+|0100\rangle$, $n_{9,3}=|1110\rangle+|1000\rangle+|0101\rangle$, $n_{9,4}=|1110\rangle+|1000\rangle$, $n_{9,5}=|1101\rangle+|0100\rangle$, $n_{9,6}=|1100\rangle$ for $i=9$.

Case $i = 10$. Here $\mathfrak{a} = \mathfrak{sl}(2,\mathbb{C})^3$ and there are 26 nilpotent $Z_{\widehat{G}}(p)^\circ$-orbits in $\mathfrak{a}_1$. Up to the action of the component group, 13 of them remain, with representatives

$$\begin{aligned}
&n_{10,1} = |1100\rangle + |1010\rangle + |0110\rangle, && n_{10,2} = |1010\rangle + |0110\rangle, && n_{10,3} = |1010\rangle + |0110\rangle + |0011\rangle,\\
&n_{10,4} = |1100\rangle + |0110\rangle, && n_{10,5} = |0110\rangle, && n_{10,6} = |0110\rangle + |0011\rangle,\\
&n_{10,7} = |1100\rangle + |0110\rangle + |0101\rangle, && n_{10,8} = |0110\rangle + |0101\rangle, && n_{10,9} = |0110\rangle + |0101\rangle + |0011\rangle,\\
&n_{10,10} = |1100\rangle + |1010\rangle, && n_{10,11} = |1010\rangle, && n_{10,12} = |1010\rangle + |0011\rangle,\\
&n_{10,13} = |0011\rangle.
\end{aligned}$$

In conclusion, we have shown:

Theorem 3.7. *For $i \in \{2,\dots,10\}$ let Σ_i be a set of G_0-conjugacy representatives of semisimple elements in $\mathfrak{h}^\circ_{\Pi_i}$ as specified in Table 2 and Remark 3.3. Up to G_0-conjugacy, the mixed type elements in $\mathfrak{g}_1$ are the elements $s + n_{i,j}$ where $i \in \{2,\dots,10\}$, $s \in \Sigma_i$, and $n_{i,j}$ as specified in Case i above.*

4. A classification up to $\mathcal{S}$-conjugacy

This section has two aims. First, we compare our classifications with the families determined by Verstraete et al. [7]; the latter families have been reconsidered and corrected by Chterental & Djoković [8]. Second, we describe our classification of nilpotent, semisimple, and mixed elements up to $\mathcal{S}$-conjugacy where $\mathcal{S} = \mathrm{Sym}_4 \ltimes \mathrm{SL}_2(\mathbb{C})^4$. We start with a preliminary section on deciding conjugacy of elements.

4.1. Deciding conjugacy.

Let $u, v \in \mathcal{H}_4$. In our classification below we need to decide whether u and v lie in the same $\widehat{G}$-orbit and, if so, to find a $g \in \widehat{G}$ with $gv = u$. A general method for this is based on the computational technique of Gröbner bases [34]: the relation $gv = u$ gives linear relations on the entries of the four matrices in g; to these relations we add the polynomials that express that the determinants of the matrices are 1. We then compute a Gröbner basis of the ideal generated by the resulting polynomials. This Gröbner basis is trivial (i.e., consists only of 1) if and only if there is no solution. If the Gröbner basis is not trivial, then in many cases it can be used to effectively solve the equations and find a solution. A related problem is to find, given a $u \in \mathcal{H}_4$, an element v in our classification to which u is conjugate to.

First, suppose u is semisimple. It follows from [18, Theorem 3] that two semisimple u and v are conjugate if and only if $\mathcal{F}(u) = \mathcal{F}(v)$, where $\mathcal{F}$ is defined in (5.1) below (it maps u to the values of the generating invariants of $\mathbb{C}[\mathfrak{g}_1]^{\widehat{G}}$). We compute $\mathcal{F}(u)$ with Table 10, and use a Gröbner basis computation to find an element v in one of the 10 semisimple classes with $\mathcal{F}(u) = \mathcal{F}(v)$; we then find a conjugating element via the above method. If u is nilpotent, then we have to perform at most 30 Gröbner basis computations to find the element in Table 7 that is conjugate to u. By computations in $\mathfrak{g}$, we can also reduce the number of candidates of nilpotent elements in Table 7 that are possibly conjugate to u. For example, conjugate elements have centralisers in $\mathfrak{g}_0$ of the same dimension. Moreover, the theory of $\mathfrak{sl}_2$-triples can be used to reduce the number of candidates, see [21, §8.3.2]. If u is mixed, then we first identify its semisimple part with an element in our classification; subsequently we deal with the nilpotent part.

4.2. Classification results.

Before we describe our classification, we first recall the classification in [8] in the language of our paper:

Theorem 4.1 (Theorem 3.6 in [8]). *The $\mathcal{S}$-orbits on $\mathfrak{g}_1$ are classified by the nine families $D_1,\dots,D_9$ in Table 4. Elements belonging to different families are not equivalent under $\mathcal{S}$-operations. However, within the same family, different families of the parameters may give elements belonging to the same $\mathcal{S}$-orbit.*

The next theorem identifies in which of these nine families our G_0-orbit representatives lie, up to $\mathcal{S}$-conjugacy; we also present a new, complete and irredundant classification up to $\mathcal{S}$-conjugacy.

fam.	elements
D_1	$S_1(a,b,c,d)+N_1$ where $S_1(a,b,c,d)=\frac{a+d}{2}u_1+\frac{b-c}{2}u_2+\frac{b+c}{2}u_3+\frac{a-d}{2}u_4$ and $N_1=0$
D_2	$S_2(a,b,c)+N_2$ where $S_2(a,b,c)=\frac{a+c}{2}u_1+\frac{b-c}{2}u_2+\frac{b+c}{2}u_3+\frac{a-c}{2}u_4$ and $N_2=\frac{\imath}{2}\left(u_3+u_4-u_2-u_1+\lvert 1110\rangle+\lvert 0001\rangle+\lvert 1000\rangle+\lvert 0111\rangle-\lvert 1101\rangle-\lvert 0010\rangle-\lvert 1011\rangle-\lvert 0100\rangle\right)$
D_3	$S_3(a,b)+N_3$ where $S_3(a,b)=\frac{a}{2}u_1+\frac{b}{2}u_2+\frac{b}{2}u_3+\frac{a}{2}u_4$ and $N_3=\frac{1}{2}\left(u_3-u_2+\lvert 0010\rangle+\lvert 1101\rangle-\lvert 1110\rangle-\lvert 0001\rangle\right)$
D_4	$S_4(a,b)+N_4$ where $S_4(a,b)=\frac{a+b}{2}u_1+bu_3+\frac{a-b}{2}u_4$ and $N_4=\imath(\lvert 1001\rangle-\lvert 0110\rangle)+\frac{1}{2}\left(\lvert 1101\rangle+\lvert 0100\rangle+\lvert 1011\rangle+\lvert 0010\rangle-\lvert 1110\rangle-\lvert 0001\rangle-\lvert 1000\rangle-\lvert 0111\rangle\right)$
D_5	$S_5(a)+N_5$ where $S_5(a)=au_1+au_3$ and $N_5=2\imath\left(\lvert 0001\rangle+\lvert 0110\rangle-\lvert 1011\rangle\right)$
D_6	$S_6(a)+N_6$ where $S_6(a)=\frac{a}{2}u_1+\frac{a}{2}u_2+\frac{a}{2}u_3+\frac{a}{2}u_4$ and $N_6=\frac{\imath+1}{2}(\lvert 0010\rangle+\lvert 1101\rangle-u_2)+\frac{\imath-1}{2}(\lvert 1110\rangle+\lvert 0001\rangle-u_3)-\frac{\imath}{2}(\lvert 1011\rangle+\lvert 0100\rangle+\lvert 1000\rangle+\lvert 0111\rangle-u_1-u_4)$
D_7	S_7+N_7 where $S_7=0$ and $N_7=(\lvert 1010\rangle-\lvert 1001\rangle+\lvert 0011\rangle+\lvert 0000\rangle)+(\imath+1)(\lvert 0110\rangle+\lvert 0101\rangle)-\imath(\lvert 1011\rangle+\lvert 1000\rangle+\lvert 0010\rangle-\lvert 0001\rangle)$
D_8	S_8+N_8 where $S_8=0$ and $N_8=\frac{\imath+1}{2}u_1-\frac{\imath-1}{2}u_4+\frac{\imath-1}{2}(\lvert 1110\rangle+\lvert 0001\rangle)-\frac{\imath+1}{2}(\lvert 1101\rangle+\lvert 0010\rangle)$ $+\frac{1}{2}(\lvert 1011\rangle+\lvert 0110\rangle+\lvert 0101\rangle+\lvert 1000\rangle)+\frac{1-2\imath}{2}(\lvert 0111\rangle+\lvert 1010\rangle+\lvert 1001\rangle+\lvert 0100\rangle)$
D_9	S_9+N_9 where $S_9=0$ and $N_9=\frac{1}{2}(\lvert 1111\rangle+\lvert 1100\rangle+\lvert 1011\rangle+\lvert 1000\rangle+\imath\lvert 1110\rangle+\imath\lvert 1101\rangle-\imath\lvert 1010\rangle+\imath\lvert 1001\rangle)$ $+\frac{1}{2}(\lvert 0111\rangle+\lvert 0100\rangle+\lvert 0011\rangle+\lvert 0000\rangle+\imath\lvert 0110\rangle+\imath\lvert 0101\rangle-\imath\lvert 0010\rangle+\imath\lvert 0001\rangle)$

TABLE 4. The nine families of Theorem 4.1 with parameters $a,b,c,d\in\mathbb{C}$.

Theorem 4.2. a) *Up to $\mathcal{S}$-conjugacy, the nilpotent orbits in $\mathfrak{g}_1$ are the elements $N_1,\dots,N_9$ in Table 4.*

b) *Up to $\mathcal{S}$-conjugacy, the semisimple orbits in $\mathfrak{g}_1$ are the elements in Table 5.*

c) *Up to $\mathcal{S}$-conjugacy, the mixed elements in $\mathfrak{g}_1$ are the elements in Table 6.*

The right column in Table 5 and the second column in Table 6 indicate to which family D_i (as in Table 4) the element is $\mathcal{S}$-conjugate to. Tables 3, 6, and 8 contain information about the centralisers in $\widehat{G}$.

Proof. As before, all the direct computations mentioned in this proof have been carried out in GAP [23] and its interface to Singular [24]. We briefly comment on our approach; let $s,t\in\mathfrak{g}_1$ and let $\sigma\in\mathrm{Sym}_4$. By abuse of notation, we also denote by σ the induced automorphism of $\mathfrak{g}$, see Remark 3.1. It is straightforward to compute the image $\sigma(s)$. As in the proof of Proposition 3.6, we use Gröbner basis techniques to determine $\widehat{G}$-conjugacy of $\sigma(s)$ and t: for example, if $g=(A,B,C,D)\in\widehat{G}$ is a general element with 16 indeterminates $a_{ij},b_{i,j},c_{ij},d_{ij}$, then the command `HasTrivialGroebnerBasis` allows us to decide quickly whether a solution to $g(\sigma(s))=t$ exists. This approach can also be used if s and t are semisimple or mixed elements defined by parameters λ_i and λ_i': if the Gröbner basis is trivial, then the elements are not conjugate; if nontrivial, then the elements are *potentially conjugate*. In the latter situation one still has to determine whether a solution exists that satisfies the conditions on the parameters λ_i and λ_i'. As explained below, we usually reduce $\widehat{G}$-conjugacy testing to testing of W-conjugacy, see Proposition 2.5; the latter is a finite explicit calculation.

a) Table 4 yields nine elements $N_1,\dots,N_9$, and a direct calculation shows that they are all nilpotent. Another direct computation (using Gröbner bases) shows that all these elements are not $\mathcal{S}$-conjugate, as expected by Theorem 4.1. It has been determined in [10] that there are 31 nilpotent G_0-orbits in $\mathfrak{g}_1$; the corresponding classification over the reals has been presented in [35]. In Table 7 (left) we list representatives for the nilpotent orbits (taken from [35, Table I]) and determine (using Gröbner bases) to which nilpotent element $N_1,\dots,N_9$ the element is $\mathcal{S}$-conjugate to; the claim follows.

b) Recall that $u_1,\dots,u_4$ span the Cartan subspace $\mathfrak{h}$, which shows that all the elements $S_1,\dots,S_9$ in Table 4 are semisimple. By Theorem 3.2, every semisimple element is conjugate to an element in family D_1. It remains to reduce our classification of semisimple elements (as given in Table 2) up to $\mathcal{S}$-conjugacy. Due to Remark 3.1, it suffices to consider elements in $\mathfrak{h}^\circ_{\Pi_i}$ for $i \in \{1,2,3,4,7,10\}$. First, we note that if $s \in \mathfrak{h}^\circ_{\Pi_i}$ and $t \in \mathfrak{h}^\circ_{\Pi_j}$ with distinct $i, j \in \{1,2,3,4,7,10\}$, then s and t are not $\mathcal{S}$-conjugate: this follows because s and t have centralisers of different dimensions (see Table 2) and because the permutation action of every $\sigma \in \mathrm{Sym}_4$ on $\mathfrak{g}_1$ extends to Lie algebra automorphisms of $\mathfrak{g}$ (cf. Remark 3.1). Thus, it remains to determine when $s, t \in \mathfrak{h}^\circ_{\Pi_i}$ are $\mathcal{S}$-conjugate; note that G_0-conjugacy is already determined in Table 2. We also note that Sym_4 stabilises $\mathfrak{h}$, so G_0-conjugacy of Sym_4-conjugate elements in $\mathfrak{h}$ can be decided by considering the action of W, see Proposition 2.5. For example, consider $i = 3$. Elements $s = \lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3)$ and $t = \lambda_1'(u_1 - u_2) + \lambda_2'(u_1 - u_3)$ are G_0-conjugate if and only if $(\lambda_1', \lambda_2') = \pm(\lambda_1, \lambda_2)$. We now consider every Sym_4-conjugate of s, for example $s' = \lambda_1(u_1 - u_4) + \lambda_2(u_1 - u_2)$, and then determine λ_1' and λ_2' such that s' is W-conjugate to $\lambda_1'(u_1 - u_2) + \lambda_2'(u_1 - u_3)$. In this particular example, $(\lambda_1', \lambda_2') = (\lambda_1 + \lambda_2, -\lambda_1)$, which shows that $\lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3)$ is $\mathcal{S}$-conjugate to $(\lambda_1 + \lambda_2)(u_1 - u_2) - \lambda_1(u_1 - u_3)$. Doing this for all Sym_4-conjugates of semisimple representatives in $\mathfrak{h}^\circ_{\Pi_i}$ allows us to determine the conditions for $\mathcal{S}$-conjugacy; the result is listed in Table 5.

c) Let $x = s + n_{i,j}$ be a mixed element as in Theorem 3.7. Due to Remark 3.1, up to $\mathcal{S}$-conjugacy, we can assume that $i \in \{2,3,4,7,10\}$. (There are no mixed elements for $i = 1$.) As in part a), we first determine to which nilpotent element $N_1, \ldots, N_9$ the element $n_{i,j}$ is $\mathcal{S}$-conjugate to; this is listed in Table 7 (right). If we have determined that $n_{i,j}$ is $\mathcal{S}$-conjugate to N_k, then we use Gröbner basis computations to verify that x is indeed $\mathcal{S}$-conjugate to an element of the form $S_k + N_k$, hence, up to $\mathcal{S}$-conjugacy, x lies in family D_k. Recall that Sym_4 preserves $\mathfrak{h}$ and the action of every $\sigma \in \mathrm{Sym}_4$ on $\mathfrak{g}_1$ extends to a Lie algebra automorphism of $\mathfrak{g}$. In particular, it follows that $\sigma(x) = \sigma(s) + \sigma(n_{i,j})$ is the Jordan decomposition of $\sigma(x)$. The centraliser information in Table 2 now implies that the only $\mathcal{S}$-conjugacies between elements of the form $s + n_{i,j}$ (with $i \in \{2,3,4,7,10\}$, $s \in \mathfrak{h}^\circ_{\Pi_i} \setminus \{0\}$, and permissible j) are between elements whose semisimple parts lie in the same component $\mathfrak{h}^\circ_{\Pi_i}$. Thus, it remains to decide $\mathcal{S}$-conjugacy of elements $x = s + n_{i,j}$ and $y = t + n_{i,\ell}$ where $i \in \{2,3,4,7,10\}$, $s, t \in \mathfrak{h}^\circ_{\Pi_i}$ as in Table 2, and $n_{i,j}$ and $n_{i,\ell}$ in the same family D_k. In this situation, explicit Gröbner basis computations show that if $s = t$, then x and y are $\mathcal{S}$-conjugate. It therefore remains to consider $\mathcal{S}$-conjugacy between elements

$$x = s + n_{i,j} \quad \text{and} \quad y = t + n_{i,j}$$

with $s, t \in \mathfrak{h}^\circ_{\Pi_i}$ as in Table 2; by what is said in the previous sentence, for each i we only have to consider one j for each class D_k. For this we consider every possible Sym_4-conjugate of x, say $x' = \sigma(x) = \sigma(s) + \sigma(n_{i,j})$, and check whether x' is potentially G_0-conjugate to an element $t + n_{i,j}$ with $t \in \mathfrak{h}^\circ_{\Pi_i}$: for this we first check whether $\sigma(n_{i,j})$ is potentially conjugate to $n_{i,j}$, and if so, then we test the same for $\sigma(x)$ and y. If the test is positive, then we check whether $\sigma(s)$ is W-conjugate to an element t, cf. the proof of part b). We briefly comment on each case.

If $i = 7$, then $s = \lambda_1 u_1$ and $t = \lambda_1' u_1$, and a direct computation shows that x and y are $\mathcal{S}$-conjugate if and only if $s = \pm t$; the same holds for $i = 10$. If $i = 4$, then $s = \lambda_1 u_1 + \lambda_2 u_4$ and $t = \lambda_1' u_1 + \lambda_2' u_4$. A computation shows that $\sigma(x)$ and y are potentially G_0-conjugate if and only if $\sigma(s) = s$. More precisely, if $j = 3$, then $\sigma(x) = s + \sigma(n_{4,3})$, and the latter can be shown to be G_0-conjugate to $s + n_{4,3}$. If $j = 1$, then $\sigma(x) = s + \sigma(n_{4,1})$ is G_0-conjugate to $s + n_{4,1}$. In conclusion, for $i = 4$ it follows that x and y are $\mathcal{S}$-conjugate if and only if s and t are G_0-conjugate. If $i = 2$, then for every permutation σ, the element $\sigma(x) = \sigma(s) + \sigma(n_{2,1})$ has mixed type and must be G_0-conjugate to some $t + n_{2,1}$; this follows from the centraliser dimension in Table 2 and Theorem 3.7. We can determine the possible transformations $s \to \sigma(s) \to t$ by using the same computations as in b). Now consider $i = 3$. One can show that every Sym_4-conjugate of s is G_0-conjugate to an element in $\mathfrak{h}^\circ_{\Pi_3}$ as in Table 2; since the possible nilpotent parts $n_{3,1}$ and $n_{3,2}$ lie in different families D_k, they are not G_0-conjugate, thus, if $x = s + n_{3,i}$, then each $\sigma(x)$ is G_0-conjugate to an element $t + n_{3,i}$ with $t \in \mathfrak{h}^\circ_{\Pi_3}$ as in Table 2, and we determine the possible transformations $s \to \sigma(s) \to t$ as in b). All the results are listed in Table 6. $\square$

element	component	conditions	family
$\lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\lambda_4u_4$	$\mathfrak{h}^\circ_{\Pi_1}$	with $\lambda_1,\ldots,\lambda_4\neq 0$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$ up to the action of PQ^i with $i\in\{0,1,2\}$ where Q as in Remark 3.3d) and P is any 4×4 signed permutation matrix	D_1
$\lambda_1u_1+\lambda_2u_2+\lambda_3u_3$	$\mathfrak{h}^\circ_{\Pi_2}$	with $\lambda_1,\lambda_2,\lambda_3\neq 0$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\}$ up to the action of 3×3 signed permutation matrices	D_1
$\lambda_1(u_1-u_2)+\lambda_2(u_1-u_3)$	$\mathfrak{h}^\circ_{\Pi_3}$	with $\lambda_1,\lambda_2\neq 0$ and $\lambda_1\neq -\lambda_2$ up to action of $\langle\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right)\rangle\cong \mathrm{Dih}_6$	D_1
$\lambda_1u_1+\lambda_2u_4$	$\mathfrak{h}^\circ_{\Pi_4}$	with $\lambda_1,\lambda_2\neq 0$ and $\lambda_1\neq\pm\lambda_2$ up to the action of $\langle\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\rangle\cong \mathrm{Dih}_4$	D_1
$\lambda_1(u_1-u_4)$	$\mathfrak{h}^\circ_{\Pi_7}$	with $\lambda_1\neq 0$ up to the action of $\langle(-1)\rangle$	D_1
λ_1u_1	$\mathfrak{h}^\circ_{\Pi_{10}}$	with $\lambda_1\neq 0$ up to the action of $\langle(-1)\rangle$	D_1

TABLE 5. The classification of semisimple elements up to $\mathcal{S}$-conjugacy; the action of each matrix group is on the vector of parameters (λ_1), (λ_1,λ_2), etc. The corresponding centraliser of an element in $\mathfrak{h}^\circ_{\Pi_i}$ is given in Row i of Table 3.

element	fam.	identity component Z°	preimages of generators of Z/Z°
$s+n_{2,1}$	D_2	1	$(-I,-I,I,I),(-I,I,-I,I),(L,L,L,L)$
$s+n_{3,1}$	D_2	$\{(L(a')^\intercal,L(a')^\intercal,L(a'),L(a'))\ :\ a'\in\mathbb{C}\}$	$(-I,-I,I,I),(-I,I,-I,I),(L,L,L,L)$
$s+n_{3,2}$	D_4	1	$(-I,-I,I,I),(-I,I,-I,I),(-I,-I,-I,-I)$
$s+n_{4,1}$	D_3	1	$(-I,-I,I,I),(-I,I,-I,I),(L,L,L,L)$
$s+n_{4,3}$	D_2	$\{(D(a)^{-1},D(a),D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$
$s+n_{7,1}$	D_4	$\{(L(a')^{-1},L(a')^{-\intercal},L(a')^\intercal,L(a'))\ :\ a'\in\mathbb{C}\}$	$(-I,-I,I,I),(-I,I,-I,I),(-I,-I,-I,-I)$
$s+n_{7,2}$	D_5	1	$(-I,-I,I,I),(-I,I,-I,I),(-I,-I,-I,-I)$
$s+n_{7,4}$	D_3	$\{(L(a'),L(a')^\intercal,D(b)^{-1},D(b))\ :\ a'\in\mathbb{C},b\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I),(L,L,J,J)$
$s+n_{7,6}$	D_2	$(D(a^{-1},c'),D(a,c')^\intercal,D(a^{-1},b')^\intercal,D(a,b'))$ with $a\in\mathbb{C}^\times$ and $b',c'\in\mathbb{C}$	$(-I,-I,I,I),(-I,I,-I,I)$
$s+n_{10,1}$	D_6	1	$(-I,-I,I,I),(-I,I,-I,I),(L,L,L,L)$
$s+n_{10,2}$	D_3	$\{(D(a)^{-1},D(a)^{-1},D(a),D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$
$s+n_{10,5}$	D_2	$\{(D(a)^{-1},D(b)^{-1},D(b),D(a))\ :\ a,b\in\mathbb{C}^\times\}$	$(-I,-I,I,I)$

TABLE 6. The classification of mixed elements up to $\mathcal{S}$-conjugacy; for each listed element $s+n_{i,j}$ the semisimple part $s\in\mathfrak{h}^\circ_{\Pi_i}$ is as given in Table 5. Last two columns describe their centralisers, where the notation is from (3.1).

5. INVARIANTS

The aim of this section is to describe the invariant ring $R=\mathbb{C}[\mathfrak{g}_1]^{\widehat{G}}$. Let $B=\{b_1,\ldots,b_{16}\}$ be the basis of $\mathfrak{g}_1$ such that $b_1=|1111\rangle, b_2=|1110\rangle, b_3=|1101\rangle, b_4=|1100\rangle,\ldots$ in lexicographical ordering. Let $\mathbb{C}[\mathfrak{g}_1]$ be the ring of polynomial functions on $\mathfrak{g}_1$. We identify $\mathbb{C}[\mathfrak{g}_1]$ with the polynomial ring $\mathbb{C}[x_1,\ldots,x_{16}]$ using the basis B, so $f\in\mathbb{C}[x_1,\ldots,x_{16}]$ is identified with the polynomial function on $\mathfrak{g}_1$ that maps $p\in\mathfrak{g}_1$ to $f(c_1,\ldots,c_{16})$ where $c_1,\ldots,c_{16}$ are the coefficients of p with respect to B.

The group $\widehat{G}$ acts on $\mathbb{C}[\mathfrak{g}_1]$ by $g\cdot f(x)=f(g^{-1}\cdot x)$, and the invariant ring $\mathbb{C}[\mathfrak{g}_1]^{\widehat{G}}$ consists of all polynomials $f\in\mathbb{C}[\mathfrak{g}_1]$ such that $g\cdot f=f$ for all $g\in\widehat{G}$. The invariants are interesting in our context because they are invariant on orbits. By a celebrated theorem of Hilbert, $\mathbb{C}[\mathfrak{g}_1]$ is finitely generated. Vinberg has proved a generalization of Chevalley's restriction theorem, see [18, Theorem 7] or [25, Theorem 3.62], showing that the restriction map $\mathbb{C}[\mathfrak{g}_1]^{\widehat{G}}\to\mathbb{C}[\mathfrak{h}]^W$ is an isomorphism. Moreover, the degrees

orbit	representative	$\mathcal{S}$-conjugate to
1	$\|1100\rangle$	N_2 in D_2
2	$\|1100\rangle+\|0000\rangle$	N_3 in D_3
3	$\|1100\rangle+\|1001\rangle$	N_3 in D_3
4	$\|1100\rangle+\|1010\rangle$	N_3 in D_3
5	$\|1101\rangle+\|0100\rangle$	N_3 in D_3
6	$\|1110\rangle+\|0100\rangle$	N_3 in D_3
7	$\|1110\rangle+\|1101\rangle$	N_3 in D_3
8	$\|1101\rangle+\|0100\rangle+\|1000\rangle$	N_6 in D_6
9	$\|1110\rangle+\|0100\rangle+\|1000\rangle$	N_6 in D_6
10	$\|1110\rangle+\|1101\rangle+\|1000\rangle$	N_6 in D_6
11	$\|1110\rangle+\|1101\rangle+\|0100\rangle$	N_6 in D_6
12	$\|0101\rangle+\|1100\rangle+\|1001\rangle+\|0000\rangle$	N_9 in D_9
13	$\|0110\rangle+\|1100\rangle+\|1010\rangle+\|0000\rangle$	N_9 in D_9
14	$\|1111\rangle+\|1100\rangle+\|1001\rangle+\|1010\rangle$	N_9 in D_9
15	$\|0111\rangle+\|1110\rangle+\|1101\rangle+\|0100\rangle$	N_9 in D_9
16	$\|1110\rangle+\|1101\rangle+\|0100\rangle+\|1000\rangle$	N_4 in D_4
17	$\|1110\rangle+\|1101\rangle+\|0000\rangle$	N_5 in D_5
18	$\|1110\rangle+\|0100\rangle+\|1001\rangle$	N_5 in D_5
19	$\|1101\rangle+\|0100\rangle+\|1010\rangle$	N_5 in D_5
20	$\|0101\rangle+\|1110\rangle+\|1000\rangle$	N_5 in D_5
21	$\|0110\rangle+\|1101\rangle+\|1000\rangle$	N_5 in D_5
22	$\|1111\rangle+\|0100\rangle+\|1000\rangle$	N_5 in D_5
23	$\|1110\rangle+\|0100\rangle+\|0000\rangle+\|1001\rangle$	N_8 in D_8
24	$\|0110\rangle+\|1101\rangle+\|1000\rangle+\|0000\rangle$	N_8 in D_8
25	$\|1111\rangle+\|0100\rangle+\|1000\rangle+\|1001\rangle$	N_8 in D_8
26	$\|1111\rangle+\|0100\rangle+\|1000\rangle+\|1010\rangle$	N_8 in D_8
27	$\|0101\rangle+\|1110\rangle+\|0000\rangle+\|1001\rangle$	N_7 in D_7
28	$\|0110\rangle+\|1101\rangle+\|0000\rangle+\|1010\rangle$	N_7 in D_7
29	$\|1111\rangle+\|0100\rangle+\|1001\rangle+\|1010\rangle$	N_7 in D_7
30	$\|1111\rangle+\|0110\rangle+\|0101\rangle+\|1000\rangle$	N_7 in D_7
31	0	

element $n_{i,j}$	$\mathcal{S}$-conjugate to
$n_{2,1}$	N_2 in D_2
$n_{3,1}$	N_2 in D_2
$n_{3,2}$	N_4 in D_4
$n_{4,1}$	N_3 in D_3
$n_{4,2}$	N_3 in D_3
$n_{4,3}$	N_2 in D_2
$n_{4,4}$	N_2 in D_2
$n_{7,1}$	N_4 in D_4
$n_{7,2}$	N_5 in D_5
$n_{7,3}$	N_5 in D_5
$n_{7,4}$	N_3 in D_3
$n_{7,5}$	N_3 in D_3
$n_{7,6}$	N_2 in D_2
$n_{10,1}$	N_6 in D_6
$n_{10,2}$	N_3 in D_3
$n_{10,3}$	N_6 in D_6
$n_{10,4}$	N_3 in D_3
$n_{10,5}$	N_2 in D_2
$n_{10,6}$	N_3 in D_3
$n_{10,7}$	N_6 in D_6
$n_{10,8}$	N_3 in D_3
$n_{10,9}$	N_6 in D_6
$n_{10,10}$	N_3 in D_3
$n_{10,11}$	N_2 in D_2
$n_{10,12}$	N_3 in D_3
$n_{10,13}$	N_2 in D_2

TABLE 7. Complex nilpotent orbits (left) and nilpotent elements $n_{i,j}$ from Theorem 3.7 (right).

fam.	i	identity component Z°	preimages of generators of Z/Z°
N2	1	$\{(D(b^{-1}cd,a'),D(b,b'),D(c,c')^\intercal,D(d,d')^\intercal)$ with $a',b',c',d'\in\mathbb{C}$ and $b,c,d\in\mathbb{C}^\times$	
N3	2	$\{(B^{-\intercal},B,D(d^{-1},c')^\intercal,D(d,d')^\intercal)\ :\ B\in\mathrm{SL}(2,\mathbb{C}),d\in\mathbb{C}^\times,c',d'\in\mathbb{C}\}$	$(-I,I,-I,I)$
N4	16	$\{(L(b'+c'+d')^{-1},L(b'),L(c')^\intercal,L(d')^\intercal)\ :\ b',c',d'\in\mathbb{C}\}$	$(-I,-I,I,I),(-I,I,-I,I),(-L,L,L,L)$
N5	17	$\{(D(b)^{-1},D(b),L(d')^{-\intercal},L(d')^\intercal)\ :\ b\in\mathbb{C}^\times,d\in\mathbb{C}\}$	$(-I,I,-I,I),(-I,I,I,-I)$
N6	8	$(D(d^{-1},-(b'+d')),D(d^{-1},b'),D(d^{-1},c')^\intercal,D(d,d')^\intercal)$ with $d\in\mathbb{C}^\times$ and $b',c',d'\in\mathbb{C}$	$(-I,-I,I,I),(-I,I,-I,I)$
N7	27	1	$(-I,-I,I,I),(-I,I,-I,I),(-I,I,I,-I)$
N8	23	$\{(L(a'),I,L(a')^\intercal,L(a')^\intercal)\ :\ a'\in\mathbb{C}\}$	$(-I,-I,I,I),(-I,I,-I,I),(-I,I,I,-I)$
N9	12	$(M(c,d)^{-1}M(a,b)^{-1},M(c,d),L(u)^\intercal,M(a,b))$ with $a,b,c,d,u\in\mathbb{C}$ and $a^2=1+b^2$ and $c^2=1+d^2$	$(-I,I,-I,I),(L,L,L,L)$

TABLE 8. The centralisers $Z=Z_{\widehat{G}}(e)$ where e is the representative of the nilpotent orbit labelled i in Table 7; the notation is explained in (3.1).

of the generating invariants of $\mathbb{C}[\mathfrak{h}]^W$ are known to be $2,4,4,6$ (this can be read from Table 1 by setting

pol.	list of monomials
H	-8.9, 7.10, 6.11, -5.12, 4.13, -3.14, -2.15, 1.16
L	4.7.10.13, -4.7.9.14, -4.6.11.13, 4.6.9.15, 4.5.11.14, -4.5.10.15, -3.8.10.13, 3.8.9.14, 3.6.12.13, -3.6.9.16, -3.5.12.14, 3.5.10.16, 2.8.11.13, -2.8.9.15, -2.7.12.13, 2.7.9.16, 2.5.12.15, -2.5.11.16, -1.8.11.14, 1.8.10.15, 1.7.12.14, -1.7.10.16, -1.6.12.15, 1.6.11.16
M	-6.7.10.11, 6.7.9.12, 5.8.10.11, -5.8.9.12, 4.6.11.13, -4.6.9.15, -4.5.11.14, 4.5.9.16, -3.6.12.13, 3.6.10.15, 3.5.12.14, -3.5.10.16, -2.8.11.13, 2.8.9.15, 2.7.11.14, -2.7.9.16, -2.3.14.15, 2.3.13.16, 1.8.12.13, -1.8.10.15, -17.12.14, 1.7.10.16, 1.4.14.15, -1.4.13.16
D	-4.6.8.9.11.13, 4.6.8.9.9.15, -4.6.7.10.11.13, 4.6.7.9.12.13, 4.6.7.9.11.14, -4.6.7.9.9.16, 4.6.6.11.11.13, -4.6.6.9.11.15, 4.5.8.10.11.13, -4.5.8.9.10.15, -4.5.7.9.12.14, 4.5.7.9.10.16, -4.5.6.11.12.13, -4.5.6.11.11.14, 4.5.6.10.11.15, 4.5.6.9.11.16, 4.5.5.11.12.14, -4.5.5.10.11.16, 4.4.6.11.13.13, -4.4.6.9.13.15, -4.4.5.11.13.14, 4.4.5.9.14.15, 3.6.8.10.11.13, -3.6.8.9.10.15, -3.6.7.9.12.14, 3.6.7.9.10.16, -3.6.6.11.12.13, 3.6.6.9.12.15, -3.5.8.10.12.13, -3.5.8.10.11.14, 3.5.8.10.10.15, 3.5.8.9.12.14, 3.5.7.10.12.14, -3.5.7.10.10.16, 3.5.6.12.12.13, 3.5.6.11.12.14, -3.5.6.10.12.15, -3.5.6.9.12.16, -3.5.5.12.12.14, 3.5.5.10.12.16, -3.4.6.12.13.13, -3.4.6.11.13.14, 3.4.6.10.13.15, 3.4.6.9.13.16, 3.4.5.12.13.14, 3.4.5.11.14.14, -3.4.5.10.14.15, -3.4.5.9.14.16, 3.3.6.12.13.14, -3.3.6.10.13.16, -3.3.5.12.14.14, 3.3.5.10.14.16, 2.8.8.9.11.13, -2.8.8.9.9.15, -2.7.8.9.12.13, -2.7.8.9.11.14, 2.7.8.9.10.15, 2.7.8.9.9.16, 2.7.7.9.12.14, -2.7.7.9.10.16, -2.6.8.11.11.13, 2.6.8.9.11.15, 2.6.7.11.12.13, -2.6.7.9.12.15, 2.5.8.11.11.14, -2.5.8.10.11.15, 2.5.8.9.12.15, -2.5.8.9.11.16, -2.5.7.11.12.14, 2.5.7.10.11.16, -2.4.8.11.13.13, 2.4.8.9.13.15, 2.4.7.11.13.14, -2.4.7.9.14.15, -2.4.6.11.13.15, 2.4.6.9.15.15, 2.4.5.11.13.16, -2.4.5.9.15.16, 2.3.8.12.13.13, -2.3.8.10.13.15, 2.3.8.9.14.15, -2.3.8.9.13.16, -2.3.7.12.13.14, 2.3.7.10.13.16, 2.3.6.11.13.16, -2.3.6.9.15.16, 2.3.5.12.14.15, -2.3.5.12.13.16, -2.3.5.11.14.16, 2.3.5.9.16.16, 2.2.8.11.13.15, -2.2.8.9.15.15, -2.2.7.11.13.16, 2.2.7.9.15.16, -1.8.8.10.11.13, 1.8.8.9.10.15, 1.7.8.10.12.13, 1.7.8.10.11.14, -1.7.8.10.10.15, -1.7.8.9.10.16, -1.7.7.10.12.14, 1.7.7.10.10.16, 1.6.8.11.12.13, -1.6.8.9.12.15, -1.6.7.12.12.13, 1.6.7.10.12.15, -1.6.7.10.11.16, 1.6.7.9.12.16, -1.5.8.11.12.14, 1.5.8.10.11.16, 1.5.7.12.12.14, -1.5.7.10.12.16, 1.4.8.11.13.14, -1.4.8.9.14.15, -1.4.7.11.14.14, 1.4.7.10.14.15, -1.4.7.10.13.16, 1.4.7.9.14.16, 1.4.6.12.13.15, 1.4.6.11.14.15, -1.4.6.11.13.16, -1.4.6.10.15.15, -1.4.5.12.14.15, 1.4.5.10.15.16, -1.3.8.12.13.14, 1.3.8.10.13.16, 1.3.7.12.14.14, -1.3.7.10.14.16, -1.3.6.12.14.15, 1.3.6.10.15.16, 1.3.5.12.14.16, -1.3.5.10.16.16, -1.2.8.12.13.15, -1.2.8.11.14.15, 1.2.8.10.15.15, 1.2.8.9.15.16, 1.2.7.12.13.16, 1.2.7.11.14.16, -1.2.7.10.15.16, -1.2.7.9.16.16, 1.1.8.12.14.15, -1.1.8.10.15.16, -1.1.7.12.14.16, 1.1.7.10.16.16

TABLE 9. Generators of $\mathbb{C}[\mathfrak{g}_1]^{\widehat{G}}$: each generator is the sum of the listed monomials, where $i_1.i_2.i_3\ldots$ stands for $x_{i_1}x_{i_2}x_{i_3}\ldots$, and $-i_1.i_2.i_3\ldots$ represents $-x_{i_1}x_{i_2}x_{i_3}\ldots$

$n=4$ in the fourth row and recalling the isomorphism $\mathrm{SO}(4,\mathbb{C})=\mathrm{SL}(2,\mathbb{C})^2$). It follows that $\mathbb{C}[\mathfrak{g}_1]^{\widehat{G}}$ is generated by four homogeneous invariants of degrees $2,4,4,6$. Formulas for generating invariants have been determined by Luque & Thibon [36]; in this reference they are denoted H, L, M, D_{xt}. In Table 9 we give their explicit form, where we write D instead of D_{xt}. We have checked the correctness of these expressions by computer in the following way: Let $\mathbb{C}[\mathfrak{g}_1]_k$ denote the space of homogeneous polynomials of degree k. There is a canonical isomorphism of $\widehat{G}$-modules $\mathbb{C}[\mathfrak{g}_1]_k\to\mathrm{Sym}^k(\mathfrak{g}_1^*)$, where $\mathfrak{g}_1^*$ denotes the dual module of $\mathfrak{g}_1$. Under this isomorphism, every invariant spans a trivial 1-dimensional submodule. For $k=2,4,4,6$ we determined the trivial 1-dimensional submodules of $\mathrm{Sym}^k(\mathfrak{g}_1^*)$ by linear algebra methods using the Lie algebra of $\widehat{G}$; this way we found the same invariants as Luque & Thibon. We now define

$$\mathcal{F}\colon\mathfrak{g}_1\to\mathbb{C}^4,\quad \mathcal{F}(s)=(H(s),L(s),M(s),D(s)). \tag{5.1}$$

For a 4-tuple $v\in\mathbb{C}^4$ denote by $U_v=\{s\in\mathfrak{g}_1:\mathcal{F}(s)=v\}$ the corresponding fibre of $\mathcal{F}$; all these fibres partition $\mathfrak{g}_1$. Recall that $e\in\mathfrak{g}_1$ is nilpotent if and only if there are $g_1,g_2,\ldots\in\widehat{G}$ with $\lim_{i\to\infty}g_i(e)=0$, see [18, Proposition 1]. Since $\mathcal{F}$ is polynomial, this implies that $\mathcal{F}(0)=\lim_{i\to\infty}\mathcal{F}(g_i(e))=\mathcal{F}(e)$. In particular, if $p+e\in U_v$ is a mixed element, so $[p,e]=0$, then we can assume that each $g_i\in Z_{\widehat{G}}(p)$, and hence $\mathcal{F}(p+e)=\mathcal{F}(g_i(p+e))=\mathcal{F}(p+g_i(e))$, with limit $\mathcal{F}(p)=v$. By [18, Theorem 3], each fibre U_v consists of a single semisimple orbit $\widehat{G}p$ (with $p\in\mathfrak{g}_1$ semisimple such that $\mathcal{F}(p)=v$) along with the mixed orbits that have an element in $\widehat{G}p$ as their semisimple part; in particular, each fiber is the union of finitely many orbits, cf. [18, Theorem 4]. The different values $\mathcal{F}(s)$ with $s\in\mathfrak{h}^\circ_{\Pi_i}$ are listed in Table 10. Furthermore, in Table 11 we list generators of the ideal of the polynomial relations between these values.

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i	**invariant values $\mathcal{F}(s)$**
1	$(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2, -\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_4^2-\lambda_3^2\lambda_4^2, \lambda_1^2\lambda_2^2-\lambda_1^2\lambda_4^2-\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_4^2,$ $\lambda_1^4\lambda_2^2+\lambda_1^2\lambda_2^4-\lambda_1^2\lambda_2^2\lambda_3^2-\lambda_1^2\lambda_2^2\lambda_4^2-\lambda_1^2\lambda_3^2\lambda_4^2-\lambda_2^2\lambda_3^2\lambda_4^2+\lambda_3^4\lambda_4^2+\lambda_3^2\lambda_4^4)$
2	$(\lambda_1^2+\lambda_2^2+\lambda_3^2, -\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2, \lambda_1^2\lambda_2^2-\lambda_2^2\lambda_3^2, \lambda_1^4\lambda_2^2+\lambda_1^2\lambda_2^4-\lambda_1^2\lambda_2^2\lambda_3^2)$
3	$(2\lambda_1^2+2\lambda_1\lambda_2+2\lambda_2^2, -\lambda_1^4-2\lambda_1^3\lambda_2+2\lambda_1\lambda_2^3+\lambda_2^4, \lambda_1^4+2\lambda_1^3\lambda_2, 2\lambda_1^6+6\lambda_1^5\lambda_2+6\lambda_1^4\lambda_2^2+2\lambda_1^3\lambda_2^3)$
4	$(\lambda_1^2+\lambda_2^2, 0, -\lambda_1^2\lambda_2^2, 0)$
5	$(\lambda_1^2+\lambda_2^2, \lambda_1^2\lambda_2^2, 0, 0)$
6	$(\lambda_1^2+\lambda_2^2, -\lambda_1^2\lambda_2^2, \lambda_1^2\lambda_2^2, \lambda_1^4\lambda_2^2+\lambda_1^2\lambda_2^4),$
7	$(2\lambda_1^2, 0, -\lambda_1^4, 0)$
8	$(2\lambda_1^2, \lambda_1^4, 0, 0)$
9	$(2\lambda_1^2, -\lambda_1^4, \lambda_1^4, 2\lambda_1^6)$
10	$(\lambda_1^2, 0, 0, 0)$

TABLE 10. Invariant values $\mathcal{F}(s)$ for $s \in \mathfrak{h}^\circ_{\Pi_i}$ with parameters $\lambda_1, \ldots, \lambda_4$ as in Table 2.

i	**relations**
2	$H^5LMD-H^4L^2M^2-H^4LD^2+H^4MD^2-8H^3L^2MD+8H^3LM^2D+8H^2L^3M^2-8H^2L^2M^3$ $-H^3D^3+8H^2L^2D^2-46H^2LMD^2+8H^2M^2D^2+16HL^3MD+64HL^2M^2D+16HLM^3D-16L^4M^2$ $-32L^3M^3-16L^2M^4+36HLD^3-36HMD^3-16L^3D^2-24L^2MD^2+24LM^2D^2+16M^3D^2+27D^4$
3	$H^3D-2H^2LM-4HLD+4HMD+8L^2M-8LM^2-18D^2,$ $H^4-8H^2L+8H^2M-24HD+16L^2+16LM+16M^2$
4	$D,\ L$
5	$D,\ M$
6	$L+M,\ HM-D$
7	$D,\ L,\ H^2+4M$
8	$D,\ M,\ H^2-4L$
9	$M^3-\frac{1}{4}D^2,\ L+M,\ HD-4M^2,\ HM-D,\ H^2-4M$
10	$D,\ M,\ L$

TABLE 11. Generators of the ideal of the polynomial relations between the evaluated invariants for the i-th semisimple family. The elements H, L, M, D correspond to the first to fourth entries in the coordinate vector given in the corresponding row of Table 10.

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