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Testing nonlinear electrodynamics in waveguides: the effect of magnetostatic fields on the transmitted power

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Abstract
In Born–Infeld theory and other nonlinear electrodynamics, the presence of a magnetostatic field modifies the dispersion relation and the energy velocity of waves propagating in a hollow waveguide. As a consequence, the transmitted power along a waveguide suffers slight changes when a magnetostatic field is switched on and off. This tiny effect could be better tested by operating the waveguide at a frequency close to the cutoff frequency.

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1. Introduction

Born–Infeld electrodynamics [1–4] was created with the aim of healing the divergence of the point-like charge self-energy, but nowadays it has become the object of great interest because of its relation with the low energy dynamics of strings and branes [5–10]. As a nonlinear extension of Maxwell's theory, Born–Infeld electrodynamics has the quality of being the sole extension free of birefringence (i.e. the wave vector of a propagating perturbation results to be univocally determined) [11–14]. Few exact solutions of Born–Infeld theory are known, but some issues concerning the wave propagation have been thoroughly studied. Namely, it is known that the propagation velocity is c for free perturbations traveling in vacuum, but it is lower than c if an external field is present [11, 13, 15]; in other words, the external fields affect the dispersion relation. The dispersion relation is also affected by superposition of free waves. In fact, although the Born–Infeld free waves are identical to those of Maxwell's theory, the Born–Infeld nonlinearity causes interactions among free waves that influence the dispersion relation. This subject was studied in a previous paper by solving the Born–Infeld

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field equations in a cavity where stationary waves are formed [16]. Since a stationary wave is the superposition of free waves bouncing backward and forward, the interaction between them leads the wave amplitude to enter the dispersion relation in a waveguide. In this paper we will show that even a magnetostatic field affects the dispersion relation of waves propagating in a hollow waveguide by increasing the energy velocity. As a consequence, the transmitted power along the waveguide could be controlled by means of magnetostatic fields. Although this nonlinear electrodynamics effect should be very tiny, it has a better chance of being tested when the waveguide works at a frequency close to the cutoff frequency.

2. Born–Infeld field

As in Maxwell's electromagnetism, the Born–Infeld field $F_{\mu\nu}$ is derived from a potential: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (in geometric language, the 2-form F is exact: $F = \mathrm{d}A$). This condition cancels the curl of the electric field **E** and the divergence of the magnetic field **B**:

$$\partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} = 0, \tag{1}$$

i.e. $\mathrm{d}F = 0$. The Born–Infeld field differs from the Maxwell field in the dynamic equations, which are written in terms of the tensor

$$\mathcal{F}_{\mu\nu} = \frac{F_{\mu\nu} - \frac{P}{b^2}\,{}^*F_{\mu\nu}}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}}, \tag{2}$$

where S and P are the scalar and pseudoscalar field invariants:

$$S = \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2}(|\mathbf{B}|^2 - |\mathbf{E}|^2) \tag{3}$$

$$P = \tfrac{1}{4}{}^* F_{\mu\nu} F^{\mu\nu} = \mathbf{E}\cdot\mathbf{B} \tag{4}$$

(${}^*F_{\mu\nu}$ is the dual field tensor, i.e. the tensor resulting from exchanging the roles of **E** and $-$**B**). Born–Infeld dynamical equations are

$$\partial_\nu \mathcal{F}^{\mu\nu} = 0, \tag{5}$$

(i.e. $\mathrm{d}\,{}^*\mathcal{F} = 0$) which is obtained from the Born–Infeld Lagrangian

$$L[A_\mu] = -\frac{b^2}{4\pi}\left(1 - \sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}\right). \tag{6}$$

The constant b in equations (2) and (6) is a new universal constant with units of field that controls the scale for passing from Maxwell's theory to the nonlinear Born–Infeld regime, in the same way as the light speed c is the velocity scale that indicates the range of validity of Newtonian mechanics. The Maxwell Lagrangian and its related dynamical equations are recovered in the limit $b \to \infty$, or in regions where the field is small compared with b. Besides, Born–Infeld solutions having $S = 0 = P$ ('free waves') also solve Maxwell's equations.

3. Stationary waves

For our purposes, we will concentrate in those Born–Infeld waves that can be written as $F = \mathrm{d}[u(t,x)] \wedge \mathrm{d}y$, i.e.

$$F = \frac{\partial u(t,x)}{\partial t}\,\mathrm{d}t \wedge \mathrm{d}y + \frac{\partial u(t,x)}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y, \tag{7}$$

where the symbol $\wedge$ is the antisymmetrized tensor product. Expression (7) means

$$cE_y = F_{ty} = \partial u/\partial t = -F_{yt} \tag{8}$$

$$B_z = -F_{xy} = -\partial u/\partial x = F_{yx}, \tag{9}$$

the rest of the components $F_{\mu\nu}$ being zero. So the pseudoscalar invariant P vanishes for the proposed solution. The field (7) accomplishes equation (1), since dF is identically null. We will choose the function $u(t, x)$ to satisfy boundary conditions suitable for a rectangular box:

$$E_y(t, x = 0) = 0 = E_y(t, x = d). \tag{10}$$

By substituting the field (7) in equation (5), one obtains the dynamical equation for $u(t, x)$:

$$\mathcal{BI}u(t,x) \equiv \left[1 + \frac{1}{b^2}\left(\frac{\partial u}{\partial x}\right)^2\right]\frac{\partial^2 u}{\partial t^2} - \frac{2}{b^2}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t\partial x} - c^2\left[1 - \frac{1}{c^2b^2}\left(\frac{\partial u}{\partial t}\right)^2\right]\frac{\partial^2 u}{\partial x^2} = 0. \tag{11}$$

This is the so-called Born–Infeld equation, which can be independently derived from the scalar field Lagrangian $L[u] \propto (1 - b^{-2}\eta^{\mu\nu}\partial_\mu u\partial_\nu u)^{1/2}$.

4. BI-guided waves in the presence of a magnetostatic field

In [16] we have used the scheme of section 3 to study waves propagating in a rectangular waveguide along the z-axis. Although the field (7) does not depend on z—so the field (7) would be just a stationary wave bouncing backward and forward between two opposite walls of the guide—the propagation along the guide can be introduced by transforming the field with a Lorentz boost in the z-direction. This procedure would convert the solution (7) into a transverse electric propagating mode (note that a boost along z does not modify the boundary conditions (10)). In this paper we will study the nonlinear effects produced by the presence of a magnetostatic field B_ℓ along the waveguide. Equipped with the knowledge of the solution for a stationary wave between parallel plates with $B_\ell = 0$ [16], we prepare the solution with three free parameters—α, β and Ω—that will allow us to fit equation (11):

$$\begin{aligned} u(t,x) = {}& \frac{E}{\kappa}\left[\cos(\Omega t) + \frac{E^2}{32b^2}\cos(3\Omega t)\right]\left[\sin(\kappa x) + \frac{E^2}{32b^2}\sin(3\kappa x)\right] \\ & - B_\ell\left[x + \alpha\frac{E^2}{\kappa b^2}\sin(\beta\kappa x)\right] + O(b^{-4}), \end{aligned} \tag{12}$$

where $\Omega = \kappa c(1 + \varepsilon b^{-2})$. In the Maxwellian limit, $(b \to \infty)$ $u(t, x)$ represents a stationary wave plus a uniform magnetostatic field B_ℓ. The boundary condition (10) implies $\kappa = n\pi/d$. The α-term is a modulation of the magnetic field caused by the stationary wave. By replacing this solution in the Born–Infeld equation (11), one obtains

$$\begin{aligned} \mathcal{BI}u(t,x) = {}& -\frac{c^2\kappa E}{2b^2}\big[\big(E^2 + 2B_\ell^2 + 4\varepsilon\big)\cos(\kappa ct)\sin(\kappa x) \\ & - 2B_\ell E(\sin(2\kappa x) - \alpha\beta^2\sin(\beta\kappa x))\big] + O(b^{-4}). \end{aligned} \tag{13}$$

Therefore, in order that $u(t, x)$ in equation (12) be a solution of equation (11) at the considered order, the values of Ω, α and β should be

$$\Omega = \kappa c\left(1 - \frac{E^2 + 2B_\ell^2}{4b^2}\right), \qquad \alpha = \frac{1}{4}, \qquad \beta = 2. \tag{14}$$

As mentioned, a Lorentz boost along z will transform this field in a transversal electric mode propagating in the waveguide. Since the solution does not depend on y, the wave will result in a TE$_{n0}$ mode. The Lorentz boost will not modify x and y in equation (7), but dt will be replaced by $\gamma(V)(\mathrm{d}t' - Vc^{-2}\,\mathrm{d}z')$:

$$F = \frac{\partial u(t,x)}{\partial t}\gamma(V)\,\mathrm{d}t' \wedge \mathrm{d}y - \frac{\partial u(t,x)}{\partial t}\gamma(V)Vc^{-2}\,\mathrm{d}z' \wedge \mathrm{d}y + \frac{\partial u(t,x)}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y. \tag{15}$$

Besides, the phase Ωt will change to $\Omega\gamma(V)(t' - Vc^{-2}z')$, which means that the components of the wave vector are

$$k_{t'} = \omega = \Omega\gamma(V), \qquad k_{z'} = \Omega\gamma(V)Vc^{-2}, \tag{16}$$

so the dispersion relation is

$$\omega^2 = k_{z'}^2 c^2 + \Omega^2 \tag{17}$$

The energy velocity V can be recovered from equations (16) and (17)) as

$$V = \frac{c^2 k_{z'}}{\omega} = \frac{\partial\omega}{\partial k_{z'}}. \tag{18}$$

For a given frequency ω, the wave number $k_{z'}$ and the energy velocity V depend on the wave amplitude E and the magnetostatic field B_ℓ through the functional form of Ω (see equations (14) and (17)). In particular, equation (17) tells us that the minimum frequency that propagates in the waveguide is

$$\omega_{\text{cutoff}} = \Omega_{\min} = \Omega(n=1) = \frac{\pi c}{d}\left(1 - \frac{E^2 + 2B_\ell^2}{4b^2}\right) + O(b^{-4}). \tag{19}$$

The presence of B_ℓ in the cutoff frequency (a typical nonlinear effect) offers a way for controlling the energy flux in the guide. Namely, a given frequency ω could be larger than ω_{cutoff} when the magnetostatic field B_ℓ is on, so the wave propagates. But the same ω could become lower than ω_{cutoff} if B_ℓ is turned off. Thus, one could allow the wave propagate or not by switching on and off the magnetostatic field along the waveguide.

5. The transmitted power

We will calculate the energy flux in the waveguide. The energy flux per unit of time and area along the z-direction is the component $T_{t'}^{z'}$ of the energy–momentum tensor. The non-diagonal components of $T^{\mu\nu}$ in Born–Infeld electrodynamics are particularly simple (see for instance [17]):

$$T_{t'}^{z'} = -\frac{1}{4\pi}F_{t'\mu}\mathcal{F}^{z'\mu}. \tag{20}$$

In the case under study, it is

$$T_{t'}^{z'} = -\frac{1}{4\pi}\frac{F_{t'\mu}F^{z'\mu}}{\sqrt{1+\frac{2S}{b^2}}} = -\frac{1}{4\pi}\frac{F_{t'y}F_{z'y}}{\sqrt{1+\frac{2S}{b^2}}} = \frac{\gamma(V)^2 V}{4\pi c^2}\frac{(\partial u/\partial t)^2}{\sqrt{1+\frac{2S}{b^2}}} = \frac{\omega k_{z'}}{4\pi\Omega^2}\frac{(\partial u/\partial t)^2}{\sqrt{1+\frac{2S}{b^2}}}. \tag{21}$$

Therefore, the time-averaged transmitted power in the waveguide is

$$\mathcal{P} = \int \mathrm{d}x\,\mathrm{d}y\left\langle T_{t'}^{z'}\right\rangle = \frac{\omega k_{z'}\,\text{Area}\,E^2}{16\pi\kappa^2}\left(1 + \frac{3E^2}{32b^2} + O(b^{-4})\right), \tag{22}$$

where $\langle\,\rangle$ means the integration in a period divided by the period. If the Born–Infeld constant b goes to infinity, then the Maxwellian result for the transmitted power is recovered. The Born–Infeld correction is expected to be very weak. However, if one manages to operate the waveguide near the cutoff frequency, then one could benefit from the fact that $\partial k_{z'}/\partial\omega$

diverges at $\omega_{\rm cutoff}$. This implies that the nonlinear contributions to $k_{z'}$ are amplified near the cutoff frequency. In fact, according to equations (17) and (19), $k_{z'}$ is

$$k_{z'} = c^{-1}\sqrt{\omega^2-\Omega_{\rm min}^2} = c^{-1}\sqrt{\omega^2-\left(\frac{\pi c}{d}\right)^2} + \frac{\pi^2 c}{4d^2b^2}\frac{E^2+2B_\ell^2}{\sqrt{\omega^2-\left(\frac{\pi c}{d}\right)^2}} + O(b^{-4}). \tag{23}$$

Thus, the slight change in the transmitted power along the waveguide caused by switching on and off the longitudinal magnetostatic field B_ℓ can be approximated as

$$\frac{\Delta\mathcal{P}}{\mathcal{P}} = \frac{\Delta k_{z'}}{k_{z'}} = \frac{1}{2}\frac{B_\ell^2/b^2}{\left(\frac{\omega d}{\pi c}\right)^2-1} + O(b^{-4}), \tag{24}$$

the approximation being valid if expression (24) is much smaller than 1.

6. Conclusion

According to equation (24), by tuning the wave frequency ω very close to the cutoff frequency $\omega_{\rm cutoff} \simeq \pi c/d$, one could improve the chance of revealing the tiny nonlinear effect on the transmitted power and thus determining the Born–Infeld constant b. This fine tuning could be achieved by moving the power sensor in search of the frame where the wave number is nearly zero (so the energy velocity V is nearly zero). The detection of the effect (24) on the transmitted power would constitute a clear manifestation of the nonlinear behavior in the propagation of electromagnetic waves in vacuum. Of course, since we have solved the Born–Infeld theory at the order b^{-2}, the obtained results are also valid for any nonlinear electrodynamics having the same form at that order: $L = (1/4\pi)[S - b^{-2}(S^2+P^2)/2] + O(b^{-4})$. In particular, since the pseudoscalar P vanishes for solution (7), our results are shared with the Euler–Heisenberg Lagrangian, the weak-field limit for the one-loop approximation of QED [18–20],

$$L_{\rm EH} = \frac{1}{4\pi}\left[S - 4\mu\left(S^2+\frac{7}{4}P^2\right)\right], \tag{25}$$

μ being

$$\mu = \frac{2\alpha^2}{45 m_e c^2}\left(\frac{\hbar}{m_e c}\right)^3, \tag{26}$$

where α is the fine-structure constant. In fact, those solutions having $P = 0$ are common to both theories provided that the Born–Infeld constant b is identified with $1/\sqrt{4\mu}$, which has a magnitude of 10^{20} V m^{-1} [21].

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