A Note on Scalar Field Theory in AdS_3/CFT_2

Pablo Minces

Instituto de Astronomía y Física del Espacio (IAFE),
C.C.67 - Suc. 28, 1428 Buenos Aires, Argentina.
minces@iafe.uba.ar

Abstract

We consider a scalar field theory in AdS_{d+1}, and introduce a formalism on surfaces at equal values of the radial coordinate. In particular, we define the corresponding conjugate momentum. We compute the Noether currents for isometries in the bulk, and perform the asymptotic limit on the corresponding charges. We then introduce Poisson brackets at the border, and show that the asymptotic values of the bulk scalar field and the conjugate momentum transform as conformal fields of scaling dimensions Δ_- and Δ_+, respectively, where $\Delta_\pm$ are the standard parameters giving the asymptotic behavior of the scalar field in AdS. Then we consider the case $d = 2$, where we obtain two copies of the Virasoro algebra, with vanishing central charge at the classical level. An AdS_3/CFT_2 prescription, giving the commutators of the boundary CFT in terms of the Poisson brackets at the border, arises in a natural way. We find that the boundary CFT is similar to a generalized ghost system. We introduce two different ground states, and then compute the normal ordering constants and quantum central charges, which depend on the mass of the scalar field and the AdS radius. We discuss certain implications of the results.

1 Introduction

An intensive study of diverse theoretical aspects of Anti-de Sitter (AdS) spaces has been carried out since the proposal [1] of the existence of a duality between a supergravity theory on AdS and a Conformal Field Theory (CFT) living at its boundary. In addition, the precise AdS/CFT prescription given in [2][3], where the partition function of the AdS theory is identified with the generating functional of the dual CFT, has allowed to perform several explicit checks and calculations.

In this context, the scalar field theory on AdS space is an interesting toy model which allows to analyze diverse aspects of the AdS/CFT correspondence and exhibits some subtle properties, so that it has received considerable attention in the literature. For instance, the early works [4][5] (see also [6]) showed that it possesses the interesting property of having two different kinds of normalizable modes, thus giving rise to two possible quantizations in the AdS bulk. This happens for masses of the scalar field in the range

$$m_{BF}^2 \ < \ m^2 \ < \ m_{BF}^2 \ + \ \frac{1}{l^2} \ , \tag{1}$$

where

$$m_{BF}^2 \ = \ -\frac{d^2}{4l^2} \ ,$$

is the Breitenlohner-Freedman mass. Here $d+1$ is the dimension of the AdS_{d+1} space and l is the AdS radius. The Breitenlohner-Freedman bound reads

$$m^2 \ \geq \ m_{BF}^2 \ , \tag{2}$$

and solutions below it correspond to tachyons in AdS. Throughout this note, we will consider masses of the scalar field in the range (1).[1]

We will propose here a new approach to the formulation of scalar field theory in the AdS/CFT correspondence, which we hope allows to gain further insight in the way both theories relate to each other, and will lead to find interesting new results. Even when some calculations will be performed for AdS spaces of generic $d+1$ dimensions, our main focus here will be on the $d=2$ case.

In order to set our notation, we point out that throughout this note we will consider the Euclidean representation of AdS_{d+1} in Poincaré coordinates, described by the half space $x^0>0$, $x^i \in \mathbf{R}$ with metric

$$ds^2 = \frac{l^2}{x_0^2} \sum_{\mu=0}^{d} dx^\mu dx^\mu \ . \tag{3}$$

In particular, the boundary of the AdS space is located at $x^0 \to 0$.

[1]We will exclude the particular case $m^2 \ = \ m_{BF}^2$ from our analysis.

The behavior of the scalar field close to the border is of the form[2]

$$\Phi(\epsilon, \vec{x}) \ = \ \epsilon^{\Delta_+}(\alpha(\vec{x}) + O(\epsilon^2)) \ + \ \epsilon^{\Delta_-}(\beta(\vec{x}) + O(\epsilon^2)) \ , \tag{4}$$

where $\epsilon \ = \ x^0$ is taken to be small. Here we have

$$\Delta_\pm \ = \ \frac{d}{2} \ \pm \ l\sqrt{m^2 \ - \ m_{BF}^2} \ . \tag{5}$$

The usual procedure is to require boundary conditions that set to zero half of the modes of the field near the boundary, thus giving rise to two possible quantizations.

From the AdS/CFT point of view, we expect to find two different CFTs at the border. One of them, corresponding to a dual operator of conformal dimension Δ_+, was reproduced through the prescription in [2] (see also [7][8] for further analysis). The other one, corresponding to the conformal dimension Δ_-, was found to be obtained by performing a Legendre transformation to the original generating functional in the theory with conformal dimension Δ_+ [9] (see also [10] for previous results). Related issues involve double-trace perturbations and the role of boundary conditions for scalar field theory in Anti-de Sitter space (see e.g. [11]-[44]).

Following the standard AdS/CFT prescription in [2][3], most of the literature dealing with the scalar field theory in the AdS/CFT correspondence focuses on correlation functions. As expected, this has proven a fruitful approach. However, in this note we would like to consider the particular case of three dimensional AdS space and shift the focus to the information we could get about the boundary two dimensional CFT by considering the 'conserved' charges of the theory in the bulk. The motivation is as follows. We know that isometries of the AdS_3 background correspond to global conformal transformations at the boundary. Taking this into account, our proposal here is to identify the generators of global conformal transformations of the boundary CFT with the asymptotic expressions of the charges in the bulk, and then perform a proper expansion on such generators in order to compute the Virasoro generators of the theory. This idea is somehow similar in spirit to that analyzed in [45] (see also e.g. [46][47][26] for the inclusion of the scalar field theory into the analysis), but we will consider it here in the context of a formulation where we make use of the fixed metric (3) and choose x^0 to play a special role, so that we foliate the space on surfaces at equal values of it, as we will discuss shortly.

Now, once in possession of the expressions of the Virasoro generators of the boundary CFT, we aim at proposing a proper prescription which allows to compute the Virasoro algebra including the corresponding central charge, which is in principle expected to depend on the mass of the scalar field and the AdS radius, in a similar way as the conformal dimensions of the boundary CFT operators do through the usual AdS/CFT prescription in [2][3]. The computation of the Virasoro algebra in the asymptotic limit would constitute a non trivial result which could be considered as a consistency check

[2]We point out that the quantization in [4][5] was performed in global coordinates. We take the results which are relevant to our present purposes.

on our calculations. In addition, to compute the corresponding central charge would provide, in addition, some interesting new information on the boundary CFT and on the way both theories in different dimensions relate to each other. Besides, it is expected that some other information on the boundary CFT could be obtained, e.g. from the requirement for the central charge to be positive, or from the specific form the Virasoro generators would have in terms of the asymptotic expressions of quantities in the bulk. At a more speculative level, possible applications e.g. statistical or in black hole physics could also exist, but we will not address this issue here.

Now, the computation of the generators of the boundary CFT or the calculation of their corresponding algebra could in principle be performed only after developing a formalism where all the information in the bulk is mapped to the boundary. In particular, this concerns the above mentioned fact that, for masses in the range (1), there are two possible quantizations of the bulk scalar field. We should be able to take this information to the boundary. In order to do this, there is the important observation to be made that not only the bulk field, but also the corresponding canonical momentum in a formulation where the radial coordinate x^0 (see (3)) plays the role of 'time', should be taken into account. A formalism which makes use of this canonical momentum was presented in [24][36]. The fact that its inclusion is required follows from results in [41][14][23] (a discussion of other aspects of this issue is postponed to footnote 10 since it will better be considered after the introduction of some results and notation). The point is that the inclusion of both the field and the momentum is required in order to have the complete information about both boundary CFTs corresponding to both quantizations in the bulk.

Motivated by the discussion above, we will then consider the radial coordinate x^0 to play a special role, and introduce a formulation on surfaces at equal values of it. In particular, we will define the corresponding conjugate momentum. In this way, when carefully performing the limit $\epsilon \to 0$ we will locate on the surface $x^0 = 0$, where Poisson brackets of 'conserved' charges may be computed. Such charges will be obtained from the Noether currents corresponding to isometries on the AdS bulk, and we will show them to be finite in the limit $\epsilon \to 0$, provided the action is supplemented by a proper surface term.

In order to perform certain consistency checks and introduce aspects of the formalism, in Section 2 we will first focus in the generic case of AdS_{d+1}. On the one hand, we will verify that the asymptotic charges actually generate the global conformal algebra in d dimensions. From well known results in the literature we also expect, in addition, that the asymptotic values of the bulk scalar field and the momentum, which we will call Φ_0 and Π_0, should transform as conformal fields of scaling dimensions Δ_- and Δ_+, respectively. Using the Poisson brackets, we will verify that the asymptotic charges actually realize these required properties too. Even when expected, these results are non trivial, and any of them can be considered as a consistency check on our formalism.

We point out that, before the checks above are performed, we will have to consider the interesting and important issue of the well definiteness of the limit $\epsilon \to 0$ when it is performed on the charges. We will show this to hold when the action for the scalar field is supplemented by precisely the same (already known in the literature) boundary term which makes the action to be finite in such limit, thus adding to the consistency of the formalism.

Having performed all the checks above, we will then in Section 3 focus on the case of AdS_3, where the main results of this note will be obtained. Here the boundary will be described in terms of complex holomorphic and antiholomorphic coordinates, and the definition of the charges will involve contour integrals. Once again, we will perform the limit $\epsilon \to 0$, and then, by Laurent expanding the asymptotic charges we will obtain two series of coefficients L_n and $\bar{L}_n$ $(n \in \mathbf{Z})$. On the one hand, we will show these coefficients to satisfy the expected but non trivial result of giving rise to two copies of the Virasoro algebra, with vanishing central charge at the classical level. On the other hand, we will show Φ_0 and Π_0 to transform as conformal fields of weights $(\frac{\Delta_-}{2}, \frac{\Delta_-}{2})$ and $(\frac{\Delta_+}{2}, \frac{\Delta_+}{2})$, respectively, in agreement with the previous results in Section 2. Motivated by this, we will also propose mode expansions for Φ_0 and Π_0, and reproduce again the same results as before, this time in terms of modes. Once again, any of the results above can also be considered as a non trivial check on the formalism.

At this point, we will put aside the role of Φ_0 and Π_0 as the asymptotic values of the bulk fields, and treat them as conformal fields with the given weights, living in the boundary CFT. We will then consider aspects of the quantization of such CFT. The motivation to attempt this is that, as we will see, a prescription relating the Poisson brackets at $x^0 = 0$ to commutators on the boundary CFT will arise in a natural way. This will be a non trivial prescription, since it will relate Poisson brackets in the asymptotic limit of a three dimensional theory to commutators in a two dimensional one. This will be allowed by the property of the asymptotic Poisson brackets of being computed at equal values of the distance to the origin of the complex plane (in a manner to be illustrated later), as well as, simultaneously, at the surface of fixed $x^0 = 0$. This property will give the asymptotic Poisson brackets a meaning from the point of view of the boundary CFT, and will be inherited from the definition of the charges using contour integrals. It is also what will allow us to go one step further in Section 3 than in the generic AdS_{d+1} case of Section 2, where we will deal with a $d+1$ dimensional theory only.

These calculations will lead us to find expressions for the generators and commutators of the theory which, exception made of the fact that the fields will not factorize in the holomorphic and antiholomorphic parts, are surprisingly similar to the corresponding ones in generalized ghost systems.

Then, we will introduce two different ground states, which are not SL_2 invariant, and that correspond to choosing the zero mode of which one of the fields, Φ_0 or Π_0, is grouped with the lowering operators. This should correspond to the two possible quantizations in the bulk found in [4][5]. The fact that we will find two different quantizations, as expected, can be considered as a last non trivial check on our formalism.

In both cases we will find the same normal ordering constants and quantum central charges, which are given by $L_0|0\rangle = \bar{L}_0|0\rangle = \frac{l^2m^2}{8}|0\rangle$ and $c = \bar{c} = 2+3l^2m^2$, respectively. As expected, they depend on the mass of the bulk field and the AdS radius, in an analogous way as the conformal dimensions do. Thus, our formalism allows to find interesting new information on the boundary CFT corresponding to scalar field on AdS_3, and on the way both theories in different dimensions relate to each other. We point out that, as we will show, the normal ordering constant and central charge are in the ranges $-\frac{1}{8} < \frac{l^2m^2}{8} < 0$ and $-1 < c < 2$, respectively. In particular, we will show that the requirement for the central charge to be positive sets $l^2m^2 > -\frac{2}{3}$. This condition is more restrictive than the Breitenlohner-Freedman bound. It is a new and interesting result, which seems to be detected only from the boundary point of view. The precise meaning of this and the way in which it could be red from the bulk point of view deserve more studies.

A surprising result that we will also find will be that, from the fact that the boundary CFT will be similar to a generalized ghost system, and from the explicit dependence of the central charge with the mass of the scalar field, we will conclude that the mass of the bulk scalar field seems to play the role of a background charge for the boundary CFT. We believe that further insight into this interesting issue could be obtained by performing a bosonization program on the boundary CFT. However, we will no longer pursue this issue here.

2 Asymptotic limit in AdS_{d+1}

The action of a massive, minimally coupled scalar field theory in AdS_{d+1} is of the form

$$I_0 = \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \, \sqrt{g} \, \left(g^{\mu\nu}\partial_\mu\Phi \, \partial_\nu\Phi \; + \; m^2\Phi^2 \right) \quad , \tag{6}$$

where m is the mass of the scalar field and $g_{\mu\nu}$ is given by (3). Our conventions are that Greek indices $\mu, \nu, ...$ take the values from 0 to d. The equation of motion reads $(\nabla^2 - m^2)\Phi = 0$.

We consider the space as foliated by a family of surfaces ∂M_ϵ defined by $x^0 = \epsilon$, and with outward pointing unit normal vector

$$n_\mu = (-l\epsilon^{-1}, \mathbf{0}) \ .$$

In particular, the boundary ∂M of the AdS space is located at $\epsilon \to 0$. Actually, the action includes terms which diverge in such limit, and in order to take care of them we supplement I_0 with a proper surface term, which will not introduce any changes in the equation of motion. We take the action to be

$$I \; = \; I_0 \; + \; \sigma \, I_S \ , \tag{7}$$

where σ is a coefficient and

$$I_S = \int_{\mathcal{M}} d^{d+1}x \, \sqrt{g} \, \nabla_\mu(n^\mu\Phi^2) \ \ . \tag{8}$$

In fact, the choice of σ for which there are no divergent terms in the action, for $\epsilon \to 0$, and provided we consider the range (1), is given by [14][41][3]

$$\sigma = \frac{\Delta_-}{2l} \quad . \tag{9}$$

However, we will maintain a generic value of σ for a while, because it will be interesting to see how (9) will arise again in a different context.

We consider isometries of the AdS background, i.e. coordinate transformations $x^\mu \to x^\mu + \delta x^\mu$ such as $\delta g_{\mu\nu} = 0$. The variation of the action (7) is of the form

$$\delta I \sim \int_{\partial\mathcal{M}_\epsilon} d^d x \, \sqrt{h} \, n_\mu \, J^\mu \; ,$$

where $h_{\mu\nu}$ is the induced metric and J^μ is the Noether current.[4] The calculations are similar to those in [41][5] and we find

$$J^\mu \;=\; \Lambda^\mu{}_\nu \, \delta x^\nu \; , \tag{10}$$

where

$$\begin{aligned} \Lambda_{\mu\nu} \;&=\; \Theta_{\mu\nu} \;+\; \sigma \left[g_{\mu\nu} \nabla_\alpha (n^\alpha \Phi^2) \;-\; 2 \, n_\mu \Phi \, \partial_\nu \Phi \right] \; , \\ \Theta_{\mu\nu} \;&=\; -\partial_\mu \Phi \, \partial_\nu \Phi \;+\; \frac{1}{2} \, g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \Phi \, \partial_\beta \Phi \;+\; m^2 \Phi^2) \; . \end{aligned} \tag{11}$$

We write

$$I \;=\; \int_{\mathcal{M}} d^{d+1} x \, \sqrt{g} \, \mathcal{L} \; ,$$

[3]See also [23][27][29]. In [24][36] Eq.(9) has been derived using the conjugate momentum in a formulation on surfaces at equal values of x^0. Here we will focus on the well definiteness of the limit $\epsilon \to 0$ when it is performed on the charges. Actually (9) works well in the asymptotic limit of both the charges and the action, as expected, and as already suggested by the results in [41].

[4]The standard approach would be to consider the following definition of the conserved current $\hat{J}^\mu \sim T^\mu{}_\nu \delta x^\nu$, where $T_{\mu\nu}$ is the usual energy-momentum tensor of the scalar field obtained by performing an infinitesimal variation on the metric, $g_{\mu\nu} \to g_{\mu\nu} \,+\, \delta g_{\mu\nu}$, and writing

$$\delta_g I \;=\; \frac{1}{2} \int d^{d+1} x \, \sqrt{g} \, T_{\mu\nu} \, \delta g^{\mu\nu} \quad .$$

As emphasized in [41], J^μ and $\hat{J}^\mu$ are inequivalent. The reasons why we make use of J^μ instead of $\hat{J}^\mu$ are, on the one hand, that, given that we want to compute the generators of the asymptotic symmetries corresponding to isometries in the bulk, then it seems the natural choice to make, and, on the other hand, the arguments presented in detail in [41]. In particular, $\hat{J}^\mu$ does not contain the information on (9), and this happens so because it is unsensitive to the addition of the surface term (8) to the action (see [41] for additional details).

[5]Actually only the Noether current for time displacements in global coordinates was computed in [41], but at this stage the formal calculations are similar, even when we are considering a different metric and generic isometries.

where the Lagrangian is given by

$$\mathcal{L} \;=\; \frac{1}{2}\,\Big(g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi \;+\; m^2\Phi^2\Big) \;+\; \sigma\,\nabla_\mu(n^\mu\Phi^2)\;\;.$$

As discussed above, an important point regarding our calculations is that the usual equal time formalism will be replaced here by a formulation on surfaces at equal values of the radial coordinate x^0.[6] We introduce the conjugate momentum

$$\Pi \;=\; \sqrt{g}\,\frac{\partial\mathcal{L}}{\partial(\partial_n\Phi)} \;=\; \sqrt{g}\,(\partial_n\Phi \;+\; 2\,\sigma\,\Phi)\;\;, \tag{12}$$

where $\partial_n\Phi = n^\mu\partial_\mu\Phi = -\frac{x^0}{l}\partial_0\Phi$ is the normal derivative.[7] So we can write

$$\begin{aligned} \Lambda^0{}_i \;&=\; \frac{\Pi}{\sqrt{g}}\,\frac{x^0}{l}\,\partial_i\Phi\;\;, \\ \Lambda^0{}_0 \;&=\; 2\sigma\,\frac{\Pi}{\sqrt{g}}\,\Phi \;-\; \frac{\Pi^2}{2g} \;+\; \frac{1}{2}\,\partial^i\Phi\,\partial_i\Phi \;+\; \frac{1}{2l}\,(lm^2 \;+\; 2\sigma d \;-\; 4\sigma^2 l)\,\Phi^2\;, \end{aligned} \tag{13}$$

where we have adopted the convention that Latin indices i,j,... take the values from 1 to d.

So far, we have considered generic isometries of the AdS background. In order to determine the Noether currents (10) we need to explicitly write δx^μ in terms of Killing vectors. These are given by [39]

$$\begin{aligned} \xi^0_T \;&=\; 0 \quad , \qquad \xi^i_T \;=\; a^i\;; \\ \xi^0_R \;&=\; 0 \quad , \qquad \xi^i_R \;=\; m^i{}_j x^j \qquad (m_{ji} \;=\; -m_{ij})\;; \\ \xi^\mu_D \;&=\; \alpha x^\mu\;; \\ \xi^0_S \;&=\; 2x^0x^ib^i\;, \quad \xi^i_S \;=\; 2x^ix^jb^j - x^jx^jb^i - x^0x^0b^i\;\;. \end{aligned} \tag{14}$$

Here ξ^μ_T, ξ^μ_R, ξ^μ_D and ξ^μ_S act at the boundary as translations, rotations, dilations and special conformal transformations, respectively. This suggests to consider that the charges

$$Q = \int_{\partial\mathcal{M}_\epsilon} d^dx\,\sqrt{g}\,J^0\;, \tag{15}$$

act at the boundary on $\epsilon \to 0$ as generators of conformal transformations.

[6]This will eventually allow us to consider the boundary at $x^0 = 0$ as a particular choice among such surfaces, and compute Poisson brackets on it.

[7]Eq.(12) can also be written in the following form

$$\Pi \;=\; -\sqrt{h}\,\left(\partial_0\Phi \;-\; \frac{2\,l\,\sigma}{x^0}\,\Phi\right)\;\;,$$

which, exception made of the second term in the r.h.s., is closer to the usual notation. Here $h_{\mu\nu}$ is the induced metric.

Using (10, 14, 15) we find the following charges

$$P_i \;=\; \int_{\partial\mathcal{M}_\epsilon} d^dx\,\sqrt{g}\,\Lambda^0{}_i \qquad , \qquad M_i^{\;j} \;=\; \int_{\partial\mathcal{M}_\epsilon} d^dx\,\sqrt{g}\,(\Lambda^0{}_i\, x^j \;-\; \Lambda^{0j}\, x_i) \;\; ,$$
$$D \;=\; \int_{\partial\mathcal{M}_\epsilon} d^dx\,\sqrt{g}\,\Lambda^0{}_\mu\, x^\mu \;, \qquad K^i \;=\; \int_{\partial\mathcal{M}_\epsilon} d^dx\,\sqrt{g}\,(2\,\Lambda^0{}_\mu\, x^\mu\, x^i \;-\; \Lambda^{0i}\, x_\mu\, x^\mu) \;. \tag{16}$$

The next step is to compute the charges above in the limit $\epsilon \to 0$. We assume that the asymptotic behavior of the scalar field is given by the lowest order term in (4), namely

$$\Phi(\epsilon,\vec{x}) \;\sim\; \epsilon^{\Delta_-}\;\Phi_0(\vec{x}) \;. \tag{17}$$

Then from (12) the conjugate momentum should approach the boundary as

$$\Pi(\epsilon,\vec{x}) \;\sim\; \left(2\,\sigma \;-\; \frac{\Delta_-}{l}\right)\;\epsilon^{-\Delta_+-1}\;\Phi_0(\vec{x}) \;. \tag{18}$$

Now, when plugging (17, 18) into (16) we see that in general the charges diverge in the limit $\epsilon \to 0$. However, we notice from (18) that when σ is chosen as in (9) the $O(\epsilon^{-\Delta_+-1})$ term does not contribute. In such case, the asymptotic behavior of Π should be obtained by plugging the next lowest order term of (4) into (12). Such next order term of (4) has to be chosen among the possibilities $O(\epsilon^{\Delta_-+2})$ and $O(\epsilon^{\Delta_+})$. Since we are considering the range (1) then the lowest order of the two is $O(\epsilon^{\Delta_+})$, and using (12) this gives

$$\Pi(\epsilon,\vec{x}) \;\sim\; \epsilon^{-\Delta_--1}\;l\;\Pi_0(\vec{x}) \;, \tag{19}$$

which replaces (18). Notice that (9) simplifies (13) to

$$\Lambda^0{}_i \;=\; \frac{\Pi}{\sqrt{g}}\,\frac{x^0}{l}\,\partial_i\Phi \;\;,$$
$$\Lambda^0{}_0 \;=\; \frac{\Delta_-}{l}\,\frac{\Pi}{\sqrt{g}}\,\Phi \;-\; \frac{\Pi^2}{2g} \;+\; \frac{1}{2}\,\partial^i\Phi\,\partial_i\Phi \;\;, \tag{20}$$

and the conjugate momentum (12) now reads

$$\Pi \;=\; \sqrt{g}\,\left(\partial_n\Phi \;+\; \frac{\Delta_-}{l}\,\Phi\right) \;. \tag{21}$$

Plugging (17, 19, 20) into (16) and taking into account (1) we see that the charges are finite in the limit $\epsilon \to 0$.[8] Thus, from now on we choose σ to be given by (9) and the

[8]In particular, notice that only the first term in the r.h.s of the expression of $\Lambda^0{}_0$ (see (20)) will contribute, since the other two terms are of higher order in ϵ. In addition, $\Lambda^0{}_i$ has a non-vanishing contribution.

expressions (20, 21), together with the asymptotic behaviors (17, 19), are assumed.[9]

A comment is in order. Notice that the asymptotic behaviors of Π and Φ were obtained using the first and second contributions to the r.h.s. of (4), respectively. This suggests that, in the asymptotic limit, the scaling dimension of Π should be Δ_+, whereas Φ should have dimension Δ_-. This fact will explicitly be verified shortly, and will play an important role in what follows.[10]

Now, taking the limit $\epsilon \to 0$ and using (1, 17, 19, 20) we find the following expressions for the charges (16) evaluated on the boundary at $x^0 = 0$

$$\begin{aligned} \tilde{P}_i &= -\int d^dx\, \Pi_0\, \partial_i \Phi_0 \quad , \qquad \tilde{M}_i^{\,j} \;=\; -\int d^dx\, \Pi_0\, (x^j\, \partial_i \;-\; x_i\, \partial^j)\, \Phi_0 \;\; , \\ \tilde{D} &= -\int d^dx\, \Pi_0\, (x^i\, \partial_i \;+\; \Delta_-)\, \Phi_0 \;\; , \\ \tilde{K}^i &= -\int d^dx\, \Pi_0\, [2\, x^i\, (x^j\, \partial_j + \Delta_-) \;-\; x^j\, x_j\, \partial^i]\, \Phi_0 \;\; , \end{aligned} \tag{22}$$

where $\tilde{P}_i \equiv \lim_{\epsilon \to 0} P_i$ and similar definitions are assumed for the remaining charges.

As we have pointed out before, we are considering here a formulation where x^0 plays a special role and we step on surfaces at equal values of it. In particular, we have now positioned on the boundary at $x^0 = 0$, where we consider the Poisson brackets

$$\{\Pi_0(\vec{x}), \Phi_0(\vec{y})\}_{P.B.} \;=\; \delta^d(\vec{x} - \vec{y}) \;\; , \tag{23}$$

and

$$\{\Pi_0(\vec{x}), \Pi_0(\vec{y})\}_{P.B.} \;=\; \{\Phi_0(\vec{x}), \Phi_0(\vec{y})\}_{P.B.} \;=\; 0 \;\; , \tag{24}$$

which give

$$\begin{aligned} \{\tilde{P}_i\, ,\; \Phi_0\}_{P.B.} &= -\,\partial_i \Phi_0 \quad , \qquad \{\tilde{M}_i^{\,j}\, ,\; \Phi_0\}_{P.B.} \;=\; -\,(x^j\, \partial_i \;-\; x_i\, \partial^j)\, \Phi_0 \;\; , \\ \{\tilde{D}\, ,\; \Phi_0\}_{P.B.} &= -\,(x^i\, \partial_i \;+\; \Delta_-)\, \Phi_0 \quad , \\ \{\tilde{K}^i\, ,\; \Phi_0\}_{P.B.} &= -\,[2\, x^i\, (x^j\, \partial_j + \Delta_-) \;-\; x^j\, x_j\, \partial^i]\, \Phi_0 \;\; . \end{aligned} \tag{25}$$

[9] The choice (9) was found in [41][14][23] to correspond to the case where the divergent local terms in the asymptotic expression of the action vanish, and, in addition, to make the canonical energy computed in global coordinates to be conserved, positive and finite for 'irregular' modes propagating in the bulk. Here the choice (9) arises again, now making the charges to be finite in the asymptotic limit.

[10] The facts that Π approaches the boundary as in (19), and that it should transform with scaling dimension Δ_+ in the asymptotic limit, were shown already in the calculations in [41][14][23] (see also [40] for previous related results). In particular, in the notation in [23], it was shown that, once (9) is chosen, the field '$\psi^{(1)}$' given by $\psi^{(1)} = \partial_n \Phi + \frac{\Delta_-}{l} \Phi$ (i.e. $\Pi = \sqrt{g}\psi^{(1)}$, see (21)) approaches the boundary as $\psi^{(1)}(\epsilon, \vec{x}) \sim \epsilon^{\Delta_+} \psi^{(1)}(\vec{x})$, from which (19) follows. In addition, it was shown that $\psi^{(1)}$ couples, through the standard AdS/CFT prescription given in [2][3][9], with a modification explained in [41], to a boundary conformal operator of dimension Δ_-, thus indicating that $\psi^{(1)}$, and by extension Π, should have scaling dimension Δ_+ in the asymptotic limit. On the other hand, the interpretation of Π as the conjugate momentum to Φ (see (12)) was given in [24][36].

In this way, the asymptotic charges $\tilde{P}_i$, $\tilde{M}_i{}^j$, $\tilde{D}$ and $\tilde{K}^i$ respectively generate translations, rotations, dilations and special conformal transformations at the boundary. Notice, in addition, that Φ_0 has scaling dimension Δ_-, as expected.

Now, performing integrations by parts in (22) and using (5), we also find

$$\begin{aligned}
\{\tilde{P}_i \,,\, \Pi_0\}_{P.B.} &= -\,\partial_i \Pi_0 \quad , \qquad \{\tilde{M}_i{}^j \,,\, \Pi_0\}_{P.B.} \;=\; -\,(x^j\,\partial_i \,-\, x_i\,\partial^j)\,\Pi_0 \ , \\
\{\tilde{D} \,,\, \Pi_0\}_{P.B.} &= -\,(x^i\,\partial_i \,+\, \Delta_+)\,\Pi_0 \quad , \\
\{\tilde{K}^i \,,\, \Pi_0\}_{P.B.} &= -\,[2\,x^i\,(x^j\,\partial_j + \Delta_+) \,-\, x^j\,x_j\,\partial^i]\,\Pi_0 \ ,
\end{aligned} \tag{26}$$

thus showing that Π_0 has scaling dimension Δ_+, as anticipated. In the following section, this result will motivate us to treat Φ_0 and Π_0 as independent fields with the given dimensions.

Using (23, 24), the asymptotic charges (22) can analogously be shown to satisfy the global conformal algebra in d dimensions, as expected. This result, as well as (25) and (26), can be considered as non trivial checks on our formalism.

In this way, we have computed the explicit expressions of the asymptotic charges which generate the conformal transformations at the boundary. They are written in terms of the asymptotic values of the bulk scalar field and the conjugate momentum, namely Φ_0 and Π_0, which in turn behave as conformal fields with scaling dimensions Δ_- and Δ_+, respectively.

3 Asymptotic limit in AdS_3

We now turn our attention to the particular case of $d = 2$, where we expect to find the local conformal algebra and Virasoro generators in the asymptotic limit. We also expect to compute the corresponding central charge and find related information.

We consider the following change of variables

$$(x^0 \,,\, x^1 \,,\, x^2) \;\longrightarrow\; (x^0 \,,\, z \,,\, \bar{z}) \ ,$$

where the complex variables z and $\bar{z}$ are given by

$$z \;=\; x^1 \,+\, i\,x^2 \qquad , \qquad \bar{z} \;=\; x^1 \,-\, i\,x^2 \ . \tag{27}$$

The status of z and $\bar{z}$ is similar here to the usual one in standard two dimensional CFT, in that we extend the range of x^1 and x^2 to the complex plane, and so (27) is understood as just a change of independent variables. In this way, $\bar{z}$ is not the complex conjugate of z, namely z^*. On the other hand, the physical space is the 'real' surface $\bar{z} = z^*$ where we recover $x^1, x^2 \in \mathbf{R}$.

The non-vanishing components of the metric are

$$g_{00} \;=\; \frac{l^2}{x_0^2} \qquad , \qquad g_{z\bar{z}} \;=\; g_{\bar{z}z} \;=\; \frac{l^2}{2x_0^2} \ , \tag{28}$$

and the Killing vectors (14) now read

$$\begin{array}{lllllllll}
\xi^0_T & = & 0 & , & \xi^z_T & = & a^1 + i\,a^2 & , & \xi^{\bar z}_T = a^1 - i\,a^2 \ ; \\
\xi^0_R & = & 0 & , & \xi^z_R & = & -\,i\,m^1{}_2\,z & , & \xi^{\bar z}_R = i\,m^1{}_2\,\bar z \ ; \\
\xi^0_D & = & \alpha\,x^0 & , & \xi^z_D & = & \alpha\,z & , & \xi^{\bar z}_D = \alpha\,\bar z \ ; \\
\xi^0_S & = & \multicolumn{7}{l}{x^0\,[b^1\,(z + \bar z) - i\,b^2\,(z - \bar z)] \ ,} \\
\xi^z_S & = & \multicolumn{7}{l}{b^1\,(z^2 - x_0^2) - i\,b^2\,(z^2 + x_0^2) \ ,} \\
\xi^{\bar z}_S & = & \multicolumn{7}{l}{b^1\,(\bar z^2 - x_0^2) + i\,b^2\,(\bar z^2 + x_0^2) \ .}
\end{array} \tag{29}$$

The formulation is again on surfaces at equal values of the radial coordinate x^0. We generically define the charges at $x^0 = \epsilon$ as

$$Q = \oint_0 dz \oint_0 d\bar z\,\sqrt{g}\,J^0 \ , \tag{30}$$

where the integrations are counterclockwise around $z = 0$ and $\bar z = 0$. The Noether currents and the conjugate momentum are formally given by (10, 11) and (12) (supplemented by (9)), but we now make use of the metric (28) and the expression (29) for the Killing vectors. Notice that (1, 5) now reduce to

$$\Delta_\pm = 1 \pm \sqrt{1 + l^2 m^2} \ , \tag{31}$$

with

$$-1 < l^2 m^2 < 0 \ , \tag{32}$$

so that

$$0 < \Delta_- < 1 \quad , \quad 1 < \Delta_+ < 2 \ . \tag{33}$$

In particular we have the following useful relation

$$\Delta_+ + \Delta_- = 2 \ . \tag{34}$$

The calculations are analogous to those performed in the previous section, and we write here the results. If we consider the choice (9) then the asymptotic behaviors of Φ and Π are similar to those in (17, 19), namely

$$\Phi(\epsilon, z, \bar z) \sim \epsilon^{\Delta_-}\,\Phi_0(z, \bar z) \quad , \quad \Pi(\epsilon, z, \bar z) \sim \epsilon^{-\Delta_- - 1}\,l\,\Pi_0(z, \bar z) \ . \tag{35}$$

We find the following charges in the asymptotic limit $\epsilon \to 0$ (here we define $\tilde P_i \equiv \lim_{\epsilon\to 0} P_i$ and so on)

$$\begin{array}{lllllll}
\tilde P_1 & = & -i\,(G_T + \bar G_T) & , & \tilde P_2 & = & G_T - \bar G_T \ , \\
\tilde M & = & -\,(G_{DR} - \bar G_{DR}) & , & \tilde D & = & -i\,(G_{DR} + \bar G_{DR}) \ , \\
\tilde K^1 & = & -i\,(G_S + \bar G_S) & , & \tilde K^2 & = & -\,(G_S - \bar G_S) \ ,
\end{array} \tag{36}$$

where (from now on we define $\partial \equiv \frac{\partial}{\partial z}$ and $\bar\partial \equiv \frac{\partial}{\partial \bar z}$)

$$
\begin{aligned}
G_T &= -i \oint_0 dz \oint_0 d\bar z \, \Pi_0 \, \partial\Phi_0 \ , \\
G_{DR} &= -i \oint_0 dz \oint_0 d\bar z \, \Pi_0 \left(z \, \partial\Phi_0 \ + \ \frac{1}{2} \, \Delta_- \, \Phi_0 \right) \ , \\
G_S &= -i \oint_0 dz \oint_0 d\bar z \, \Pi_0 \, z \, (z \, \partial\Phi_0 \ + \ \Delta_- \, \Phi_0) \ ,
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
\bar G_T &= -i \oint_0 d\bar z \oint_0 dz \, \Pi_0 \, \bar\partial\Phi_0 \ , \\
\bar G_{DR} &= -i \oint_0 d\bar z \oint_0 dz \, \Pi_0 \left(\bar z \, \bar\partial\Phi_0 \ + \ \frac{1}{2} \, \Delta_- \, \Phi_0 \right) \ , \\
\bar G_S &= -i \oint_0 d\bar z \oint_0 dz \, \Pi_0 \, \bar z \, (\bar z \, \bar\partial\Phi_0 \ + \ \Delta_- \, \Phi_0) \ .
\end{aligned}
\tag{38}
$$

Here the integration is performed counterclockwise over circles around $z = 0$ and $\bar z = 0$. We will show later that G_T and $\bar G_T$ generate translations at the boundary, whereas G_{DR} and $\bar G_{DR}$ correspond to both dilations and rotations. In addition, G_S and $\bar G_S$ generate special conformal transformations.

Now, integrating by parts and using (34) we get[11]

$$
\begin{aligned}
G_T &= -\frac{i}{2} \oint_0 dz \oint_0 d\bar z \, (\Delta_+ \, \partial\Phi_0 \, \Pi_0 \ - \ \Delta_- \, \Phi_0 \, \partial\Pi_0) \ , \\
G_{DR} &= -\frac{i}{2} \oint_0 dz \, z \oint_0 d\bar z \, (\Delta_+ \, \partial\Phi_0 \, \Pi_0 \ - \ \Delta_- \, \Phi_0 \, \partial\Pi_0) \ , \\
G_S &= -\frac{i}{2} \oint_0 dz \, z^2 \oint_0 d\bar z \, (\Delta_+ \, \partial\Phi_0 \, \Pi_0 - \ \Delta_- \, \Phi_0 \, \partial\Pi_0) \ ,
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
\bar G_T &= -\frac{i}{2} \oint_0 d\bar z \oint_0 dz \, (\Delta_+ \, \bar\partial\Phi_0 \, \Pi_0 \ - \ \Delta_- \, \Phi_0 \, \bar\partial\Pi_0) \ , \\
\bar G_{DR} &= -\frac{i}{2} \oint_0 d\bar z \, \bar z \oint_0 dz \, (\Delta_+ \, \bar\partial\Phi_0 \, \Pi_0 \ - \ \Delta_- \, \Phi_0 \, \bar\partial\Pi_0) \ , \\
\bar G_S &= -\frac{i}{2} \oint_0 d\bar z \, \bar z^2 \oint_0 dz \, (\Delta_+ \, \bar\partial\Phi_0 \, \Pi_0 \ - \ \Delta_- \, \Phi_0 \, \bar\partial\Pi_0) \ .
\end{aligned}
\tag{40}
$$

We may write

$$
\begin{aligned}
G_T &= \frac{1}{2\pi i} \oint_0 dz \, T(z) \ , \quad G_{DR} = \frac{1}{2\pi i} \oint_0 dz \, z \, T(z) \ , \quad G_S = \frac{1}{2\pi i} \oint_0 dz \, z^2 \, T(z), \\
\bar G_T &= \frac{1}{2\pi i} \oint_0 d\bar z \, \bar T(\bar z) \ , \quad \bar G_{DR} = \frac{1}{2\pi i} \oint_0 d\bar z \, \bar z \, \bar T(\bar z) \ , \quad \bar G_S = \frac{1}{2\pi i} \oint_0 d\bar z \, \bar z^2 \, \bar T(\bar z),
\end{aligned}
$$

[11]In these and the following calculations we assume the product $\Pi_0(z,\bar z)\Phi_0(z,\bar z)$ to admit a Laurent expansion in z and $\bar z$ (even when Π_0 and Φ_0 are not separately required to satisfy this property), so in particular we have $\oint_0 dz \, \partial(z^n \Pi_0 \Phi_0) \ = \ \oint_0 d\bar z \, \bar\partial(\bar z^n \Pi_0 \Phi_0) \ = \ 0$ ($n \in \mathbf{Z}$) (notice that since we are considering contour integrals then z and $\bar z$ can be treated as independent variables). The explicit series expansions for Π_0 and Φ_0 which satisfy these requirements will be introduced later, after a proper motivation.

where

$$T(z) \;=\; \pi \oint_0 d\bar{z}\, (\Delta_+ \,\partial\Phi_0\, \Pi_0 - \;\Delta_-\, \Phi_0\, \partial\Pi_0) \;\;,$$
$$\bar{T}(\bar{z}) \;=\; \pi \oint_0 dz\, (\Delta_+ \,\bar{\partial}\Phi_0\, \Pi_0 - \;\Delta_-\, \Phi_0\, \bar{\partial}\Pi_0) \;\;.$$

Now we Laurent expand

$$T(z) \;=\; \sum_{n\in\mathbf{Z}} L_n\, z^{-n-2} \qquad , \qquad \bar{T}(\bar{z}) \;=\; \sum_{n\in\mathbf{Z}} \bar{L}_n\, \bar{z}^{-n-2} \;\;, \tag{41}$$

where

$$L_n \;=\; -\frac{i}{2} \oint_0 dz\, z^{n+1} \oint_0 d\bar{z}\, (\Delta_+ \,\partial\Phi_0\, \Pi_0 \;-\; \Delta_-\, \Phi_0\, \partial\Pi_0) \;\;, \tag{42}$$

and

$$\bar{L}_n \;=\; -\frac{i}{2} \oint_0 d\bar{z}\, \bar{z}^{n+1} \oint_0 dz\, (\Delta_+ \,\bar{\partial}\Phi_0\, \Pi_0 \;-\; \Delta_-\, \Phi_0\, \bar{\partial}\Pi_0) \;\;. \tag{43}$$

In particular, notice from (39, 40) that

$$\begin{aligned} G_T \;&=\; L_{-1} \quad , \quad G_{DR} \;=\; L_0 \quad , \quad G_S \;=\; L_1 \;\;, \\ \bar{G}_T \;&=\; \bar{L}_{-1} \quad , \quad \bar{G}_{DR} \;=\; \bar{L}_0 \quad , \quad \bar{G}_S \;=\; \bar{L}_1 \;\;. \end{aligned} \tag{44}$$

As in the previous section, we are considering here a formulation where x^0 plays a special role and we step on surfaces at equal values of it. By performing the limit $\epsilon \to 0$ we have positioned on the boundary at $x^0 = 0$, where we consider the Poisson brackets

$$\{\Pi_0(z,\bar{z})\;, \Phi_0(w,\bar{w})\}_{P.B.} \;=\; \delta(z-w)\bar{\delta}(\bar{z}-\bar{w}) \;\;, \tag{45}$$

and

$$\{\Pi_0(z,\bar{z})\;, \Pi_0(w,\bar{w})\}_{P.B.} \;=\; \{\Phi_0(z,\bar{z})\;, \Phi_0(w,\bar{w})\}_{P.B.} \;=\; 0 \;\;. \tag{46}$$

Here the δ-function satisfies (in the particular case where the integration contours are circles around the origin, as considered here)

$$f(w) \;=\; \oint_{0\;,\;|z|=|w|} dz\; f(z)\; \delta(z-w) \quad ,$$
$$-\partial f(w) \;=\; \oint_{0\;,\;|z|=|w|} dz\; f(z)\; \partial_z\delta(z-w) \quad ,$$

where f is a generic function, and the integration is over a circle satisfying $|z| = |w|$. So the Poisson brackets are computed at equal values of the distance to the origin of the complex plane, in a sense that we illustrate with the following example (see (42))

$$\begin{aligned} \{L_n\;,\; \Phi_0(w,\bar{w})\}_{P.B.} = -\,\frac{i}{2} & \oint_{0\;,\;|z|=|w|} dz\; z^{n+1} \oint_{0\;,\;|\bar{z}|=|\bar{w}|} d\bar{z} \\ & \times\; [\Delta_+\, \partial\Phi_0(z,\bar{z})\; \delta(z-w)\; \bar{\delta}(\bar{z}-\bar{w}) \;-\; \Delta_-\; \Phi_0(z,\bar{z})\; \partial_z\delta(z-w)\; \bar{\delta}(\bar{z}-\bar{w})]. \end{aligned} \tag{47}$$

Here we are integrating over counterclockwise circles around the origin of the complex plane and satisfying $|z| = |w|$ and $|\bar{z}| = |\bar{w}|$. It is this property of the asymptotic Poisson brackets, of being computed at equal values of the distance to the origin of the complex plane, that will give them a meaning from the point of view of the boundary two dimensional CFT.

Now using Eqs.(34, 42, 43, 45, 46) we get

$$
\begin{aligned}
i\{L_n\ ,\ \Phi_0(z,\bar{z})\}_{P.B.} &= \frac{\Delta_-}{2}\ (n+1)\ z^n\ \Phi_0(z,\bar{z})\ +\ z^{n+1}\ \partial\Phi_0(z,\bar{z})\ \ , \\
i\{\bar{L}_n\ ,\ \Phi_0(z,\bar{z})\}_{P.B.} &= \frac{\Delta_-}{2}\ (n+1)\ \bar{z}^n\ \Phi_0(z,\bar{z})\ +\ \bar{z}^{n+1}\ \bar{\partial}\Phi_0(z,\bar{z})\ \ ,
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
i\{L_n\ ,\ \Pi_0(z,\bar{z})\}_{P.B.} &= \frac{\Delta_+}{2}\ (n+1)\ z^n\ \Pi_0(z,\bar{z})\ +\ z^{n+1}\ \partial\Pi_0(z,\bar{z})\ \ , \\
i\{\bar{L}_n\ ,\ \Pi_0(z,\bar{z})\}_{P.B.} &= \frac{\Delta_+}{2}\ (n+1)\ \bar{z}^n\ \Pi_0(z,\bar{z})\ +\ \bar{z}^{n+1}\ \bar{\partial}\Pi_0(z,\bar{z})\ \ .
\end{aligned}
\tag{49}
$$

These expressions suggest the identification of the coefficients L_n and $\bar{L}_n$ with generators of the local conformal group on the boundary of AdS_3, and of the asymptotic values of the bulk scalar field and the conjugate momentum, namely Φ_0 and Π_0, respectively, with conformal fields with the following weights

$$
\begin{aligned}
\Phi_0 \ &\longrightarrow\ h_{\Phi_0}\ =\ \bar{h}_{\Phi_0}\ =\ \frac{\Delta_-}{2}\ \ , \\
\Pi_0 \ &\longrightarrow\ h_{\Pi_0}\ =\ \bar{h}_{\Pi_0}\ =\ \frac{\Delta_+}{2}\ \ .
\end{aligned}
\tag{50}
$$

In particular, these results are consistent with those in the previous section (see (25, 26)) where the global conformal group in d dimensions was found and we computed the scaling dimensions (i.e. $h_{\Phi_0} + \bar{h}_{\Phi_0}$ and $h_{\Pi_0} + \bar{h}_{\Pi_0}$ in the particular case of $d = 2$) of Φ_0 and Π_0. This can be considered as a consistency check. Notice, also, that both fields Φ_0 and Π_0 have spin zero ($h_{\Phi_0} - \bar{h}_{\Phi_0} = h_{\Pi_0} - \bar{h}_{\Pi_0} = 0$). In addition, from (44) and (48, 49) we see that (G_T, G_{DR}, G_S) and $(\bar{G}_T, \bar{G}_{DR}, \bar{G}_S)$ correspond to the global conformal group and that, as anticipated, G_T and $\bar{G}_T$ generate translations at the boundary, whereas G_{DR} and $\bar{G}_{DR}$ generate both dilations and rotations, and G_S, as well as $\bar{G}_S$, correspond to special conformal transformations.

We also find the expected results[12]

$$
i\{L_n\ ,\ L_m\}_{P.B.}\ =\ (n-m)\ L_{n+m}\ \ ,
\tag{51}
$$

[12]This is more easily computed by integrating by parts (see footnote 11) e.g. in (42) and using (34) in order to write

$$
iL_n\ =\ A_n\ +\ \frac{\Delta_-}{2}\ (n+1)\ B_n\ \ ,
$$

$$i\{\bar{L}_n \ , \ \bar{L}_m\}_{P.B.} \ = \ (n-m) \ \bar{L}_{n+m} \quad , \tag{52}$$

and

$$\{L_n \ , \ \bar{L}_m\}_{P.B.} \ = \ 0 \quad . \tag{53}$$

In this way, after performing the asymptotic limit we have obtained two copies of the Virasoro algebra, with vanishing central charges at the classical level. These results, as well as (48) and (49), can be considered as consistency checks on the formalism. As a further additional check, Eqs.(51, 52, 53) can also be obtained using mode expansions. We write[13]

$$\begin{aligned}
\Phi_0(z,\bar{z}) \ &= \ \frac{1}{2\pi} \sum_{n,m\in\mathbf{Z}} z^{-n-\frac{\Delta_-}{2}} \ \bar{z}^{-m-\frac{\Delta_-}{2}} \ \Phi_{n,m} \quad , \\
\Pi_0(z,\bar{z}) \ &= \ \frac{1}{2\pi} \sum_{n,m\in\mathbf{Z}} z^{-n-\frac{\Delta_+}{2}} \ \bar{z}^{-m-\frac{\Delta_+}{2}} \ \Pi_{n,m} \quad ,
\end{aligned} \tag{54}$$

where

$$\begin{aligned}
\Phi_{n,m} \ &= \ -\frac{1}{2\pi} \oint_0 dz \ z^{n+\frac{\Delta_-}{2}-1} \oint_0 d\bar{z} \ \bar{z}^{m+\frac{\Delta_-}{2}-1} \ \Phi_0(z,\bar{z}) \quad , \\
\Pi_{n,m} \ &= \ -\frac{1}{2\pi} \oint_0 dz \ z^{n+\frac{\Delta_+}{2}-1} \oint_0 d\bar{z} \ \bar{z}^{m+\frac{\Delta_+}{2}-1} \ \Pi_0(z,\bar{z}) \quad .
\end{aligned} \tag{55}$$

The extra powers of $\frac{\Delta_-}{2}$ and $\frac{\Delta_+}{2}$ appearing in (54) are motivated by (50).

From (34), (45, 46) and (55) we find

$$\{\Pi_{n,m} \ , \ \Phi_{r,s}\}_{P.B.} \ = \ - \ \delta_{n+r,0} \ \bar{\delta}_{m+s,0} \quad , \tag{56}$$

where

$$A_n \ = \ \oint_0 dz \ z^{n+1} \oint_0 d\bar{z} \ \partial\Phi_0 \ \Pi_0 \quad , \quad B_n \ = \ \oint_0 dz \ z^n \oint_0 d\bar{z} \ \Phi_0 \ \Pi_0 \quad .$$

Then by computing

$$\{A_n \ , \ A_m\}_{P.B.} \ = \ (n-m) \ A_{n+m} \quad , \quad \{A_n \ , \ B_m\}_{P.B.} \ = \ -m \ B_{n+m} \quad , \quad \{B_n \ , \ B_m\}_{P.B.} \ = \ 0 \ ,$$

Eq.(51) follows. In a similar way (52) and (53) can also be obtained. With illustrative purposes, notice that the integration contours are taken as, e.g.,

$$\begin{aligned}
\{A_n \ , \ A_m\}_{P.B.} \ = \ &\oint_0 dw \ w^{m+1} \oint_0 d\bar{w} \oint_{0,|z|=|w|} dz \ z^{n+1} \oint_{0,|\bar{z}|=|\bar{w}|} d\bar{z} \\
&\times [-\partial\Phi_0(w,\bar{w}) \ \Pi_0(z,\bar{z}) \ \partial_z\delta(z-w) \ \bar{\delta}(\bar{z}-\bar{w}) \ + \ \partial\Phi_0(z,\bar{z}) \ \Pi_0(w,\bar{w}) \ \partial_w\delta(z-w) \ \bar{\delta}(\bar{z}-\bar{w})].
\end{aligned}$$

[13]Using (34) we see that these mode expansions meet the requirements in footnote 11. There is the formal aspect of the well definiteness of the product of two Laurent expansions, but, since this will not affect the physics, then we will not consider it here.

and

$$\{\Pi_{n,m}\ ,\ \Pi_{r,s}\}_{P.B.} \ = \ \{\Phi_{n,m}\ ,\ \Phi_{r,s}\}_{P.B.} \ = \ 0 \ . \tag{57}$$

On the other hand, plugging (54) into (42, 43) and using (34) we get

$$\begin{aligned} L_n &= i \sum_{r,s\in\mathbf{Z}} \left(n\,\frac{\Delta_-}{2} \ - \ r\right) \Phi_{r,s}\, \Pi_{n-r,-s} \quad , \\ \bar{L}_n &= i \sum_{r,s\in\mathbf{Z}} \left(n\,\frac{\Delta_-}{2} \ - \ s\right) \Phi_{r,s}\, \Pi_{-r,n-s} \quad . \end{aligned} \tag{58}$$

Now, using (56, 57, 58) and performing some straightforward algebra we reproduce Eqs.(51, 52, 53) again, as expected.

We also find the following results

$$\begin{aligned} i\{L_n\ ,\ \Phi_{r,s}\}_{P.B.} &= \left[n\left(\frac{\Delta_-}{2} \ - \ 1\right) - r\right] \Phi_{n+r,s} \quad , \\ i\{\bar{L}_n\ ,\ \Phi_{r,s}\}_{P.B.} &= \left[n\left(\frac{\Delta_-}{2} \ - \ 1\right) - s\right] \Phi_{r,n+s} \quad , \end{aligned} \tag{59}$$

and

$$\begin{aligned} i\{L_n\ ,\ \Pi_{r,s}\}_{P.B.} &= \left[n\left(\frac{\Delta_+}{2} \ - \ 1\right) - r\right] \Pi_{n+r,s} \quad , \\ i\{\bar{L}_n\ ,\ \Pi_{r,s}\}_{P.B.} &= \left[n\left(\frac{\Delta_+}{2} \ - \ 1\right) - s\right] \Pi_{r,n+s} \quad , \end{aligned} \tag{60}$$

which are consistent with (48, 49).

On the basis of the results above, we interpret the coefficients $L_n,\ \bar{L}_n$ in (58) to act as the Virasoro generators of the boundary CFT. Notice that Hermitian conjugation

$$\Phi^\dagger_{n,m} \ = \ \Phi_{-n,-m} \qquad , \qquad \Pi^\dagger_{n,m} \ = \ \Pi_{-n,-m} \ , \tag{61}$$

gives in (58) the expected properties

$$L^\dagger_n \ = \ L_{-n} \qquad , \qquad \bar{L}^\dagger_n \ = \ \bar{L}_{-n} \ . \tag{62}$$

These results motivate us to consider aspects of the quantization of the boundary CFT and attempt to compute the quantum central charge. From now on, Φ_0 and Π_0 will be treated as conformal fields of the boundary CFT having weights as given by (50), their role as the asymptotic values of fields living in the bulk being put aside. Notice that Eqs.(48, 49) and (59, 60) suggest to consider the following AdS_3/CFT_2 prescription

$$\{\ ,\ \}_{P.B.} \ \longrightarrow \ -i\,[\ ,\ ] \ , \tag{63}$$

where [,] are commutators on the boundary CFT, consistent with the radial quantization procedure.[14] In particular, this gives in (56, 57)

$$[\Pi_{n,m} \ , \ \Phi_{r,s}] \ = \ - \, i \, \delta_{n+r,0} \, \bar{\delta}_{m+s,0} \ , \tag{64}$$

and

$$[\Pi_{n,m} \ , \ \Pi_{r,s}] \ = \ [\Phi_{n,m} \ , \ \Phi_{r,s}] \ = \ 0 \ . \tag{65}$$

We point out that the reason why (63) makes sense is the fact that the asymptotic Poisson brackets are computed at equal values of the distance to the origin of the complex plane, as discussed above and illustrated e.g. by (47). It is this property, which is inherited from the definition of the charges using contour integrals (see (30)), that causes the asymptotic Poisson brackets to have a meaning from the point of view of the two dimensional boundary CFT, and is what allows us to go here one step further than in the generic AdS_{d+1} case analyzed in the previous section.

From now on, we will simultaneously consider the following two alternative sets of definitions

$$\begin{gathered} b_{n,m} \ \equiv \ i\Phi_{n,m} \qquad , \qquad c_{n,m} \ \equiv \ \Pi_{n,m} \ , \\ \lambda \ \equiv \ \frac{\Delta_-}{2} \ , \end{gathered} \tag{66}$$

or else

$$\begin{gathered} b_{n,m} \ \equiv \ -i\Pi_{n,m} \qquad , \qquad c_{n,m} \ \equiv \ \Phi_{n,m} \ , \\ \lambda \ \equiv \ \frac{\Delta_+}{2} \ . \end{gathered} \tag{67}$$

Choosing any of the definitions (66) or (67), and using (34), we see that Eqs.(58, 61, 64) can be written in the following way

$$\begin{aligned} L_n \ &= \ \sum_{r,s\in\mathbf{Z}} (n \, \lambda \ - \ r) \, b_{r,s} \, c_{n-r,-s} \ , \\ \bar{L}_n \ &= \ \sum_{r,s\in\mathbf{Z}} (n \, \lambda \ - \ s) \, b_{r,s} \, c_{-r,n-s} \ , \end{aligned} \tag{68}$$

$$b^\dagger_{n,m} \ = \ -b_{-n,-m} \qquad , \qquad c^\dagger_{n,m} \ = \ c_{-n,-m} \ , \tag{69}$$

$$[c_{n,m} \ , \ b_{r,s}] \ = \ \delta_{n+r,0} \, \bar{\delta}_{m+s,0} \ . \tag{70}$$

[14]The expression 'radial quantization' is not to be confused with the radial coordinate of AdS_3, since it refers to the standard quantization procedure in two dimensional CFTs, where operators within correlation functions are radially ordered.

Exception made of the fact that the fields are neither purely holomorphic nor antiholomorphic, this is strikingly similar to a generalized ghost system.[15]

We would like to compute the quantum central charge in the two copies of the Virasoro algebra

$$[L_n\ ,\ L_m]\ =\ (n-m)\ L_{n+m}\ +\ \frac{c}{12}\ n\ (n^2\ -\ 1)\ \delta_{n+m,0}\ \ , \tag{71}$$

and

$$[\bar{L}_n\ ,\ \bar{L}_m]\ =\ (n-m)\ \bar{L}_{n+m}\ +\ \frac{\bar{c}}{12}\ n\ (n^2\ -\ 1)\ \bar{\delta}_{n+m,0}\ \ . \tag{72}$$

We define the following ground state

$$b_{0,0}|0\rangle\ =\ 0 \qquad , \qquad b_{n,m}|0\rangle\ =\ c_{n,m}|0\rangle\ =\ 0 \qquad (n>0\ \text{ or }\ m>0)\ \ , \tag{73}$$

which is not SL_2 invariant. Here we have grouped $b_{0,0}$ with the lowering operators and $c_{0,0}$ with the raising ones. Notice that, in fact, this gives two possible ground states, corresponding to the choices (66, 67). This should correspond to the two possible quantizations in the bulk found in [4][5]. The fact that we find two possible quantizations, as expected, could be considered as an additional consistency check on our formalism.

We introduce a normal ordering in (68) where lowering operators are placed to the right. We find

$$L_n|0\rangle\ =\ \bar{L}_n|0\rangle\ =\ 0 \qquad (n>0)\ \ . \tag{74}$$

The result

$$L_1L_{-1}|0\rangle\ =\ \bar{L}_1\bar{L}_{-1}|0\rangle\ =\ -\lambda(1-\lambda)|0\rangle\ \ ,$$

gives the following normal ordering constant for the coefficients L_0 and $\bar{L}_0$

$$L_0|0\rangle\ =\ \bar{L}_0|0\rangle\ =\ -\frac{1}{2}\ \lambda\ (1-\lambda)|0\rangle\ \ . \tag{75}$$

Using any of the choices (66) or (67), together with (31), we obtain

$$L_0|0\rangle\ =\ \bar{L}_0|0\rangle\ =\ \frac{l^2m^2}{8}|0\rangle\ \ . \tag{76}$$

Notice from (32) that this constant is in the range

$$-\frac{1}{8}\ <\ \frac{l^2m^2}{8}\ <0\ \ . \tag{77}$$

Now computing

$$L_2L_{-2}|0\rangle\ =\ \bar{L}_2\bar{L}_{-2}|0\rangle\ =\ [(2\lambda-1)^2-\ 4\lambda(1-\lambda)]|0\rangle\ \ ,$$

[15] The existence of the analogous of a ghost current will not be discussed here.

we get the central charges

$$c \;=\; \bar{c} \;=\; 3(2\lambda - 1)^2 -\; 1\ . \tag{78}$$

Using any of the choices (66) or (67), together with (31), we find

$$c \;=\; \bar{c} \;=\; 2 \;+\; 3l^2m^2\ \ . \tag{79}$$

We also notice from (32) that the central charges are in the range

$$-1 \;<\; c \;<\; 2 \qquad , \qquad -1 \;<\; \bar{c} \;<\; 2\ . \tag{80}$$

As seen in (76, 79) both the normal ordering constant and quantum central charge depend on the mass of the bulk field and on the AdS radius, in an analogous way as the conformal dimensions do.

A striking result, which is obtained from the facts that the boundary CFT is similar to a generalized ghost system (see (68, 69, 70)), and from the explicit dependence of the central charge with the mass (see (79)) is that the mass of the bulk scalar field seems to play the role of a background charge for the boundary CFT. In order to shed some light into this susprising result, it would be interesting to perform a bosonization program on the boundary CFT. However, this issue will no longer be considered here.

Notice also from (79) that the requirement for the central charge to be positive gives

$$l^2m^2 \;>\; -\frac{2}{3}\ \ . \tag{81}$$

This is more restrictive than the Breitenlohner-Freedman bound (see (2)). This new interesting result seems to be detected only from the boundary point of view, and the way it could be red from the bulk point of view remains to be investigated.

We also point out again that we have found two possible choices for the ground state, corresponding to (66, 67), together with (73), which should in turn correspond to the two possible quantizations in the bulk found in [4][5], and that this result could be considered as a last consistency check on the formalism that we have developed here.

It would be interesting to investigate the generalization of the present formalism to the case of interacting scalar field in the bulk. For instance, we could consider a polynomial interaction

$$I \;=\; \int d^3x\,\sqrt{g}\left[\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi \;+\; m^2\Phi^2\right) + \sum_{n\geq 3}\frac{\alpha_n}{n!}\,\Phi^n\right]\ \ .$$

Notice that, exception made of the asymptotic behavior (4), we have not made use of the explicit form of the solution to the equation of motion in this formalism. This property is expected to simplify the calculations. On the other hand, we could consider a perturbative approach where the solution to the free scalar field case is inserted into modified 'conserved' charges, thus possibly giving rise to modified theories at the boundary.

Another interesting issue is the inclusion of gravity in the present formalism. We would like to analyze the case of asymptotically anti-de Sitter spaces. At a more speculative level, there is the interesting possibility that the results we have computed here correspond to a certain limit of a gravity theory. This is suggested by the facts that we have found a full Virasoro symmetry algebra in the asymptotic limit of AdS_3, and further, that the corresponding energy-momentum tensor (see e.g.(41)) is in principle expected to correspond to a massless spin 2 field in the bulk. It is possible that such investigations could allow us to establish a connection among the formalism developed here and the results in [45], and, in addition, to investigate possible applications in black hole physics.

Finally, it also would be interesting to consider possible extensions of our formalism to higher dimensional Anti-de Sitter spaces, and to values of l^2m^2 outside the range (1).

Acknowledgments: It is a pleasure to thank W. Baron, C. Cardona, G. Giribet, S. Iguri, L. Nicolás and C. Núñez for very useful comments and discussions, and M. Schvellinger for encouragement. This work was supported by CONICET.

References

[1] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", Adv. Theor. Math. Phys. **2** (1998) 231; Int. J. Theor. Phys. **38** (1999) 1113.

[2] E. Witten, "Anti De Sitter Space and Holography", Adv. Theor. Math. Phys. **2** (1998) 253.

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge Theory Correlators from Non-Critical String Theory", Phys. Lett. **B428** (1998) 105.

[4] P. Breitenlohner and D. Z. Freedman, "Stability in Gauged Extended Supergravity", Ann. Phys. **144** (1982) 249.

[5] P. Breitenlohner and D. Z. Freedman, "Positive Energy in Anti-de Sitter Backgrounds and Gauged Extended Supergravity", Phys. Lett. **B115** (1982) 197.

[6] L. Mezincescu and P. K. Townsend, "Stability at a Local Maximum in Higher Dimensional Anti-de Sitter Space and Applications to Supergravity", Ann. Phys. **160** (1985) 406.

[7] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, "Correlation functions in the CFT_d/AdS_{d+1} correspondence", Nucl. Phys. **B546** (1999) 96.

[8] W. Mück and K. S. Viswanathan, "Conformal Field Theory Correlators from Classical Scalar Field Theory on AdS_{d+1}", Phys. Rev. **D58** (1998) 041901.

[9] I. R. Klebanov and E. Witten, "AdS/CFT Correspondence and Symmetry Breaking", Nucl. Phys. **B556** (1999) 89.

[10] V. Balasubramanian, P. Kraus and A. Lawrence, "Bulk vs. Boundary Dynamics in Anti-de Sitter Spacetime", Phys. Rev. **D59** (1999) 046003.

[11] E. Witten, "Multi-Trace Operators, Boundary Conditions, And AdS/CFT Correspondence", arXiv:hep-th/0112258.

[12] M. Berkooz, A. Sever and A. Shomer, "Double-trace Deformations, Boundary Conditions and Spacetime Singularities", JHEP **0205** (2002) 034.

[13] W. Mück, "An improved correspondence formula for AdS/CFT with multi-trace operators", Phys. Lett. **B531** (2002) 301.

[14] P. Minces, "Multi-Trace Operators and the Generalized AdS/CFT Prescription", Phys. Rev. **D68** (2003) 024027.

[15] A. C. Petkou, "Boundary multi-trace deformations and OPEs in AdS/CFT correspondence", JHEP **0206** (2002) 009.

[16] A. Sever and A. Shomer, "A Note on Multi-trace Deformations and AdS/CFT", JHEP **0207** (2002) 027.

[17] S. S. Gubser and I. Mitra, "Double-trace operators and one-loop vacuum energy in AdS/CFT", Phys. Rev. **D67** (2003) 064018.

[18] S. S. Gubser and I. R. Klebanov, "A universal result on central charges in the presence of double-trace deformations", Nucl. Phys. **B656** (2003) 23.

[19] E. Witten, "SL(2,Z) Action On Three-Dimensional Conformal Field Theories With Abelian Symmetry", arXiv:hep-th/0307041.

[20] Y. Satoh and J. Troost, "On time-dependent AdS/CFT", JHEP **0301** (2003) 027.

[21] J. Troost, "A Note on causality in the bulk and stability on the boundary", Phys. Lett. **B578** (2004) 210.

[22] D. Nolland, "AdS/CFT boundary conditions, multi-trace perturbations, and the c-theorem", Phys. Lett. **B584** (2004) 192.

[23] P. Minces, "Bound States in the AdS/CFT Correspondence", Phys. Rev. **D70** (2004) 025011.

[24] I. Papadimitriou and K. Skenderis, “AdS/CFT correspondence and Geometry”, arXiv:hep-th/0404176.

[25] K. Zoubos, “A Conformally Invariant Holographic Two-Point Function on the Berger Sphere”, JHEP **0501** (2005) 031.

[26] S. Hollands, A. Ishibashi and D. Marolf, “Comparison between various notions of conserved charges in asymptotically AdS-spacetimes”, Class. Quant. Grav. **22** (2005) 2881.

[27] D. Marolf, “States and Boundary Terms: Subtleties of Lorentzian AdS/CFT”, JHEP **0505** (2005) 042.

[28] O. Aharony, M. Berkooz and B. Katz, “Non-Local Effects of Multi-Trace Deformations in the AdS/CFT Correspondence”, JHEP **0510** (2005) 097.

[29] D. Marolf and S. Ross, “Boundary Conditions and Dualities: Vector Fields in AdS/CFT”, JHEP **0611** (2006) 085.

[30] T. Hartman and L. Rastelli, “Double-Trace Deformations, Mixed Boundary Conditions and Functional Determinants in AdS/CFT”, JHEP **0801** (2008) 019.

[31] D. Marolf and S. F. Ross, “Reversing Renormalization-Group Flows with AdS/CFT”, JHEP **0805** (2008) 055.

[32] E. Kiritsis, “Product CFTs, gravitational cloning, massive gravitons and the space of gravitational duals”, JHEP **0611** (2006) 049.

[33] M. Banados, A. Schwimmer and S. Theisen, “Remarks on Resonant Scalars in the AdS/CFT Correspondence”, JHEP **0609** (2006) 058.

[34] I. Papadimitriou, “Non-Supersymmetric Membrane Flows from Fake Supergravity and Multi-Trace Deformations”, JHEP **0702** (2007) 008.

[35] D. E. Diaz and H. Dorn, “Partition functions and double-trace deformations in AdS/CFT”, JHEP **0705** (2007) 046.

[36] I. Papadimitriou, “Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT”, JHEP **0705** (2007) 075.

[37] D. E. Diaz, “Polyakov formulas for GJMS operators from AdS/CFT”, JHEP **0807** (2008) 103.

[38] E. Kiritsis and V. Niarchos, “(Multi)Matrix Models and Interacting Clones of Liouville Gravity”, JHEP **0808** (2008) 044.

[39] G. E. Arutyunov and S. A. Frolov, “On the origin of supergravity boundary terms in the AdS/CFT correspondence”, Nucl. Phys. **B544** (1999) 576.

[40] P. Minces and V. O. Rivelles, “Scalar Field Theory in the AdS/CFT Correspondence Revisited”, Nucl. Phys. **B572** (2000) 651.

[41] P. Minces and V. O. Rivelles, “Energy and the AdS/CFT Correspondence”, JHEP **0112** (2001) 010.

[42] E. D’Hoker and D. Z. Freedman, “Supersymmetric Gauge Theories and the AdS/CFT Correspondence”, arXiv:hep-th/0201253.

[43] A. Ishibashi and R. M. Wald, “Dynamics in Non-Globally-Hyperbolic Static Spacetimes III: Anti-de Sitter Spacetime”, Class. Quant. Grav. **21** (2004) 2981.

[44] T. Andrade, M. Banados and F. Rojas, “Variational Methods in AdS/CFT”, Phys. Rev. **D75** (2007) 065013.

[45] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three Dimensional Gravity”, Comm. Math. Phys. **104** (1986) 207.

[46] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, “Black holes and asymptotics of 2+1 gravity coupled to a scalar field”, Phys. Rev. **D65** (2002) 104007.

[47] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, “Asymptotically Anti-de Sitter spacetimes and scalar fields with a logarithmic branch”, Phys. Rev. **D70** (2004) 044034.