

A matrix formula for Schur complements of nonnegative selfadjoint linear relations

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Abstract. If a nonnegative selfadjoint linear relation A in a Hilbert space and a closed subspace \mathcal{S} are assumed to satisfy that the domain of A is invariant under the orthogonal projector onto \mathcal{S} , then A admits a particular matrix representation with respect to the decomposition $\mathcal{S} \oplus \mathcal{S}^\perp$. This matrix representation of A is used to give explicit formulae for the Schur complement of A on \mathcal{S} as well as the \mathcal{S} -compression of A .

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1. Introduction

Given a nonnegative selfadjoint linear relation A in a Hilbert space \mathcal{H} and a closed subspace \mathcal{S} of \mathcal{H} , it is not always the case that A admits a 2×2 block matrix representation with respect to the decomposition $\mathcal{S} \oplus \mathcal{S}^\perp$. On the other hand, if it does, the matrix representation need not be unique. Results on this subject can be found in [8, 14, 11, 6, 10]. Under the hypothesis that $\text{dom}(A)$ (the domain of A) is an invariant subspace for the orthogonal projection onto \mathcal{S} , $P_{\mathcal{S}}$ (that is $\mathcal{D}_1 := P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$), we show that A can be represented by a 2×2 block matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c and d are linear relations. Furthermore, A admits a specific representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ similar to the one for bounded operators (cf. [1], [7, Lema A.1]), in the sense that a and d in this decomposition are nonnegative selfadjoint linear relations and there exists a contraction $g : \mathcal{S}^\perp \rightarrow \mathcal{S}$ such that $b = a^{1/2}g d^{1/2}|_{P_{\mathcal{S}^\perp}(\text{dom}(A))}$ and $c = d^{1/2}g^* a^{1/2}|_{P_{\mathcal{S}}(\text{dom}(A))}$.

In [3], Arlinskiĭ proves that for \leq the forms order [12, 4], the maximum of the following set of nonnegative selfadjoint linear relations,

$$\{X : 0 \leq X \leq A, \text{ran}(X) \subseteq \mathcal{S}^\perp\}$$

always exists and he defines the Schur complement of the relation A with respect to \mathcal{S} , $A_{/\mathcal{S}}$, as this maximum. Under the invariance condition mentioned above, we give a matrix formula for $A_{/\mathcal{S}}$ in terms of the matrix coefficients of A ; namely,

$$A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & T^*T \end{pmatrix}$$

with $T := \overline{D_g d^{1/2}|_{P_{\mathcal{S}^\perp}(\text{dom}(A))}}$, where $D_g := (1 - g^*g)^{1/2}$ is the defect operator associated to the matrix representation of A . We also give an alternate proof of the existence of the Schur complement. This formula is an extension of the well known formula by Anderson and Trapp for bounded operators [1]. We also define the \mathcal{S} -compression $A_{\mathcal{S}}$ of A . If we assume further that $\overline{\text{dom}(A^{1/2})}$ is an invariant subspace of the orthogonal projection $P_{\mathcal{L}}$, where $\mathcal{L} := \overline{A^{1/2}(\mathcal{D}_1)} \cap \overline{\text{dom}(A)}$, then we obtain Pekarev-type formulae for $A_{/\mathcal{S}}$ and $A_{\mathcal{S}}$ [15], and we show that $A = A_{\mathcal{S}} + A_{/\mathcal{S}}$.

The paper is organized as follows. In Section 2 we outline some background material, primarily on linear relations. Section 3 is devoted to the problem of representing a selfadjoint linear relation A as a 2×2 relation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the decomposition $\mathcal{S} \oplus \mathcal{S}^\perp$. In Proposition 3.5, we prove that the relation A admits a 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ if and only if its operator part A_0 admits a block matrix representation with respect to $\overline{\mathcal{D}_1}$ plus the extra condition $\mathcal{S} \ominus \mathcal{D}_1 \subseteq \text{mul}(A)$ (the multivalued part of A). The main result of this section is Theorem 3.11, where this matrix representation of A is fully described when A is nonnegative. In Section 4, we again use the matrix representation of the nonnegative selfadjoint linear relation A to derive formulae for the Schur complement and compression of A .

2. Preliminaries

Throughout, all spaces are complex and separable Hilbert spaces. As usual, the direct sum of two subspaces \mathcal{M} and \mathcal{N} of a Hilbert space \mathcal{H} is indicated by $\mathcal{M} + \mathcal{N}$ and the orthogonal direct sum by $\mathcal{M} \oplus \mathcal{N}$. The orthogonal complement of a subspace $\mathcal{M} \subseteq \mathcal{H}$ is written as \mathcal{M}^\perp , or $\mathcal{H} \ominus \mathcal{M}$ interchangeably. The symbol $P_{\mathcal{M}}$ denotes the orthogonal projection with range \mathcal{M} .

The space of everywhere defined bounded linear operators from \mathcal{H} to \mathcal{K} is written as $L(\mathcal{H}, \mathcal{K})$, or $L(\mathcal{H})$ when $\mathcal{H} = \mathcal{K}$. The identity operator on \mathcal{H} is written as 1 , or $1_{\mathcal{H}}$ if it is necessary to disambiguate.

The notion of Schur complement (or shorted operator) of A to \mathcal{S} for a nonnegative selfadjoint operator $A \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace, was introduced by M.G. Krein [13]. When \leq is the usual order in $L(\mathcal{H})$, he proved that the set $\{X \in L(\mathcal{H}) : 0 \leq X \leq A \text{ and } \text{ran}(X) \subseteq \mathcal{S}^\perp\}$ has a maximum element, which he defined as the Schur complement $A_{/\mathcal{S}}$ of A to \mathcal{S} . This notion was later rediscovered by Anderson and Trapp [1]. If A is represented as the 2×2 block matrix $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ with respect to the decomposition of $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$, they established the formula

$$A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & d - y^*y \end{pmatrix}$$

where y is the unique solution of the equation $b = a^{1/2}x$ such that the range inclusion $\text{ran}(y) \subseteq \overline{\text{ran}}(a)$ holds.

Although familiarity with the theory of linear relations is presumed, some background material from [9] is summarized below.

A linear relation (l.r.) from a Hilbert space \mathcal{H} to a Hilbert space \mathcal{K} is a linear subspace T of the cartesian product $\mathcal{H} \times \mathcal{K}$. The domain, range, null space or kernel and multivalued part of T is denoted by $\text{dom}(T)$, $\text{ran}(T)$, $\text{ker}(T)$ and $\text{mul}(T)$, respectively. When $\text{mul}(T) = \{0\}$, T is an operator; in this case, the operator T is uniquely determined by $Tx = y$ for $(x, y) \in T$.

The sum of two linear relations T and S from \mathcal{H} to \mathcal{K} is the linear relation defined by

$$T + S := \{(x, y + z) : (x, y) \in T \text{ and } (x, z) \in S\}.$$

The componentwise sum is the linear relation defined by

$$T \hat{+} S := \{(x_1 + x_2, y + z) : (x_1, y) \in T \text{ and } (x_2, z) \in S\}.$$

The componentwise sum of T and S with $T \perp S$ is denoted by $T \hat{\oplus} S$. Let T be a linear relation from \mathcal{H} to a Hilbert space \mathcal{E} and let S be a linear relation from \mathcal{E} to \mathcal{K} then the product ST is a linear relation from \mathcal{H} to \mathcal{K} defined by

$$ST := \{(x, y) : (x, z) \in T \text{ and } (z, y) \in S \text{ for some } z \in \mathcal{E}\}.$$

If $T \in L(\mathcal{H}, \mathcal{E})$ then $(x, y) \in ST$ if and only if $(Tx, y) \in S$.

The closure of a linear relation from \mathcal{H} to \mathcal{K} is the closure of the linear subspace in $\mathcal{H} \times \mathcal{K}$, when the product is provided with the product topology. The closure of an operator need not be an operator; if it is then one speaks of a *closable operator*. The relation T is called *closed* when it is closed as a subspace of $\mathcal{H} \times \mathcal{K}$. The *adjoint* relation from \mathcal{K} to \mathcal{H} is defined by

$$T^* := JT^\perp = (JT)^\perp,$$

where $J(x, y) = (y, -x)$. The adjoint is automatically a closed linear relation and, if \overline{T} denotes the closure of T , then $\overline{T} = T^{**} := (T^*)^*$. By definition, it is immediate that $\overline{T^*} = T^*$. Clearly,

$$T^* = \{(x, y) \in \mathcal{K} \times \mathcal{H} : \langle g, x \rangle = \langle f, y \rangle \text{ for all } (f, g) \in T\}.$$

Hence $\text{mul}(T^*) = \text{dom}(T)^\perp$ and $\text{ker}(T^*) = \text{ran}(T)^\perp$. Then, if T is closed both $\text{ker}(T)$ and $\text{mul}(T)$ are closed subspaces.

Let T be a linear relation from \mathcal{H} to a Hilbert space \mathcal{E} and let S be a linear relation from \mathcal{E} to \mathcal{K} then

$$T^*S^* \subset (ST)^* \tag{2.1}$$

and there is equality in (2.1) if $S \in L(\mathcal{E}, \mathcal{K})$. If T and S are linear relations from \mathcal{H} to \mathcal{K} then

$$T^* + S^* \subset (T + S)^* \tag{2.2}$$

and there is equality in (2.2) if $S \in L(\mathcal{H}, \mathcal{K})$.

Let T be a (not necessarily closed) linear relation in \mathcal{H} . Define $T_0 := T \cap (\overline{\text{dom}}(T) \times \overline{\text{dom}}(T^*))$ and $T_{\text{mul}} := \{0\} \times \text{mul}(T)$. Then T_0 is a closable operator from $\overline{\text{dom}}(T)$ to $\overline{\text{dom}}(T^*)$ [11].

Theorem 2.1 ([11, Theorem 3.9]). *Let T be a (not necessarily closed) linear relation in \mathcal{H} . If there exists a linear relation B in \mathcal{H} such that*

$$T = B \hat{+} T_{\text{mul}}, \quad \text{ran}(B) \subseteq \overline{\text{dom}}(T^*), \quad (2.3)$$

then the sum in (2.3) is direct and B is a closable operator which coincides with T_0 . In particular, the decomposition of T in (2.3) is unique.

Hence if T admits a componentwise sum decomposition of the form (2.3) then, since $\overline{\text{dom}}(T^*) = \text{mul}(\overline{T})^\perp \subseteq \text{mul}(T)^\perp$, it follows that

$$T = T_0 \hat{+} T_{\text{mul}}. \quad (2.4)$$

We say that T is *decomposable* if T admits the componentwise sum decomposition (2.3), or equivalently, (2.4).

In particular, if T is a closed linear relation in \mathcal{H} then $\text{mul}(T) = \text{dom}(T^*)^\perp$ and T is decomposable and (2.4) is valid. In this case, T_0 is a closed operator from $\overline{\text{dom}}(T)$ to $\overline{\text{dom}}(T^*)$ and T_{mul} is a closed linear relation. Also, $\text{dom}(T_0) = \text{dom}(T)$ and $\text{ran}(T_0) \subseteq \overline{\text{dom}}(T^*)$. The operator part T_0 is densely defined in $\overline{\text{dom}}(T)$ and maps into $\overline{\text{dom}}(T^*)$. The operator parts T_0 and $(T^*)_0$ are connected by

$$(T_0)^\times = (T^*)_0, \quad (2.5)$$

where $(T_0)^\times$ denotes the adjoint of T_0 when viewed as an operator from $\overline{\text{dom}}(T)$ to $\overline{\text{dom}}(T^*)$.

A linear relation T in \mathcal{H} is *symmetric* if $T \subset T^*$, *selfadjoint* if $T = T^*$ and *nonnegative* if $\langle y, x \rangle \geq 0$ for all $(x, y) \in T$. If T is a nonnegative selfadjoint linear relation we write $T \geq 0$.

Lemma 2.2. *Let T be a closed linear relation in \mathcal{H} and suppose that $T = T_0 \hat{+} T_{\text{mul}}$ as in (2.4). Then T is selfadjoint if and only if $\overline{\text{dom}}(T^*) = \overline{\text{dom}}(T)$ and T_0 is a selfadjoint operator in $\overline{\text{dom}}(T)$.*

Proof. If $T = T^*$ then clearly $\overline{\text{dom}}(T^*) = \overline{\text{dom}}(T)$ and, by (2.5), $(T_0)^\times = (T^*)_0 = T_0$ [9]. Conversely, suppose that $\overline{\text{dom}}(T^*) = \overline{\text{dom}}(T)$ and T_0 is a selfadjoint operator in $\overline{\text{dom}}(T)$. Then $\text{mul}(T) = \text{dom}(T^*)^\perp = \text{dom}(T)^\perp = \text{mul}(T^*)$ and, by (2.5), $(T^*)_0 = (T_0)^\times = T_0$. So that

$$T^* = (T^*)_0 \hat{+} (\{0\} \times \text{mul}(T^*)) = T_0 \hat{+} (\{0\} \times \text{mul}(T)) = T.$$

□

Next a well-known result due to von Neumann (see [16, Proposition 3.18]) is extended to closed linear relations:

Theorem 2.3 ([9, Lemma 2.4]). *Let T be a closed linear relation in \mathcal{H} . Then T^*T is a nonnegative selfadjoint linear relation in \mathcal{H} . Furthermore,*

$$T^*T = T^*T_0 = T_0^*T_0, \quad (2.6)$$

where T_0 is the operator part of T . In particular

$$\ker(T^*T) = \ker(T) = \ker(T_0) \text{ and } \text{mul}(T^*T) = \text{mul}(T^*) = \text{mul}(T_0^*). \quad (2.7)$$

Also, the operator part of T^*T is

$$(T^*T)_0 = (T^*)_0 T_0 = (T_0)^\times T_0. \quad (2.8)$$

Let $T \geq 0$ be a linear relation in \mathcal{H} . Since T is selfadjoint (and therefore closed), $\text{mul}(T) = \overline{\text{dom}(T)}^\perp$. Hence $\mathcal{H} = \overline{\text{dom}(T)} \oplus \text{mul}(T)$. In this case T can be written as $\underline{T} = T_0 \hat{\oplus} T_{\text{mul}}$ where, by Lemma 2.2, T_0 is a nonnegative selfadjoint operator in $\overline{\text{dom}(T)}$. For $T \geq 0$, the (unique) nonnegative selfadjoint *square root* of T is defined by

$$T^{1/2} := T_0^{1/2} \hat{\oplus} (\{0\} \times \text{mul}(T)),$$

where $T_0^{1/2}$ is the square root of T_0 [5]. Then, $\text{mul}(T^{1/2}) = \text{mul}(T)$, $T_0^{1/2} = (T^{1/2})_0$ and $\overline{\text{dom}(T)} = \overline{\text{dom}(T^{1/2})}$ [9, Lemma 2.5]. Also, by (2.7),

$$\ker(T) = \ker(T^{1/2}) = \ker(T_0). \quad (2.9)$$

There is a natural ordering for nonnegative selfadjoint relations in \mathcal{H} . For two nonnegative selfadjoint relations A and B , we write $A \leq B$ if

$$\text{dom}(B_0^{1/2}) \subseteq \text{dom}(A_0)^{1/2} \text{ and } \|A_0^{1/2}u\| \leq \|B_0^{1/2}u\|, \text{ for all } u \in \text{dom}(B_0^{1/2}). \quad (2.10)$$

The following is a result given in [9, Theorem 3.4]; we include its proof for the sake of completeness.

Lemma 2.4. *Let A, B be nonnegative selfadjoint linear relations such that $A \leq B$. Then, there exists a contraction $W \in L(\overline{\text{dom}(B)}, \overline{\text{dom}(A)})$ such that*

$$WB_0^{1/2} \subset A_0^{1/2} \quad (2.11)$$

where A_0 and B_0 are the operator parts of A and B , respectively.

Proof. Since $A \leq B$, $\text{dom}(B_0^{1/2}) \subseteq \text{dom}(A_0)^{1/2}$ and

$$\|A_0^{1/2}u\| \leq \|B_0^{1/2}u\|, \quad (2.12)$$

for every $u \in \text{dom}(B_0^{1/2})$. Define the linear relation

$$W := \{(B_0^{1/2}h, A_0^{1/2}h) : h \in \text{dom}(B_0^{1/2})\}.$$

If $(x, y) \in W$ then $(x, y) = (B_0^{1/2}h, A_0^{1/2}h)$ for some $h \in \text{dom}(B_0^{1/2})$. Then, by (2.12),

$$\|y\| = \|A_0^{1/2}h\| \leq \|B_0^{1/2}h\| = \|x\|.$$

So that W is a contraction from $\text{ran}(B_0^{1/2})$ to $\text{ran}(A_0^{1/2})$. Then W has a unique extension named again W from $\overline{\text{ran}(B_0^{1/2})} \subseteq \overline{\text{dom}(B)}$ to $\overline{\text{ran}(A_0^{1/2})} \subseteq \overline{\text{dom}(A)}$. Defining W as zero in $\overline{\text{dom}(B)} \ominus \text{ran}(B_0^{1/2})$, the result follows. \square

If T is a linear relation in $\mathcal{H} \times \mathcal{K}$ and \mathcal{S} is a subspace of $\text{dom}(T)$ then

$$T|_{\mathcal{S}} := \{(x, y) \in T : x \in \mathcal{S}\} \text{ and } T(\mathcal{S}) := \{y : (x, y) \in T \text{ for some } x \in \mathcal{S}\}.$$

A linear subspace \mathcal{D} of $\text{dom}(T)$ is a *core* of T if the set $T|_{\mathcal{D}}$ is dense in T , in which case $\overline{T(\mathcal{D})} = \overline{\text{ran}T}$. If T admits the sum decomposition $T = T_0 \hat{\oplus} T_{\text{mul}}$ as in (2.4) and \mathcal{D} is a core of T_0 then \mathcal{D} is a core of T . If T is a selfadjoint linear relation in \mathcal{H} and \mathcal{D} is a core of T then $(T|_{\mathcal{D}})^* = T$.

3. Matrix decomposition of nonnegative selfadjoint relations

Let \mathcal{S} be a closed subspace of \mathcal{H} and let $a \subseteq \mathcal{S} \times \mathcal{S}$, $b \subseteq \mathcal{S}^\perp \times \mathcal{S}$, $c \subseteq \mathcal{S} \times \mathcal{S}^\perp$ and $d \subseteq \mathcal{S}^\perp \times \mathcal{S}^\perp$ be linear relations. In [10, Definition 5.1], the linear relation in $\mathcal{H} \times \mathcal{H}$ generated by the blocks a , b , c and d is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \left\{ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} w_1 + z_1 \\ w_2 + z_2 \end{pmatrix} \right) : \begin{array}{l} (x_1, w_1) \in a, (x_2, z_1) \in b \\ (x_1, w_2) \in c, (x_2, z_2) \in d \end{array} \right\}.$$

On the other hand, given a linear relation A in \mathcal{H} and \mathcal{S} a closed subspace of \mathcal{H} , we say that A admits a 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ if there exist blocks $a \subseteq \mathcal{S} \times \mathcal{S}$, $b \subseteq \mathcal{S}^\perp \times \mathcal{S}$, $c \subseteq \mathcal{S} \times \mathcal{S}^\perp$ and $d \subseteq \mathcal{S}^\perp \times \mathcal{S}^\perp$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In this case, it is easy to check that:

1. $\text{dom}(a) \cap \text{dom}(c) = \mathcal{S} \cap \text{dom}(A)$ and $\text{dom}(b) \cap \text{dom}(d) = \mathcal{S}^\perp \cap \text{dom}(A)$.
2. $\text{mul}(a) + \text{mul}(b) = \mathcal{S} \cap \text{mul}(A)$ and $\text{mul}(c) + \text{mul}(d) = \mathcal{S}^\perp \cap \text{mul}(A)$.

Lemma 3.1. *Let \mathcal{M} and \mathcal{S} be subspaces of \mathcal{H} with \mathcal{S} closed. Then the following are equivalent:*

- (i) $P_{\mathcal{S}}(\mathcal{M}) \subseteq \mathcal{M}$;
- (ii) $\mathcal{M} = \mathcal{S} \cap \mathcal{M} \oplus \mathcal{S}^\perp \cap \mathcal{M}$;
- (iii) $P_{\mathcal{S}}(\mathcal{M}) = \mathcal{S} \cap \mathcal{M}$.

Theorem 3.2 (cf. [10, Theorem 5.1]). *Let A be a linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} . Then the following are equivalent:*

- (i) A admits a 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$;
- (ii) $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ and $P_{\mathcal{S}}(\text{mul}(A)) \subseteq \text{mul}(A)$;
- (iii) A admits a representation as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (3.1)$$

where $a := P_{\mathcal{S}}A|_{\mathcal{S}}$, $b := P_{\mathcal{S}}A|_{\mathcal{S}^\perp}$, $c := P_{\mathcal{S}^\perp}A|_{\mathcal{S}}$ and $d := P_{\mathcal{S}^\perp}A|_{\mathcal{S}^\perp}$.

Lemma 3.3. *Let A be a selfadjoint linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} . If $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ then $P_{\mathcal{S}}(\text{mul}(A)) \subseteq \text{mul}(A)$.*

Proof. Since A is selfadjoint, $\text{mul}(A) = \text{dom}(A)^\perp$. Let $y \in \text{mul}(A)$. Then, for all $h \in \text{dom}(A)$

$$\langle P_{\mathcal{S}}y, h \rangle = \langle y, P_{\mathcal{S}}h \rangle = 0,$$

because $P_{\mathcal{S}}h \in \text{dom}(A)$. Therefore $P_{\mathcal{S}}y \in \text{dom}(A)^\perp = \text{mul}(A)$. □

Let A be a selfadjoint linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} . Define

$$\mathcal{D}_1 := \mathcal{S} \cap \text{dom}(A), \quad \mathcal{D}_2 := \mathcal{S}^\perp \cap \text{dom}(A), \quad (3.2)$$

$$\mathcal{M}_1 := \mathcal{S} \cap \text{mul}(A) \text{ and } \mathcal{M}_2 := \mathcal{S}^\perp \cap \text{mul}(A). \quad (3.3)$$

If $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ then, by Lemmas 3.1 and 3.3,

$$\text{dom}(A) = \mathcal{D}_1 \oplus \mathcal{D}_2 \text{ and } \text{mul}(A) = \mathcal{M}_1 \oplus \mathcal{M}_2, \quad (3.4)$$

and A admits a 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$.

Define $\mathcal{N}_i := \overline{\mathcal{D}_i}$, for $i = 1, 2$. Clearly, $\overline{\text{dom}}(A) = \mathcal{N}_1 \oplus \mathcal{N}_2$.

Lemma 3.4. *Let A be a selfadjoint linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} . Then, the following are equivalent:*

- (i) $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$;
- (ii) $P_{\mathcal{M}_1}(\text{dom}(A)) \subseteq \text{dom}(A)$ and $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$;
- (iii) $P_{\mathcal{N}_2}(\text{dom}(A)) \subseteq \text{dom}(A)$ and $\mathcal{S}^\perp = \mathcal{N}_2 \oplus \mathcal{M}_2$.

In this case, $\mathcal{N}_1 = \mathcal{S} \cap \overline{\text{dom}}(A)$ and $\mathcal{N}_2 = \mathcal{S}^\perp \cap \overline{\text{dom}}(A)$.

Proof. (i) \Leftrightarrow (ii): If $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ then by (3.4), $\overline{\text{dom}}(A) = \mathcal{N}_1 \oplus \mathcal{N}_2$, $\mathcal{D}_1 = \mathcal{N}_1 \cap \text{dom}(A)$ and $\mathcal{D}_2 = \mathcal{N}_2 \cap \text{dom}(A)$. Therefore

$$\text{dom}(A) = \mathcal{N}_1 \cap \text{dom}(A) \oplus \mathcal{N}_2 \cap \text{dom}(A). \quad (3.5)$$

Hence $P_{\mathcal{M}_1}(\text{dom}(A)) = \mathcal{D}_1 \subseteq \text{dom}(A)$.

Also $\overline{\text{dom}}(A) \subseteq (\mathcal{S} \ominus \mathcal{N}_1)^\perp$ or, equivalently, $\mathcal{S} \ominus \mathcal{N}_1 \subseteq \text{mul}(A)$. In fact, $(\mathcal{S} \ominus \mathcal{N}_1)^\perp = \mathcal{S}^\perp \oplus \mathcal{N}_1 \supseteq \mathcal{N}_2 \oplus \mathcal{N}_1 = \overline{\text{dom}}(A)$. Hence

$$\mathcal{S} = \mathcal{N}_1 \oplus (\mathcal{S} \ominus \mathcal{N}_1) \subseteq \mathcal{N}_1 \oplus (\mathcal{S} \cap \text{mul}(A)) = \mathcal{N}_1 \oplus \mathcal{M}_1 \subseteq \mathcal{S}.$$

Conversely, suppose that $P_{\mathcal{M}_1}(\text{dom}(A)) \subseteq \text{dom}(A)$ and $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$. Then $P_{\mathcal{S}} = P_{\mathcal{N}_1} + P_{\mathcal{M}_1}$. Since $\text{dom}(A) \subseteq \text{mul}(A)^\perp \subseteq \mathcal{M}_1^\perp$, it follows that

$$P_{\mathcal{S}}(\text{dom}(A)) = (P_{\mathcal{N}_1} + P_{\mathcal{M}_1})(\text{dom}(A)) = P_{\mathcal{N}_1}(\text{dom}(A)) \subseteq \text{dom}(A).$$

(i) \Leftrightarrow (iii): It follows as (i) \Leftrightarrow (ii) using that $P_{\mathcal{S}^\perp}(\text{dom}(A)) \subseteq \text{dom}(A)$.

In this case, $\mathcal{N}_1 = \mathcal{S} \cap \overline{\text{dom}}(A)$. The inclusion $\mathcal{N}_1 = \mathcal{S} \cap \overline{\text{dom}}(A) \subseteq \mathcal{S} \cap \text{dom}(A)$ always holds. Conversely, if $x \in \mathcal{S} \cap \overline{\text{dom}}(A)$ write $x = x_1 + x_2$, with $x_1 \in \mathcal{N}_1$ and $x_2 \in \mathcal{N}_2$. Then $x_2 = x - x_1 \in \mathcal{S} \cap \mathcal{S}^\perp$. So that $x_2 = 0$. Likewise, $\mathcal{N}_2 = \mathcal{S}^\perp \cap \overline{\text{dom}}(A)$. \square

Now, suppose that the selfadjoint linear relation A is written as

$$A = A_0 \hat{\oplus} A_{\text{mul}}, \quad (3.6)$$

where A_0 is the selfadjoint operator part of A in $\overline{\text{dom}}(A)$.

Proposition 3.5. *Let A be a selfadjoint linear relation in \mathcal{H} , let \mathcal{S} be a closed subspace of \mathcal{H} and suppose that A is written as in (3.6). Then A admits a 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ if and only if A_0 admits a 2×2 block matrix representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$ and $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$, where $\mathcal{N}_1 = \overline{\mathcal{D}_1}$, $\mathcal{N}_2 = \overline{\mathcal{D}_2}$, and $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{M}_1 are defined as in (3.2) and (3.3).*

Proof. If A admits a 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$, by Theorem 3.2, $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Then, by Lemma 3.4, equation (3.5) follows and $P_{\mathcal{N}_1 // \mathcal{N}_2}(\text{dom}(A_0)) \subseteq \text{dom}(A_0)$, where $P_{\mathcal{N}_1 // \mathcal{N}_2}$ is the orthogonal projection onto \mathcal{N}_1 in $L(\text{dom}(A_0))$. Therefore, by Theorem 3.2 the linear operator A_0 admits a 2×2 block matrix representation (in $\text{dom}(A_0)$) with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$ and, by Lemma 3.4, $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$. Conversely, if the linear operator A_0 admits a 2×2 block matrix representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$, by Theorem 3.2, $P_{\mathcal{N}_1 // \mathcal{N}_2}(\text{dom}(A_0)) \subseteq \text{dom}(A_0)$. So that, by Lemma 3.1, equation (3.5) follows. Then, $P_{\mathcal{N}_1}(\text{dom}(A)) \subseteq$

$\text{dom}(A)$ and, since $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$, by Lemma 3.4, $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Hence, by Theorem 3.2, A admits a 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$. \square

Corollary 3.6. *Let A be a selfadjoint linear relation in \mathcal{H} , let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ and suppose that A is written as in (3.6).*

If A_0 admits the representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$, $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, then A admits the representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$,

$$A = \begin{pmatrix} a_0 \hat{\oplus} (\{0\} \times \mathcal{M}'_1) & b_0 \hat{\oplus} (\{0\} \times \mathcal{M}''_1) \\ c_0 \hat{\oplus} (\{0\} \times \mathcal{M}'_2) & d_0 \hat{\oplus} (\{0\} \times \mathcal{M}''_2) \end{pmatrix},$$

where $\mathcal{M}'_1, \mathcal{M}''_1$ are subspaces of \mathcal{S} and $\mathcal{M}'_2, \mathcal{M}''_2$ are subspaces of \mathcal{S}^\perp such that $\mathcal{M}'_1 + \mathcal{M}''_1 = \mathcal{M}_1$ and $\mathcal{M}'_2 + \mathcal{M}''_2 = \mathcal{M}_2$.

Conversely, if A admits the representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A_0 admits the representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$,

$$A_0 = \begin{pmatrix} P_{\mathcal{N}_1}a & P_{\mathcal{N}_1}b \\ P_{\mathcal{N}_2}c & P_{\mathcal{N}_2}d \end{pmatrix}.$$

Proof. Suppose that A_0 admits the representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}.$$

Set $a := a_0 \hat{\oplus} \{0\} \times \mathcal{M}'_1$, $b := b_0 \hat{\oplus} \{0\} \times \mathcal{M}''_1$, $c := c_0 \hat{\oplus} \{0\} \times \mathcal{M}'_2$, $d := d_0 \hat{\oplus} \{0\} \times \mathcal{M}''_2$. Since $\mathcal{M}'_1, \mathcal{M}''_1 \subseteq \mathcal{M}_1$, $\mathcal{M}'_2, \mathcal{M}''_2 \subseteq \mathcal{M}_2$, $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$ and $\mathcal{S}^\perp = \mathcal{N}_2 \oplus \mathcal{M}_2$, it is clear that $a \subseteq \mathcal{S} \times \mathcal{S}$, $b \subseteq \mathcal{S}^\perp \times \mathcal{S}$, $c \subseteq \mathcal{S} \times \mathcal{S}^\perp$ and $d \subseteq \mathcal{S}^\perp \times \mathcal{S}^\perp$. Also,

$$\begin{aligned} \text{dom} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \text{dom}(a) \cap \text{dom}(c) \oplus \text{dom}(b) \cap \text{dom}(d) \\ &= \text{dom}(a_0) \cap \text{dom}(c_0) \oplus \text{dom}(b_0) \cap \text{dom}(d_0) \\ &= \mathcal{N}_1 \cap \text{dom}(A) \oplus \mathcal{N}_2 \cap \text{dom}(A) \\ &= \mathcal{D}_1 \oplus \mathcal{D}_2 = \text{dom}(A), \end{aligned}$$

and

$$\begin{aligned} \text{mul} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \text{mul}(a) + \text{mul}(b) \oplus \text{mul}(c) + \text{mul}(d) \\ &= \mathcal{M}'_1 + \mathcal{M}''_1 \oplus \mathcal{M}'_2 + \mathcal{M}''_2 = \mathcal{M}_1 \oplus \mathcal{M}_2 = \text{mul}(A). \end{aligned}$$

Let $(x, y) \in A = A_0 \hat{\oplus} (\{0\} \times \text{mul}(A))$. Then there exists $m \in \text{mul}(A)$ such that $(x, y) = (x, A_0x) + (0, m)$. Then $x = x_1 + x_2$ for some $x_1 \in \mathcal{D}_1$ and $x_2 \in \mathcal{D}_2$ and $m = m_1 + m_2$ for some $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$. Since $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$, there exist $m'_1 \in \mathcal{M}'_1$, $m''_1 \in \mathcal{M}''_1$, $m'_2 \in \mathcal{M}'_2$ and $m''_2 \in \mathcal{M}''_2$ such that $m_1 = m'_1 + m''_1$

and $m_2 = m'_2 + m''_2$. Then

$$\begin{aligned} (x, y) &= (x, A_0x) + (0, m) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) + \left(0, \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a_0x_1 + b_0x_2 + m_1 \\ c_0x_1 + d_0x_2 + m_2 \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a_0x_1 + b_0x_2 + m'_1 + m''_1 \\ c_0x_1 + d_0x_2 + m'_2 + m''_2 \end{pmatrix} \right). \end{aligned}$$

Now, since $(x_1, a_0x_1 + m'_1) = (x_1, a_0x_1) + (0, m'_1) \in a$, $(x_2, b_0x_2 + m''_1) = (x_2, b_0x_2) + (0, m''_1) \in b$, $(x_1, c_0x_1 + m'_2) = (x_1, c_0x_1) + (0, m'_2) \in c$ and $(x_2, d_0x_2 + m''_2) = (x_2, d_0x_2) + (0, m''_2) \in d$, it follows that $(x, y) \in \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence, $A \subset \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and, since $\text{dom}(A) = \text{dom} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\text{mul}(A) = \text{mul} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, by [9, Corollary 2.2], $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Conversely, suppose that A is represented as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set $a_0 := P_{\mathcal{N}_1}a$, $b_0 := P_{\mathcal{N}_1}b$, $c_0 := P_{\mathcal{N}_2}c$ and $d_0 := P_{\mathcal{N}_2}d$. Then a_0 is an operator in \mathcal{N}_1 . In fact, if $(0, y) \in a_0$, then there exists $z \in \mathcal{S}$ such that $(0, z) \in a$ and $y = P_{\mathcal{N}_1}z$. Therefore, $z \in \text{mul}(a) \subseteq \mathcal{M}_1 \perp \mathcal{N}_1$ and then $y = 0$. Analogously, b_0 , c_0 and d_0 are operators. Also,

$$\begin{aligned} \text{dom} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} &= \text{dom}(a_0) \cap \text{dom}(c_0) \oplus \text{dom}(b_0) \cap \text{dom}(d_0) \\ &= \text{dom}(a) \cap \text{dom}(c) \oplus \text{dom}(b) \cap \text{dom}(d) \\ &= \mathcal{D}_1 \oplus \mathcal{D}_2 = \text{dom}(A_0). \end{aligned}$$

Let $(x, y) \in \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. Then $x = x_1 + x_2 \in \mathcal{D}_1 \oplus \mathcal{D}_2 \subseteq \overline{\text{dom}}(A)$ and $y = \begin{pmatrix} a_0x_1 + b_0x_2 \\ c_0x_1 + d_0x_2 \end{pmatrix} \in \mathcal{N}_1 \oplus \mathcal{N}_2 = \overline{\text{dom}}(A)$.

Set $w_1 := a_0x_1$ and $z_1 := b_0x_2$. Then $(x_1, w_1) \in a_0 = P_{\mathcal{N}_1}a$ and $(x_2, z_1) \in b_0 = P_{\mathcal{N}_1}b$. Then, there exists $s_1 \in \mathcal{S}$ such that $(x_1, s_1) \in a$ and $w_1 = P_{\mathcal{N}_1}s_1$, and there exists $t_1 \in \mathcal{S}$ such that $(x_2, t_1) \in b$ and $z_1 = P_{\mathcal{N}_1}t_1$. Recall that $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$ then $P_{\mathcal{N}_1} + P_{\mathcal{M}_1} = P_{\mathcal{S}}$ so that

$$w_1 = P_{\mathcal{N}_1}s_1 = s_1 - P_{\mathcal{M}_1}s_1 \text{ and } z_1 = P_{\mathcal{N}_1}t_1 = t_1 - P_{\mathcal{M}_1}t_1.$$

Hence, since $P_{\mathcal{M}_1}s_1 + P_{\mathcal{M}_1}t_1 \in \mathcal{M}_1 = \text{mul}(a) + \text{mul}(b)$, there exist $m_1 \in \text{mul}(a)$ and $n_1 \in \text{mul}(b)$ such that $P_{\mathcal{M}_1}s_1 + P_{\mathcal{M}_1}t_1 = m_1 + n_1$. Then $(0, m_1) \in a$ and $(0, n_1) \in b$. Therefore $w_1 + z_1 = (s_1 - m_1) + (t_1 - n_1)$ and

$$(x_1, s_1 - m_1) = (x_1, s_1) - (0, m_1) \in a \text{ and } (x_2, t_1 - n_1) = (x_2, t_1) - (0, n_1) \in b.$$

Similarly, set $w_2 := c_0x_1$ and $z_2 := d_0x_2$. Then, there exist $s_2, t_2 \in \mathcal{S}^\perp$, $m_2 \in \text{mul}(c)$ and $n_2 \in \text{mul}(d)$ such that $w_2 + z_2 = (s_2 - m_2) + (t_2 - n_2)$, $(x_1, s_2 - m_2) \in c$ and

$(x_2, t_2 - n_2) \in d$. Therefore,

$$(x, y) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} w_1 + z_1 \\ w_2 + z_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} (s_1 - m_1) + (t_1 - n_1) \\ (s_2 - m_2) + (t_2 - n_2) \end{pmatrix} \right) \in A.$$

Hence, $(x, y) \in A \cap (\overline{\text{dom}}(A) \times \overline{\text{dom}}(A)) = A_0$. Then, $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \subset A_0$ and, since $\text{dom} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \text{dom}(A_0)$, it follows that $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. \square

Corollary 3.7. *Let A be a selfadjoint linear relation in \mathcal{H} , let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ and suppose that A admits the representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\text{dom}(a) \subseteq \text{dom}(c)$ and $\text{mul}(b) \subseteq \text{mul}(a)$ then*

$$a = P_{\mathcal{N}_1} a \hat{\oplus} (\{0\} \times \text{mul}(a)).$$

Similar results can be stated for b, c and d .

Proof. By Corollary 3.6, A admits the representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$,

$$A = \begin{pmatrix} P_{\mathcal{N}_1} a \hat{\oplus} (\{0\} \times \text{mul}(a)) & P_{\mathcal{N}_1} b \hat{\oplus} (\{0\} \times \text{mul}(b)) \\ P_{\mathcal{N}_2} c \hat{\oplus} (\{0\} \times \text{mul}(c)) & P_{\mathcal{N}_2} d \hat{\oplus} (\{0\} \times \text{mul}(d)) \end{pmatrix}.$$

Set $\tilde{a} := P_{\mathcal{N}_1} a \hat{\oplus} (\{0\} \times \text{mul}(a))$, $\tilde{b} := P_{\mathcal{N}_1} b \hat{\oplus} (\{0\} \times \text{mul}(b))$, $\tilde{c} := P_{\mathcal{N}_2} c \hat{\oplus} (\{0\} \times \text{mul}(c))$ and $\tilde{d} := P_{\mathcal{N}_2} d \hat{\oplus} (\{0\} \times \text{mul}(d))$.

Clearly, $\text{dom}(a) = \text{dom}(\tilde{a})$ and $\text{mul}(a) = \text{mul}(\tilde{a})$. Let $(x, y) \in a$ then there exists $y' \in \mathcal{S}^\perp$ such that $(x, y') \in c$ because $\text{dom}(a) \subseteq \text{dom}(c)$. So that

$$(x, y) = \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y+0 \\ y'+0 \end{pmatrix} \right) \in A = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}.$$

Then $(x, y) = \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ y' \end{pmatrix} \right) = \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} w+z \\ w'+z' \end{pmatrix} \right)$ with $(x, w) \in \tilde{a}$, $(x, w') \in \tilde{c}$, $(0, z) \in \tilde{b}$ and $(0, z') \in \tilde{d}$.

Then $(0, z) \in \text{mul}(\tilde{b}) = \text{mul}(b) \subseteq \text{mul}(a) = \text{mul}(\tilde{a})$ so that, $(0, z) \in \tilde{a}$. Hence

$$(x, y) = (x, w + z) = (x, w) + (0, z) \in \tilde{a}.$$

Then $a \subseteq \tilde{a}$ and since $\text{dom}(a) = \text{dom}(\tilde{a})$ and $\text{mul}(a) = \text{mul}(\tilde{a})$, by [9, Corollary 2.2], $a = \tilde{a} = P_{\mathcal{N}_1} a \hat{\oplus} (\{0\} \times \text{mul}(a))$. The analogous results for b, c and d follow in a similar way. \square

Next we focus on describing the matrix decompositions of nonnegative selfadjoint linear relations (operators).

The following lemmas are needed for the proof of Proposition 3.10.

Lemma 3.8. *Let A be a selfadjoint linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Consider the matrix representation of A as in (3.1). Then a and d are symmetric linear relations, $c \subset b^*$ and a, b, c and d are decomposable linear relations with (unique) decompositions: $a = P_{\mathcal{N}_1} a \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $b = P_{\mathcal{N}_1} b \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $c = P_{\mathcal{N}_2} c \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ and $d = P_{\mathcal{N}_2} d \hat{\oplus} (\{0\} \times \mathcal{M}_2)$.*

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrix representation of A with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ given by Theorem 3.2. From Lemma 3.4, $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$ and $\mathcal{S}^\perp = \mathcal{N}_2 \oplus \mathcal{M}_2$. Write $A = A_0 \hat{\oplus} A_{\text{mul}}$ as in (3.6). Then, by Corollary 3.6, A_0 admits the matrix representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \quad (3.7)$$

where $a_0 := P_{\mathcal{N}_1}a$, $b_0 := P_{\mathcal{N}_1}b$, $c_0 := P_{\mathcal{N}_2}c$ and $d_0 := P_{\mathcal{N}_2}d$. Since $\text{dom}(a) = \text{dom}(c) = \mathcal{D}_1$ and $\text{mul}(a) = \text{mul}(b) = \mathcal{M}_1$, by Corollary 3.7, $a = a_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$. Likewise, $b = b_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $c = c_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ and $d = d_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$.

Define

$$\hat{A}_0 := \begin{pmatrix} a_0^\times & c_0^\times \\ b_0^\times & d_0^\times \end{pmatrix}$$

with $\text{dom}(\hat{A}_0) = \text{dom}(a_0^\times) \cap \text{dom}(b_0^\times) \oplus \text{dom}(c_0^\times) \cap \text{dom}(d_0^\times)$, where a_0^\times denotes the adjoint of a_0 when viewed as an operator from \mathcal{N}_1 to \mathcal{N}_1 , likewise b_0^\times , c_0^\times and d_0^\times .

Since A is selfadjoint, $\overline{A_0} = A_0^\times$, where A_0^\times denotes the adjoint of A_0 when viewed as an operator from $\text{dom}(A)$ to $\text{dom}(A)$. Then A_0^\times admits a matrix decomposition with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$. Then, by [6, Theorem 2.2], $A_0 = A_0^\times = \hat{A}_0$. So that

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} a_0^\times & c_0^\times \\ b_0^\times & d_0^\times \end{pmatrix} = \hat{A}_0.$$

Then

$$a_0 \subset a_0^\times, \quad d_0 \subset d_0^\times, \quad b_0 \subset c_0^\times \text{ and } c_0 \subset b_0^\times.$$

So that a_0 and d_0 are symmetric operators on \mathcal{N}_1 and \mathcal{N}_2 , respectively, and b_0 and c_0 are closable operators. Also, since a_0, b_0, c_0, d_0 are closable operators, by Theorem 2.1, a, b, c and d are decomposable with (unique) decompositions: $a = P_{\mathcal{N}_1}a \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $b = P_{\mathcal{N}_1}b \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $c = P_{\mathcal{N}_2}c \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ and $d = P_{\mathcal{N}_2}d \hat{\oplus} (\{0\} \times \mathcal{M}_2)$.

Let us see that $a \subset a^*$. Let $(x_1, w_1) \in a$, then $x_1 \in \mathcal{D}_1$ and there exists $m_1 \in \mathcal{M}_1$ such that

$$(x_1, w_1) = (x_1, a_0x_1) + (0, m_1).$$

Also, let $(f, g) \in a$, then $f \in \mathcal{D}_1$ and there exists $m \in \mathcal{M}_1$ such that

$$(f, g) = (f, a_0f) + (0, m).$$

Hence

$$\begin{aligned} \langle g, x_1 \rangle_{\mathcal{H}} &= \langle a_0f + m, x_1 \rangle_{\mathcal{H}} = \langle a_0f, x_1 \rangle_{\mathcal{H}} = \langle a_0f, x_1 \rangle_{\mathcal{N}_1} \\ &= \langle a_0^\times f, x_1 \rangle_{\mathcal{N}_1} = \langle f, a_0x_1 \rangle_{\mathcal{N}_1} = \langle f, a_0x_1 + m_1 \rangle_{\mathcal{H}} = \langle f, w_1 \rangle_{\mathcal{H}}. \end{aligned}$$

Then $(x_1, w_1) \in a^*$. Likewise, $d \subset d^*$ and $c \subset b^*$. □

By the proof of the last lemma, $A \subset \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$ and, by [10, Proposition 6.1], the other inclusion always holds. So that A admits the matrix representation

$$A = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

Lemma 3.9 (cf. [12, Chapter VI], [4, Lemma 5.3.1]). *Let A be a nonnegative symmetric linear relation in \mathcal{H} . If A_F is the Friedrichs extension of A , then $\text{dom}(A)$ is a core of $A_F^{1/2}$ and $\text{mul}(A_F) = \text{mul}(A^*)$.*

Proposition 3.10. *Let $A \geq 0$ be a linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Then A admits the 2×2 block matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$*

$$\begin{pmatrix} a_F & b \\ c & d_F \end{pmatrix} \quad (3.8)$$

where a_F and d_F are the Friedrichs extensions of $a := P_{\mathcal{S}}A|_{\mathcal{S}}$ and $d := P_{\mathcal{S}^\perp}A|_{\mathcal{S}^\perp}$, respectively, $b := P_{\mathcal{S}}A|_{\mathcal{S}^\perp}$, $c := P_{\mathcal{S}^\perp}A|_{\mathcal{S}}$ are decomposable linear relations and $c \subset b^*$.

Moreover, if A is written as in (3.6) then A_0 admits the matrix representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$:

$$A_0 = \begin{pmatrix} (a_F)_0 & b_0 \\ c_0 & (d_F)_0 \end{pmatrix} \quad (3.9)$$

where $(a_F)_0$ and $(d_F)_0$ are the nonnegative selfadjoint operator parts of a_F and d_F , respectively and $a_F = (a_F)_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $b = b_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $c = c_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ and $d = (d_F)_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$, where $b_0 = P_{\mathcal{N}_1}b$, $c_0 = P_{\mathcal{N}_2}c$ and $(a_F)_0$ and $(d_F)_0$ are the Friedrichs extensions of $a_0 = P_{\mathcal{N}_1}a$ and $d_0 = P_{\mathcal{N}_2}d$, respectively.

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrix representation of A with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ as in Lemma 3.8. Since $A \geq 0$, it follows that a and d are nonnegative symmetric linear relations.

Also, by Corollaries 3.6 and 3.7, if A is written as in (3.6) then A_0 admits the matrix representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$: $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, where $a = a_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $b = b_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $c = c_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ and $d = d_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$.

Let a_F and d_F be the Friedrichs extensions of a and d , respectively. By Lemma 3.9, $\text{dom}(a) = \mathcal{D}_1$ is a core of $a_F^{1/2}$ and $\text{dom}(d) = \mathcal{D}_2$ is a core of $d_F^{1/2}$.

Set

$$A' := \begin{pmatrix} a_F & b \\ c & d_F \end{pmatrix}.$$

Then $\text{dom}(A') = \text{dom}(a_F) \cap \text{dom}(c) \oplus \text{dom}(b) \cap \text{dom}(d_F) = \mathcal{D}_1 \oplus \mathcal{D}_2 = \text{dom}(A)$, because $\text{dom}(c) = \mathcal{D}_1$ and $\text{dom}(b) = \mathcal{D}_2$. Also, $\text{mul}(A') = \text{mul}(a_F) + \text{mul}(b) \oplus$

$\text{mul}(c) + \text{mul}(d_F) = \mathcal{M}_1 \oplus \mathcal{M}_2 = \text{mul}(A)$, because $\text{mul}(a_F) = \text{mul}(a^*) = \text{dom}(a)^\perp = \mathcal{M}_1$, $\text{mul}(b) = \mathcal{M}_1$, $\text{mul}(d_F) = \text{mul}(d^*) = \text{dom}(d)^\perp = \mathcal{M}_2$ and $\text{mul}(c) = \mathcal{M}_2$. But, since $A \subset A'$, it follows that

$$A = A' = \begin{pmatrix} a_F & b \\ c & d_F \end{pmatrix}.$$

Since a_F and d_F are selfadjoint, a_F and d_F are decomposable and $a_F = (a_F)_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$ and $d_F = (d_F)_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ where $(a_F)_0$ and $(d_F)_0$ are the nonnegative selfadjoint operator parts of a_F and d_F , respectively.

Let us see that $(a_F)_0$ is the Friedrichs extension of a_0 and $(d_F)_0$ is the Friedrichs extension of d_0 , cf. [4, Theorem 5.3.3]. Since a is a nonnegative symmetric linear relation in \mathcal{S} , the form \mathfrak{t}_a given by $\mathfrak{t}_a[u, v] := \langle u', v \rangle$ for $(u, u'), (v, v') \in a$ with $\text{dom}(\mathfrak{t}_a) = \text{dom}(a)$, is nonnegative and closable, [4, Lemma 5.1.17]. Also, by the proof of Lemma 3.8, a_0 is a nonnegative symmetric linear operator on \mathcal{N}_1 , then the form \mathfrak{t}_{a_0} given by $\mathfrak{t}_{a_0}[u, v] := \langle a_0 u, v \rangle$ for $u, v \in \text{dom}(a_0)$, with $\text{dom}(\mathfrak{t}_{a_0}) = \text{dom}(a_0)$, is nonnegative and closable. But

$$\mathfrak{t}_a = \mathfrak{t}_{a_0}.$$

In fact, it is clear that $\text{dom}(\mathfrak{t}_{a_0}) = \text{dom}(\mathfrak{t}_a)$. Let $u, v \in \text{dom}(\mathfrak{t}_a) = \text{dom}(a)$ then there exist $u', v' \in \mathcal{H}$ such that $(u, u'), (v, v') \in a$. Then $u' = a_0 u + m$ for some $m \in \mathcal{M}_1 \perp \mathcal{N}_1$. Then

$$\mathfrak{t}_a[u, v] = \langle a_0 u + m, v \rangle = \langle a_0 u, v \rangle = \mathfrak{t}_{a_0}[u, v],$$

because $v \in \mathcal{D}_1$. Hence, the closures of the forms coincide, i.e., $\overline{\mathfrak{t}_a} = \overline{\mathfrak{t}_{a_0}}$. Then, by the Second Representation Theorem [4, Theorem 5.1.23],

$$\overline{\mathfrak{t}_a}[u, v] = \left\langle (a_F)_0^{1/2} u, (a_F)_0^{1/2} v \right\rangle$$

for every $u, v \in \text{dom}((a_F)_0^{1/2}) = \text{dom}(\overline{\mathfrak{t}_a})$ and

$$\overline{\mathfrak{t}_{a_0}}[u, v] = \left\langle (a_0)_F^{1/2} u, (a_0)_F^{1/2} v \right\rangle$$

for every $u, v \in \text{dom}((a_0)_F^{1/2}) = \text{dom}(\overline{\mathfrak{t}_{a_0}})$, where $(a_0)_F$ is the Friedrichs extension of a_0 . So that $(a_F)_0 = (a_0)_F$. Likewise, $(d_F)_0 = (d_0)_F$. Then, $a_0 \subset (a_F)_0$, $d_0 \subset (d_F)_0$ and, by Lemma 3.9, $\text{dom}(a_0) = \mathcal{D}_1$ is a core of $(a_F)_0^{1/2}$ and $\text{dom}(d_0) = \mathcal{D}_2$ is a core of $(d_F)_0^{1/2}$. Then

$$A_0 \subset A'' := \begin{pmatrix} (a_F)_0 & b_0 \\ c_0 & (d_F)_0 \end{pmatrix}.$$

But, $\text{dom}(A'') = \text{dom}((a_F)_0) \cap \text{dom}(c_0) \oplus \text{dom}(b_0) \cap \text{dom}((d_F)_0) = \mathcal{D}_1 \oplus \mathcal{D}_2 = \text{dom}(A_0)$, because $\text{dom}(c_0) = \mathcal{D}_1$ and $\text{dom}(b_0) = \mathcal{D}_2$. Then $A_0 = A''$. \square

Theorem 3.11. *Let $A \geq 0$ be a linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Then A admits a matrix decomposition in \mathcal{H} with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$,*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.10}$$

such that:

1. a and d are nonnegative selfadjoint linear relations with $\mathcal{D}_1 \subseteq \text{dom}(a)$, $\mathcal{D}_2 \subseteq \text{dom}(d)$, $\mathcal{D}_2 = \text{dom}(b)$, $\mathcal{D}_1 = \text{dom}(c)$, and $c \subset b^*$;
2. \mathcal{D}_1 is a core of $a^{1/2}$ and \mathcal{D}_2 is a core of $d^{1/2}$;
3. there exists a contraction $g : \mathcal{S}^\perp \rightarrow \mathcal{S}$ such that

$$b = a^{1/2}g d^{1/2}|_{\mathcal{D}_2} \text{ and } c = d^{1/2}g^* a^{1/2}|_{\mathcal{D}_1}.$$

Proof. Items 1 and 2 are proved in Proposition 3.10.

3 : Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the block matrix representation of A given in (3.8).

From Lemma 3.4, $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$ and $\mathcal{S}^\perp = \mathcal{N}_2 \oplus \mathcal{M}_2$. Write $A = A_0 \hat{\oplus} A_{\text{mul}}$ as in (3.6). Then, by Proposition 3.10, A_0 admits the matrix representation with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$:

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix},$$

where a_0 and d_0 are the nonnegative selfadjoint operator parts of a and d , respectively, \mathcal{D}_1 is a core of $a_0^{1/2}$, \mathcal{D}_2 is a core of $d_0^{1/2}$, $a = a_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $b = b_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $c = c_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ and $d = d_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$.

Since $A \geq 0$, then A_0 is a nonnegative selfadjoint operator on $\overline{\text{dom}}(A)$. Then

$$\left\langle A_0^{1/2}h, A_0^{1/2}k \right\rangle = \langle A_0h, k \rangle \text{ for every } h, k \in \text{dom}(A),$$

because $A_0 = A_0^{1/2}A_0^{1/2}$. In particular, for every $h_1 \in \mathcal{D}_1$

$$\left\langle A_0^{1/2}h_1, A_0^{1/2}h_1 \right\rangle = \langle A_0h_1, h_1 \rangle = \langle a_0h_1, h_1 \rangle = \left\langle a_0^{1/2}h_1, a_0^{1/2}h_1 \right\rangle.$$

Then the map $a_0^{1/2}(\mathcal{D}_1) \rightarrow A_0^{1/2}(\mathcal{D}_1)$,

$$a_0^{1/2}h_1 \mapsto A_0^{1/2}h_1$$

can be extended to a partial isometry V_1 on all of \mathcal{N}_1 , with initial space $\overline{a_0^{1/2}(\mathcal{D}_1)} = \overline{\text{ran}}(a_0^{1/2})$ (where we used that \mathcal{D}_1 is a core of $a_0^{1/2}$), so that $\ker(V_1) = \ker(a_0^{1/2})$, and final space $\overline{A_0^{1/2}(\mathcal{D}_1)}$. Therefore

$$V_1 a_0^{1/2} = A_0^{1/2} \text{ on } \mathcal{D}_1. \quad (3.11)$$

So, for every $h_2 \in \mathcal{D}_2$ and $k_1 \in \mathcal{D}_1$,

$$\begin{aligned} \langle b_0h_2, k_1 \rangle &= \langle A_0h_2, k_1 \rangle = \left\langle A_0^{1/2}h_2, A_0^{1/2}k_1 \right\rangle = \left\langle A_0^{1/2}h_2, V_1 a_0^{1/2}k_1 \right\rangle \\ &= \left\langle V_1^* A_0^{1/2}h_2, a_0^{1/2}k_1 \right\rangle. \end{aligned}$$

Therefore, $V_1^* A_0^{1/2}h_2 \in \text{dom}((a_0^{1/2})^\times)$ and $(a_0^{1/2})^\times V_1^* A_0^{1/2}h_2 = b_0h_2$. Since $a_0^{1/2}$ is selfadjoint and the above holds for any $h_2 \in \mathcal{D}_2$, it follows that

$$b_0 = a_0^{1/2}V_1^* A_0^{1/2} \text{ on } \mathcal{D}_2.$$

Likewise, there exists a partial isometry V_2 in \mathcal{N}_2 with initial space $\overline{d_0^{1/2}(\mathcal{D}_2)}$ and final space $\overline{A_0^{1/2}(\mathcal{D}_2)}$, such that

$$V_2 d_0^{1/2} = A_0^{1/2} \text{ on } \mathcal{D}_2 \text{ and } c_0 = d_0^{1/2} V_2^* A_0^{1/2} \text{ on } \mathcal{D}_1.$$

Then

$$b_0 h_2 = a_0^{1/2} V_1^* A_0^{1/2} h_2 = a_0^{1/2} V_1^* V_2 d_0^{1/2} h_2 \text{ for every } h_2 \in \mathcal{D}_2.$$

Set $f := V_1^* V_2$. Then f is a contraction from \mathcal{N}_1 to \mathcal{N}_2 such that $b_0 = a_0^{1/2} f d_0^{1/2}$ on \mathcal{D}_2 . Likewise, $c_0 = d_0^{1/2} f^* a_0^{1/2}$ on \mathcal{D}_1 .

Using that $\mathcal{S}^\perp = \mathcal{N}_2 \oplus \mathcal{M}_2$, f has an extension, again a contraction from \mathcal{S}^\perp to \mathcal{S} , named g such that $gx = 0$ for every $x \in \mathcal{M}_2$. Let $(x, y) \in a^{1/2} g d^{1/2}|_{\mathcal{D}_2}$. Then there exists $z \in \mathcal{S}^\perp$ such that $(x, z) \in d^{1/2}|_{\mathcal{D}_2}$ and $(z, y) \in a^{1/2} g$. Then

$$(x, z) = (x, d_0^{1/2} x) + (0, m_2)$$

for some $m_2 \in \mathcal{M}_2$ and so $z = d_0^{1/2} x + m_2$. Also, since $(z, y) \in a^{1/2} g$, it follows that $(gz, y) \in a^{1/2}$. Then $(gz, y) = (gz, a_0^{1/2} gz) + (0, m_1)$ for some $m_1 \in \mathcal{M}_1$. Then, since $m_2 \in \ker(g)$ and $d_0^{1/2} x \in \mathcal{N}_2$,

$$y = a_0^{1/2} gz + m_1 = a_0^{1/2} g(d_0^{1/2} x + m_2) + m_1 = a_0^{1/2} f d_0^{1/2} x + m_1 = b_0 x + m_1.$$

Hence,

$$(x, y) = (x, b_0 x) + (0, m_1) \in b.$$

Conversely, suppose that $(x, y) \in b$, then $x \in \mathcal{D}_2$ and

$$(x, y) = (x, b_0 x) + (0, m_1) = (x, a_0^{1/2} f d_0^{1/2} x) + (0, m_1)$$

for some $m_1 \in \mathcal{M}_1$ and so $y = a_0^{1/2} f d_0^{1/2} x + m_1$. Set $z := d_0^{1/2} x \in \mathcal{N}_2$ then $(x, z) = (x, d_0^{1/2} x) \in d^{1/2}|_{\mathcal{D}_2}$. Also,

$$\begin{aligned} (gz, y) &= (gz, a_0^{1/2} f d_0^{1/2} x) + (0, m_1) = (gz, a_0^{1/2} fz) + (0, m_1) \\ &= (gz, a_0^{1/2} gz) + (0, m_1) \in a^{1/2}. \end{aligned}$$

So that $(z, y) \in a^{1/2} g$ and then $(x, y) \in a^{1/2} g d^{1/2}|_{\mathcal{D}_2}$.

Likewise, $c = d^{1/2} g^* a^{1/2}|_{\mathcal{D}_1}$. □

Corollary 3.12. *Let $A \geq 0$ be a linear operator in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the block matrix representation of A given in (3.8). Set $Z := \begin{pmatrix} a^{1/2}|_{\mathcal{D}_1} & 0 \\ 0 & d^{1/2}|_{\mathcal{D}_2} \end{pmatrix}$ and $W := \begin{pmatrix} 1 & f \\ 0 & (1 - f^* f)^{1/2} \end{pmatrix} \in L(\mathcal{H})$, where $f : \mathcal{S}^\perp \rightarrow \mathcal{S}$ is the contraction in the proof of Theorem 3.11. Then the operator WZ is closable and*

$$A = (WZ)^* WZ = (WZ)^* \overline{WZ}.$$

Proof. Define $\Gamma := W^*W = \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix}$. Then $\Gamma \in L(\mathcal{H})$ and $\Gamma \geq 0$, because f is a contraction, and Z is a densely defined operator with $\text{dom}(Z) = \mathcal{D}_1 \oplus \mathcal{D}_2$. Since \mathcal{D}_1 is a core of $a^{1/2}$ and \mathcal{D}_2 is a core of $d^{1/2}$,

$$Z^* = \begin{pmatrix} a^{1/2} & 0 \\ 0 & d^{1/2} \end{pmatrix}.$$

Consider the operator $Z^*\Gamma Z$. Then

$$\text{dom}(Z^*\Gamma Z) = \mathcal{D}_1 \oplus \mathcal{D}_2.$$

Clearly, $\text{dom}(Z^*\Gamma Z) \subseteq \text{dom}(Z) = \mathcal{D}_1 \oplus \mathcal{D}_2$. On the other hand, take $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{D}_1 \oplus \mathcal{D}_2$, then

$$\Gamma Z \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix} \begin{pmatrix} a^{1/2}h_1 \\ d^{1/2}h_2 \end{pmatrix} = \begin{pmatrix} a^{1/2}h_1 + fd^{1/2}h_2 \\ f^*a^{1/2}h_1 + d^{1/2}h_2 \end{pmatrix}.$$

Since $b = a^{1/2}fd^{1/2}$ on \mathcal{D}_2 and $a^{1/2}(\mathcal{D}_1) \subseteq \text{dom}(a^{1/2})$, it follows that $a^{1/2}h_1 + fd^{1/2}h_2 \in \text{dom}(a^{1/2})$. Likewise, since $c = d^{1/2}f^*a^{1/2}$ on \mathcal{D}_1 and $d^{1/2}(\mathcal{D}_2) \subseteq \text{dom}(d^{1/2})$, it follows that $f^*a^{1/2}h_1 + d^{1/2}h_2 \in \text{dom}(d^{1/2})$. Hence, $\Gamma Zh \in \text{dom}(Z^*)$ and $h \in \text{dom}(Z^*\Gamma Z)$. Then $Z^*\Gamma Z$ has matrix representation and, by [6, Theorem 2.1],

$$\begin{aligned} Z^*\Gamma Z &= \begin{pmatrix} a^{1/2} & 0 \\ 0 & d^{1/2} \end{pmatrix} \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix} \begin{pmatrix} a^{1/2}|_{\mathcal{D}_1} & 0 \\ 0 & d^{1/2}|_{\mathcal{D}_2} \end{pmatrix} \\ &= \begin{pmatrix} a|_{\mathcal{D}_1} & b \\ c & d|_{\mathcal{D}_2} \end{pmatrix} \subseteq A. \end{aligned}$$

But, since $\text{dom}(Z^*\Gamma Z) = \text{dom}(A)$ it follows that $A = Z^*\Gamma Z = Z^*W^*WZ = (WZ)^*WZ$.

If $Y := WZ$, then $\text{dom}(Y) = \text{dom}(Z) = \text{dom}(A)$. Therefore, $\text{dom}(Y^*Y) = \text{dom}(A) = \text{dom}(Y)$. Then, by [17, Theorem 5.1], $Y = WZ$ is closable. Finally,

$$A = Y^*Y = A^* = (Y^*Y)^* \supset Y^*\bar{Y} \supset Y^*Y = A.$$

□

4. The Schur complement of nonnegative selfadjoint linear relations

Let $A \geq 0$ be a linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.1}$$

be the 2×2 block matrix representation of A with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ as in Theorem 3.11. That is, a and d are nonnegative selfadjoint linear relations with $\mathcal{D}_1 \subseteq \text{dom}(a)$,

$\mathcal{D}_2 \subseteq \text{dom}(d)$, $\mathcal{D}_2 = \text{dom}(b)$, $\mathcal{D}_1 = \text{dom}(c)$, and $c \subset b^*$. Also, \mathcal{D}_1 is a core of $a^{1/2}$, \mathcal{D}_2 is a core of $d^{1/2}$ and there exists a contraction $g : \mathcal{S}^\perp \rightarrow \mathcal{S}$ such that

$$b = a^{1/2} g d^{1/2}|_{\mathcal{D}_2} \text{ and } c = d^{1/2} g^* a^{1/2}|_{\mathcal{D}_1}.$$

Write $A = A_0 \hat{\oplus} A_{\text{mul}}$ as in (3.6). Then, $a = a_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $b = b_0 \hat{\oplus} (\{0\} \times \mathcal{M}_1)$, $c = c_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ and $d = d_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2)$, where

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \quad (4.2)$$

is the 2×2 block matrix representation of A_0 with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$ given in (3.9). By Theorem 3.11, there exists a contraction $f : \mathcal{N}_2 \rightarrow \mathcal{N}_1$ such that

$$b_0 = a_0^{1/2} f d_0^{1/2}|_{\mathcal{D}_2} \text{ and } c_0 = d_0^{1/2} f^* a_0^{1/2}|_{\mathcal{D}_1}.$$

By Lemma 3.4, $\mathcal{S} = \mathcal{N}_1 \oplus \mathcal{M}_1$ and $\mathcal{S}^\perp = \mathcal{N}_2 \oplus \mathcal{M}_2$. Then $g = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$ is the matrix decomposition of $g : \mathcal{N}_2 \oplus \mathcal{M}_2 \rightarrow \mathcal{N}_1 \oplus \mathcal{M}_1$.

In order to define the Schur complement of A , consider $D_g := (1 - g^*g)^{1/2} \in L(\mathcal{S}^\perp)$ and the closed linear relation

$$T := \overline{D_g d^{1/2}|_{\mathcal{D}_2}} \subseteq \mathcal{S}^\perp \times \mathcal{S}^\perp.$$

Lemma 4.1. *Under the above hypotheses,*

$$T^*T = d_0^{1/2} D_f D_f d_0^{1/2}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2),$$

where $D_f := (1 - f^*f)^{1/2} \in L(\mathcal{N}_2)$.

Proof. The matrix decomposition of D_g with respect to $\mathcal{N}_2 \oplus \mathcal{M}_2$ is $D_g = \begin{pmatrix} D_f & 0 \\ 0 & 1 \end{pmatrix}$.

Then $D_g d_0^{1/2}|_{\mathcal{D}_2} = D_f d_0^{1/2}|_{\mathcal{D}_2} \subseteq \mathcal{N}_2 \times \mathcal{N}_2$ and, since $d^{1/2}|_{\mathcal{D}_2} = d_0^{1/2}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2)$,

$$D_g d^{1/2}|_{\mathcal{D}_2} = D_f d_0^{1/2}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2). \quad (4.3)$$

So that

$$T = \overline{D_f d_0^{1/2}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2)} = \bar{t} \hat{\oplus} (\{0\} \times \mathcal{M}_2), \quad (4.4)$$

where $t := D_f d_0^{1/2}|_{\mathcal{D}_2}$. Since $\mathcal{D}_2 \subseteq \text{dom}(T) = \text{dom}(\bar{t}) \subseteq \mathcal{N}_2$, then

$$\overline{\text{dom}(T)} = \overline{\text{dom}(\bar{t})} = \mathcal{N}_2.$$

Also,

$$T^* = (D_g d^{1/2}|_{\mathcal{D}_2})^* = (d^{1/2}|_{\mathcal{D}_2})^* D_g = d^{1/2} D_g,$$

where we used that $D_g \in L(\mathcal{S}^\perp)$ so there is equality in (2.1) and \mathcal{D}_2 is a core of $d^{1/2}$. Then

$$T^* = (d_0^{1/2} \hat{\oplus} (\{0\} \times \mathcal{M}_2)) D_g = d_0^{1/2} D_f \hat{\oplus} (\{0\} \times \mathcal{M}_2) = t^\times \hat{\oplus} (\{0\} \times \mathcal{M}_2),$$

where t^\times denotes the adjoint of t when viewed as an operator in \mathcal{N}_2 . Finally, since t is a densely defined operator in \mathcal{N}_2 , t^\times is an operator in \mathcal{N}_2 and $\text{mul}(t^\times \bar{t}) = \text{mul}(t^\times) = \{0\}$.

Therefore, by Theorem 2.3, $t^\times \bar{t}$ is a nonnegative selfadjoint linear operator in \mathcal{N}_2 and

$$\text{mul}(T^*T) = \text{mul}(T^*) = \text{dom}(T)^\perp = \mathcal{S}^\perp \ominus \mathcal{N}_2 = \mathcal{M}_2.$$

Now, suppose that $(x, y) \in T^*T$. Then $(x, z) \in T$ and $(z, y) \in T^*$ for some $z \in \mathcal{S}^\perp$. Then

$$(x, z) = (x, z') + (0, m) \text{ for some } m \in \mathcal{M}_2 \text{ and } z' \in \mathcal{N}_2 \text{ such that } (x, z') \in \bar{t},$$

$$(z, y) = (z, t^\times z) + (0, m') \text{ for some } m' \in \mathcal{M}_2.$$

Since $z \in \text{dom}(T^*) \subseteq \mathcal{N}_2$, $z' \in \text{ran}(\bar{t}) \subseteq \mathcal{N}_2$ and $z = z' + m$, it holds that $m = 0$ and $z = z'$. Then, from the fact that $(x, z) = (x, z') \in \bar{t}$ and $(z, t^\times z) \in t^\times$ it follows that $(x, t^\times z) \in t^\times \bar{t}$. Hence, since $y = t^\times z + m'$,

$$(x, y) = (x, t^\times z) + (0, m') \in t^\times \bar{t} \hat{\oplus} (\{0\} \times \mathcal{M}_2).$$

Therefore

$$T^*T \subset t^\times \bar{t} \hat{\oplus} (\{0\} \times \mathcal{M}_2). \quad (4.5)$$

By Theorem 2.3, T^*T is a nonnegative selfadjoint linear relation in \mathcal{S}^\perp . Then T^*T admits a unique decomposition as in (2.4):

$$T^*T = (T^*T)_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2),$$

where $(T^*T)_0$ is a selfadjoint operator in $\overline{\text{dom}(T^*T)} = \mathcal{N}_2$. By (4.5), $(T^*T)_0 \subset t^\times \bar{t}$ and, since $(T^*T)_0$ and $t^\times \bar{t}$ are selfadjoint operators in \mathcal{N}_2 , equality holds, i.e., $(T^*T)_0 = t^\times \bar{t}$. Hence

$$T^*T = (T^*T)_0 \hat{\oplus} (\{0\} \times \mathcal{M}_2) = d_0^{1/2} \overline{D_f d_0^{1/2}}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2).$$

□

Consider the set

$$\mathcal{M}(A, \mathcal{S}^\perp) := \{X \text{ l.r. in } \mathcal{H} : 0 \leq X \leq A, \text{ran}(X) \subseteq \mathcal{S}^\perp\}.$$

In [3], Arlinskii proved that the set $\mathcal{M}(A, \mathcal{S}^\perp)$ has a maximum element and defined the Schur complement of A to \mathcal{S} denoted by $A_{/\mathcal{S}}$ as the maximum of $\mathcal{M}(A, \mathcal{S}^\perp)$. In what follows we give an alternate proof of the existence of the Schur complement as well as a formula for $A_{/\mathcal{S}}$ using the matrix decomposition of A when $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$.

Theorem 4.2. *Let A be a linear relation in \mathcal{H} , let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ and consider the matrix representation of A with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ in (4.1). Then the set $\mathcal{M}(A, \mathcal{S}^\perp)$ has a maximum element $A_{/\mathcal{S}}$. Moreover,*

$$A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & T^*T \end{pmatrix},$$

where $T := \overline{D_g d^{1/2}}|_{\mathcal{D}_2}$.

Proof. Write $A = A_0 \hat{\oplus} A_{\text{mul}(A)}$ and set $C := \begin{pmatrix} 0 & 0 \\ 0 & T^*T \end{pmatrix}$. Then $\text{ran}(C) = \text{ran}(T^*T) = \text{ran}((T^*T)_0) \oplus \mathcal{M}_2 \subseteq \mathcal{N}_2 \oplus \mathcal{M}_2 = \mathcal{S}^\perp$ and $C^* = C \geq 0$. Suppose that T is written as $T = T_0 \hat{\oplus} (\{0\} \times \text{mul}(T))$ as in (2.4). Let C_0 be the operator part of C then, by [9, Proposition 2.7],

$$\left\langle C_0^{1/2} u, C_0^{1/2} v \right\rangle = \langle T_0 u_2, T_0 v_2 \rangle,$$

for every $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \text{dom}(C_0^{1/2}) = \mathcal{S} \oplus \text{dom}(T_0)$.

Then, since $\mathcal{D}_2 \subseteq \text{dom}(T) = \text{dom}(T_0)$

$$\text{dom}(A) = \mathcal{D}_1 \oplus \mathcal{D}_2 \subseteq \mathcal{S} \oplus \text{dom}(T_0) = \text{dom}(C_0^{1/2}).$$

Let (4.2) be the matrix decomposition of A_0 (in $\overline{\text{dom}}(A)$) with respect to $\mathcal{N}_1 \oplus \mathcal{N}_2$. Let V_1 and V_2 be the partial isometries given in the proof of Theorem 3.11 such that

$$V_1 a_0^{1/2} = A_0^{1/2} \text{ on } \mathcal{D}_1 \text{ and } V_2 d_0^{1/2} = A_0^{1/2} \text{ on } \mathcal{D}_2,$$

and $f = V_1^* V_2$. Then, by Corollary 3.12, $A_0 = Z^* \Gamma Z$, where $\Gamma = \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix}$ and

$Z = \begin{pmatrix} a_0^{1/2}|_{\mathcal{D}_1} & 0 \\ 0 & d_0^{1/2}|_{\mathcal{D}_2} \end{pmatrix}$. Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{D}_1 \oplus \mathcal{D}_2$. Then

$$\begin{aligned} \langle A_0 h, h \rangle &= \left\langle \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix} \begin{pmatrix} a_0^{1/2} h_1 \\ d_0^{1/2} h_2 \end{pmatrix}, \begin{pmatrix} a_0^{1/2} h_1 \\ d_0^{1/2} h_2 \end{pmatrix} \right\rangle \\ &\geq \left\langle \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix} /_{\mathcal{N}_1} \begin{pmatrix} a_0^{1/2} h_1 \\ d_0^{1/2} h_2 \end{pmatrix}, \begin{pmatrix} a_0^{1/2} h_1 \\ d_0^{1/2} h_2 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 - f^* f \end{pmatrix} \begin{pmatrix} a_0^{1/2} h_1 \\ d_0^{1/2} h_2 \end{pmatrix}, \begin{pmatrix} a_0^{1/2} h_1 \\ d_0^{1/2} h_2 \end{pmatrix} \right\rangle \\ &= \langle D_f d_0^{1/2} h_2, D_f d_0^{1/2} h_2 \rangle = \|t h_2\|^2. \end{aligned}$$

Let us see that

$$\|t h_2\|^2 \geq \|T_0 h_2\|^2.$$

In fact, $(h_2, t h_2) \in t \subseteq T$. Since $T = T_0 \hat{\oplus} (\{0\} \times \text{mul}(T))$,

$$(h_2, t h_2) = (h_2, T_0 h_2) + (0, z)$$

for some $z \in \text{mul}(T)$. Then $t h_2 = T_0 h_2 + z$. Since $T_0 h_2 \in \text{ran}(T_0) \subseteq \overline{\text{dom}}(T^*) \subseteq \text{mul}(T)^\perp$ and $\mathcal{D}_1 \subseteq \mathcal{S}$ it follows that

$$\|t h_2\|^2 = \|T_0 h_2\|^2 + \|z\|^2 \geq \|T_0 h_2\|^2 = \|C_0^{1/2} h\|^2.$$

Then

$$\langle A_0 h, h \rangle = \|A_0^{1/2} h\|^2 \geq \|C_0^{1/2} h\|^2 \text{ for every } h \in \text{dom}(A).$$

Since $\text{dom}(A)$ is a core for $A_0^{1/2}$, by [16, Lemma 10.10], it follows that $\text{dom}(A_0^{1/2}) \subseteq \text{dom}(C_0^{1/2})$ and $\|A_0^{1/2} h\| \geq \|C_0^{1/2} h\|$ for every $h \in \text{dom}(A_0^{1/2})$. Hence, $A \geq C$. So that

$$C \in \mathcal{M}(A, \mathcal{S}^\perp).$$

Let $X \in \mathcal{M}(A, \mathcal{S}^\perp)$. Then, by Lemma 2.4, there exists a contraction $W \in L(\mathcal{H})$ such that

$$X_0^{1/2} \supset W A_0^{1/2},$$

where X_0 is the operator part of X . Recall that X_0 is a nonnegative selfadjoint linear operator in $\overline{\text{dom}}(X)$. Also, if $h_2 \in \mathcal{D}_2 \subseteq \text{dom}(A) = \text{dom}(A_0) \subseteq \text{dom}(A_0^{1/2})$,

$$X_0^{1/2}h_2 = WA_0^{1/2}h_2 = WV_2d_0^{1/2} = W'd_0^{1/2}h_2,$$

with $W' = WV_2$. Also, since $X \leq A$, we have that $\text{dom}(A) \subseteq \text{dom}(A_0^{1/2}) \subseteq \text{dom}(X_0^{1/2})$ and

$$\left\langle X_0^{1/2}h, X_0^{1/2}h \right\rangle \leq \left\langle A_0^{1/2}h, A_0^{1/2}h \right\rangle = \langle A_0h, h \rangle \text{ for every } h \in \text{dom}(A).$$

Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{D}_1 \oplus \mathcal{D}_2$. Then, since $\mathcal{D}_1 \subseteq \mathcal{S} \subseteq \ker(X) = \ker(X_0)$,

$$\begin{aligned} \left\langle X_0^{1/2}h, X_0^{1/2}h \right\rangle &= \left\langle X_0^{1/2}h_2, X_0^{1/2}h_2 \right\rangle = \left\langle W'd_0^{1/2}h_2, W'd_0^{1/2}h_2 \right\rangle \\ &= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & W'^*W' \end{pmatrix} \begin{pmatrix} 0 \\ d_0^{1/2}h_2 \end{pmatrix}, \begin{pmatrix} 0 \\ d_0^{1/2}h_2 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 0 & 0 \\ 0 & W'^*W' \end{pmatrix} \begin{pmatrix} a_0^{1/2}h_1 \\ d_0^{1/2}h_2 \end{pmatrix}, \begin{pmatrix} a_0^{1/2}h_1 \\ d_0^{1/2}h_2 \end{pmatrix} \right\rangle \\ &\leq \left\langle A_0h, h \right\rangle = \left\langle \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix} \begin{pmatrix} a_0^{1/2}h_1 \\ d_0^{1/2}h_2 \end{pmatrix}, \begin{pmatrix} a_0^{1/2}h_1 \\ d_0^{1/2}h_2 \end{pmatrix} \right\rangle. \end{aligned}$$

Since \mathcal{D}_1 is a core of $a_0^{1/2}$ and \mathcal{D}_2 is a core of $d_0^{1/2}$, we have that $\overline{a_0^{1/2}(\mathcal{D}_1)} = \overline{\text{ran}(a_0^{1/2})}$ and $\overline{d_0^{1/2}(\mathcal{D}_2)} = \overline{\text{ran}(d_0^{1/2})}$. Also, $\ker(d_0^{1/2}) = \ker(V_2) \subseteq \ker(W') \cap \ker(f)$ and $\ker(a_0^{1/2}) \subseteq \ker(f^*)$. Hence, by the last inequality, it follows that

$$0 \leq \begin{pmatrix} 0 & 0 \\ 0 & W'^*W' \end{pmatrix} \leq \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix},$$

where the inequality holds in the Hilbert space $\overline{\text{dom}}(A) = \mathcal{N}_1 \oplus \mathcal{N}_2$. Therefore

$$\begin{pmatrix} 0 & 0 \\ 0 & W'^*W' \end{pmatrix} \leq \begin{pmatrix} 1 & f \\ f^* & 1 \end{pmatrix}_{/\mathcal{N}_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 - f^*f \end{pmatrix}.$$

So that $W'^*W' \leq 1 - f^*f$. Then

$$\begin{aligned} \left\langle X_0^{1/2}h, X_0^{1/2}h \right\rangle &= \left\langle W'd_0^{1/2}h_2, W'd_0^{1/2}h_2 \right\rangle \\ &\leq \left\langle (1 - f^*f)^{1/2}d_0^{1/2}h_2, (1 - f^*f)^{1/2}d_0^{1/2}h_2 \right\rangle \\ &= \left\langle D_f d_0^{1/2}h_2, D_f d_0^{1/2}h_2 \right\rangle = \|D_f d_0^{1/2}h_2\|^2 = \|th_2\|^2. \end{aligned}$$

Next we show that $C \geq X$. Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \text{dom}(C_0^{1/2}) = \mathcal{S} \oplus \text{dom}(T_0)$. Then $h_2 \in \text{dom}(T_0)$. So that there exists $k \in \mathcal{N}_2$ such that $(h_2, k) \in T_0 \subset T$. Since T_0 is an operator, it follows that $k = T_0h_2$. Also, since $(h_2, k) \in T = \overline{D_g d^{1/2}|_{\mathcal{D}_2}}$, there exists a sequence $(h_n, y_n)_{n \geq 1} \in D_g d^{1/2}|_{\mathcal{D}_2}$ such that $\lim_{n \rightarrow \infty} (h_n, y_n) = (h_2, k)$.

Since $(h_n, y_n) \in D_g d^{1/2}|_{\mathcal{D}_2} = D_f d_0^{1/2}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2)$ for every $n \in \mathbb{N}$, then $h_n \in \mathcal{D}_2$ and, for every $n \in \mathbb{N}$, there exists $m_n \in \mathcal{M}_2$ such that

$$(h_n, y_n) = (h_n, D_f d_0^{1/2} h_n) + (0, m_n).$$

Then, $\lim_{n \rightarrow \infty} h_n = h_2$ and $\lim_{n \rightarrow \infty} D_f d_0^{1/2} h_n + m_n = k$. But, since $D_f d_0^{1/2} h_n \in \mathcal{N}_2$ for every $n \in \mathbb{N}$ and $k \in \mathcal{N}_2 \perp \mathcal{M}_2$, it follows that $\lim_{n \rightarrow \infty} m_n = 0$ and then $\lim_{n \rightarrow \infty} D_f d_0^{1/2} h_n = \lim_{n \rightarrow \infty} t h_n = k$. From

$$\|X_0^{1/2} h_n\|^2 \leq \|t h_n\|^2 \text{ for every } n \in \mathbb{N},$$

it follows that $(X_0^{1/2} h_n)_{n \geq 1}$ is a Cauchy sequence (so it converges). From the fact that $X_0^{1/2}$ is a closed operator, $h_2 \in \text{dom}(X_0^{1/2})$ and $\lim_{n \rightarrow \infty} X_0^{1/2} h_n = X_0^{1/2} h_2$. Then, since $\mathcal{S} \subseteq \ker(X_0) = \ker(X_0^{1/2}) \subseteq \text{dom}(X_0^{1/2})$,

$$\text{dom}(C_0^{1/2}) = \mathcal{S} \oplus \text{dom}(T_0) \subseteq \text{dom}(X_0^{1/2}).$$

Therefore, since $h_1 \in \ker(X_0^{1/2})$,

$$\begin{aligned} \|X_0^{1/2} h\| &= \|X_0^{1/2} h_2\| = \lim_{n \rightarrow \infty} \|X_0^{1/2} h_n\| \\ &\leq \lim_{n \rightarrow \infty} \|t h_n\| = \|k\| = \|T_0 h_2\| = \|C_0^{1/2} h\|. \end{aligned}$$

□

Remark. Suppose that $A \geq 0$ is (a densely defined) operator in \mathcal{H} . If $X \in \mathcal{M}(A, \mathcal{S}^\perp)$ then X is an operator in \mathcal{H} . In fact, if $X \in \mathcal{M}(A, \mathcal{S}^\perp)$ then $\text{dom}(A^{1/2}) \subseteq \text{dom}(X^{1/2})$ and then

$$\text{mul}(X) = \text{mul}(X^{1/2}) = \text{dom}(X^{1/2})^\perp \subseteq \text{dom}(A^{1/2})^\perp = \text{mul}(A) = \{0\}.$$

In this case, $\mathcal{N}_1 = \overline{\mathcal{D}_1} = \mathcal{S}$ and $\mathcal{M}_1 = \mathcal{M}_2 = \{0\}$. So that $f = g$, $d = d_0$, $T^*T = t^{\times \bar{t}}$ and,

$$A/S = \begin{pmatrix} 0 & 0 \\ 0 & T^*T \end{pmatrix} = \max \{X \text{ l.o. in } \mathcal{H} : 0 \leq X \leq A, \text{ran}(X) \subseteq \mathcal{S}^\perp\}.$$

In a similar way, we now define A_S the *compression* of A . For this, consider the row linear relation

$$S := (a^{1/2}|_{\mathcal{D}_1} \quad g d^{1/2}|_{\mathcal{D}_2}) \subseteq \mathcal{H} \times \mathcal{S}$$

with $\text{dom}(S) = \mathcal{D}_1 \oplus \mathcal{D}_2 = \text{dom}(A)$. Define A_S by

$$A_S := S^* \bar{S}.$$

Then, by Theorem 2.3, A_S is a nonnegative selfadjoint linear relation \mathcal{H} .

Lemma 4.3. *Under the above hypotheses, \bar{S} is decomposable and*

$$A_S = s^{\times \bar{s}} \hat{\oplus} (\{0\} \times \text{mul}(A)),$$

where $s : \mathcal{D} \rightarrow \mathcal{N}_1$ is the closable linear operator defined by

$$s := \left(a_0^{1/2}|_{\mathcal{D}_1} \quad f d_0^{1/2}|_{\mathcal{D}_2} \right) \quad (4.6)$$

and s^\times is the adjoint of s when viewed as an operator from $\overline{\text{dom}}(S)$ to $\overline{\text{dom}}(S^*)$.

Proof. Since $a^{1/2}|_{\mathcal{D}_1} = a_0^{1/2}|_{\mathcal{D}_1} \hat{\oplus} (\{0\} \times \mathcal{M}_1)$ and $g d^{1/2}|_{\mathcal{D}_2} = f d_0^{1/2}|_{\mathcal{D}_2}$, it follows that

$$S = s \hat{\oplus} (\{0\} \times \mathcal{M}_1).$$

In fact, it is clear that $\text{ran}(s) \subseteq \mathcal{N}_1$ and, since $\text{mul}(g d^{1/2}|_{\mathcal{D}_2}) = \{0\}$, $\text{mul}(S) = \text{mul}(a^{1/2}|_{\mathcal{D}_1}) + \text{mul}(g d^{1/2}|_{\mathcal{D}_2}) = \mathcal{M}_1$ and $\text{dom}(S) = \text{dom}(A) = \text{dom}(s)$. Also, if $(h, y) \in S$ then $(h, y) = \left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, y_1 + y_2 \right)$ where $h_1 \in \mathcal{D}_1$, $h_2 \in \mathcal{D}_2$ and $(h_1, y_1) \in a^{1/2}|_{\mathcal{D}_1}$ and $(h_2, y_2) \in g d^{1/2}|_{\mathcal{D}_2} = f d_0^{1/2}|_{\mathcal{D}_2}$. So that $(h_1, y_1) = (h_1, a_0^{1/2} h_1) + (0, m_1)$ for some $m_1 \in \mathcal{M}_1$ and $y_2 = f d_0^{1/2} h_2$. Hence

$$\begin{aligned} (h, y) &= \left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, y_1 + y_2 \right) \\ &= \left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, a_0^{1/2} h_1 + f d_0^{1/2} h_2 \right) + (0, m_1) \in s \hat{\oplus} (\{0\} \times \mathcal{M}_1). \end{aligned}$$

Then $S \subset s \hat{\oplus} (\{0\} \times \mathcal{M}_1)$ and, by [9, Corollary 2.2], $S = s \hat{\oplus} (\{0\} \times \mathcal{M}_1)$.

The row operator s is closable, in fact, $s^\times = \begin{pmatrix} a_0^{1/2} \\ d_0^{1/2} f^* \end{pmatrix}$ and, as $a_0^{1/2}(\mathcal{D}_1) \subseteq \text{dom}(a_0^{1/2}) \cap \text{dom}(d_0^{1/2} f^*)$ and $\ker(a_0^{1/2}) \subseteq \text{dom}(a_0^{1/2}) \cap \ker(f^*)$,

$$\text{dom}(s^\times) \supseteq a_0^{1/2}(\mathcal{D}_1) \oplus \ker(a_0^{1/2})$$

which is dense in \mathcal{N}_1 . Then \bar{s} is an operator. Moreover, by Theorem 2.1, \bar{S} is decomposable and

$$\bar{S} = \bar{s} \hat{\oplus} (\{0\} \times \mathcal{M}_1).$$

Also, since \mathcal{D}_1 is a core of $a^{1/2}$ and \mathcal{D}_2 is a core of $d^{1/2}$, it follows that

$$S^* = \begin{pmatrix} a^{1/2} \\ d^{1/2} g^* \end{pmatrix},$$

$\text{mul}(A_S) = \text{mul}(S^*) = \text{mul}(a^{1/2}) \oplus \text{mul}(d^{1/2} g^*) = \mathcal{M}_1 \oplus \mathcal{M}_2 = \text{mul}(A)$ and, by Theorem 2.3, the operator part of $S^* \bar{S}$ is $(S^* \bar{S})_0 = ((\bar{S})_0)^\times (\bar{S})_0 = s^\times \bar{s}$. Then

$$A_S = s^\times \bar{s} \hat{\oplus} (\{0\} \times \text{mul}(A)).$$

□

Let V_1 be the partial isometry given in the proof of Theorem 3.11. Then

$$s = V_1^* A_0^{1/2} \text{ on } \text{dom}(A). \quad (4.7)$$

Proposition 4.4. *Let $A \geq 0$ be a linear relation in \mathcal{H} and let S be a closed subspace of \mathcal{H} such that $P_S(\text{dom}(A)) \subseteq \text{dom}(A)$. Then*

$$A \geq A_S.$$

Proof. Suppose that $(A_S)_0$ is the operator part of A_S then, by [9, Proposition 2.7],

$$\left\langle (A_S)_0^{1/2}u, (A_S)_0^{1/2}v \right\rangle = \left\langle (\bar{S})_0u, (\bar{S})_0v \right\rangle = \langle \bar{s}u, \bar{s}v \rangle,$$

for every $u, v \in \text{dom}((A_S)_0^{1/2}) = \text{dom}((\bar{S})_0) = \text{dom}(\bar{s})$. Then

$$\text{dom}(A) = \text{dom}(s) \subseteq \text{dom}((A_S)_0^{1/2}).$$

Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{D}_1 \oplus \mathcal{D}_2 = \text{dom}(s)$. Then, by (4.7),

$$\|(A_S)_0^{1/2}h\| = \|\bar{s}h\| = \|sh\| = \|V_1^*A_0^{1/2}h\| \leq \|A_0^{1/2}h\| = \|A_0^{1/2}h\|.$$

Hence, since $\text{dom}(A)$ is a core of $A^{1/2}$, by [16, Lemma 10.10], $A \geq A_S$. \square

Define

$$\mathcal{L} := \overline{A^{1/2}(\mathcal{D}_1)} \cap \overline{\text{dom}(A)}.$$

In the following we show that if the positive relations A and $A^{1/2}$ admit a matrix representation with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ and $\mathcal{L} \oplus \mathcal{L}^\perp$, respectively, then

$$A = A_S + A_{/S}.$$

Lemma 4.5. *Let $A \geq 0$ be a linear relation in \mathcal{H} and let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Consider the matrix representation of A with respect to $\mathcal{S} \oplus \mathcal{S}^\perp$ in (4.1). Then the following are equivalent:*

- (i) $P_{\mathcal{L}}(A^{1/2}(\text{dom}(A))) \subseteq \text{dom}(A^{1/2})$;
- (ii) $\text{dom}(d^{1/2}g^*g d^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2$;
- (iii) $\text{dom}(d^{1/2}D_g^2 d^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2$.

In this case, the linear relation $D_g d^{1/2}$ is decomposable.

Proof. Since $A^{1/2}(\text{dom}(A)) = A_0^{1/2}(\text{dom}(A)) \oplus \text{mul}(A)$ and $\text{mul}(A) \subseteq \mathcal{L}^\perp$, it follows that

$$P_{\mathcal{L}}(A^{1/2}(\text{dom}(A))) = P_{\mathcal{L}}(A_0^{1/2}(\text{dom}(A)) \oplus \text{mul}(A)) = P_{\mathcal{L}}(A_0^{1/2}(\text{dom}(A))). \quad (4.8)$$

Let V_1 and V_2 be the partial isometries given in the proof of Theorem 3.11.

Then $f = V_1^*V_2$ and, since $\mathcal{L} = \overline{A_0^{1/2}(\mathcal{D}_1)}$, $P_{\mathcal{L}} = V_1V_1^*$. Also,

$$A_0^{1/2}|_{\text{dom}(A)} = \left(V_1a_0^{1/2}|_{\mathcal{D}_1} \quad V_2d_0^{1/2}|_{\mathcal{D}_2} \right),$$

and $A_0^{1/2} = \begin{pmatrix} a_0^{1/2}V_1^* \\ d_0^{1/2}V_2^* \end{pmatrix}$, so that $\text{dom}(A_0^{1/2}) = \text{dom}(a_0^{1/2}V_1^*) \cap \text{dom}(d_0^{1/2}V_2^*)$. Then

$$P_{\mathcal{L}}(A_0^{1/2}(\mathcal{D}_2)) \subseteq \text{dom}(A_0^{1/2}) \Leftrightarrow \text{dom}(d_0^{1/2}f^*f d_0^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2. \quad (4.9)$$

In fact, by Theorem 3.11,

$$V_1^*P_{\mathcal{L}}(A_0^{1/2}(\mathcal{D}_2)) = f d_0^{1/2}(\mathcal{D}_2) \subseteq \text{dom}(a_0^{1/2})$$

and

$$V_2^*P_{\mathcal{L}}(A_0^{1/2}(\mathcal{D}_2)) = f^*f d_0^{1/2}(\mathcal{D}_2).$$

Then (4.9) follows.

Since $gd^{1/2}|_{\mathcal{D}_2} = fd_0^{1/2}|_{\mathcal{D}_2}$ we have that

$$d^{1/2}g^*gd^{1/2}|_{\mathcal{D}_2} = d_0^{1/2}f^*fd_0^{1/2}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2). \quad (4.10)$$

Then (i) \Leftrightarrow (ii) follows from (4.8) and (4.9).

Applying (4.3), it can be seen that

$$d^{1/2}D_g^2d^{1/2}|_{\mathcal{D}_2} = d_0^{1/2}D_f^2d_0^{1/2}|_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2). \quad (4.11)$$

By (4.10), $\text{dom}(d^{1/2}g^*gd^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2$ if and only if $\text{dom}(d_0^{1/2}f^*fd_0^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2$. Then (ii) \Leftrightarrow (iii) follows from (4.11) and from the fact that $f^*fd_0^{1/2}|_{\mathcal{D}_2} + D_f^2d_0^{1/2}|_{\mathcal{D}_2} = d_0^{1/2}|_{\mathcal{D}_2}$.

Since equation (4.3) holds and $\text{mul}(D_gd^{1/2}|_{\mathcal{D}_2}) = \mathcal{M}_2$ to see that $D_gd^{1/2}$ is decomposable it is sufficient to prove that the operator $D_f d_0^{1/2}$ is closable [11, Theorem 3.10]. In fact, let $(y_n)_{n \geq 1} \subseteq \mathcal{D}_2$ be such that $y_n \rightarrow 0$ and $D_f d_0^{1/2}y_n \rightarrow h$. Then, for every $h_2 \in \mathcal{D}_2$,

$$\begin{aligned} \left\langle h, D_f d_0^{1/2}h_2 \right\rangle &= \lim_{n \rightarrow \infty} \left\langle D_f d_0^{1/2}y_n, D_f d_0^{1/2}h_2 \right\rangle \\ &= \lim_{n \rightarrow \infty} \left\langle y_n, d_0^{1/2}D_f^2d_0^{1/2}h_2 \right\rangle = 0, \end{aligned}$$

where we used that, by (4.9), $\text{dom}(d_0^{1/2}D_f^2d_0^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2$. Then $h \in \overline{\text{ran}}(D_f d_0^{1/2}) \cap \text{ran}(D_f d_0^{1/2})^\perp$ and $h = 0$. \square

Theorem 4.6. *Let $A \geq 0$ be a linear relation in \mathcal{H} , let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$. Then the following are equivalent:*

- (i) $\text{dom}(A) \subseteq \text{dom}(A_{\mathcal{S}})$;
- (ii) $P_{\mathcal{L}}(A^{1/2}(\text{dom}(A))) \subseteq \text{dom}(A^{1/2})$;
- (iii) $A = A_{\mathcal{S}} + A_{/\mathcal{S}}$.

Proof. (i) \Rightarrow (ii): Let us see that $\text{dom}(d_0^{1/2}f^*fd_0^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2$. In fact, let $h_2 \in \mathcal{D}_2$ then $h_2 \in \text{dom}(A_{\mathcal{S}}) = \text{dom}(s^\times \bar{s})$, where s is as in (4.6), and $s^\times \bar{s}$ is the operator part of $A_{\mathcal{S}}$. Since $h_2 \in \mathcal{D}_2 \subseteq \text{dom}(s)$ and s is closable, it follows that

$$\bar{s}h_2 = sh_2 = fd_0^{1/2}h_2 \in \text{dom}(s^\times) = \text{dom}(d_0^{1/2}) \cap \text{dom}(d_0^{1/2}f^*).$$

Hence $h_2 \in \text{dom}(d_0^{1/2}f^*fd_0^{1/2}|_{\mathcal{D}_2})$. Then, by (4.8) and (4.9), $P_{\mathcal{L}}(A^{1/2}(\text{dom}(A))) \subseteq \text{dom}(A^{1/2})$.

(ii) \Rightarrow (iii): By the proof of Lemma 4.5,

$$\text{dom}(d_0^{1/2}f^*fd_0^{1/2}|_{\mathcal{D}_2}) = \text{dom}(d_0^{1/2}D_f^2d_0^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2.$$

Also, since

$$g^*gd^{1/2}|_{\mathcal{D}_2} + D_g^2d^{1/2}|_{\mathcal{D}_2} = d^{1/2}|_{\mathcal{D}_2}$$

and $\text{dom}(d^{1/2}g^*gd^{1/2}|_{\mathcal{D}_2}) = \text{dom}(d^{1/2}D_g^2d^{1/2}|_{\mathcal{D}_2}) = \mathcal{D}_2 \subseteq \text{dom}(d^{1/2})$ (see Lemma 4.5), it follows that

$$d^{1/2}g^*gd^{1/2}|_{\mathcal{D}_2} + d^{1/2}D_g^2d^{1/2}|_{\mathcal{D}_2} = d|_{\mathcal{D}_2}. \quad (4.12)$$

Next we show that

$$s^\times s = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0^{1/2} f^* f d_0^{1/2} |_{\mathcal{D}_2} \end{pmatrix}.$$

Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{D}_1 \oplus \mathcal{D}_2$. Then

$$s^\times s h = \begin{pmatrix} a_0^{1/2} (a_0^{1/2} h_1 + f d_0^{1/2} h_2) \\ d_0^{1/2} f^* (a_0^{1/2} h_1 + f d_0^{1/2} h_2) \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0^{1/2} f^* f d_0^{1/2} |_{\mathcal{D}_2} \end{pmatrix} h,$$

where the last equality follows from the fact that, since $f d_0^{1/2} h_2 \in \text{dom}(d_0^{1/2} f^*)$, it is possible to distribute. Then, since $d^{1/2} g^* g d^{1/2} |_{\mathcal{D}_2} = d_0^{1/2} f^* f d_0^{1/2} |_{\mathcal{D}_2} \hat{\oplus} (\{0\} \times \mathcal{M}_2)$, it follows that

$$A_S \supset s^\times s \hat{\oplus} (\{0\} \times \text{mul}(A)) = \begin{pmatrix} a & b \\ c & d^{1/2} g^* g d^{1/2} |_{\mathcal{D}_2} \end{pmatrix}. \quad (4.13)$$

Clearly,

$$A_{/S} \supset \begin{pmatrix} 0 & 0 \\ 0 & d^{1/2} D_g^2 d^{1/2} |_{\mathcal{D}_2} \end{pmatrix}.$$

Then, by [10, Lemma 5.5] and (4.12),

$$\begin{pmatrix} a & b \\ c & d^{1/2} g^* g d^{1/2} |_{\mathcal{D}_2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d^{1/2} D_g^2 d^{1/2} |_{\mathcal{D}_2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d |_{\mathcal{D}_2} \end{pmatrix} = A. \quad (4.14)$$

Hence $A_S + A_{/S} \supset A$ and, by (2.2),

$$A = A^* \supset (A_S + A_{/S})^* \supset (A_S)^* + (A_{/S})^* = A_S + A_{/S} \supset A.$$

So that $A = A_S + A_{/S}$.

(iii) \Rightarrow (i): It is straightforward. □

For a nonnegative operator $A \in L(\mathcal{H})$ and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, Pekarev [15] showed that the Schur complement $A_{/S}$ can be expressed as $A_{/S} = A^{1/2} P_{\mathcal{L}^\perp} A^{1/2}$ where $\mathcal{L} = \overline{A^{1/2}(\mathcal{S})}$. In what follows, we extend this formula for a linear relation $A \geq 0$ in \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ and $P_{\mathcal{L}}(A^{1/2}(\text{dom}(A))) \subseteq \text{dom}(A^{1/2})$.

Corollary 4.7. *Let $A \geq 0$ be a linear relation in \mathcal{H} , let \mathcal{S} be a closed subspace of \mathcal{H} such that $P_{\mathcal{S}}(\text{dom}(A)) \subseteq \text{dom}(A)$ and $P_{\mathcal{L}}(A^{1/2}(\text{dom}(A))) \subseteq \text{dom}(A^{1/2})$. Then*

$$A_{/S} = A^{1/2} \overline{P_{\mathcal{L}^\perp} A^{1/2} |_{\text{dom}(A)}}, \quad A_S = A^{1/2} \overline{P_{\mathcal{L}} A^{1/2} |_{\text{dom}(A)}}.$$

Proof. Let $h = h_1 + h_2 \in \mathcal{D}_1 \oplus \mathcal{D}_2$. Then

$$\begin{aligned} \|th_2\|^2 &= \left\langle D_f d_0^{1/2} h_2, D_f d_0^{1/2} h_2 \right\rangle = \left\langle (1 - f^* f) d_0^{1/2} h_2, d_0^{1/2} h_2 \right\rangle \\ &= \left\langle V_2^* (1 - V_1 V_1^*) V_2 d_0^{1/2} h_2, d_0^{1/2} h_2 \right\rangle = \left\langle (1 - P_{\mathcal{L}}) A_0^{1/2} h_2, A_0^{1/2} h_2 \right\rangle \\ &= \left\langle P_{\mathcal{L}^\perp} A_0^{1/2} h_2, A_0^{1/2} h_2 \right\rangle = \|P_{\mathcal{L}^\perp} A_0^{1/2} h\|^2, \end{aligned}$$

where we used that $P_{\mathcal{L}^\perp} A_0^{1/2} h = P_{\mathcal{L}^\perp} A_0^{1/2} h_2$, because $A_0^{1/2} h_1 \in \mathcal{L}$. Then, since t is closable (see Lemma 4.5), $P_{\mathcal{L}^\perp} A_0^{1/2}|_{\text{dom}(A)}$ is also closable. Set $W := \overline{P_{\mathcal{L}^\perp} A_0^{1/2}|_{\text{dom}(A)}}$. Then, since $P_{\mathcal{L}^\perp} A_0^{1/2}|_{\text{dom}(A)} = P_{\mathcal{L}^\perp} A_0^{1/2}|_{\text{dom}(A)} \hat{\oplus} (\{0\} \times \text{mul}(A))$, it follows that

$$W = \overline{P_{\mathcal{L}^\perp} A_0^{1/2}|_{\text{dom}(A)}} \hat{\oplus} (\{0\} \times \text{mul}(A)). \quad (4.15)$$

Moreover, since t and $P_{\mathcal{L}^\perp} A_0^{1/2}|_{\text{dom}(A)}$ are closable operators, by (4.4) and (4.15), it follows that the operator part of T is $T_0 = \bar{t}$ and the operator part of W is $W_0 = P_{\mathcal{L}^\perp} A_0^{1/2}|_{\text{dom}(A)}$. Also,

$$\mathcal{L}^\perp \cap \text{dom}(A_0^{1/2}) \subseteq A_0^{-1/2}(\mathcal{S}^\perp) := \{y \in \text{dom}(A_0^{1/2}) : A_0^{1/2}y \in \mathcal{S}^\perp\}.$$

In fact, let $y \in \mathcal{L}^\perp \cap \text{dom}(A_0^{1/2})$. Then, for every $h_1 \in \mathcal{D}_1$,

$$0 = \left\langle y, A_0^{1/2} h_1 \right\rangle = \left\langle A_0^{1/2} y, h_1 \right\rangle.$$

So that

$$A_0^{1/2} y \in \mathcal{D}_1^\perp = (\mathcal{S}^\perp \oplus \mathcal{M}_1) \cap \overline{\text{dom}(A)} \subseteq \mathcal{S}^\perp$$

because $\mathcal{M}_1 = \mathcal{S} \cap \text{mul}(A)$. Then

$$\text{ran}(W_0^* W_0) \subseteq \mathcal{S}^\perp. \quad (4.16)$$

In fact, let $y \in \text{ran}(W_0^* W_0)$. Then, since $\text{ran}(W_0) \subseteq \mathcal{L}^\perp$, it follows that

$$y = W_0^* W_0 x = A_0^{1/2} W_0 x,$$

for some $x \in \text{dom}(W_0^* W_0)$. Then $W_0 x \in \mathcal{L}^\perp \cap \text{dom}(A_0^{1/2}) \subseteq A_0^{-1/2}(\mathcal{S}^\perp)$ and $y = A_0^{1/2} W_0 x \in \mathcal{S}^\perp$. So that, by (4.16), $\mathcal{S} \subseteq \ker(W_0^* W_0) = \ker(W_0) \subseteq \text{dom}(W_0)$, where we used Theorem 2.3. Hence

$$h \in \text{dom}(W_0) \Leftrightarrow P_{\mathcal{S}^\perp} h \in \text{dom}(T_0) \text{ and } \|W_0 h\| = \|T_0 P_{\mathcal{S}^\perp} h\|. \quad (4.17)$$

Now we show that

$$A_{/S} = \begin{pmatrix} 0 & 0 \\ 0 & T^* T \end{pmatrix} = W^* W = A^{1/2} \overline{P_{\mathcal{L}^\perp} A^{1/2}|_{\text{dom}(A)}},$$

where for the last equality we used that $\text{ran}(\overline{P_{\mathcal{L}^\perp} A^{1/2}|_{\text{dom}(A)}}) \subseteq \mathcal{L}^\perp$.

Suppose that $(W^* W)_0$ is the operator part of $W^* W$ then, by [9, Proposition 2.7],

$$\left\langle (W^* W)_0^{1/2} u, (W^* W)_0^{1/2} v \right\rangle = \langle W_0 u, W_0 v \rangle,$$

for every $u, v \in \text{dom}((W^* W)_0^{1/2}) = \text{dom}(W_0)$.

Suppose that $(A_{/S})_0$ is the operator part of $A_{/S}$. Let $h \in \text{dom}((A_{/S})_0^{1/2}) = \mathcal{S} \oplus \text{dom}(T_0)$ then $h = h_1 + h_2$ with $h_1 \in \mathcal{S}$ and $h_2 \in \text{dom}(T_0)$. Then, by (4.17), $h \in \text{dom}(W_0)$. Conversely, if $h \in \text{dom}(W_0)$, by (4.17), $P_{\mathcal{S}^\perp} h \in \text{dom}(T_0)$. Then $h = P_{\mathcal{S}} h + P_{\mathcal{S}^\perp} h \in \mathcal{S} \oplus \text{dom}(T_0) = \text{dom}((A_{/S})_0^{1/2})$.

Also, if $h \in \text{dom}((W^* W)_0^{1/2}) = \text{dom}(W_0) = \text{dom}((A_{/S})_0^{1/2})$, it follows that $h = h_1 + h_2 \in \mathcal{S} \oplus \mathcal{S}^\perp$ and, by (4.17),

$$\|(A_{/S})_0^{1/2} h\| = \|T_0 h_2\| = \|W_0 h\| = \|(W^* W)_0^{1/2} h\|.$$

Then $A_{/S} = W^*W$.

Finally, by (4.7),

$$V_1 s = P_{\mathcal{L}} A_0^{1/2} |_{\text{dom}(A)}.$$

Then, since s is closable and V_1 is a partial isometry, the operator $P_{\mathcal{L}} A_0^{1/2} |_{\text{dom}(A)}$ is closable and

$$\overline{s} = \overline{V_1^* P_{\mathcal{L}} A_0^{1/2} |_{\text{dom}(A)}} = \overline{V_1^* P_{\mathcal{L}} A_0^{1/2} |_{\text{dom}(A)}}.$$

So that

$$s^{\times} \overline{s} = A_0^{1/2} P_{\mathcal{L}} V_1 V_1^* P_{\mathcal{L}} A_0^{1/2} |_{\text{dom}(A)} = A_0^{1/2} P_{\mathcal{L}} P_{\mathcal{L}} A_0^{1/2} |_{\text{dom}(A)},$$

and, since $\text{ran}(P_{\mathcal{L}} A^{1/2} |_{\text{dom}(A)}) \subseteq \mathcal{L}$,

$$\begin{aligned} A^{1/2} P_{\mathcal{L}} P_{\mathcal{L}} A^{1/2} |_{\text{dom}(A)} &= A^{1/2} P_{\mathcal{L}} A^{1/2} |_{\text{dom}(A)} = A_0^{1/2} P_{\mathcal{L}} A_0^{1/2} |_{\text{dom}(A)} \hat{\oplus} (\{0\} \times \text{mul}(A)) \\ &= s^{\times} \overline{s} \hat{\oplus} (\{0\} \times \text{mul}(A)) = A_S. \end{aligned}$$

□

Corollary 4.8. *Let $A \geq 0$ be a linear relation in \mathcal{H} and let S be a closed subspace of \mathcal{H} . If A and $A^{1/2}$ admit a matrix representation with respect to $S \oplus S^{\perp}$ and $\mathcal{L} \oplus \mathcal{L}^{\perp}$, respectively, then*

$$A_{/S} = A^{1/2} P_{\mathcal{L}^{\perp}} A^{1/2} |_{\text{dom}(A)}, \quad A_S = A^{1/2} P_{\mathcal{L}} A^{1/2} |_{\text{dom}(A)}, \quad \text{and} \quad A = A_S + A_{/S}.$$

Proof. By Theorem 3.2, $P_S(\text{dom}(A)) \subseteq \text{dom}(A)$ and $P_{\mathcal{L}}(\text{dom}(A^{1/2})) \subseteq \text{dom}(A^{1/2})$. Then, since $A^{1/2}(\text{dom}(A)) \subseteq \text{dom}(A^{1/2}) \oplus \text{mul}(A)$, it follows that

$$P_{\mathcal{L}}(A^{1/2}(\text{dom}(A))) \subseteq P_{\mathcal{L}}(\text{dom}(A^{1/2})) \subseteq \text{dom}(A^{1/2}).$$

Then, the result follows from Corollary 4.7 and Theorem 4.6. □

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