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THE EFFECT OF RANDOM COUPLING COEFFICIENTS ON DECOHERENCE

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The aim of this letter is to analyze the effect on decoherence of the randomness of the coupling coefficients involved in the interaction Hamiltonian. By studying the spin-bath model with computer simulations, we show that such randomness greatly improves the "efficiency" of decoherence and, then, its physical meaning deserves to be considered.

Keywords: Quantum decoherence; spin-bath model; random coupling; spin-bath.

1. Introduction

Environment induced decoherence (EID) is usually tested by means of computer simulations on models whose coefficients are, in general, taken as random with the idea of simulating a generic situation (see, for example, Refs. 1–6). This widely spread strategy is reasonable for the coefficients of the particles' states, but it is not natural for the coupling coefficients involved in the interaction Hamiltonian: there is no reason to assume that particles of the same nature interact with each other with different strengths. The point would be irrelevant if the randomness of the coupling coefficients did not affect the overall phenomenon. The aim of this letter is to show that this is not the case: the randomness of the coupling coefficients greatly improves the "efficiency" of decoherence. This result has to be taken as a warning against the uncritical use of random coupling coefficients for drawing conclusions about decoherence. We will argue for this claim by means of the analysis of a well-known model.

2. The Spin-Bath Model

This is a very simple model that has been exactly solved in previous paper (see Ref. 1). We will study it from the general theoretical framework for decoherence

presented in a previous work.[7] Let us consider a closed system $U = S + E$ where (i) the system S is a spin-1/2 particle P represented in the Hilbert space $\mathcal{H}_S$, and (ii) the environment E is composed of N spin-1/2 particles P_i, each one represented in its own Hilbert space $\mathcal{H}_i$. The complete Hilbert space of the composite system U is $\mathcal{H} = \mathcal{H}_S \otimes \left(\bigotimes_{i=1}^{N} \mathcal{H}_i\right)$. In the particle P, the two eigenstates of the spin operator $S_{S,\mathbf{v}}$ in direction $\mathbf{v}$ are $|\Uparrow\rangle$ and $|\Downarrow\rangle$, such that $S_{S,\mathbf{v}}\,|\Uparrow\rangle = \frac{1}{2}\,|\Uparrow\rangle$ and $S_{S,\mathbf{v}}\,|\Downarrow\rangle = -\frac{1}{2}\,|\Downarrow\rangle$. In each particle P_i, the two eigenstates of the corresponding spin operator $S_{i,\mathbf{v}}$ in direction $\mathbf{v}$ are $|\uparrow_i\rangle$ and $|\downarrow_i\rangle$, such that $S_{i,\mathbf{v}}\,|\uparrow_i\rangle = \frac{1}{2}\,|\uparrow_i\rangle$ and $S_{i,\mathbf{v}}\,|\downarrow_i\rangle = -\frac{1}{2}\,|\downarrow_i\rangle$. Therefore, a pure initial state of U reads

$$|\psi_0\rangle = (a\,|\Uparrow\rangle + b\,|\Downarrow\rangle) \otimes \left(\bigotimes_{i=1}^{N} (\alpha_i\,|\uparrow_i\rangle + \beta_i\,|\downarrow_i\rangle)\right), \tag{1}$$

where the coefficients a, b, α_i, β_i are such that $|a|^2 + |b|^2 = 1$ and $|\alpha_i|^2 + |\beta_i|^2 = 1$ are satisfied. The self-Hamiltonians H_S and H_E of S and E, respectively, are taken to be zero. Then the total Hamiltonian $H = H_S + H_E + H_{SE}$ of the composite system U results (see Refs. 1 and 2)

$$H = H_{SE} = S_{S,\mathbf{v}} \otimes \sum_{i=1}^{N} 2 g_i S_{i,\mathbf{v}} \otimes \left(\bigotimes_{j\neq i}^{N} I_j\right), \tag{2}$$

where I_j is the identity operator on the subspace $\mathcal{H}_j$, $S_{S,\mathbf{v}} = \frac{1}{2}\left(|\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow|\right)$ and $S_{i,\mathbf{v}} = \frac{1}{2}\left(|\uparrow_i\rangle\langle\uparrow_i| - |\downarrow_i\rangle\langle\downarrow_i|\right)$. Under the action of $H = H_{SE}$, the state $|\psi_0\rangle$ evolves as $|\psi(t)\rangle = a\,|\Uparrow\rangle\,|\mathcal{E}_\Uparrow(t)\rangle + b\,|\Downarrow\rangle\,|\mathcal{E}_\Downarrow(t)\rangle$ where $|\mathcal{E}_\Uparrow(t)\rangle = |\mathcal{E}_\Downarrow(-t)\rangle$ and

$$|\mathcal{E}_\Uparrow(t)\rangle = \bigotimes_{i=1}^{N} \left(\alpha_i\, e^{-i g_i t/2}\;|\uparrow_i\rangle + \beta_i\, e^{i g_i t/2}\,|\downarrow_i\rangle\right). \tag{3}$$

An observable of U, $O \in \mathcal{O} = \mathcal{H} \otimes \mathcal{H}$, reads

$$O = \begin{pmatrix} s_{\Uparrow\Uparrow}\,|\Uparrow\rangle\langle\Uparrow| \\ +s_{\Uparrow\Downarrow}\,|\Uparrow\rangle\langle\Downarrow| \\ +s_{\Downarrow\Uparrow}\,|\Downarrow\rangle\langle\Uparrow| \\ +s_{\Downarrow\Downarrow}\,|\Downarrow\rangle\langle\Downarrow| \end{pmatrix} \otimes \left(\bigotimes_{i=1}^{N} \begin{pmatrix} \epsilon^{(i)}_{\uparrow\uparrow}\,|\uparrow_i\rangle\langle\uparrow_i| \\ +\epsilon^{(i)}_{\downarrow\downarrow}\,|\downarrow_i\rangle\langle\downarrow_i| \\ +\epsilon^{(i)}_{\downarrow\uparrow}\,|\downarrow_i\rangle\langle\uparrow_i| \\ +\epsilon^{(i)}_{\uparrow\downarrow}\,|\uparrow_i\rangle\langle\downarrow_i| \end{pmatrix}\right). \tag{4}$$

In the typical situation studied by the EID approach, the system of interest S is simply the particle P. Therefore, the relevant observables O_R are obtained from Eq. (4) by making $\epsilon^{(i)}_{\uparrow\uparrow} = \epsilon^{(i)}_{\downarrow\downarrow} = 1$ and $\epsilon^{(i)}_{\uparrow\downarrow} = 0$:

$$O_R = \left(\sum_{s,s'=\Uparrow,\Downarrow} s_{ss'}|s\rangle\langle s'|\right) \otimes \left(\bigotimes_{i=1}^{N} I_i\right) = O_S \otimes I_E\,. \tag{5}$$

The expectation value of these observables in the state $|\psi(t)\rangle$ is given by

$$\langle O_R\rangle_{\psi(t)} = |a|^2 s_{\Uparrow\Uparrow} + |b|^2 s_{\Downarrow\Downarrow} + 2\,\mathrm{Re}[a b^* s_{\Downarrow\Uparrow}\, r(t)]\,, \tag{6}$$

where (see Ref. 2)

$$r(t) = \langle \mathcal{E}_{\Downarrow}(t) \, | \mathcal{E}_{\Uparrow}(t) \rangle = \prod_{i=1}^{N} \left[|\alpha_i|^2 e^{-ig_i t} + |\beta_i|^2 e^{ig_i t} \right] . \tag{7}$$

It is clear that the time-behavior of $\langle O_R \rangle_{\psi(t)}$ depends on the behavior of $r(t)$. In order to know such a behavior, we performed numerical simulations for

$$\begin{aligned} |r(t)|^2 &= \prod_{i=1}^{N} (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2 |\beta_i|^2 \cos 2g_i t) \\ &= \prod_{i=1}^{N} f_i(t) \, , \end{aligned} \tag{8}$$

where the $|\alpha_i|^2$ were obtained from a random-number generator, and the $|\beta_i|^2$ were computed as $|\beta_i|^2 = 1 - |\alpha_i|^2$. But before presenting the computer simulations, we will study the Poincaré time for this model.

3. Analyzing the Poincaré Time

Each $f_i(t)$ of Eq. (8) comes back to its initial value for the first time at a time t_{Pi}, such that $2g_i t_{Pi} = 2\pi \Rightarrow t_{Pi} = \pi/g_i$. In turn, $|r(t)|^2$ comes back to its initial value when all the $f_i(t)$ do it. Therefore, the Poincaré time t_{P} of this model is the time when all the $f_i(t)$ come back to their initial values for the first time. Let us consider three cases:

(a) All the g_i have the same value: $g_i = g$, for all i. So, the mean value is $\overline{g_i} = g$. In this case, all the $f_i(t)$ come back to their initial values at the same time $t_{Pi} = \pi/g$. Therefore, the Poincaré time is $t_{\mathrm{P}} = \pi/g = \pi/\overline{g_i}$, and it does not depend on the number of particles N.

(b) All the g_i are such that $g_i = n_i g_{\min}$, with $n_i \in \mathbb{N}$. In this case, t_{P} is the largest t_{Pi}, corresponding to the smallest g_i, $g_{\min}$: $t_{\mathrm{P}} = \pi/g_{\min}$. So, given a $g_{\min}$, the Poincaré time t_{P} does not depend on N. Since the mean value is $\overline{g_i} > g_{\min}$, then $t_{\mathrm{P}} > \pi/\overline{g_i}$.

(c) All the g_i are random. Since in the computations the g_i are rational numbers, we can express them as $g_i = p_i/q_i$, with $p_i, q_i \in \mathbb{N}$. If we make $t_{\mathrm{P}} = \pi Q$, the number Q has to be such that $Q = n_i q_i/p_i$ for all i, with $n_i \in \mathbb{N}$. Then, $n_i = Q \, p_i/q_i$. Since the p_i and q_i are random natural numbers, the least Q that guarantees that n_i is a natural number for all i is $Q = \prod_{i=1}^{N} q_i$. Therefore,

$$t_{\mathrm{P}} = \pi Q = \pi \prod_{i=1}^{N} q_i \, . \tag{9}$$

In turn, Q is larger than any q_i, larger as N increases. Then, for N large and for any $g_i = p_i/q_i$, the Poincaré time is $t_{\mathrm{P}} = \pi Q \gg \pi q_i > \pi q_i/p_i = \pi/g_i = t_{Pi}$, and also $t_{\mathrm{P}} \gg \pi/\overline{g_i}$. Moreover, the order to magnitude of t_{P} can be estimated

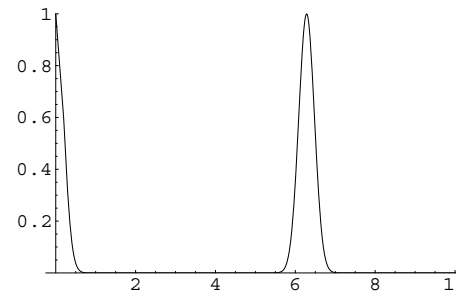

Fig. 1. Plot of $|r(t)|^2$ given by Eq. (8), for $N = 100$ and $g_i = g = 0.5$.

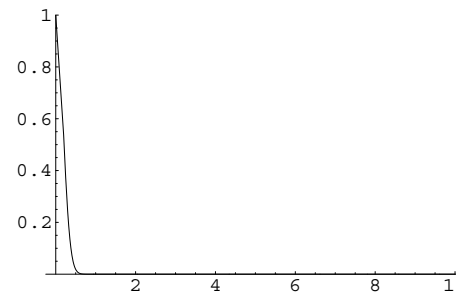

Fig. 2. Plot of $|r(t)|^2$ given by Eq. (8), for $N = 100$ and random $g_i \in [0.4, 0.6]$.

as $t_{\mathrm{P}} = \pi Q \sim \pi \overline{q_i}^N \gg (\pi/\overline{g_i})^N$: when the coupling coefficients are random, the Poincaré time increases exponentially with the number of particles.

Case 1. Homogeneous environment. Let us consider the case where all the particles of the environment are of the same kind. In this case, the reasonable assumption is that the particle P interacts in the same way with all the environmental particles and, as a consequence, $g_i = g$ for all i (Case (a)). The time-behavior of $|r(t)|^2$ of Eq. (8), for $N = 100$ and $g_i = g = 0.5$, is plotted in Fig. 1: we can see that the Poincaré time $t_{\mathrm{P}} = \pi/g = \pi/0.5 \simeq 6.28$ is not sufficiently larger than the decoherence time to consider the result as an effective decoherence that may lead to classicality.

Now let us consider the case that P does not interact in the same way with all the environmental particles. In particular, we will assume that the coupling coefficients are random in an interval $[\overline{g_i} - \Delta g, \overline{g_i} + \Delta g]$ around the mean value $\overline{g_i}$. The time-behavior of $|r(t)|^2$ of Eq. (8), for $N = 100$ and g_i random, with $\overline{g_i} = 0.5$ and $\Delta g = 0.1$, is plotted in Fig. 2. In this case, the Poincaré time is much larger than $(\pi/\overline{g_i})^N = (6.28)^{100}$ (Case (c)), a value that can be considered infinite for all practical purposes. Therefore, as Fig. 2 shows, it can be legitimately mentioned that the particle P decoheres and may become classical.

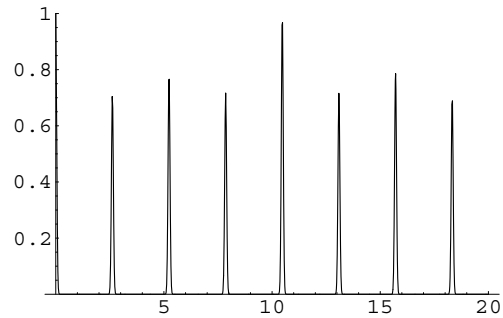

Fig. 3. Plot of $|r(t)|^2$ given by Eq. (10), for $N = 100$, $N_1 = 91$, $N_2 = N_3 = N_4 = 3$, $g_1 = 2.4$, $g_2 = 1.2$, $g_3 = 0.6$ and $g_4 = 0.3$.

Case 2. Nonhomogeneous environment. In this case we will consider an environment where particles are not all of the same kind, and the particles of each kind j interact with P through their own coupling coefficient g_j: N_1 particles through g_1, N_2 particles through $g_2, \ldots$ and N_p particles through g_p, such that $\sum_{j=1}^{p} N_j = N$. Then, the $|r(t)|^2$ of Eq. (8) can be rewritten as:

$$|r(t)|^2 = \prod_{i=1}^{N} f_i(t)$$

$$= \left(\prod_{i=1}^{N_1} f_{i,g_1}(t)\right)\left(\prod_{i=1}^{N_2} f_{i,g_2}(t)\right)\cdots\left(\prod_{i=1}^{N_p} f_{i,g_p}(t)\right), \tag{10}$$

where each particular product is the contribution of each kind of particles. In particular, we will consider a situation where the environment is composed of N particles that are almost all of the same kind, with the exception of a slight "contamination" of particles of different kinds. The time-behavior of $|r(t)|^2$ of Eq. (10), for $N = 100$, $N_1 = 91$, $N_2 = N_3 = N_4 = 3$, $g_1 = 2.4$, $g_2 = 1.2$, $g_3 = 0.6$ and $g_4 = 0.3$, is plotted in Fig. 3. Since all the g_i are such that $g_i = n_i g_{\min} = n_i\ g_4$, this situation corresponds to Case (b). Then, the Poincaré time can be easily computed as $t_{\mathrm{P}} = \pi/g_{\min} = \pi/0.3 \simeq 10.43$. Therefore, in Fig. 3, the peak in 10.43 is the Poincaré time, but the peaks around it are *not* due to the recurrence of $|r(t)|^2$. This means that the particle P in interaction with this '"contaminated" environment does not decohere.

Now we will consider that P does not interact in the same way with the particles of the same kind j, but the coupling coefficients g_{ji} are random in the intervals $[\overline{g_{ji}} - \Delta g_j, \overline{g_{ji}} + \Delta g_j]$ around the corresponding mean value $\overline{g_{ji}}$: N_1 particles with $g_{1i} \in [\overline{g_{1i}} - \Delta g_1, g_{1i} + \Delta g_1]$, N_2 particles with $g_{2i} \in [\overline{g_{2i}} - \Delta g_2, \overline{g_{2i}} + \Delta g_2], \ldots$ and N_p particles with $g_{pi} \in [\overline{g_{pi}} - \Delta g_p, \overline{g_{pi}} + \Delta g_p]$, such that $\sum_{j=1}^{p} N_j = N$. Then, $|r(t)|^2$ of Eq. (8) can be rewritten as:

$$|r(t)|^2 = \prod_{i=1}^{N} f_i(t)$$

$$= \left(\prod_{i=1}^{N_1} f_{i,g_{1i}}(t)\right)\left(\prod_{i=1}^{N_2} f_{i,g_{2i}}(t)\right)\cdots\left(\prod_{i=1}^{N_p} f_{i,g_{pi}}(t)\right). \qquad (11)$$

In particular, we will study a situation similar to the previous one with respect to the values of N and of the N_j, and where the g_j used there become the mean values $\overline{g_{ji}}$. Moreover, in all the cases the Δg_j were selected as approximately 30% of the corresponding mean value $\overline{g_{ji}}$. The time-behavior of $|r(t)|^2$ of Eq. (11), for $N = 100$, $N_1 = 91$, $N_2 = N_3 = N_4 = 3$, $\overline{g_{1i}} = 2.4$, $\overline{g_{2i}} = 1.2$, $\overline{g_{3i}} = 0.6$, $\overline{g_{4i}} = 0.3$, $\Delta g_1 = 0.8$, $\Delta g_2 = 0.4$, $\Delta g_3 = 0.2$ and $\Delta g_4 = 0.1$, is plotted in Fig. 4, where we can see the drastic decoherence of the model and the decoherence time.

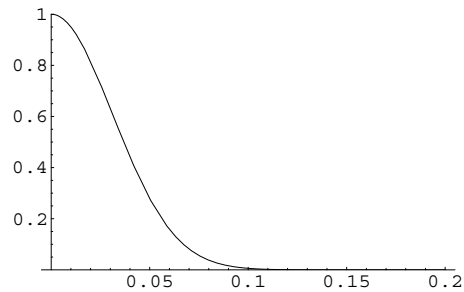

Fig. 4. Plot of $|r(t)|^2$ given by Eq. (11), for $N = 100$, $N_1 = 91$, $N_2 = N_3 = N_4 = 3$, and random $g_{1i} \in [1.6, 3.2]$, $g_{2i} \in [0.8, 1.6]$, $g_{3i} \in [0.4, 0.8]$, and $g_{4i} \in [0.2, 0.4]$.

4. Conclusions

By means of computer simulations we have proved the great influence that the randomness of the coupling coefficients exerts on decoherence. Such an influence appears under two forms: (i) the huge increasing of the Poincaré time, and (ii) the strong damping off of the peaks that preclude decoherence. Then, in both cases the change from constant coefficients to random coefficients greatly improves the "efficiency"of decoherence. It is worth stressing that, through the first effect may be expected, the second effect is not foreseeable: it is not easy to explain *a priori* why the randomness of the coupling coefficients washes off the high peaks present in the non-random case.

On the other hand, a realistic environment is usually composed by a limited number of different kinds of particles, and it is reasonable to assume that the particle of interest P interacts with all the particles of the same kind with the same strength. So, the models with random coupling coefficients are not realistic. As we have pointed out in the Introduction, this would not be relevant if the randomness of the coefficients had no significant effect. But now we know that such randomness has a dramatic influence on decoherence, whether it is expected or not. Therefore,

the criterion for the selection of the coupling coefficients has to be carefully analyzed in each model, in order to avoid conclusions drawn from unphysical results.

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