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# A NOTE ON ÉTALE REPRESENTATIONS FROM NILPOTENT ORBITS

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Dedicated to the memory of Professor E. B. Vinberg

**Abstract**

A linear étale representation of a complex algebraic group $G$ is given by a complex algebraic $G$-module $V$ such that $G$ has a Zariski-open orbit in $V$ and $\dim G = \dim V$. A current line of research investigates which reductive algebraic groups admit such étale representations, with a focus on understanding common features of étale representations. One source of new examples arises from the classification theory of nilpotent orbits in semisimple Lie algebras. We survey what is known about reductive algebraic groups with étale representations and then discuss two classical constructions for nilpotent orbit classifications due to Vinberg and to Bala and Carter. We determine which reductive groups and étale representations arise in these constructions and we work out in detail the relation between these two constructions.

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## 1. Introduction

The problem of deciding which Lie groups are *affinely flat* (meaning they admit a left-invariant affine connection with zero curvature and torsion) has been studied for the past five decades, but a complete classification seems out of reach. In this paper we describe certain such groups that occur implicitly in the classification theory of nilpotent orbits. In doing so, we also take the opportunity to clarify the relations between different approaches to this classification theory. This connects to Ernest Vinberg's work in two ways. The first is through Vinberg's work on homogeneous cones [23] and his development of *left-symmetric algebras*, which are an algebraic way of characterising affinely flat Lie groups through an additional algebraic structure on their Lie algebras. The second is through Vinberg's classification theory [24] of

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nilpotent orbits in semisimple Lie algebras, which uses a construction of certain Lie algebras of affinely flat reductive groups.

We focus on this second aspect in this paper. The original work on nilpotent orbit classifications by Vinberg [24] and by Bala and Carter [2, 3] is quite involved, and one aim of this paper is to provide a more accessible and self-contained description of some of the main constructions in these papers. In addition to that, our discussion explains in detail the relation between these constructions. We also discuss an interesting connection to a construction of étale representations by Gyoja [15]. Our subject's broad appeal comes from its relevance to research areas such as Lie theory, invariant theory and representation theory. Moreover, it has applications in mathematical physics. For example, certain flat affine connections on Lie groups correspond to the classical Yang–Baxter equation (see Bordemann [6]). Nilpotent orbit classifications, on the other hand, are useful tools in the study of supergravities and quantum information theory (see Dietrich *et al.* [11, 12]).

We start with a brief survey on reductive algebraic groups with étale representations (Section 1.1) and then describe the results and structure of this paper (Section 1.2).

**1.1. Flat Lie groups and étale representations.** The existence question for left-invariant flat affine connections on Lie groups first arose during the investigation of which Lie groups act (simply) transitively on an affine space (see Auslander [1]) or, more generally, on subdomains of affine space, such as convex homogeneous cones (see Vinberg [23, Section I.6]). In these situations, the affinely flat Lie groups are always solvable. Indeed, geodesic completeness of the left-invariant flat affine connection is equivalent to the underlying simply connected Lie group acting simply transitively on an affine space, and this in turn is equivalent to the Lie group being solvable (see Milnor [19, Theorem 3.2]). It was conjectured for some time that all nilpotent Lie groups are affinely flat, but this was eventually refuted by Benoist [5].

A seminal paper by Medina Perea [18] highlighted the equivalence of the existence of left-invariant flat affine connections on a Lie group, of left-symmetric products on its Lie algebra, and of *affine étale representations* of the Lie group or its Lie algebra. The latter are representations by affine transformations on an affine space of the same dimension as the group, such that there is an open orbit for the group action; these étale representations are a crucial tool in the analysis of affinely flat Lie groups. As they are best studied on Lie algebras, a Lie algebra is called *affinely flat* if it is the Lie algebra of an affinely flat group.

Eventually, there was interest in the (necessarily geodesically incomplete) affine structures on reductive algebraic groups. An example of an affinely flat structure on a reductive group is the general linear group $\mathrm{GL}(n)$ where the flat structure is induced through its multiplication action on the affine space of $n \times n$ matrices. The case of the reductive Lie algebra $\mathfrak{gl}(n)$ was treated comprehensively by Baues [4] from the perspective of left-symmetric algebras, and Burde [7, 8] showed that $\mathrm{GL}(n)$ is the only complex or split real affinely flat Lie group with one-dimensional centre and simple commutator subgroup.

For a reductive group, every representation by affine transformations is in fact equivalent to a linear representation (see Milnor [19, Lemma 2.3]), so we can and will henceforth take all étale representations for reductive groups or algebras to be linear representations. This frames the question for étale representations as a special case of the question for *prehomogeneous modules* for reductive algebraic groups. The latter is a module $(G, \varrho, V)$, where $G$ is a reductive algebraic group $G$ acting on a finite-dimensional complex vector space $V$ through an algebraic representation $\varrho\colon G \to \mathrm{GL}(V)$ such that the $G$-action has a Zariski-open orbit on $V$; the points of this open orbit are said to be *in general position* in $V$. Then necessarily $\dim G \geqslant \dim V$ and we have an étale representation precisely if $\dim G = \dim V$. In this case we also call $(G, \varrho, V)$ an *étale module*. Clearly, for étale modules, the stabiliser in $G$ of any point in the open orbit is a finite subgroup. In terms of Lie algebras $\mathfrak{g}$, an étale representation is one where the action of $\mathfrak{g}$ on a point in general position yields a vector space isomorphism between $\mathfrak{g}$ and $V$; in particular, $\dim \mathfrak{g} = \dim V$ and the stabiliser subalgebra at a point in general position is trivial.

Since several classification results for prehomogeneous modules are available, this framework is well suited to the study of affinely flat reductive groups. The classification of irreducible prehomogeneous modules by Sato and Kimura [21] allows us to find all irreducible étale representations for reductive groups simply by comparing dimensions in the Sato–Kimura classification. It is important to emphasise that nonirreducible étale modules cannot in general be written as a direct sum of irreducible étale modules. Some examples of nonirreducible prehomogeneous modules are given by Kimura *et al.* [16, 17]. These partial classifications were summarised by Burde and Globke [9], who also discuss some obstructions to the existence of flat affine structures on reductive groups. Unfortunately, these examples do not point to a sensible structure theory for étale modules of reductive algebraic groups. On the other hand, it is remarkable that almost all simple factors appearing for the groups in these modules are of Lie type A, with the only exception being an occasional factor of type $\mathrm{C}_2$.

With a comprehensive structure theory seeming far away, it is natural to ask which simple factors can appear in affinely flat reductive algebraic groups. This was further pursued in Burde *et al.* [10], where étale representations for certain reductive groups with one factor of either type $\mathrm{B}_n$, $\mathrm{C}_n$ or $\mathrm{D}_n$ for arbitrary $n$ were constructed. The examples with a factor of type $\mathrm{C}_n$ were particularly interesting, as their étale representations have not only finite, but trivial stabiliser. In particular, they provide a counterexample to a conjecture of Popov on the Zariski cancellation problem [20]. Moreover, in [10] it was shown that no reductive group with simple factors of type $\mathrm{F}_4$ or $\mathrm{E}_8$ admits an étale representation.

**1.2. Overview of this paper.** We focus on the étale representations for reductive algebraic groups arising in the classification of nilpotent orbits in semisimple Lie algebras, in particular, the classifications of Vinberg [24] and Bala and Carter [2, 3]. We show in Proposition 2.1 that these groups are subject to certain restrictions, notably that all their simple factors are either special linear or orthogonal groups.

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In light of known examples of groups with symplectic groups as simple factors, this shows that étale modules of this type are a proper subclass of the étale modules for general reductive algebraic groups. The classification methods of Vinberg and of Bala and Carter share some common ideas, and the second aim of this paper is to provide concise descriptions of these methods and to explain how they are related. In Section 2 we first look at Vinberg's construction of carrier algebras for nilpotent elements in a graded Lie algebra. From the classification of simple carrier algebras we determine the types of reductive groups for which étale modules arise by this method. In Section 3, we show how Bala and Carter find minimal Levi subalgebras for a given nilpotent element in a semisimple Lie algebra and explain how it relates to Vinberg's carrier algebras (see Proposition 3.2 and its corollary). In fact, we see that the two approaches coincide for $\mathbb{Z}$-graded algebras. Lastly, we discuss Gyoja's method [15], which describes how a prehomogeneous module $(G, \varrho, V)$ for a reductive algebraic group can be used to construct an étale module $(G', \varrho', V')$ for a reductive subgroup $G' \leq G$ and a quotient module $V'$ of $V$. We show in Proposition 4.1 how this method generalises the constructions of Vinberg and Bala and Carter.

**1.3. Notation.** All Lie algebras $\mathfrak{g}$ we consider are defined over the field of complex numbers. The centraliser of a subset $X \subseteq \mathfrak{g}$ in $\mathfrak{g}$ is $\mathfrak{z}_{\mathfrak{g}}(X) = \{y \in \mathfrak{g} : [y, X] = \{0\}\}$ and the normaliser is $\mathfrak{n}_{\mathfrak{g}}(X) = \{y \in \mathfrak{g} : [y, X] \subseteq \mathrm{span}_{\mathbb{C}}(X)\}$. An element $x \in \mathfrak{g}$ is *nilpotent* (or *semisimple*) if its adjoint representation $\mathrm{ad}(x)$ on $\mathfrak{g}$ is nilpotent (or semisimple). An algebraic group $G$ is *reductive* if its maximal unipotent normal subgroup is trivial. Here we call a Lie algebra $\mathfrak{g}$ *reductive* if it is the Lie algebra of a reductive algebraic group, but other definitions exist (see [22, Section 20.5]). In this case $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is the semisimple commutator subalgebra of $\mathfrak{g}$ and $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$. Let $n > 0$ be an integer and $\mathbb{Z}_n = \{0, \ldots, n-1\}$, or $n = \infty$ and $\mathbb{Z}_\infty = \mathbb{Z}$. A Lie algebra $\mathfrak{g}$ is *$\mathbb{Z}_n$-graded* if $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_n} \mathfrak{g}_i$, where each $\mathfrak{g}_i \leq \mathfrak{g}$ is a subspace and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all $i, j$; here $\mathfrak{g}_k = \mathfrak{g}_{k \bmod n}$ for all $k \in \mathbb{Z}$. Note that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$.

## 2. Vinberg's carrier algebras

Vinberg [24] studied complex semisimple Lie algebras graded by an arbitrary abelian group. However, the first step in his analysis is to restrict to a subalgebra graded by a cyclic group, so we will only consider this case. Let $\mathfrak{g}$ be a $\mathbb{Z}_n$-graded semisimple Lie algebra, where $n > 0$ is an integer or $n = \infty$. If $n$ is finite, then such a grading is the eigenspace decomposition of a Lie algebra automorphism of order $n$. If $n = \infty$, then the $\mathbb{Z}$-grading of $\mathfrak{g}$ comes from a derivation $\varphi$ that acts as multiplication by $i$ on each $\mathfrak{g}_i$. For semisimple $\mathfrak{g}$, this derivation is inner, that is, $\varphi = \mathrm{ad}(h)$ for a unique *defining element* $h \in \mathfrak{g}_0$.

**2.1. Carrier algebras.** Carrier algebras for $\mathfrak{g}$ are constructed as follows. For a nonzero nilpotent $e \in \mathfrak{g}_1$ choose an $\mathfrak{sl}_2$-triple $(h, e, f)$ where $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$; this means $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Let $\mathfrak{t}_0$ be a maximal toral subalgebra of the centraliser of $(h, e, f)$ in $\mathfrak{g}_0$ and define $\mathfrak{t} = \mathbb{C}h \oplus \mathfrak{t}_0$. Equivalently, $\mathfrak{t}$ is a maximal

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toral subalgebra of the normaliser of $\mathbb{C}e$ in $\mathfrak{g}_0$ (see [13, Lemma 30]), where $\mathbb{C}e$ denotes the $\mathbb{C}$-span of $e$. Now let $\lambda\colon \mathfrak{t} \to \mathbb{C}$ such that $[t, e] = \lambda(t)e$ for all $t \in \mathfrak{t}$, and define the $\mathbb{Z}$-graded algebra $\mathfrak{g}(\mathfrak{t}, e)$ by

$$\mathfrak{g}(\mathfrak{t}, e) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}(\mathfrak{t}, e)_k \quad \text{with } \mathfrak{g}(\mathfrak{t}, e)_k = \{x \in \mathfrak{g}_k : [t, x] = k\lambda(t)x \text{ for all } t \in \mathfrak{t}\}. \tag{2.1}$$

The derived subalgebra of $\mathfrak{g}(\mathfrak{t}, e)$ is the *carrier algebra* of $e$, denoted by

$$\mathfrak{c}(e) = [\mathfrak{g}(\mathfrak{t}, e), \mathfrak{g}(\mathfrak{t}, e)].$$

It is $\mathbb{Z}$-graded with the induced grading; note that $e \in \mathfrak{c}(e)_1$. This carrier algebra of $e$ is unique up to conjugacy under the adjoint group $G_0$ of $\mathfrak{g}_0$; one therefore also speaks of *the* carrier algebra of $e$ in $\mathfrak{g}$. Moreover, two nonzero nilpotent elements of $\mathfrak{g}_1$ are $G_0$-conjugate if and only if their carrier algebras are $G_0$-conjugate, which makes carrier algebras a useful tool for classifying nilpotent orbits (see Vinberg [24, Section 4]). For details on the classification of nilpotent orbits in *real* semisimple Lie algebras using carrier algebras defined over the real field we refer to Dietrich *et al.* [12, 13].

Vinberg [24, Theorem 4] showed that every carrier algebra is semisimple $\mathbb{Z}$-graded with $\mathfrak{c}(e)_k \leq \mathfrak{g}_k$ for each $k \in \mathbb{Z}$, and that carrier algebras are characterised by the following three conditions:

(V1) $\dim \mathfrak{c}(e)_0 = \dim \mathfrak{c}(e)_1$;
(V2) $\mathfrak{c}(e)$ is normalised by a maximal toral subalgebra of $\mathfrak{g}_0$;
(V3) $\mathfrak{c}(e)$ is not a proper subalgebra of a reductive $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$ of the same rank.

Moreover, [24, Theorem 2] shows that $e$ is in *generic* (or *general*) *position* in $\mathfrak{c}(e)_1$, that is, $[\mathfrak{c}(e)_0, e] = \mathfrak{c}(e)_1$, so (V1) states that the adjoint action of $\mathfrak{c}(e)_0$ on $\mathfrak{c}(e)_1$ yields an étale representation for $\mathfrak{c}(e)_0$.

Only property (V1) is intrinsic to $\mathfrak{c}(e)$, whereas (V2) and (V3) are determined by its embedding in the ambient Lie algebra $\mathfrak{g}$. Thus, to describe the Lie algebras that can appear as carrier algebras for nilpotent elements in semisimple Lie algebras, one must merely classify $\mathbb{Z}$-graded Lie algebras with (V1); we call such an algebra an *abstract carrier algebra*. Every abstract carrier algebra is a direct sum of simple abstract carrier algebras, so to describe the possible étale modules $(\mathfrak{c}(e)_0, \mathrm{ad}, \mathfrak{c}(e)_1)$ coming from semisimple carrier algebras, it is sufficient to focus on simple abstract carrier algebras in Lie algebras. In the next section we follow Djokovič's description [14] (based on work by Vinberg [24]) of the classification of all simple abstract carrier algebras. Using a different terminology, Bala and Carter [2] have also obtained a classification for the classical case. These classifications determine the following proposition.

PROPOSITION 2.1. *A reductive Lie algebra $\mathfrak{g}_0$ admitting an étale representation coming from the adjoint action of a nilpotent element is a direct sum of the degree-*0 *components of $j \geqslant 1$ simple abstract carrier algebras. As such, the semisimple part of $\mathfrak{g}_0$, if nontrivial, has simple factors of type* A *and at most $j$ factors of type* B *or* D.

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*The centre of* $\mathfrak{g}_0$ *has dimension higher than j, unless all the simple abstract carrier algebras involved have weighted Dynkin diagrams of types in* $\{\mathrm{A}_1, \mathrm{E}_8^{(11)}, \mathrm{F}_4^{(4)}, \mathrm{G}_2^{(2)}\}$ *as defined in* [14, *Table II*]*, in which case the centre of* $\mathfrak{g}_0$ *has dimension j.*

From Burde *et al.* [10] we know that there exist étale representations for reductive algebraic groups with a simple factor of type C. This shows the following corollary.

COROLLARY 2.2. *There are étale representations for reductive Lie algebras that do not come from the adjoint action of a nilpotent element.*

**2.2. Simple abstract carrier algebras Lie algebras.** Recall that the grading of a semisimple $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ with defining element $h \in \mathfrak{g}$ is determined by $\mathfrak{g}_k = \{x \in \mathfrak{g} : [h, x] = kx\}$ for $k \in \mathbb{Z}$. Two $\mathbb{Z}$-graded Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ with defining elements $h$ and $h'$ are isomorphic if there is a Lie algebra isomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ with $\varphi(h) = h'$. Djokovič [14] classified, up to isomorphism, semisimple $\mathbb{Z}$-graded Lie algebras in terms of weighted Dynkin diagrams. Let $\mathfrak{h} \leq \mathfrak{g}$ be a maximal toral subalgebra containing $h$, with corresponding root system $\Phi$. Let $\Pi$ be a basis of simple roots such that $\alpha(h) \geqslant 0$ for every $\alpha \in \Pi$. Let $\Delta(\mathfrak{g})$ be the Dynkin diagram of $\mathfrak{g}$ with respect to $\mathfrak{h}$, with vertices labelled by $\Pi$, and to each vertex $\alpha \in \Pi$ attach the integer $\alpha(h)$. The resulting weighted Dynkin diagram is denoted by $\Delta(\mathfrak{g}, h)$. Now [14, Theorem 1] shows that there is a bijection between (isomorphism classes of) $\mathbb{Z}$-graded semisimple Lie algebras $(\mathfrak{g}, h)$ and (isomorphism classes of) weighted Dynkin diagrams $\Delta(\mathfrak{g}, h)$.

In the following, let $(\mathfrak{g}, h)$ be simple $\mathbb{Z}$-graded and define $\deg\alpha \in \mathbb{Z}$ for $\alpha \in \Phi$ by $x_\alpha \in \mathfrak{g}_{\deg\alpha}$, where $x_\alpha \in \mathfrak{g}$ is a root vector corresponding to $\alpha$. If $\deg\alpha = k$, then $\alpha(h)x_\alpha = [h, x_\alpha] = kx_\alpha$, hence $\alpha(h) = k$; this shows that $\deg\alpha = \alpha(h)$. If $r_k$ is the number of roots with degree $k$, then $\mathfrak{g}$ is an abstract carrier algebra if and only if $\dim\mathfrak{h} + r_0 = r_1$. It is shown in [14, page 374] that if $\mathfrak{g}$ is an abstract carrier algebra, then $\deg\alpha \in \{0, 1\}$ for every simple root $\alpha \in \Pi$. So for the classification it remains to determine the weighted Dynkin diagrams with weights $\{0, 1\}$ such that $\dim\mathfrak{h}_0 + r_0 = r_1$. The reductive subalgebra $\mathfrak{g}_0$ is then given by the subdiagram consisting of the vertices with weight 0. To illustrate the method, we include the full proof for type A. To keep the exposition short, for the other types we only describe the results and refer to [14, Section 4], [2, Section 3] and [24, page 30] for more details.

If $\mathfrak{g}_0 = \mathfrak{h}$ is a maximal toral subalgebra, then $r_0 = 0$ and all labels in the weighted diagram are 1; one says that $\mathfrak{g}$ is *principal*. In this case the 0-component of the carrier algebra is abelian.

*Type* A. Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$. The diagonal matrix $h = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})$ is the defining element with $\lambda_1 \geqslant \cdots \geqslant \lambda_{n+1}$. Consider the root system $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j < n\}$ and basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ where each $\varepsilon_i$ maps $h$ to $\lambda_i$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Let $k = \lambda_1 - \lambda_{n+1}$ and, for $i = 0, \ldots, k$, let $d_i$ be the number of $\lambda_r$ with $\lambda_r = \lambda_1 - i$. A root $\pm(\varepsilon_i - \varepsilon_j)$ has degree 0 if and only if $\lambda_i = \lambda_j = \lambda_1 - r$ for some $r$, and for each $r$ there are $d_r(d_r - 1)$ possibilities for $\varepsilon_i$ and $\varepsilon_j$. This implies that $r_0 = \sum_{j=0}^{k} d_j(d_j - 1)$. In a similar way $r_1 = \sum_{j=0}^{k-1} d_j d_{j+1}$, and now a direct calculation shows that $n + r_0 = r_1$ if and only if

$$(d_0 - d_1)^2 + (d_1 - d_2)^2 + \cdots + (d_{k-1} - d_k)^2 + (d_0^2 - 1) + (d_k^2 - 1) = 0;$$

to see the latter, note that $d_0 + \cdots + d_k = n + 1$. In conclusion, $\mathfrak{g}$ is an abstract carrier algebra if and only if $d_0 = \cdots = d_k = 1$ and $k = n$, which is equivalent to $\mathfrak{g}$ being principal.

*Types* B *and* D. Let $\mathfrak{g} = \mathfrak{so}(m, \mathbb{C})$ be realised as $\mathfrak{g} = \{X \in \mathfrak{gl}(m, \mathbb{C}) : X^\top J = -JX\}$ where $J$ is the matrix with 1s on its anti-diagonal and 0s elsewhere, and either $m = 2n + 1$ (with $n \geqslant 2$) or $m = 2n$ (with $n \geqslant 4$). Write the defining element as $h = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_m$. Since $hJ = -Jh$, each $-\lambda_i = \lambda_{m+1-i}$. It has been shown that there are $s \geqslant 1$ and integers $k_1 > \cdots > k_s \geqslant 0$ such that, as multisets,

$$\{\lambda_1, \ldots, \lambda_m\} = \{k_i, k_i - 1, \ldots, 1 - k_i, -k_i : 1 \leq i \leq s\} \tag{2.2}$$

and $m = (2k_1 + 1) + \cdots + (2k_s + 1)$; note that 0 occurs $s$ times in $\{\lambda_1, \ldots, \lambda_m\}$ and 1 occurs at least $s - 1$ times, and so on. Conversely, for any such integers $k_1 > \cdots > k_s \geqslant 0$ with $m = (2k_1 + 1) + \cdots + (2k_s + 1)$ there is a defining element $h$ whose eigenvalues satisfy (2.2). To determine the labelled Dynkin diagrams, one chooses the diagonal matrices in $\mathfrak{g}$ as maximal toral subalgebra, and then the following statements hold.

If $m = 2n + 1$, then $s$ is odd and $\lambda_{n+1} = 0$. The corresponding simple abstract carrier algebra $B(k_1, \ldots, k_s)$ of type $\mathrm{B}_n$ has a weighted Dynkin diagram with labels $\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n$, where $\lambda_n$ is the label of the shorter root; see [14, Figure 5]. In that figure the last label is given as $2\lambda_n$ which is a typo (see [2, pages 410–412]). If $s = 1$, then $\{\lambda_1, \ldots, \lambda_m\} = \{n, n - 1, \ldots, 1 - n, -n\}$ and $B(n)$ is principal. If $s = 3$, then $\lambda_{n+1}, \lambda_n = 0$ and $0 < \lambda_{n-1}$, implying that the semisimple part of $\mathfrak{g}_0$ is a direct sum of algebras of type A. If $s \geqslant 5$, then $\lambda_{n+1}, \lambda_n, \lambda_{n-1} = 0$ and that semisimple part is a direct sum of algebras of type A and one algebra of type B.

If $m = 2n$, then $s$ is even and $\lambda_n = 0$. The corresponding abstract carrier algebra $D(k_1, \ldots, k_s)$ of type $\mathrm{D}_n$ has a weighted Dynkin diagram with labels $\lambda_1 - \lambda_2, \ldots, \lambda_{n-2} - \lambda_{n-1}, \lambda_{n-1}, \lambda_{n-1}$, where $\lambda_{n-2} - \lambda_{n-1}$ is the label of the vertex of degree 3 connected to the two vertices of degree 1 with label $\lambda_{n-1}$ (see [14, Figure 6]). If $s \geqslant 6$, then $\lambda_n, \lambda_{n-1}, \lambda_{n-2} = 0$ and the semisimple part of $\mathfrak{g}_0$ is a direct sum of algebras of type A and one algebra of type D. If $s = 2$ and $k_2 > 0$, or $s = 4$, then that semisimple part is a direct sum of algebras of type A; if $s = 2$ and $k_2 = 0$, then $D(n - 1, 0)$ is principal.

*Type* C. Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ be realised as $\mathfrak{g} = \{X \in \mathfrak{gl}(2n, \mathbb{C}) : X^\top S = -SX\}$ where $S$ has the identity matrix $I_n$ and the negative $-I_n$ on its anti-diagonal. The simple abstract carrier algebras have the form $C(k_1, \ldots, k_s)$ and the construction is similar to those for type B and D. Here we can assume the defining element is $h = \mathrm{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0$ and, as multisets, $\{\pm\lambda_1, \ldots, \pm\lambda_n\} = \{k_i - \frac{1}{2}, k_i - \frac{3}{2}, \ldots, \frac{3}{2} - k_i, \frac{1}{2} - k_i : 1 \leq i \leq s\}$ for some $k_1 > k_2 > \cdots > k_s > 0$ with $n = k_1 + \cdots + k_n$. If one chooses the diagonal matrices in $\mathfrak{g}$ as

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maximal toral subalgebra, then the Dynkin diagram of $C(k_1, \dots, k_s)$ has labels $\lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n, 2\lambda_n$, where $2\lambda_n$ is attached to the longer root (see [14, Figure 7]). Since $\lambda_n \neq 0$, we have $2\lambda_n = 1$, and so the semisimple part of $\mathfrak{g}_0$ is a direct sum of algebras of type A. If $s = 1$, then $C(n)$ is principal.

*Exceptional types.* A direct calculation yields the abstract carrier algebras $\mathfrak{g}$ of exceptional types $G_2$, $F_4$, $E_6$, $E_7$, $E_8$; the semisimple part of $\mathfrak{g}_0$ is always a sum of Lie algebras of type A (see [24, Table 1]).

*The centre of* $\mathfrak{g}_0$. Let $\mathfrak{g}_0$ be as before and write $\mathfrak{g}_0 = \mathfrak{z} \oplus \mathfrak{s}$ where $\mathfrak{z}$ is the centre and $\mathfrak{s}$ is semisimple. It follows from [24, page 19] that $\dim \mathfrak{z} = \operatorname{rk} \mathfrak{g}_0 - \operatorname{rk} \mathfrak{s} = \operatorname{rk} \mathfrak{g} - \operatorname{rk} \mathfrak{s}$. Since $\operatorname{rk} \mathfrak{s}$ equals the number of labels 0 in the weighted Dynkin diagram of $\mathfrak{g}$, the dimension of $\mathfrak{z}$ equals the number of labels 1. For example, if $\mathfrak{g} = B(5, 2, 1)$ with rank 9, then $\lambda_1, \dots, \lambda_9 = 5, 4, 3, 2, 2, 1, 1, 1, 0$, yielding labels $1, 1, 1, 0, 1, 0, 0, 1, 0$; thus, $\dim \mathfrak{z} = 5$. From the above classification, it follows that $\dim \mathfrak{z} = 1$ if and only if $\mathfrak{g}$ has type $A_1$ or if $\mathfrak{g}$ is the $\mathbb{Z}$-graded algebra $E_8^{(11)}$, $F_4^{(4)}$ or $G_2^{(2)}$ as defined in [14, Table II].

## 3. Bala and Carter's construction

Bala and Carter [2, 3] classified the nilpotent orbits in a complex simple Lie algebra using a construction very similar to Vinberg's, without the assumption that the Lie algebras are graded. Just like Vinberg's construction, this yields an étale representation for a certain reductive subalgebra of $\mathfrak{g}$. In this section we review some of these results and show how the approaches by Vinberg and by Bala and Carter are related.

**3.1. Minimal Levi subalgebras.** First, we recall a few definitions. Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}$ be a reductive Lie algebra with $\mathfrak{s}$ semisimple and $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ the centre. A *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$, and a subalgebra of $\mathfrak{g}$ is a *parabolic subalgebra* if it contains a Borel subalgebra of $\mathfrak{g}$. Every parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a semidirect product $\mathfrak{p} = \mathfrak{m} \ltimes \mathfrak{n}$ of a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{p}$, all of whose elements are nilpotent, and a reductive subalgebra $\mathfrak{m}$. A parabolic subalgebra is *distinguished* if $\dim \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] = \dim \mathfrak{m}$. Any reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ arising in this way for some parabolic subalgebra of $\mathfrak{g}$ is a *Levi subalgebra* in $\mathfrak{g}$. Its commutator $\mathfrak{c} = [\mathfrak{m}, \mathfrak{m}]$ is *semisimple of parabolic type*. Bala and Carter defined the terms above for semisimple $\mathfrak{g}$, but they carry over without change to reductive $\mathfrak{g}$.

For semisimple $\mathfrak{g}$, it is shown in [3, Theorem 6.1] that the classification of nilpotent orbits is equivalent to the classification of conjugacy classes of pairs $(\mathfrak{c}, \mathfrak{q}_\mathfrak{c})$, where $\mathfrak{c}$ is semisimple subalgebra of parabolic type in $\mathfrak{g}$ and $\mathfrak{q}_\mathfrak{c}$ is a distinguished parabolic subalgebra of $\mathfrak{c}$. For a nonzero nilpotent element $e \in \mathfrak{g}$ with $\mathfrak{sl}_2$-triple $(h, e, f)$, define $\mathfrak{g}_k = \{x \in \mathfrak{g} : [h, x] = kx\}$ for $k \in \mathbb{Z}$; this furnishes $\mathfrak{g}$ with a $\mathbb{Z}$-grading. With this grading, $e \in \mathfrak{g}_2$. The element $e$ is *distinguished in* $\mathfrak{g}$ if $\operatorname{ad}(e) \colon \mathfrak{g}_0 \to \mathfrak{g}_2$ is an isomorphism, that is, if $e$ is in general position. If $e$ is not distinguished in $\mathfrak{g}$, then [3, Propositions 5.3 and 5.4] tell us how to construct a semisimple subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ in which $e$ is distinguished:

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if $\mathfrak{h}_0$ is a maximal toral subalgebra of the centraliser of $(h, e, f)$ in $\mathfrak{g}$, then

$$\mathfrak{m} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_0) \quad \text{and} \quad \mathfrak{c} = [\mathfrak{m}, \mathfrak{m}] \tag{3.1}$$

are a minimal Levi subalgebra of $\mathfrak{g}$ containing $e$ and a semisimple subalgebra of parabolic type, respectively, such that $e$ is distinguished in $\mathfrak{c}$. The pair corresponding to $e$ can be chosen to be $(\mathfrak{c}, \mathfrak{q}_{\mathfrak{c}})$, where $\mathfrak{q}_{\mathfrak{c}}$ is the Jacobson–Morozov parabolic (see [2, Proposition 4.3] and [3, Theorem 6.1]). If $G$ is a semisimple algebraic group with Lie algebra $\mathfrak{g}$, then $\mathfrak{c}$ is determined uniquely up to the action by $Z_G(e)$, the centraliser of $e$ in $\mathrm{Ad}_{\mathfrak{g}}(G)$. Since $e$ is distinguished in $\mathfrak{c}$, the $\mathbb{Z}$-grading of its $\mathfrak{sl}_2$-triple in $\mathfrak{c}$ yields an étale representation for the adjoint action of the reductive subalgebra $\mathfrak{c}_0$ on the subspace $\mathfrak{c}_2$ by evaluation at $e$. The adjoint action of $\mathfrak{g}$ integrates to that of $G$, and thus we obtain an étale representation of the reductive group with Lie algebra $\mathfrak{c}_0$ on the space $\mathfrak{c}_2$.

The Borel and parabolic subalgebras of a reductive $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ as above are precisely $\mathfrak{z} \oplus \mathfrak{b}$ and $\mathfrak{z} \oplus \mathfrak{p}$, respectively, with $\mathfrak{b} \leq \mathfrak{s}$ a Borel subalgebra and $\mathfrak{p} \leq \mathfrak{s}$ parabolic. It follows that the Levi subalgebras of $\mathfrak{g}$ are precisely $\mathfrak{z} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a Levi subalgebra of $\mathfrak{s}$; moreover, $\mathfrak{m}$ is a minimal Levi subalgebra containing $e$ in $\mathfrak{s}$ if and only if $\mathfrak{z} \oplus \mathfrak{m}$ is a minimal Levi subalgebra containing $e$ in $\mathfrak{g}$.

**3.2. Relation to carrier algebras.** We compare the Bala and Carter construction with Vinberg's carrier algebras. Vinberg starts with a semisimple $\mathbb{Z}_n$-graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_n} \mathfrak{g}_i$; recall that we allow $\mathbb{Z}_\infty = \mathbb{Z}$ here. Let $e \in \mathfrak{g}_1$ be nonzero nilpotent with $\mathfrak{sl}_2$-triple $(h, e, f)$ such that $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$ and define $\mathfrak{g}(\mathfrak{t}, e)$ as in (2.1); as mentioned before, $\mathfrak{t}$ is a maximal toral subalgebra of the normaliser $\mathfrak{n}_{\mathfrak{g}_0}(e)$ and $\lambda \colon \mathfrak{t} \to \mathbb{C}$ is defined by $[t, e] = \lambda(t)e$. Let $h_0 = \frac{1}{2}h$ define the $\mathbb{Z}$-graded algebra

$$\mathfrak{g}(h_0) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(h_0)_k \quad \text{with } \mathfrak{g}(h_0)_k = \{x \in \mathfrak{g}_k : [h_0, x] = kx\}.$$

Note that $\mathfrak{t} \leq \mathfrak{g}(h_0)_0$. It follows from [24, Lemmas 1 and 2] that $\mathfrak{g}(h_0)$ and $\mathfrak{g}(\mathfrak{t}, e)$ are both reductive. Recall that $\mathfrak{t} = \mathbb{C}h \oplus \mathfrak{t}_0$, where $\mathfrak{t}_0$ is a maximal toral subalgebra of $\mathfrak{z}_{\mathfrak{g}_0}(h, e, f)$. More precisely, we can state the following lemma.

Lemma 3.1. *We have* $\mathfrak{t}_0 = \ker \lambda = \mathfrak{z}(\mathfrak{g}(\mathfrak{t}, e))$ *and* $\mathfrak{g}(\mathfrak{t}, e) = \mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{t}_0)$.

Proof. Write $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}(\mathfrak{t}, e))$. Recall that $\mathfrak{t} = \mathbb{C}h \oplus \mathfrak{t}_0$, so $\ker \lambda = \mathfrak{t}_0$ follows from $[h, e] = 2e$. Clearly, if $t \in \ker \lambda$, then $[t, y] = 0$ for each $y \in \mathfrak{g}(\mathfrak{t}, e)_k$, so $t \in \mathfrak{z}$. Since $\mathfrak{z}$ commutes with the defining element of $\mathfrak{g}(\mathfrak{t}, e)$, we have $\mathfrak{z} \leq \mathfrak{g}(\mathfrak{t}, e)_0$. Thus, $\mathfrak{z} \leq \mathfrak{n}_{\mathfrak{g}_0}(e)$ and therefore $\mathfrak{z} \leq \mathfrak{t}$. This implies $\mathfrak{z} \leq \ker \lambda$, hence $\mathfrak{z} = \ker \lambda$. Suppose $x \in \mathfrak{g}(h_0)$ centralises $\mathfrak{t}_0$ and write $x = \bigoplus_{k \in \mathbb{Z}} x_k$ with each $x_k \in \mathfrak{g}(h_0)_k$. Since $t_0 \leq \mathfrak{g}(h_0)_0$, it follows from $0 = [x, \mathfrak{t}_0] = \bigoplus_{k \in \mathbb{Z}} [x_k, \mathfrak{t}_0]$ that each $x_k$ centralises $\mathfrak{t}_0$. By assumption, $[h, x_k] = 2kx_k$, so $x_k \in \mathfrak{g}(\mathfrak{t}, e)_k$ and hence $\mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{t}_0) \leq \mathfrak{g}(\mathfrak{t}, e)$. Conversely, if $x \in \mathfrak{g}(\mathfrak{t}, e)_k$ then $[h, x] = 2kx$, and if $t \in \mathfrak{t}_0$ then $[t, x] = k\lambda(t)x = 0$ since $\mathfrak{t}_0 = \ker \lambda$. Thus, $x \in \mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{t}_0)_k$. □

The next proposition shows that Vinberg's construction (2.1) of $\mathfrak{g}(\mathfrak{t}, e)$ and its carrier algebra is the same as applying Bala and Carter's approach (3.1) to the $\mathbb{Z}$-graded Lie

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algebra $\mathfrak{g}(h_0)$. Below, let $G$ be the semisimple algebraic group with Lie algebra $\mathfrak{g}(h_0)$. The conjugacy up to the centraliser $Z_G(e)$ reflects the freedom in choosing an $\mathfrak{sl}_2$-triple $(h, e, f)$ for a given nonzero nilpotent element $e$.

PROPOSITION 3.2. *Let $\mathfrak{g}$ be a $\mathbb{Z}_n$-graded complex semisimple Lie algebra, where $n \in \mathbb{N} \cup \{\infty\}$, and let $e$, $h_0$, and $\mathfrak{t}$ be as above. Then $\mathfrak{g}(\mathfrak{t}, e)$ is a minimal Levi subalgebra of $\mathfrak{g}(h_0)$ containing $e$ and, up to $Z_G(e)$-conjugacy, the carrier subalgebra $\mathfrak{c}(e) = [\mathfrak{g}(\mathfrak{t}, e), \mathfrak{g}(\mathfrak{t}, e)]$ is the unique semisimple subalgebra of parabolic type in $\mathfrak{g}(h_0)$ in which $e$ is distinguished.*

PROOF. Note that $h$ stabilises each $\mathfrak{g}_k$ and $\mathfrak{g}(h_0)_k$ is the intersection of $\mathfrak{g}_k$ with the $2k$-eigenspace of $h$. Lemma 3.1 shows that $\mathfrak{g}(\mathfrak{t}, e) \le \mathfrak{g}(h_0)$ and the $\mathbb{Z}$-gradings of both algebras are determined by the eigenvalues of $\mathrm{ad}(h_0)$. The semisimple part $\mathfrak{s}$ of $\mathfrak{g}(h_0)$ is a semisimple ideal in $\mathfrak{g}(h_0)$ containing $(h, e, f)$; let $\mathfrak{a}$ be the subalgebra generated by $\{h, e, f\}$. Note that for every subset $X \subseteq \mathfrak{g}(h_0)$,

$$\mathfrak{z}_{\mathfrak{g}(h_0)}(X) = \mathfrak{z}(\mathfrak{g}(h_0)) \oplus \mathfrak{z}_{\mathfrak{s}}(X). \tag{$*$}$$

We claim that $\mathfrak{t}_0$ is a maximal toral subalgebra of $\mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{a})$: recall that $\mathfrak{t}_0$ is defined as a maximal toral subalgebra of $\mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{a})$, which is reductive by [24, page 21]. Since $\mathfrak{t}_0 \le \mathfrak{g}(h_0)_0 \le \mathfrak{g}_0$, we know that $\mathfrak{t}_0$ is also a maximal toral subalgebra in $\mathfrak{z}_{\mathfrak{g}(h_0)_0}(\mathfrak{a})$. On the other hand, $\mathfrak{z}_{\mathfrak{g}(h_0)_0}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{a})$ because elements of nonzero degree in $\mathfrak{g}(h_0)$ do not commute with the defining element $h_0 \in \mathfrak{a}$; thus, $\mathfrak{t}_0 \le \mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{a})$ is a maximal toral subalgebra. We can write $\mathfrak{t}_0 = \mathfrak{z}(\mathfrak{g}(h_0)) \oplus \mathfrak{t}_0'$, where $\mathfrak{t}_0'$ is a maximal toral subalgebra of $\mathfrak{z}_{\mathfrak{s}}(\mathfrak{a})$, and so for every subset $X \subseteq \mathfrak{g}(h_0)$,

$$\mathfrak{z}_X(\mathfrak{t}_0) = \mathfrak{z}_X(\mathfrak{t}_0'). \tag{$**$}$$

The construction in (3.1) shows that $\mathfrak{m}' = \mathfrak{z}_{\mathfrak{s}}(\mathfrak{t}_0')$ is a minimal Levi subalgebra of $\mathfrak{s}$ containing $e$, so $\mathfrak{m} = \mathfrak{z}(\mathfrak{g}(h_0)) \oplus \mathfrak{m}'$ is a minimal Levi subalgebra of $\mathfrak{g}(h_0)$ containing $e$. Now ($*$), ($**$) and Lemma 3.1 show

$$\mathfrak{m} = \mathfrak{z}(\mathfrak{g}(h_0)) \oplus \mathfrak{z}_{\mathfrak{s}}(\mathfrak{t}_0') = \mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{t}_0') = \mathfrak{z}_{\mathfrak{g}(h_0)}(\mathfrak{t}_0) = \mathfrak{g}(\mathfrak{t}, e),$$

so $\mathfrak{g}(\mathfrak{t}, e)$ is a minimal Levi subalgebra in $\mathfrak{g}(h_0)$ containing $e$. The construction in (3.1) shows that $e$ is distinguished in $[\mathfrak{m}, \mathfrak{m}]$ and the latter is semisimple of parabolic type. Since $[\mathfrak{g}(\mathfrak{t}, e), \mathfrak{g}(\mathfrak{t}, e)] = [\mathfrak{m}, \mathfrak{m}]$ is the carrier algebra, the claim follows; [3, Proposition 5.3] shows uniqueness up to $Z_G(e)$-conjugacy. □

For a $\mathbb{Z}$-graded semisimple Lie algebra $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{g}(h_0)$ where $h = 2h_0$ is the defining element (see [13, Remark 33]), so the two approaches by Vinberg and Bala and Carter coincide.

COROLLARY 3.3. *If $\mathfrak{g}$ is a $\mathbb{Z}$-graded complex semisimple Lie algebra, then up to $Z_G(e)$-conjugacy, the subalgebra $\mathfrak{g}(\mathfrak{t}, e)$ obtained by Vinberg's construction and the subalgebra $\mathfrak{m}$ obtained by Bala and Carter's construction coincide.*

Even in the situation where $\mathfrak{g}$ is given without a grading and $e$ is a nonzero nilpotent element in $\mathfrak{g}$, a choice of $h_0$ induces a $\mathbb{Z}$-grading on $\mathfrak{g}$ to which Vinberg's approach can

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be applied; this is then equivalent to Bala and Carter's approach. Note that Bala and Carter use the element $h$ rather than $h_0 = \frac{1}{2}h$ to define their grading, which leads to an additional factor of 2 in the degrees.

## 4. Gyoja's construction

Gyoja [15] described constructions of étale modules out of a given prehomogeneous module. Let $G$ be a complex reductive algebraic group with algebraic representation $\varrho\colon G \to \mathrm{GL}(V)$ on a finite-dimensional complex vector space $V$ such that $(G, \varrho, V)$ is a prehomogeneous module. Let $v \in V$ be the point in general position. The construction in [15, Theorem A] proceeds as follows. Let $G_v$ be the stabiliser of $v$ in $G$ with Cartan subgroup $T \leq G_v$, define $G' = \mathrm{N}_G(T)/T$ and let $V' = V^T$ be the set of fixed points in $V$ under $T$; then $V'$ is an étale module for the induced action of $G'$. Arising from a normaliser of a torus, $G'$ is a reductive algebraic group. The second construction [15, Theorem B] yields a procedure to obtain a super-étale module from an étale module. (This means that the stabiliser of the point in general position is trivial and not just finite.) Given an étale module $(G, \varrho, V)$ with $v \in V$ in general position and stabiliser $G_v$, choose $1 \neq h \in G_v$ and let $G'' = \mathrm{N}_G(h)$ and $V'' = V^h$, the set of fixed points for $h$ in $V$. Then $V''$ is an étale module for the induced action of $G''$ and $|G''_v| < |G_v|$. Since $\langle h \rangle$ is finite, $G''$ is also reductive. After finitely many iterations (with $G''$ instead of $G$), one obtains a super-étale module.

**4.1. Relation to carrier algebras.** Gyoja's construction was formulated for groups, but can just as well be formulated for the corresponding Lie algebras. For this let $\mathfrak{g}$ be a semisimple Lie algebra with nonzero nilpotent $e \in \mathfrak{g}$, let $(h, e, f)$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and furnish $\mathfrak{g}$ with the $\mathbb{Z}$-grading induced by $\mathrm{ad}(h_0)$, where $h_0 = \frac{1}{2}h$. Observe now that Gyoja's method [15] replicates the aspect that is of interest to us in Vinberg's and Bala and Carter's theory: the focus is on the étale action of the reductive subalgebra of degree 0 on the subspace of degree 1, ignoring the subspaces of higher degree.

PROPOSITION 4.1. *Gyoja's construction, applied to the reductive Lie algebra $\mathfrak{g}_0$, the nilpotent element $e$ and the adjoint $\mathfrak{g}_0$-module $(\mathrm{ad}, \mathfrak{g}_1)$, produces Vinberg's étale representation associated with $e \in \mathfrak{g}$.*

PROOF. The stabiliser algebra of $e$ is $\mathfrak{z}_{\mathfrak{g}_0}(e)$. We have shown that $\mathfrak{t}_0 = \ker \lambda$ (as introduced in Lemma 3.1) is a maximal toral subalgebra of $\mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{a})$, where $\mathfrak{a}$ is the subalgebra spanned by $(h, e, f)$. By [24, page 21], $\mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}_0}(e, h)$ and, since $h = 2h_0 \in \mathfrak{g}_0$, it follows that $\mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}_0}(e)$. Thus, a maximal toral subalgebra of $\mathfrak{z}_{\mathfrak{g}_0}(e)$ is $\mathfrak{t}_0 = \ker \lambda$, which takes the place of Gyoja's $\mathfrak{t}$. Now Lemma 3.1 shows that the fixed point set of $\mathfrak{t}_0$ in $\mathfrak{g}_1$ is $V' = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_0) \cap \mathfrak{g}_1 = \mathfrak{g}(\mathfrak{t}, e)_1$, with $\mathfrak{t} = \mathbb{C}h \oplus \mathfrak{t}_0$; moreover, $\mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{t}_0) = \mathfrak{g}(\mathfrak{t}, e)_0$ is reductive with centre $\mathfrak{t}_0$. Hence, $\mathfrak{g}' = \mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{t}_0)/\mathfrak{t}_0$ satisfies $\mathfrak{g}' \cong [\mathfrak{g}(\mathfrak{t}, e)_0, \mathfrak{g}(\mathfrak{t}, e)_0]$, so Gyoja's $\mathfrak{g}'$ is the 0-component of the carrier algebra of $e$ in $\mathfrak{g}$. Lastly, $V' = \mathfrak{g}(\mathfrak{t}, e)_1 = [\mathfrak{g}(\mathfrak{t}, e), \mathfrak{g}(\mathfrak{t}, e)]_1$ is the 1-component of that carrier algebra, since the centre of the reductive $\mathfrak{g}(\mathfrak{t}, e)$ is contained in $\mathfrak{g}(\mathfrak{t}, e)_0$. □

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