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TOWARDS A MEASURE OF HARMONIC COMPLEXITY IN WESTERN CLASSICAL MUSIC

MARCO BUONGIORNO NARDELLI

Department of Physics and Division of Composition Studies,
University of North Texas, Denton, TX 76203, USA

Santa Fe Institute, Santa Fe, NM 87501, USA
mbn@unt.edu

GARLAND CULBRETH

Department of Physics, University of North Texas,
Denton, TX 76203, USA
Garland.Culbreth@unt.edu

MIGUEL FUENTES

Instituto de Investigaciones Filosóficas-SADAF,
Buenos Aires, Argentina

Instituto de Sistemas Complejos de Valparaíso, Valparaíso, Chile,
Santa Fe Institute, Santa Fe, NM 87501, USA
fuentesm@santafe.edu

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We recently introduced the concept of dynamical score network to represent the harmonic progressions in any composition. Through a process of *chord slicing*, we obtain a representation of the score as a complex network, where every chord is a node and each progression (voice leading) links successive chords. In this paper, we use this representation to extract quantitative information about harmonic complexity from the analysis of the topology of these networks using state-of-the-art statistical mechanics techniques. Since complex networks support the communication of information by encoding the structure of allowed messages, we can quantify the information associated with locating specific addresses through the measure of the entropy of such network. In doing so, we then characterize properties of network topology, such as the degree distribution of a graph or the shortest paths between couples of nodes. Here, we report on two different evaluations of network entropy, diffusion entropy analysis (DEA) and the Kullback–Leibler divergence applied to the conditional degree matrix, and the measurements of complexity they provide, when applied to an extensive corpus of scores spanning 500 years of western classical music. Although the analysis is limited in scope, our results already provide quantitative evidence of an increase of such measures of harmonic complexity over the corpora we have analyzed.

Advances in Complex Systems

Evolution of harmonic complexity in western classical music

Marco Buongiorno Nardelli
Department of Physics and Division of Composition Studies, University of North Texas, Denton, TX 76203, USA
Santa Fe Institute, Santa Fe, NM 87501, USA
mbn@unt.edu

Garland Culbreth
Department of Physics, University of North Texas, Denton, TX 76203, USA
Garland.Culbreth@unt.edu

Miguel Fuentes
Instituto de Investigaciones Filosóficas-SADAF, Buenos Aires, Argentina
Instituto de Sistemas Complejos de Valparaíso, Valparaíso, Chile
Santa Fe Institute, Santa Fe, NM 87501, USA
fuentesm@santafe.edu

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Keywords: music complexity; music theory; music analysis; music composition; music information retrieval; music evolution; music innovation

1. Introduction

The quantification of music complexity is a subject of interest at the intersection of statistical mechanics of complex systems and music theory, musicology, and music information retrieval [1–9]. However, most of the studies published so far focus on measures of complexity evaluated using melodic or rhythmic content and address the statistical properties of harmony as linear superposition of pitch lines. In this paper we address the problem of harmonic complexity using a recently proposed representation of compositional harmonic spaces as complex networks[10, 11]. This unique representation allows us to correlate network topology with measures of complexity based on well known tools rooted in statistical mechanics approaches.

Network analysis methods exploit the use of graphs or networks as convenient tools for modeling relations in large data sets. If the elements of a data set are thought of as nodes, then the emergence of pairwise relations between them, edges, yields a network representation of the underlying set. Similarly to social networks, biological networks and other well-known real-world complex networks, entire dataset of musical structures can be treated as a networks, where each individual musical entity (pitch class set, chord, rhythmic progression, etc.) is represented by a node, and a pair of nodes is connected by a link if the respective two objects exhibit a certain level of connection or similarity. In our previous work [11] we have introduced the concept of score network, a network where every chord is a node and each progression links successive chords in the composition. This network can be viewed both as a as a static graph that represents all the existing chord changes in a composition, or as a dynamical system, a time series of a non-stationary signal, and as such, it can be partitioned for the automatic identification of tonal regions using time series analysis and change point detection. This dual representation (static and dynamical) offers novel ways to quantify the harmonic complexity of a single score or a full corpus without relying on comparisons with pre-determined reference sets.

The paper is organized as follows: in Sec 2 we discuss the network models and the complexity measures; in Sec. 3 we present the main results of this study, we discuss the implications of these results, and outline a road map for future studies.

2. Methods

2.1. *Network model*

In this paper we start from digitalized scores (in musicxml or MIDI format) and construct networks where nodes are the individual chords that can be extracted from the score, and edges are built between successive chords in the progression: nodes are connected if they appear as neighbours in the sequence. Naturally, nodes are visited numerous times, and the score evolution implies a directionality of the links. The networks are thus directed, and each edge will have a weight (strength) proportional to the times the link is visited. The procedure is extensively discussed in [11] and here we only outline the main steps. In order to extract the data from the scores, we use two principal software libraries : MUSICNTWRK (at www.musicntwrk.com) [12] and music21 (at web.mit.edu/music21) [13]. MUSICNTWRK is an open-source python library for pitch class set and rhythmic sequences classification and manipulation, the generation of networks in generalized music and sound spaces, deep learning algorithms for timbre recognition, and the sonification of arbitrary data

[10, 12]. music21, developed at MIT [13], is an object-oriented toolkit for analysing, searching, and transforming music in symbolic (score-based) forms of great versatility, whose modularity allows a seamless integration with MUSICNTWRK and other applications. Scores are read in musicxml format by the readSCORE function of MUSCNTWRK, where their harmonic content (and other relevant information, like in which bar the chord is found) is extracted using the music21 parser and converter (using the "chordify" method). With this we obtain easily the full sequence of pcsets, chord by chord, where each change to a new pitch results in a new chord. In Figure 1 we show the network of one score from the L. van Beethoven's corpus.

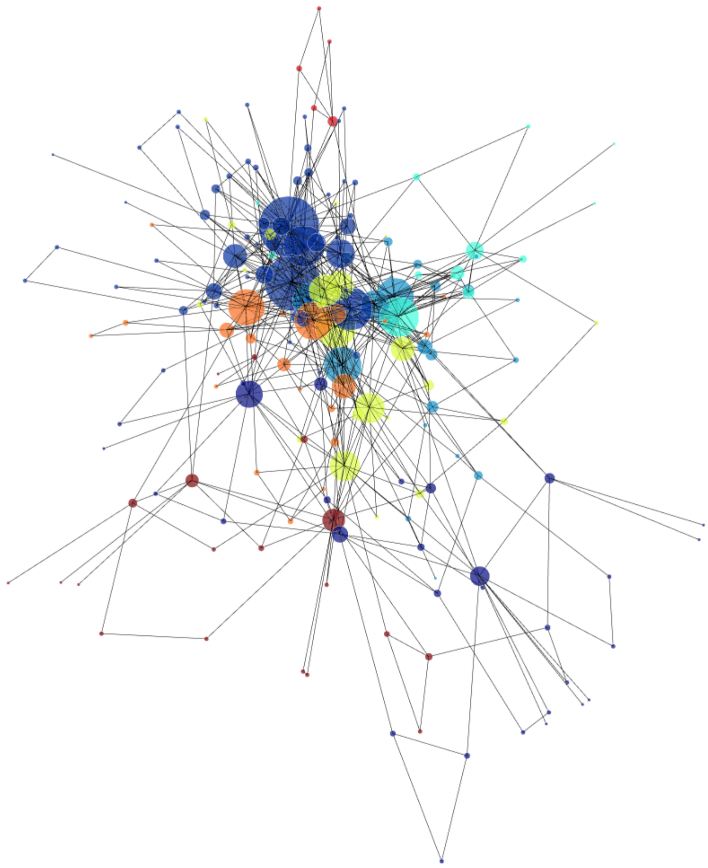

Fig. 1. Network structure of the third movement of Ludwig van Beethoven's string quartet Op. 127 n. 12. Node size is proportional to degree, colors indicate the network community structure and links correspond to voice leading (the transition from one chord to the next).

We have analyzed an extensive corpus of scores by composers spanning five centuries of western classical music: Josquin des Prez (1450-1521), G.P. da Palestrina (1525-1594), Claudio Monteverdi (1567-1643), J.S. Bach (1685-1750), J. Haydn (1732-1809), W.A. Mozart (1756-1791), L. van Beethoven (1770-1827), J. Brahms (1833-1897) and G. Mahler (1860-1911). Network data from the digital scores are available as Supplemental Information.

Once the network that characterizes each work is obtained, it is possible to construct metrics that indicate its particular characteristics. In the following we will describe two of these metrics: diffusion entropy analysis and conditional degree matrix.

2.2. *Conditional Degree Matrix*

The local metrics usually used in networks theory fall short of capturing the richness of the vast majority of natural network topologies. At the same time, one of the most commonly used (local) characteristics is a node's degree. Based on this attribute, we propose to use what has been named "conditional degree matrix" D [14] to characterize the topology of the harmonic networks. This matrix captures the classical node distribution and shows the existing architecture between the network nodes taking into account their different degrees. Each element of the matrix $D_{i,j}$ is defined as the number of nodes of degree i connected to nodes of degree j, $N_{i,j}$, divided (normalized) by the number of total nodes, N_t, that is:

$$D_{i,j} = \frac{N_{i,j}}{N_t}, \tag{1}$$

This definition produces a symmetric matrix and ensures that D is properly normalized. More generally, directed and weighted networks would result in non-symmetric matrices.

The structure of the D matrix allows us to estimate the complexity of a given network and provides more information than the classical degree distribution: D effectively acts as a probability matrix and can be the input for the evaluation of other metrics such as entropy, divergence, and complexity among others.

One of the essential properties of this matrix is that it allows us to explore the characteristics of the degree of connections of each node with its environment (its close neighborhood) in a direct way. Its importance can be understood in terms of information diffusion: the rows i of this matrix show the probability that nodes of degree i are connected with nodes of another degree j. Their frequency will finally be reflected in each of its elements $D_{i,j}$.

2.3. *Kullback-Leibler divergence*

To extract quantitative information from the network topology we use the the Kullback-Leibler (KL) divergence as metric. In both information theory and probability theory, the Kullback-Leibler divergence is used as a measure indicating the difference between two probability functions. In general terms, KL measures the expected number of extra bits or excess surprise from using Q as a model when the data distribution is P.

The Kullback-Leibler divergence for the conditional degree matrix is defined as:

$$KL = \sum_{i,j} D_{i,j} log(D_{i,j}/Q_{i,j}), , \tag{2}$$

where D is our CDM, and the reference matrix Q is defined as the mean of all the D for the whole corpus:

$$Qi, j = \frac{1}{N} \sum_{n} D^{n}_{i,j}, \tag{3}$$

and N is the total number of score networks.

Since the KL divergence quantifies how much the topology of any individual network "diverges" from the average of the reference corpus, it provides a way to quantify the difference in the distribution of observed degrees and in particular, the way in which the occurrence and distribution of hubs (as chords that are more important in the harmonic progression of a piece) characterizes the harmonic structure of the composition.

2.4. *Diffusion Entropy Analysis*

Diffusion Entropy Analysis (DEA) is a time-series analysis method for detecting temporal complexity in a dataset; such as heartbeat rhythm [15–17], a seismograph [18], or financial markets [19]. DEA uses a moving window method to convert the time-series into a diffusion trajectory, then uses the deviation of this diffusion from that of ordinary brownian motion as a measure of the temporal complexity in the data. Diffusion Entropy Analysis was first introduced by Scafetta and Grigolini [20] as a method of statistical analysis of time-series based on the Shannon entropy of the diffusion process to determine the scaling exponent of a complex dynamic system. It was later refined with the introduction of "stripes" (MDEA) by Culbreth *et al.*[21] in the context of detecting crucial events. While we refer the reader to the publications above for a full treatment of DEA, here we use the realization that the scaling of the diffusion coefficient δ obtained in DEA provides a measure of complexity of the time-series, measured through the statistics of occurrences of crucial events. Here, δ ranges between 0.5 and 1.0: for a completely non-complex process, such as a random walk, MDEA yields $\delta = 0.5$. For a process at criticality, MDEA yields $\delta = 1$. Therefore, δ represents a measure of the "strength" of the complexity present in the process: the closer δ is to 1 the closer the process is to criticality. In Fig. 4, we show a MDEA analysis of the first movement of Beethoven's string quartet Op. 127 n. 12, that was extensively discussed in [11]: $\delta \approx 0.7$ indicates a "medium" level of complexity as observed in other composition of the same time period as it will be discussed extensively in Sec. 3. MDEA analysis has been carried out using the module DEA implemented in the MUSICNTWRK library[12].

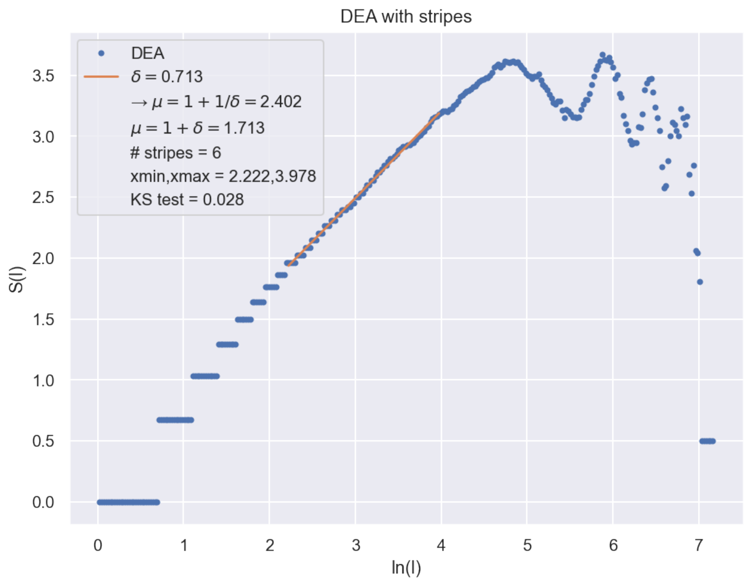

Fig. 2. Diffusion Entropy Analysis applied to the third movement of Beethoven's string quartet Op. 127 n. 12, whose network is shown in Fig. 1.

3. Results and discussion

Let's discuss first the results of the CDM analysis and what information we can extract from this metric. In general, the topology of the CDM for different composers and different time periods shows noticeable differences. From the Renaissance throughout the Common Practice period, the network topology is increasingly characterized by strongly connected hubs - a findings that points to the predominance of very clear modal/tonal regions in the harmonic evolution of a piece. Example of this can be seen in Fig. 3, where we show the CDM of three representative pieces from different time periods: in the upper panel the motet *Per Illud Ave Prolatum* by Josquin des Prez; in the middle panel the chorale from the Christmas Oratorio BWV 248 by J.S. Bach; and in the lower panel the CDM of the Scherzo from the Symphony n. 5 by G. Mahler. In these maps, hubs are seen as higher peaks in the plots (more nodes are connected to a given node of interest). While in Josquin des Prez the CDM shows a more homogeneous degree distribution, that can be associated to the characteristics of modal harmony, this topology evolves towards a clearer hub structure, as seen in J.S. Bach, with predominant chords that play a functional

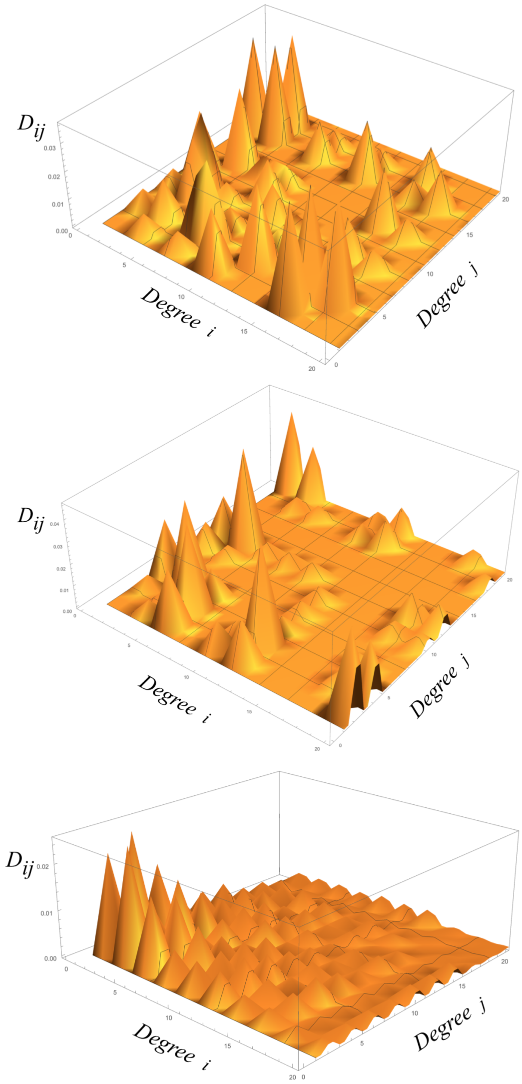

Fig. 3. D-Matrices. Top: Josquin des Prez, motet Per Illud Ave Prolatum; Middle: J.S. Bach, chorale from the Christmas Oratorio, BWV 248. Bottom: G. Mahler Symphony n.5, Part II, Scherzo.

role in the tradition of common practice harmony. Finally, in G. Mahler we see a more nuanced model of connections, corresponding to blurred tonal boundaries and increasingly chromatic harmonic progressions.

It is possible to capture this behavior using metrics such as KL divergence, introduced in previous sections. In Fig. 4 we show the KL divergence calculated

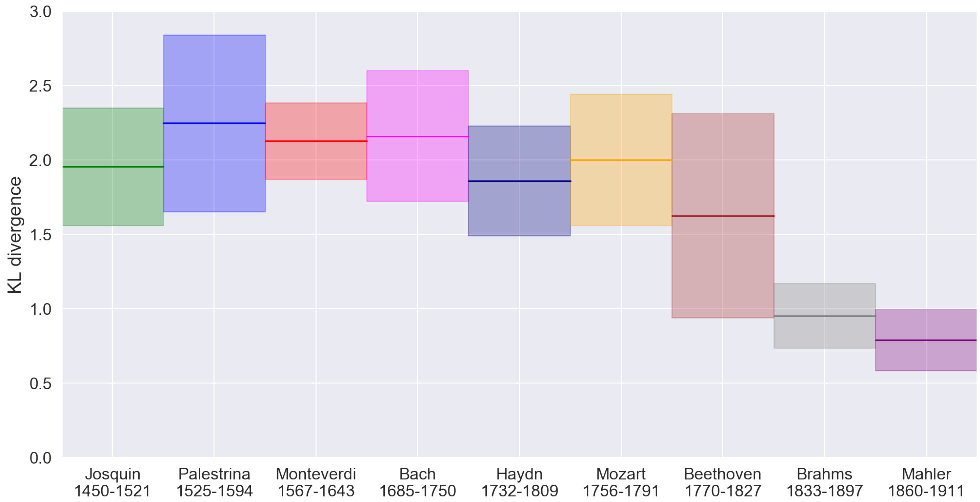

Fig. 4. Kullback-Leibler divergence. Horizontal lines are the average values of KL across the corpus of each composer; shaded areas indicate standard deviations.

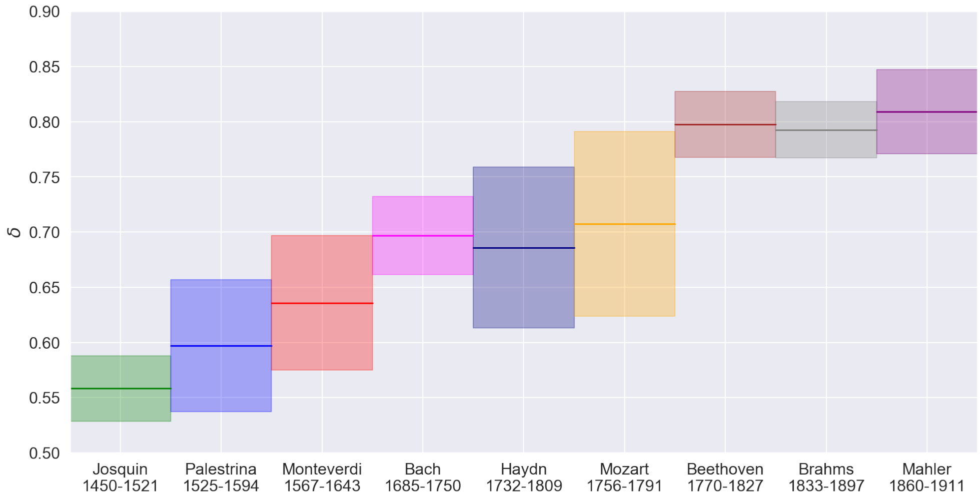

Fig. 5. Complexity for different composers as measured from the diffusion entropy analysis. Horizontal lines are the average values of δ across the corpus of each composer; shaded areas indicate standard deviations.

for our full corpus of compositions. Here the values are referenced to the "average" of the corpus, that is, we are capturing how much the topology of a given piece deviates from the cumulative average. The results confirm the observation made on the individual CDMs discussed above. There is a marked transition starting in the XIX century and culminating with the works of Brahms and Mahler. It is important to note that this metric provides a somewhat indirect measure of complexity as a relative difference between compositions.

For a more direct evaluation of complexity, we turn to the MDEA results. By applying the procedure outlined in Fig. 2 to the full corpus, we have extracted the values of δ for each piece, and collected the results in Fig. 5.

Although the data points show a wide distribution around the averages for each composer, the results point to an increase of harmonic complexity over time, a result that agrees broadly with other analysis based on different metrics. This results allow us to discriminate further among composers and different time periods. We can, in principle divide this graph in three regions. The first region corresponds to the Renaissance and Early Baroque composers, where δ is consistently low, not far from the lower limit of the random walk (0.5). Since this musical period is characterized by a modal approach to harmony, we can easily infer that modal harmony is characterized by a lower complexity, as observed in the scarcity of functional chords (functional chords are hubs in tonal harmony networks[11]), a more homogeneous distribution of node degrees, and a lack of multiple tonal centers. The second section corresponds to the Common Practice period, that shows an average complexity measure of ≈ 0.7. This is when tonal harmony has matured into an established musical language. Once again in the third section, corresponding to XIX and early XX century, harmonic complexity increases to an average of 0.8. Once more, enhanced criticality and complexity correspond to a fragmentation of the tonal harmony language towards increase chromaticism, as it was observed in the CDM data above. To further corroborate this hypothesis we evaluated the criticality index δ of I. Stravinsky's *Le Sacre du Printemp (1913)* and A. Schoenberg's *Verklärte Nacht (1899)*, two compositions that embody, in different ways, the breaking point of classical harmony, and found a δ of 0.90 and 0.86, respectively. It is important to note that these results agree with what was observed above in the behavior of the CDM discussed in Fig. 3.

To conclude we must acknowledge one limitation of the MDEA: since it is based on a time-series representation of the score, some compositions provide time-series that are too short for a meaningful statistical analysis. For instance, the corpus of the 377 chorales by J.S. Bach, a prototypical dataset for computational musicology that we use in our KL divergence analysis, is not appropriate for MDEA, since chorales provide typically less than 100 datapoints.

4. Conclusions

We have introduced two complementary measures of music complexity and analyzed score corpora across multiple centuries. Although our results point unequivocally to an increase of harmonic complexity in western classical music, our study is just an initial exploration of this fascinating topic. One of the challenges in this study was the availability of large score corpora that would make the analysis of a single composer's production more coherent. We are currently working to expand the availability of corpora and we hope to extend this work in the future. Notwithstanding its limitations, this study demonstrates that combining the abstraction of a score as network with established mathematical and statistical techniques is a powerful tool for a quantitative analysis of music production that is independent of prior musicological or music theoretic information, and opens the way to a novel interpretation of music as a dynamical process.

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