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ROLE OF THE HALL EFFECT ON THE MAGNETOROTATIONAL INSTABILITY

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Within the framework of magnetohydrodynamics, the Hall effect might become non-negligible either in fully ionized low density plasmas or in cold plasmas with a low ionization fraction . We address the role of the Hall current in the development of the magnetorotational instability. The instability criterion and the instability growth rate are derived from a one-dimensional model.

Keywords: Magnetohydrodynamics; instabilities; accretion disks.

1. Introduction

Accretion disks are formed by matter containing a significant amount of angular momentum being gravitationally attracted by compact objects. There is considerable interest in studying these astrophysical flows, since accretion is a powerful mechanism of conversion of gravitational energy into kinetic energy and radiation. Theoretical models of accretion require the generation of small-scale turbulence to enhance the effective viscosity of the flow, although the origin of turbulence is still not well understood. However if the accretion disk displays differential rotation in the presence of magnetic fields, the magnetorotational instability (MRI) appears as the most promising candidate.[1−3] The magnetorotational instability has been widely tested with numerical simulations which show that the strong turbulence generated efficiently enhances angular momentum transport.[4−6]

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Nevertheless, it seems very restrictive to assume that the generation of turbulence relies on just one single instability, with a rather specific instability condition. Therefore, it seems appropriate to continue exploring other potential sources of small-scale turbulence. In this regard, the growth of purely hydrodynamic fluctuations (even though not exponentially fast) cannot be ruled out.[7–14]

Within the framework of magnetohydrodynamics, the Hall effect might be non-negligible whenever the ion density of the plasma becomes sufficiently low. This situation can arise either in hot, fully ionized low density plasmas (e.g. Refs. 15–16 for magnetic reconnection and for dynamo mechanisms, respectively), or in cold plasmas with a low ionization fraction (e.g. Ref. 17 for protoplanetary disks). The influence of the Hall effect on the magnetorotational instability has been considered in the linear[17–18] as well as in the nonlinear regimes[19–20] (see also Refs. 22–23).

An interesting feature of the Hall effect that has not been properly addressed in the literature, is the fact that it breaks the MHD symmetry $\mathbf{B} \leftrightarrow -\mathbf{B}$. A direct consequence of this symmetry break is a strong change in the stability region in the parameter space depending on the relative alignment of the external magnetic field and the angular velocity of the disk. Also, the Hall effect qualitatively changes the criterion leading to instability in cases where the angular velocity increases outward[18–21], which esentially means that the Hall effect can destabilize any system with differential rotation. In the present paper we analyze the potential relevance of the Hall effect in a one-dimensional theoretical model. The set of equations as well as the simplifying assumptions that we make, are listed in §2. The dispersion relation that is obtained in the linear regime is shown in §3, and the role of the Hall effect on the magnetorotational instability is discussed in §4. Finally, in §5 we summarize the main conclusions of the present study.

2. General Equations

Below we list the dimensionless Hall-MHD equations in a rotating reference frame with angular velocity $\mathbf{\Omega} = \Omega_0 \hat{z}$. We use Ω_0^{-1} as our time unit, and velocities are in units of the Alfvén velocity v_A. For a fully ionized hydrogen plasma, the continuity equation is

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \ , \tag{1}$$

where n is the particle density ($n_e = n_i = n$ to guarantee charge neutrality). The equation of motion for this plasma is

$$\frac{D\mathbf{u}}{Dt} = -\nabla h + \frac{\nabla \times \mathbf{B} \times \mathbf{B}}{n} - 2\hat{z} \times \mathbf{u} \ , \tag{2}$$

where h is the enthalpy density (for a barotropic flow). The induction equation is

$$\frac{\partial \mathbf{A}}{\partial t} = \varepsilon \nabla h - \nabla \Phi + \left(\mathbf{u} - \varepsilon \frac{\nabla \times \mathbf{B}}{n} \right) \times \mathbf{B} \ , \tag{3}$$

where $\mathbf{A}$ is the vector potential, Φ is the electrostatic potential and $\varepsilon = c/\omega_{pi}L_0$ (c: speed of the light, ω_{pi}: ion plasma frequency, $L_0 = v_A/\Omega_0$) is the Hall parameter. The expresion for the enthalpy density for an ideal and isothermal gas is

$$h = \frac{c_0^2}{v_A^2}\ln(n) \ , \tag{4}$$

where c_0 is the sound speed. An important feature to keep in mind, is the fact that the relative orientation of the disk angular velocity $\mathbf{\Omega_0}$ and the external magnetic field $\mathbf{B_0}$ is related to the sign of the Hall parameter ε ($\varepsilon > 0$: aligned, and $\varepsilon < 0$: anti-aligned).

We introduce the effect of differential rotation by externally applying a linear shear flow to a small parcel of disk located at a radial distance r_0. For simplicity we also assume all fields to depend only on the $\hat{z}$-direction, i.e. $\nabla = \hat{z}\partial_z$. The external magnetic field $B_0\hat{z}$ is distorted by the applied shear flow, developing perpendicular components on both the velocity and magnetic vector fields, i.e.

$$\mathbf{B} = B_0\hat{z} + \begin{pmatrix} b_x(z,t) \\ b_y(z,t) \\ b_z(z,t) \end{pmatrix} \ , \quad \mathbf{u} = -ax\hat{y} + \begin{pmatrix} u_x(z,t) \\ u_y(z,t) \\ u_z(z,t) \end{pmatrix} \ , \tag{5}$$

where the parameter a locally describes the differential rotation profile $\Omega(r) = \Omega_0(r/r_0)^{-a}$ in a co-rotating Cartesian reference frame centered at $r = r_0$, with x oriented in the radial direction and y along the azimuthal direction. The case of keplerian differential rotation corresponds to $a = 3/2$.

3. Dispersion Relation

After linearizing the Hall MHD equations in the geometric setup described in the previous section, the system splits into two subsystems. One of them corresponds to sound waves propagating along the magnetic field (z-modes), i.e.

$$\partial_t \delta n = -\partial_z u_z \tag{6}$$

$$\partial_t u_z = -\partial_z \delta n \ . \tag{7}$$

The other subsystem corresponds to perpendicular degrees of freedom described by

$$\partial_t u_\perp = \partial_z b_\perp + \begin{pmatrix} 0 & 2 \\ (a-2) & 0 \end{pmatrix} u_\perp \tag{8}$$

$$\partial_t b_\perp = \partial_z u_\perp + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} b_\perp + \varepsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_{zz}^2 b_\perp \ , \tag{9}$$

which shows that this perpendicular part of the linear dynamics remains fully uncoupled from the $(\delta n, u_z)$ part, which is entertained in the propagation of acoustic waves.

Assuming: $u_\perp, b_\perp \sim e^{i(kz-\omega t)}$ this set of equations leads to the following dispersion relation:

$$\omega^4 - 2C_2\omega^2 + C_0 = 0 \ , \tag{10}$$

where

$$C_2(k) = \frac{\varepsilon^2}{2}k^4 + \left(1 - \frac{\varepsilon a}{2}\right)k^2 + (2-a) \tag{11}$$

$$C_0(k) = k^2\left(1 + \varepsilon(2-a)\right)\left(k^2(1+2\varepsilon) - 2a\right) \ . \tag{12}$$

4. Hall magnetorotational instability

The solutions of the dispersion relation can be written as

$$\omega_\pm^2 = C_2 \ \pm \ \sqrt{C_2^2 \ - \ C_0} \tag{13}$$

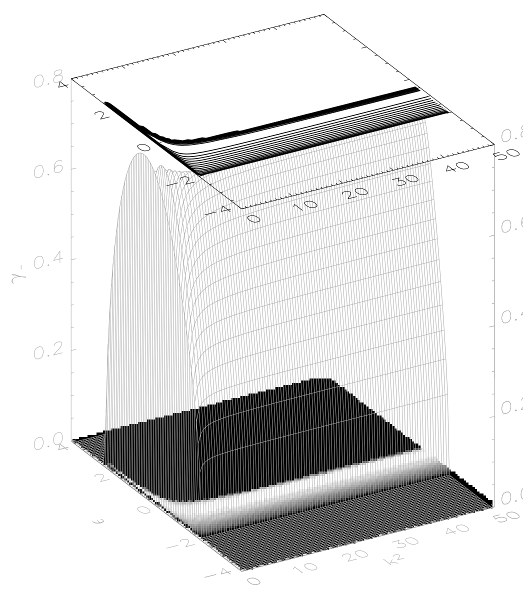

Fig. 1. Growth rate of the unstable branch as a function of the Hall parameter and wavenumber (i.e. $\gamma_-(\varepsilon, k^2)$) for a keplerian rotation profile. A contour plot is overlaid at the top and a grey scale image is at the bottom.

After exploring the solution branches ω_+^2 and ω_-^2 as a function of parameters ε and k^2 (for a given value of the differential rotation index a), we find that only ω_-^2 can lead to instability whenever $\omega_-^2 = -\gamma_-^2 < 0$ while ω_+^2 allways corresponds to oscillations. The growth rate $\gamma_-(\varepsilon, k^2)$ is shown in Fig. 1, for the keplerian case (i.e. $a = 3/2$). Contour levels of this function are also overlaid at the top.

In Fig. 2 we present the corresponding contour plots for the following three cases: sub-keplerian, keplerian, and super-keplerian. In all of them, note the asymptotic behavior of γ_- when $k^2 \to \infty$. This implies that all modes are unstable for a specific range of the Hall parameter. Also, the whole unstable region becomes appreciably larger when differential rotation goes from sub-keplerian to super-keplerian regimes. The instability region can be approximately described through a range of negative values of ε, therefore corresponding to disks where the angular velocity and magnetic field vectors are anti-aligned. For the cases where these two vectors are aligned, instability only arises in a narrow region at the very smallest wavenumbers allowed in the shearing box.

The well studied magnetorotational instability in these plots corresponds to the particular case $\varepsilon = 0$, for which the classical results are of course recovered [3]. For instance, the range of unstable wavenumbers corresponds to $0 < k^2 < 3$, and the largest growth rate in this interval is attained at $k^2 = 15/16$.

5. Conclusions

The main goal of the present study has been to address the role of the Hall effect on the magnetorotational stability of low density accretion disks. We obtained the

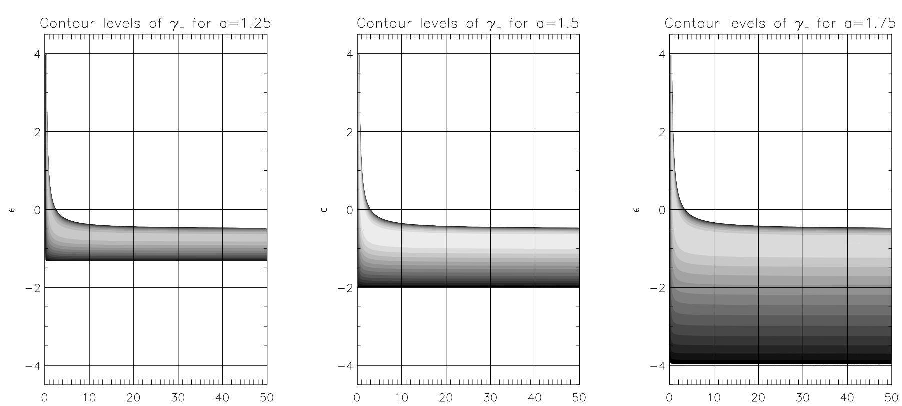

Fig. 2. Contour levels of the instability growth rate γ_- on the (ε, k^2) plane for sub-keplerian($a = 1.25$), keplerian ($a = 1.5$) and super-keplerian ($a = 1.75$) cases. Note the asymptotic behavior for $k^2 \to \infty$ in all these cases.

instability regions in the (ε, k^2) parameter space for sub-keplerian, keplerian and super-keplerian differential rotation profiles. The standard conditions for MRI are recovered in the particular case $\varepsilon = 0$. More importantly, in cases where the disk angular velocity and the external magnetic field are anti-aligned (i.e. for $\varepsilon < 0$), there is always a range of values of the Hall parameter ε for which all wavenumbers are unstable.

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