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On Blaschke products, Bloch functions and normal functions

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Abstract We prove that if G is an analytic function in the unit disc such that $G(z) \to \infty$, as $z \to 1$, and B is an infinite Blaschke product whose sequence of zeros is contained in a Stolz angle with vertex at 1 then the function $f = B \cdot G$ is not a normal function.

We prove also some results on the asymptotic cluster set of a thin Blaschke product with positive zeros which are related with the question of the existence of non-normal outer functions with restricted mean growth of the derivative.

Keywords Blaschke product · Interpolating Blaschke sequence · Thin Blaschke product · Bloch function · Normal function · Outer function · Mean Lipschitz spaces

Mathematics Subject Classification (2000) 30J45 · 30D40 · 30D45

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1 Introduction and notation

We denote by $\mathbb{D}$ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. The space of all analytic functions in $\mathbb{D}$ will be denoted by $\mathcal{H}ol(\mathbb{D})$. If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\, dt\right)^{1/p} \qquad (0 < p < \infty),$$

$$M_\infty(r, f) = \sup_{0\le t\le 2\pi} |f(re^{it})|.$$

For $0 < p \le \infty$ the Hardy space H^p consists of those functions $f \in \mathcal{H}ol(\mathbb{D})$ for which $\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$. We refer to [9] for the theory of Hardy spaces.

The space *BMOA* consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbb{D}$ (cf. [2, 10] and [14]). A function f analytic in $\mathbb{D}$ is a Bloch function if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. It is well known that

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

We mention [1] as a general reference for the theory of Bloch functions.

A function f which is meromorphic in $\mathbb{D}$ is said to be a normal function in the sense of Lehto and Virtanen [23] if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)\frac{|f'(z)|}{1 + |f(z)|^2} < \infty.$$

For simplicity, we shall let $\mathcal{N}$ denote the set of all holomorphic normal functions in $\mathbb{D}$. It is clear that any Bloch function is a normal function, that is, we have $\mathcal{B} \subset \mathcal{N}$. We refer to [1, 23] and [27] for the theory of normal functions. In particular, we remark here that if $f \in \mathcal{N}$, $\xi \in \partial\mathbb{D}$ and f has the asymptotic value L at ξ (that is, there exists a curve γ in $\mathbb{D}$ ending at ξ such that $f(z) \to L$, as $z \to \xi$ along γ) then f has the non-tangential limit L at ξ.

If a sequence of points $\{a_n\}$ in the unit disc satisfies the *Blaschke condition*: $\sum_{n=1}^\infty (1 - |a_n|) < \infty$, the corresponding Blaschke product B is defined as

$$B(z) = \prod_{n=1}^\infty \frac{|a_n|}{a_n}\frac{a_n - z}{1 - \overline{a_n} z}.$$

Such a product is analytic in $\mathbb{D}$. In fact, it is an inner function (cf. [9, Chap. 2]). If there exists $\delta > 0$ such that $\prod_{m\ne n} |\frac{a_n - a_m}{1 - \overline{a}_n a_m}| \ge \delta$, for all n, we say that the sequence $\{a_n\}$ is *uniformly separated* and that B is an *interpolating Blaschke product*. Equivalently,

$$B \text{ is an interpolating Blaschke product} \quad \Leftrightarrow \quad \inf_{n\ge 1} (1 - |a_n|^2)|B'(a_n)| > 0. \qquad (1.1)$$

We refer to [9, Chap. 9] and [10, Chap. VII] for the basic properties of interpolating Blaschke products. In particular, we recall that an exponential sequence is uniformly separated and that the converse holds if all the a_k's are positive.

2 Multiplying a normal function by a Blaschke product

It easy to see that if f is a normal function and g is an H^∞-function which is bounded away from zero then the product $h(z) = f(z)g(z)$ is also a normal function [21, Lemma 2]. However, if B is a Blaschke product and f is a normal analytic function in $\mathbb{D}$, the product $B \cdot f$ need not be normal. This was first proved by Lappan [21, Theorem 3] who used this to show that $\mathcal{N}$ is not a vector space.

Lappan's result is a consequence of the following easy fact: if B is an interpolating Blaschke product whose sequence of zeros is $\{a_n\}$ and G is an analytic function in $\mathbb{D}$ with $G(a_n) \to \infty$, then $f = B \cdot G$ is not a normal function (and hence it is not a Bloch function either). This result has been used by several authors (see [7, 12, 28, 29] and [4]) to construct distinct classes of non-normal functions.

It is natural to ask whether similar results can be obtained for certain classes of Blaschke products that are not necessarily interpolating. Using a deep result of Marshall and Sarason (see [22]), the following result was proved in [17].

Theorem A *Let B be an infinite Blaschke product whose sequence of zeros $\{a_n\}$ is contained in the radius $(0, 1)$ and let G be a Bloch function such that $G(z) \to \infty$, as $z \to 1$. Then the function $f = B \cdot G$ is not a Bloch function.*

Our first result in this paper represents an improvement of Theorem A in two senses: First, we deal with Blaschke product with zeros in a non-tangential region and, secondly, the function we construct is non-normal instead of being merely non-Bloch as in Theorem A.

Theorem 1 *Let B be an infinite Blaschke product whose sequence of zeros $\{a_n\}$ is contained in a Stolz angle with vertex at 1 and let G be analytic in $\mathbb{D}$ with $G(z) \to \infty$, as $z \to 1$. Then the function $f = B \cdot G$ is not a normal function.*

Proof Since 1 is the only limit point of the zeros of B, B can be analytically continued to a domain containing $\overline{\mathbb{D}} \setminus \{1\}$ (see Theorem 6.1 in Chap. II of [10]). Then it follows that B has non-tangential limit of absolute value 1 at all points of $\partial\mathbb{D}$ except at the point 1 and, consequently, we can find a Jordan arc γ in $\mathbb{D}$ except for its end point which is 1 such that B is bounded away from zero along γ. Since $G(z) \to \infty$, as $z \to 1$, it follows that f has the asymptotic value ∞ at 1. If f were normal then it would have the non-tangential limit ∞ at 1, but this is not true because $f(a_n) = 0$ for all n. □

3 Non-normal outer functions and mean Lipschitz spaces

Functions of the form $B \cdot F$ with B being a certain Blaschke product and F being a certain analytic function have also been used in questions about inclusions relations between a number of classical spaces of analytic functions:

If $1 \le p \le \infty$ and ϕ is a non-negative function defined in $[0, 1)$, we define $\mathcal{L}(p, \phi)$ as the space of all functions $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$M_p(r, f') = \mathrm{O}(\phi(r)), \quad \text{as } r \to 1. \tag{3.1}$$

A classical result of Privalov [9, Theorem 3.11] asserts that a function f which is analytic in $\mathbb{D}$ has a continuous extension to the closed unit disc $\overline{\mathbb{D}}$ whose boundary values are absolutely continuous on $\partial\mathbb{D}$ if and only if $f' \in H^1$. In particular, we have

$$f' \in H^1 \quad \Rightarrow \quad f \in \mathcal{A} \subset H^\infty, \tag{3.2}$$

where, $\mathcal{A}$ denotes the disc algebra. This result is sharp in a very strong sense as the following result shows.

Theorem B *Let ϕ be any positive continuous function defined in* $[0, 1)$ *with* $\phi(r) \to \infty$, *as* $r \to 1$. *Then, there exists a function* $f \in \mathcal{L}(1, \phi)$ *which is not a normal function.*

Hence, no condition on the growth of $M_1(r, f')$ other that its boundedness is enough to conclude that $f \in \mathcal{N}$. Theorem B was first proved in [12]. The proof was constructive. The constructed function f was of the form $f = BF$ where B is a Blaschke product and F is a function given by a series of analytic functions in $\mathbb{D}$ which converges uniformly on every compact subset of $\mathbb{D}$. The constructions of B and F made use in an essential way of certain sequences introduced by K.I. Oskolkov in several contexts (see, e.g., [24–26]) and were very involved. Subsequently, simpler proofs of the result have been found in [4] and [16] but all the examples of non-normal functions f with $M_1(r, f')$ tending to ∞ very slowly constructed so far are of the form $f = B \cdot F$ where F is a certain analytic function and B is a Blaschke product.

Similar questions can be considered for the spaces $\mathcal{L}(p, \phi)$, $1 < p < \infty$. These spaces are closely related with the mean Lipschitz spaces Λ^p_α, $0 < \alpha \le 1$ (cf. [9, Chap. 5], [5]) and the generalized mean Lipschitz spaces $\Lambda(p, \omega)$ (cf. [3, 4, 13]). We remark here that a classical result of Hardy and Littlewood [20] (see also Chap. 5 of [9]) asserts that, for $1 \le p < \infty$ and $0 < \alpha \le 1$, we have that

$$\Lambda^p_\alpha = \mathcal{L}(p, \phi), \quad \text{with } \phi(r) = (1 - r)^{\alpha - 1}.$$

Blasco and de Souza [3] extended this result showing that if $\omega : [0, 1] \to [0, \infty)$ is a continuous and increasing function with $\omega(0) = 0$ which satisfy the so called Dini and b_1 conditions then

$$\Lambda(p, \omega) = \mathcal{L}(p, \phi), \quad \text{with } \phi(r) = \frac{\omega(1 - r)}{1 - r}.$$

Cima and Petersen proved in [8] that $\Lambda^2_{1/2} \subset BMOA$. This result was extended by Bourdon, Shapiro and Sledd [5] showing that

$$\Lambda^p_{1/p} \subset BMOA, \quad 1 < p < \infty. \tag{3.3}$$

This has been shown to sharp in [4, 13] and [16]. Indeed, we have.

Theorem C *Suppose that* $1 \le p < \infty$ *and let* $\omega : [0, 1] \to [0, \infty)$ *be a continuous and increasing function with* $\omega(0) = 0$ *and*

$$\frac{\omega(\delta)}{\delta^{1/p}} \to \infty, \quad \text{as } \delta \to 0$$

and set $\phi(r) = \omega(1-r)/(1-r)$ $(0 < r < 1)$, *then there exists* $f \in \mathcal{L}(p, \phi)$ *which is not a normal function.*

If we assume in addition that ω *is a Dini weight and satisfies the condition* b_1 *then we can assert that there exists* $f \in \Lambda(p, \omega)$ *which is not a normal function.*

Just as in the case of Theorem B, the distinct functions f constructed so far to prove Theorem C are of the form $f = B \cdot F$ where B is a Blaschke product and F a certain analytic function in $\mathbb{D}$. Then it is natural to ask whether or not all the possible examples must be of this kind. More precisely, the following questions were raised in [15].

Question 1 *If* ϕ *is a positive continuous function defined in* $[0, 1)$ *with* $\phi(r) \to \infty$, *as* $r \to 1$, *does there exist an* outer function $f \in \mathcal{L}(1, \phi)$ *which is not a normal function?*

Question 2 *Suppose that* $1 \le p < \infty$ *and let* $\omega : [0, 1] \to [0, \infty)$ *be a continuous and increasing function with* $\omega(0) = 0$ *and* $\omega(\delta)\delta^{-1/p} \to \infty$, *as* $\delta \to 0$. *Set* $\phi(r) = \omega(1-r)/(1-r)$ $(0 < r < 1)$. *Does there exist an* outer function $f \in \mathcal{L}(p, \phi)$ *which is not a normal function?*

We refer to [9] for the definitions and basic properties of inner functions, outer functions and other related terms. We remark that Brown and Hansen [6] and Gehring [11] constructed examples of non-normal outer functions f in H^p for all $p < \infty$, but no information about the mean growth of the derivative of these functions was given.

Questions 1 and 2 remain open. Theorem 2.6 of [15] simply asserts that there exist non-Bloch outer functions in the desired spaces $\mathcal{L}(p, \phi)$:

Theorem D (i) *Let* ϕ *be a positive continuous function defined in* $[0, 1)$ *with* $\phi(r) \to \infty$, *as* $r \to 1$. *Then, there exists an outer function* $f \in \mathcal{L}(1, \phi)$ *which is not a Bloch function.*

(ii) *Suppose that* $1 \le p < \infty$ *and let* $\omega : [0, 1] \to [0, \infty)$ *be a continuous and increasing function with* $\omega(0) = 0$ *and* $\omega(\delta)\delta^{-1/p} \to \infty$, *as* $\delta \to 0$. *Set* $\phi(r) = \omega(1-r)/(1-r)$ $(0 < r < 1)$. *Then there exists an outer function* $f \in \mathcal{L}(p, \phi)$ *which is not a Bloch function*

The functions f constructed to prove Theorem D were of the form $f = (B+1)\cdot G$ where:

- $G(z) = F(z)/z$ with F a conformal mapping from $\mathbb{D}$ onto a certain simply connected domain Ω with $F(0) = 0$ and $F(z)$ tending to ∞, as $z \to 1$, slowly enough, and
- B is an infinite Blaschke product whose sequence of zeros $\{a_n\}$ is contained in the radius $(0, 1)$ and such that the counting function $n(r, B)$ grows slowly enough.

Here, as usual, for $0 < r < 1$, $n(r, B)$ denotes the number of zeros of B whose absolute value is smaller than r.

Remark 1 If the Blaschke product B could be taken satisfying the additional two conditions:

(A) there exists a curve γ in $\mathbb{D}$ ending at 1 such that $B + 1$ is bounded away from zero on γ,
(B) there exists a sequence $\{b_k\} \in (0, 1)$ with $\lim_{k\to\infty} b_k = 1$ and with the property that $B(b_k) + 1 \to 0$, as $k \to \infty$,

then combining the arguments in the proof of [15, Theorem 2.6] and those used in the proof of Theorem 1, we could deduce that the function $f = (B + 1)\cdot G$ would be non-normal and then the answers to our questions would be positive. We omit the details.

These considerations lead us to consider thin Blaschke products with positive zeros.

A uniformly separated sequence $\{a_k\}$ is said to be thin if

$$\lim_{k\to\infty} \prod_{j=1 j\neq k}^{\infty} \left| \frac{a_k - a_j}{1 - \overline{a}_j a_k} \right| = 1. \tag{3.4}$$

The Blaschke product B is called thin if its sequence of zeros $\{a_k\}$ is thin, this is equivalent to saying that

$$\lim_{k\to\infty} (1 - |a_k|^2)|B'(a_k)| = 1. \tag{3.5}$$

Proposition 1.1 of [18] asserts that if the sequence $\{a_k\}$ is contained in the radius $(0, 1)$, then $\{a_k\}$ is thin if and only if $\frac{1-a_{k+1}}{1-a_k} \to 0$, as $k \to \infty$. Thus:

(C) A Blaschke products B with positive zeros is thin if the counting function $n(r, B)$ tends to ∞ slowly enough.

Recall that the radial cluster set of a function $f \in \mathcal{H}ol(\mathbb{D})$ at the point $e^{i\theta}$ is the set of all values, w, for which there exists a sequence, $\{r_k\}$ of points in $(0, 1)$ satisfying $f(r_k e^{i\theta}) \to w$, as $k \to \infty$. Gorkin and Mortini proved the following result in [19].

Theorem E *Let B be a thin Blaschke product with positive zeros. Then the radial cluster set of B at the point* 1 *is the interval* $[-1, 1]$.

In particular, we have:

(D) If B is a thin Blaschke product with positive zeros then (B) holds.

Question 3 *Let B be an thin Blaschke product with positive zeros. Does there exist a curve Γ in $\mathbb{D}$ ending at 1 and a positive constant β such that $|B(z)+1|>\beta$, for all $z\in\Gamma$?*

Bearing in mind (C), (D) and Remark 1 it is easy to see that if the answer to Question 3 were positive then the answers to Questions 1 and 2 would be positive too. Our next result will be contained in Theorem 2 below. In particular, it implies that the answer to Question 3 is negative. Hence, questions 1 and 2 remain open.

In order to state Theorem 2 we need to introduce some notation: If S is a subset of $\mathbb{D}$ with ξ in the closure of S and f is analytic in $\mathbb{D}$ we let $C(f,S,\xi)$ be the cluster set of f at ξ along S, that is

$$w\in C(f,S,\xi) \quad\Leftrightarrow\quad \text{there exists } \{z_n\}\subset S \quad \text{with } z_n\to\xi \text{ and } f(z_n)\to w.$$

Theorem 2 *Let B be a thin Blaschke product with positive zeros. If Γ is a curve in $\mathbb{D}$ except for its end point which is 1 then $-1\in C(B,\Gamma\setminus\{1\},1)$.*

Theorem 2 will follow from the following result which may be of independent interest.

Theorem 3 *Suppose that $f\in H^\infty$, $w\in C(f,[0,1),1)\cap\partial f(\mathbb{D})$ and Ω is a simply connected domain contained in $\mathbb{D}$ satisfying the following two conditions*:

(i) $\partial\Omega\subset\mathbb{D}\cup\{1\}$.
(ii) $[x_0,1)\subset\Omega$ *for some* $x_0\in(0,1)$.

Then $w\in C(f,\partial\Omega\setminus\{1\},1)$.

Before embarking into the proof of Theorem 3 let us say that for $z\in\mathbb{C}$ and $r>0$, $D(z,r)$ will stand for the disc of radius r centered at z. Also, if S is a subset of $\mathbb{C}$, the closure of S will be denoted by $\mathrm{Cl}(S)$.

Proof of Theorem 3 Let h be a conformal mapping from $\mathbb{D}$ onto Ω and set $g=f\circ h$. The function g lies in H^∞, hence it has a finite non-tangential limit $g(e^{it})$ for almost every $t\in\mathbb{R}$. The function $g(z)-w$ is also bounded, hence it can be factored in the form

$$g(z)-w=F(z)I(z)$$

with F outer in H^∞ and I inner.

Suppose that $w\notin C(f,\partial\Omega\setminus\{1\},1)$. Then there exists $\epsilon>0$ such that

$$|g(e^{it})-w|\geq\epsilon,\quad \text{for almost every}\ \ t.$$

Then it follows that $1/F\in H^\infty$. This and the fact that $w\in C(f,[0,1),1)$ implies that the inner factor I is not constant and then $\mathbb{D}\subset\mathrm{Cl}(I(\mathbb{D}))$. This implies that there exists $\eta > 0$ such that $D(0, \eta) \subset \mathrm{Cl}((FI)(\mathbb{D})) = \mathrm{Cl}((g - w)(\mathbb{D}))$ which is equivalent to saying that $D(w, \eta) \subset \mathrm{Cl}(g(\mathbb{D}))$. Since $\mathrm{Cl}(g(\mathbb{D})) \subset \mathrm{Cl}(f(\Omega))$, this yields to $w \notin \partial f(\Omega)$ which is a contradiction. □

Proof of Theorem 2 Suppose that B is a thin Blaschke product with positive zeros and that Γ is a curve in $\mathbb{D}$ except for its end point which is 1 such that $|B(z) + 1| > \beta > 0$, for all $z \in \Gamma \setminus \{1\}$. Since $B(\overline{z}) = \overline{B(z)}$, we have that

$$|B(z) + 1| > \beta > 0, \quad z \in \left(\Gamma \cup \overline{\Gamma}\right) \setminus \{1\}.$$

(Here $\overline{\Gamma} = \{\overline{z} : z \in \Gamma\}$.) Consequently, we may assume without loss of generality that $\Gamma \subset \{\operatorname{Im} z \geq 0\}$. Then it follows easily that there exists a simple curve Γ_1 in $\mathbb{D} \cap \{\operatorname{Im} z > 0\}$ except for its end-point 1, such that

$$|B(z) + 1| > \beta/2, \quad z \in \Gamma_1 \setminus \{1\}. \tag{3.6}$$

Now we set $J = \Gamma_1 \cup \overline{\Gamma_1} \cup S$, where S is the vertical segment joining the origins of Γ_1 and $\overline{\Gamma_1}$. Then J is a Jordan curve. Let Ω be the domain inside J. Theorem E shows that -1 is in the radial cluster set of B at 1 and then it follows easily that Ω is in the conditions of Theorem 3 with $w = -1$. Then it follows that $-1 \in C(B, \Gamma_1 \setminus \{1\}, 1)$. This is in contradiction with (3.6). □

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