Wide Sense Stationary Processes Forming Frames

Juan Miguel Medina, *Member, IEEE*, and Bruno Cernuschi-Frías, *Senior Member, IEEE*

***Abstract*—In this paper, we study the question of the representation of random variables by means of frames or Riesz basis generated by stationary sequences. This concerns the representation of continuous time wide sense stationary random processes by means of discrete samples.**

***Index Terms*—Basis, frames, sampling, stationary random processes.**

I. Introduction

LET $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{A}}^{r=1\ldots n}$ be a wide sense stationary (w.s.s.) n-dimensional random process, indexed in a "time" set $\mathbb{A}$. Many statistical problems such as linear prediction, interpolation or extrapolation, or computing conditional expectations for gaussian processes [20], [7], [11], [19] are reduced to the problem of obtaining the best approximation of a random Variable $Y \in \overline{\text{span}}\mathcal{X} = H(\mathcal{X})$ in terms of an element of a closed subspace of $H(\mathcal{X})$. Some authors e.g., [17] made clear the relationship between some of these subjects and the theory of shift-invariant subspaces. A similar problem is that of reconstructing a continuous time random process/signal from discrete time samples. Analogous results to the classic Shannon sampling theorem, or its generalizations, can be given for stationary or related processes [14], [13], [20], and it may be interesting to give conditions for stable sampling and reconstruction of random signals. This could be useful in some applications such as sparse representations and compressive sampling. So it is natural to ask under what conditions Y can, in some reasonable sense, be represented in the form of an unconditionally convergent series

$$Y \sim \sum_{r=1}^{n}\sum_{t} a_r(t)X_t^r.$$

In the context of stationary processes, it is natural to formulate conditions on the spectral density, or spectral measure of the process. To treat some of these approximation problems Rozanov [20] introduced the concept of *Conditional basis*. Giving sufficient conditions [20], [21] on the spectral density, this can be strengthened to an *unconditional basis* or Riesz Basis, that makes the series unconditionally convergent. In these derivations much care is given to *minimality*, i.e., no element X_t^r of this system belongs to the closed linear span of the remaining elements. Minimal 1-dimensional processes were first introduced by Kolmogorov, and their structure can be characterized in terms of the spectral measure of the process [20], [22], [23]. Many interpolation or extrapolation problems are easier to handle when the processes involved are minimal. In the context of sampling, sufficient conditions on the spectral density are also used in e.g., [26] to construct a Wavelet Karhunen–Loéve like expansion for a w.s.s. process. The random variables obtained in this case are uncorrelated, i.e., they are orthogonal. Wavelet type expansions of random signals, and some of their properties, are also studied in [5] and [9]. However, in some applications mainly related to signal analysis, redundancy could be useful. As for example a natural way to achieve this is by means of frames. The main results are given in Section III. First, in Theorem 3.1 we give necessary and sufficient conditions in terms of the spectral measure for a stationary sequence to form a frame. This result on stability generalizes the results of Rozanov [20, Ch. 2, Secs. 7 and 11]. Second, the problem of reconstructing a random signal from its samples can be viewed as a problem of completeness of the sequence of samples in the closed linear span of the whole random process [14], [13]. So, we study conditions for the stability of these sequences of samples. We relate the previous result on frames to the case of a sequence obtained from sampling a continuous parameter process. The conditions are obtained in terms of the periodized spectral density of the process (e.g., Theorem 3.2). Finally, in Theorem 3.4 we give conditions for a sequence of samples to be a frame of the closed linear span of the whole process.

As we will see the study of conditions for stationary sequences to form frames is similar, in some way, to the problem of the characterization of frames for shift invariant subspaces (SIS) of $L^2(\mathbb{R}^d)$ ([4], [3], [2], [6]). Theorem 3.1 is an analogue to the results of Section III of [2] or Section II of [3]. In [3], the SIS are generated by more than one generator, then the conditions are given in terms of the Gramian matrix or the dual Gramian matrix. In our context, the spectral density matrix of a stationary process will play a similar role. Finally, we note that the study of conditions for frames and Riesz basis is very useful for the theory of sampling of signals, since these conditions are related to stability. For example in [16], several conditions for stable sampling in an union of shift invariant subspaces, are studied using Riesz basis. The theory of sampling in an union of subspaces gives an appropriate framework for problems related to sparse representations [16], spectrum-blind sampling of multiband signals, and compressive sampling, among other applications. In Section V, as an application, we shall see briefly that it is possible to propose a similar theory for stationary random signals.

Manuscript received April 29, 2010; revised December 15, 2010; accepted January 11, 2011. Date of current version June 22, 2011. This work was supported in part by CONICET, the University of Buenos Aires Grant 1037, and ANPCYT Grant 209-2006. Date of current version June 22, 2011.

J. M. Medina is with the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Instituo Argentino de Matemática "A.P. Calderòn," Departamento de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires, Paseo Colón 850, Buenos Aires, Argentina. (e-mail: jmedina@fi.uba.ar).

B. Cernuschi-Frías is with the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Instituo Argentino de Matemática "A.P. Calderòn," and also with the Department of Electronics, Universidad de Buenos Aires, Paseo Colón 850, Buenos Aires, Argentina (e-mail: bcf@ieee.org).

Communicated by Helmut Bölcskei, Associate Editor for Detection and Estimation.

Digital Object Identifier 10.1109/TIT.2011.2146530

II. Auxiliary Results

A. Some Generalities

Let us review some facts about wide sense stationary processes. This brief review is mainly borrowed from ([20], Chapter I). Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, let $\mathcal{X} = \left\{X_t^j\right\}_{t\in\mathbb{A}}^{j=1\ldots n}$ be a $n-dimensional$ wide sense stationary random process, where $\mathbb{A} = \mathbb{R}$ or $\mathbb{Z}$. By this we mean a family of random variables in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ (i.e., with finite variance) stationary correlated in the index t, i.e.,: $R_{ij}(t,s) = \mathbb{E}\left(X_t^i X_s^j\right) = R_{ij}(t-s)$, for all t, s. We will also assume that $\mathbb{E}X_t^k = 0$ for all t, k. In the following, $\mathbb{G} = \mathbb{R}$ if $\mathbb{A} = \mathbb{R}$ or $\mathbb{G} = [0,1)$ if $\mathbb{A} = \mathbb{Z}$.

1) Harmonic Analysis of Stationary Processes: Every stationary multidimensional process $\mathcal{X} = \left\{X_t^j\right\}_{t\in\mathbb{A}}^{j=1\ldots n}$ admits a spectral representation:

$$X_t^j = \int_{\mathbb{G}} e^{i2\pi\lambda t} d\Phi_j$$

in the form of an stochastic integral [12] with respect to a random spectral measure $\Phi = (\Phi_1, \ldots, \Phi_n)$. Moreover for each $t \in \mathbb{R}$ or $\mathbb{Z}$, X_t^r can be written as the result of the action of the (unitary) time shift operator T on $X_0^r : X_t^r = T^t X_0^r$

$$\text{where by Stone's spectral theorem :} T^t = \int_{\mathbb{G}} e^{i\lambda 2\pi t} dE(\lambda) \tag{1}$$

where the $E(\lambda)$'s are orthogonal projection operators over $H(\mathcal{X})$.

Set $\mu_{ij}(A) = \mathbb{E}(\Phi_i(A)\overline{\Phi_j(A)})$, $i, j = 1\ldots, n$. This matrix of complex measures is positive definite and we call it the *spectral measure* of the process $\mathcal{X}$. On the other hand the (cross) spectral measures μ_{rj} are also related by the following Fourier transform pairing

$$\mathbb{E}\left(X_t^r \overline{X_0^j}\right) = \int_{\mathbb{G}} e^{i\lambda 2\pi t} d\mu_{rj}. \tag{2}$$

We study some properties of the Hilbert space $H(\mathcal{X})$ which is the closed linear span of $\mathcal{X}$ in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Some properties are more easily characterized over an isometrically isomorphic space, defined as follows.

Definition 1: $\mathbf{L}^2(\mathbb{G})$ is defined as

$$\left\{ f : \mathbb{G} \to \mathbb{C}^n, f \text{ is measurable and } \int_{\mathbb{G}} f \frac{d\mathbf{M}}{d\nu} f^* d\nu < \infty \right\}$$

where $\mathbf{M}$ is a $n \times n$ positive semi definite matrix of complex measures defined by $(\mathbf{M})_{ij}(A) = \mu_{ij}(A)$ for each $A \in \mathcal{B}(\mathbb{G})$, $\nu(A) = tr(\mathbf{M})(A)$ and $\frac{d\mathbf{M}}{d\nu}$ is the matrix of Radon–Nykodym derivatives $\frac{d\mu_{ij}}{d\nu}$.

First note that always $\mu_{ij} << \nu$, so this assures the existence of the $\frac{d\mu_{ij}}{d\nu}$. In our case $\mathbf{M}$ will be the spectral measure of the process. Unless $\frac{d\mathbf{M}}{d\nu}(\lambda)$ is of full rank for almost all $\lambda[\nu]$, there may exist measurable f such that $f \neq 0$ over a set of positive ν measure with $\int_{\mathbb{G}} f \frac{d\mathbf{M}}{d\nu} f^* d\nu = 0$. Moreover if we do not distinguish between two vector functions f, g such that $\int_{\mathbb{G}} (f-g) \frac{d\mathbf{M}}{d\nu} (f-g)^* d\nu = 0$, then we can treat $\mathbf{L}^2(\mathbb{G})$ as a Hilbert space. More precisely $\mathbf{L}^2(\mathbb{G})/\{f \in \mathbf{L}^2 : f^* \in Nul(\frac{d\mathbf{M}}{d\nu}) a.e.[\nu]\}$ is a Hilbert space with the norm $\|f\|^2_{\mathbf{L}^2} = \int_{\mathbb{G}} f \frac{d\mathbf{M}}{d\nu} f^* d\nu$. The isomorphism between $\mathbf{L}^2(\mathbb{G})$ and $H(\mathcal{X})$ is given by an integral respect to the random measure Φ. That is, for every $Y \in H(\mathcal{X})$ there exists $f \in \mathbf{L}^2(\mathbb{G})$ such that $Y = \sum_{j=1}^n \int_{\mathbb{G}} f_j d\Phi_j$. In the case that all the elements $\mu_{ij} << \mathcal{L}$, where $\mathcal{L}$ is Lebesgue measure, then we call the Radon–Nykodym derivatives ϕ_{ij} with respect to $\mathcal{L}$ spectral densities, and we say that $\mathcal{X}$ has a spectral density (matrix) D of elements ϕ_{ij}. Then, the integrals involving $\mathbf{M}$ introduced before can be written as $\int_{\mathbb{G}} f D f^* d\lambda$ and so on. The same discussion made for ν and $\mathbf{M}$ also holds for this case.

Finally, it is clear that a corresponding concept of stationarity can be considered for any process $\{X_g\}_{g\in G}$, with $\mathbb{E}X_g = 0$ and $\mathbb{E}|X_g|^2 < \infty$ and the index set G a group. The results, for w.s.s. processes, involving Fourier transforms are also extended to the case when G is a LCA group [19, Ch. 1].

2) Frames and Hilbert Spaces: Let us review some of the basic results about frames and Hilbert spaces which will be used here.

Theorem 2.1 ([6], Chapter 5): If $\{g_k\}_k$ is a Riesz basis of its span then it is a frame.

We recall the following definition.

Definition 2: Let $\{g_k\}_k$ be a sequence in a Hilbert space $\mathcal{H}$, we say that $\{g_k\}_k$ is minimal if for each $j : g_j \notin \overline{\text{span}}\{g_k\}_{k\neq j}$. There is an interesting relationship between, minimal sequences and frames.

Theorem 2.2 ([6], Chapter 5): Let $\{g_k\}_k$ be a frame in a Hilbert space $\mathcal{H}$, then the following are equivalent.

i) $\{g_k\}_k$ is a Riesz basis of $\mathcal{H}$.
ii) If $\sum_k c_k g_k = 0$ for $(c_k)_k \in l^2$ then $c_k = 0$ for all k.
iii) $\{g_k\}_k$ is minimal.

There exists a useful spectral characterization of minimal sequences for stationary sequences. This result also admits an extension to processes indexed over LCA groups [23]:

Theorem 2.3 ([20, Ch. 2, Sec 11] and [23]): Let $\mathcal{X} = \{X_k^j\}_{k\in\mathbb{Z}}^{j=1\ldots n}$ be a w.s.s. process with spectral density D, if

$$\exists\, D^{-1} \text{ a.e. } [\mathcal{L}] \text{ and } \int_{[0,1)} tr(D^{-1}) d\lambda < \infty \tag{3}$$

$\Longrightarrow \mathcal{X}$ is minimal.

Remark: In [20], it is claimed, that condition 3 is necessary and sufficient, however the proof of the necessity part contains an error, see [23]. Theorem 2.3 is an immediate consequence of [23, Corollary 4.9].

Given a frame $\{g_k\}_k$ in $\mathcal{H}$, we can define the associated frame operator S defined for every $f \in \mathcal{H}$ by: $Sf = \sum_k \langle f, g_k\rangle g_k$, which is a bounded invertible operator. Frames provide stable representations by means of series expansions. However to do this it is necessary to calculate the dual frame explicitly. Given a frame in a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ often it is more convenient and more efficient to employ an iterative reconstruction method.

Algorithm: ([8], Chapter 5) Given a relaxation parameter $0 < \lambda < \frac{2}{B}$, set $\delta = \max\{|1-\lambda A|, |1-\lambda B|\}$. Let $f_0 = 0$ and define recursively: $f_{n+1} = f_n + \lambda S(f - f_n)$. Then $\lim_{n\to\infty} f_n = f$, with a geometric rate of convergence, that is $\|f - f_n\| \leq \delta^n \|f\|$.

3) Some Facts About Periodic Functions and Sampling: We will see that some properties of uniform sampling can be derived from the properties of periodic functions and measures. For this purpose it is useful to consider the quotient space $\mathbb{R}/\mathbb{Z}$. We will denote the *canonical* projection $\Pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$, the map which assigns to every $x \in \mathbb{R}$ its equivalence class $\Pi(x)$. In our derivations it is useful to make the following convention: to identify $\Pi(x)$ with its *unique* representative in the interval $[0, 1)$. That is to consider Π as the following map: $\Pi : \mathbb{R} \longrightarrow [0, 1)$, $\Pi(x) = \sum_{k\in\mathbb{Z}} \mathbf{1}_{[0,1)+k}(x)(x - k)$.

Let $\mathbb{U} \subset \mathbb{R}$ be a Borel measurable set, then, the class of Borel subsets of $\mathbb{U}$ will be denoted by $\mathcal{B}(\mathbb{U})$. If μ is a complex measure, we recall that the induced measure Π is the measure defined for every Borel set $U \subset [0, 1)$ by the formula $\mu_\Pi(U) = \mu(\Pi^{-1}(U))$. Let f be a Borel measurable 1-periodic function, i.e., $f(x) = f(x + 1)$ for every $x \in \mathbb{R}$ (we will not distinguish between two functions which are equal at almost every x μ-a.e.). Then, if we denote $f|_{[0,1)}$ the restriction of f to the interval $[0, 1)$, i.e., $(f|_{[0,1)} \circ \Pi)(x) = (f \circ \Pi)(x) = f(\Pi(x)) = f(x)$ for every $x \in \mathbb{R}$.

Given a continuous time process $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}$, with random measure Φ over $\mathbb{R}$, if we consider the sequence of samples $\{X_k^r\}_{k\in\mathbb{Z}}$ as a discrete stationary sequence, then we have two possible spectral representations for every $k \in \mathbb{Z}$:

$$X_k^j = \int_{\mathbb{R}} e^{i2\pi\lambda k} d\Phi_j = \int_{[0,1)} e^{i2\pi\lambda k} d\Phi'_j \text{ a.s.}$$

where Φ'_j is a random measure over $[0, 1)$. From this, by a density argument this is also true for 1-periodic functions, in particular if $A \in \mathcal{B}[0, 1)$:

$$\int_{\mathbb{R}} \mathbf{1}_{\Pi^{-1}(A)} d\Phi_r \int_{\mathbb{R}} \mathbf{1}_{\Pi^{-1}(A)} d\overline{\Phi_j} = \int_{[0,1)} \mathbf{1}_A d\Phi'_r \int_{[0,1)} \mathbf{1}_A d\overline{\Phi'_j} \text{ a.s.}.$$

Then, taking expected values in both sides of the equality, if we denote μ'_{rj} the cross spectral measure of the discrete sequence of samples, we have $\mu'_{rj}(A) = \mu_{rj}(\Pi^{-1}(A))$, where μ_{rj} is the cross spectral measure of the original process. On the other hand, as we have seen before, given a continuous time process $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}$, $H(\mathcal{X})$ is isomorphic to $\mathbf{L}^2(\mathbb{R})$, and this induces an isomorphism between the closed subspace $\overline{\text{span}}\,\{X_k^r\}_{k\in\mathbb{Z}}$ and the closed subspace of $\mathbf{L}^2(\mathbb{R})$: $\{f \in \mathbf{L}^2(\mathbb{R}) : f\ is\ 1-periodic\}$. Additionally, taking in account the previous discussion about periodic functions, both subspaces are isometrically isomorphic to the Hilbert space

$$\mathbf{L}^2[0,1) = \left\{ f : [0,1) \to \mathbb{C}^n, f : \int_{[0,1)} f \frac{d\mathbf{M}_\Pi}{d\nu_\Pi} f^* d\nu_\Pi < \infty \right\}.$$

This suggests that some properties may be characterized in terms of induced measures. We begin with the following result.

Proposition 2.1: Let μ be a complex measure (Borel) over $\mathbb{R}$ If $(\mu_\Pi)_s$ and $(\mu_\Pi)_{ac}$ denote the singular and absolutely continuous parts of μ_Π respectively (with respect to Lebesgue measure). Then:

a) For every Borel set U in $[0, 1)$: $(\mu_\Pi)_s(U) = \mu_s(\Pi^{-1}(U))$ and $(\mu_\Pi)_{ac}(U) = \mu_{ac}(\Pi^{-1}(U))$. The measures μ_s and μ_{ac} denote the singular and absolutely continuous parts of μ respectively. That is, the singular part of the induced measure by Π is the induced measure by Π through the singular part of μ, the same with the absolutely continuous part.

b) If $w \in L^1(\mathbb{R})$ is the Radon–Nykodym (R–N) derivative of μ_{ac} respect to Lebesgue measure, then $\sum_{k\in\mathbb{Z}} w(. + k)$ is the R-N derivative of $(\mu_\Pi)_{ac}$.

Notation: Given a complex measure μ, the total variation of μ is $|\mu|$.

Proof: a) It is sufficient to find $Z, Z' \in \mathcal{B}([0, 1))$ such that $Z \cup Z' = [0, 1)$, $Z \cap Z' = \emptyset$ and $|(\mu_s)_\Pi|(Z) = \mathcal{L}(Z') = 0$. We have, that there exists $U, U' \in \mathcal{B}(\mathbb{R})$ such that $U \cup U' = \mathbb{R}$, $U \cap U' = \emptyset$ and $|(\mu_s)|(U) = \mathcal{L}(U') = 0$. We claim that $Z' = \Pi(U')$ and $Z = [0, 1) \setminus Z'$.

To prove this fact, first note that: if μ is a complex measure, $\{A_j\}_{j=1\ldots n}$ is any (measurable) partition of $[0, 1)$ and E any measurable subset of $[0, 1)$, then $\{\Pi^{-1}(A_j)\}_{j=1\ldots n}$ is a partition of $\mathbb{R}$. So we have

$$\begin{aligned}\sum_{j=1}^n |\mu_\Pi(A_j \cap E)| &= \sum_{j=1}^n |\mu\left(\Pi^{-1}(A_j) \cap \Pi^{-1}(E)\right)| \\ &\leq |\mu|(\Pi^{-1}(E)) = |\mu|_\Pi(E).\end{aligned}$$

Then if we take the supremum over all possible partitions of $[0, 1)$, in the left-hand side of the chain of inequalities we get

$$|\mu_\Pi|(E) \leq |\mu|_\Pi(E).$$

In particular, $|(\mu_s)_\Pi|(Z) \leq |\mu_s|_\Pi(Z)$. On the other hand, $\Pi^{-1}(Z) = \mathbb{R} \setminus \Pi^{-1}(Z') = (\Pi^{-1}(Z'))^c$, but $U' \subseteq \Pi^{-1}(Z')$ then $(\Pi^{-1}(Z'))^c \subseteq (U')^c = U$. From this we have

$$|(\mu_s)_\Pi|(Z) \leq |\mu_s|_\Pi(Z) = |\mu_s|(\Pi^{-1}(Z)) \leq |\mu_s|(U) = 0.$$

On the other hand, let $I_k = [k, k+1)$, $k \in \mathbb{Z}$, then $U' = \bigcup_{k\in\mathbb{Z}} I_k \cap U'$, hence

$$Z' = \Pi(U') = \bigcup_{k\in\mathbb{Z}} \left[(I_k \cap U') - k\right].$$

Since the Lebesgue measure is invariant under translations, then $\mathcal{L}\left[(I_k \cap U') - k\right] = \mathcal{L}\left(I_k \cap U'\right) \leq \mathcal{L}(U') = 0$, and

$$\begin{aligned}\mathcal{L}(Z') &= \mathcal{L}\left(\bigcup_{k\in\mathbb{Z}} \left[(I_k \cap U') - k\right]\right) \\ &\leq \sum_{k\in\mathbb{Z}} \mathcal{L}\left[(I_k \cap U') - k\right] = 0\end{aligned}$$

so that $(\mu_s)_\Pi \perp \mathcal{L}$.

Now, we prove that $(\mu_{ac})_\Pi << \mathcal{L}$. Take $W \in \mathcal{B}[0,1)$ such that $\mathcal{L}(W)=0$. Again, by the translation invariance property of the Lebesgue measure: $\mathcal{L}(W+k)=\mathcal{L}(W)=0$, so $\mu_{ac}(W+k)=0$ for every $k \in \mathbb{Z}$, since $\mu_{ac} << \mathcal{L}$ over $\mathbb{R}$. Then

$$(\mu_{ac})_\Pi(W) = \mu_{ac}(\Pi^{-1}(W)) = \mu_{ac}\left(\bigcup_{k\in\mathbb{Z}} W+k\right) \leq \sum_{k\in\mathbb{Z}} \mu_{ac}(W+k) = 0. \quad (4)$$

Finally, we have

$$\mu_\Pi = (\mu_{ac})_\Pi + (\mu_s)_\Pi. \quad (5)$$

The equations given above, together with the uniqueness of the Lebesgue decomposition of a measure, show that 5 must be the Lebesgue decomposition of μ_Π. The result of part a) follows from this.

Part b) Immediate. ■

The following is proved in Appendix:

Lemma 2.1: Let $\mathcal{X} = \{X_k^j\}_{k\in\mathbb{Z}}^{j=1\ldots n}$ be a w.s.s. stationary in the variable k, if D is the matrix of spectral densities and if $P_{Col(D)}(\lambda)$ is the orthogonal projection matrix over $Col(D)$ for each λ, then a) The Moore–Penrose pseudo-inverse $D^\sharp$ and $P_{Col(D)}$ are measurable. b) If $0 \in \sigma(D)(\lambda)$ for all λ in a set of positive Lebesgue measure, then there exists a column of $P_{Nul(D)}$ and a measurable set A, such that $\mathcal{L}(A) > 0$ and $(P_{Nul(D)})_j(\lambda) \neq 0$ for all $\lambda \in A$. c) If $Y = H(\mathcal{X})$ admits the following representation for some $f \in \mathbf{L}^2$: $Y = \sum_{j=1}^{n} \int_{[0,1)} f_j d\Phi_j$ then for all $g \in \mathbf{L}^2$ such that $g \in Nul(D)$ a.e.: $Y = \sum_{j=1}^{n} \int_{[0,1)} (f_j+g_j)d\Phi_j$, in particular Y can be written as $Y = \sum_{j=1}^{n} \int_{[0,1)} (P_{Col(D)}f^*)_j d\Phi_j$.

Remark: The same holds in $\mathbb{R}$ (continuous parameter case) or for the derivative of the matrix measure $\mathbf{M}$ with respect to ν.

III. Main Results

Now, we can give a necessary and sufficient condition for a stationary sequence to form a frame in terms of its spectral measure. In e.g., [20, Ch. II, Sec. 7], it is proved that if a stationary sequence has an spectral density (matrix) which has all its eigenvalues inside an interval $[A,B]$ a.e. then it is a Riesz basis of its span. This stability condition entails many consequences [24]. For example: the linear predictor, for any time lag, and the innovation process are expressible as the sum of mean-convergent infinite series.

The following theorem generalizes the result of [20] in two ways. First, we obtain a similar condition, on the spectral density, for frames. And second, it is proved that Rozanov's sufficient condition of the spectral measure being absolutely continuous is also necessary. The original result of [20], [21] is contained in one of the implications of part b) of Theorem 3.1, however in our case, the result will be derived from the first result for frames. On the other hand, this theorem resembles Theorem 3.4 and [2, Prop. 3.2], or [3, Th. 2.5] for shift invariant subspaces of $L^2(\mathbb{R}^d)$.

Theorem 3.1: Let $\mathcal{X} = \{X_k^j\}_{k\in\mathbb{Z}}^{j=1\ldots n}$ be stationary in the variable k. Then a) $\mathcal{X}$ is a frame of its span $H(\mathcal{X})$ (in $L^2(\Omega,\mathcal{F},\mathbf{P})$) with constants A, B $\iff$ the (cross) spectral measures μ_{ij} verify the following conditions: i) $\mu_{ij} << \mathcal{L}$ and ii) the spectral densities matrix $D_{ij} = \frac{d\mu_{ij}}{d\mathcal{L}}$ verifies $\sigma(D)(\lambda) \subseteq \{0\} \cup [A,B]$ for almost all $\lambda \in [0,1)$ $[\mathcal{L}]$.

b) $\mathcal{X}$ is a Riesz basis with constants A, B $\iff$ i) $\mu_{ij} << \mathcal{L}$ and ii) the spectral densities matrix $D_{ij} = \frac{d\mu_{ij}}{d\mathcal{L}}$ verifies $\sigma(D)(\lambda) \subseteq [A,B]$ for almost all $\lambda \in [0,1)$ $[\mathcal{L}]$.

Proof: Part a) ($\Rightarrow$) If we suppose that $\mathcal{X}$ is a frame, then, given $Y \in H(\mathcal{X})$

$$A\mathbb{E}|Y|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{j=1}^{n} |\mathbb{E}(X_k^j\overline{Y})|^2 \leq B\mathbb{E}|Y|^2 \quad (6)$$

On the other hand we know that Y admits the following representation: $Y = \sum_{j=1}^{n} \int_{[0,1)} f_j d\Phi_j$, where $f = (f_1,\ldots,f_n) \in \mathbf{L}^2[0,1)$ and the spectral random measures verify $X_k^j = \int_{[0,1)} e^{i2\pi\lambda k} d\Phi_j$, and $\mathbb{E}(\overline{Y}X_k^j) = \sum_{j=1}^{n} \int_{[0,1)} \overline{f}_j e^{i2\pi\lambda k} d\mu_{rj}$. Recall that the latter can be written as

$$\int_{[0,1)} e^{i2\pi\lambda k} \sum_{j=1}^{n} \overline{f}_j \frac{d\mu_{rj}}{d\nu} d\nu \quad (7)$$

then

$$\sum_{k\in\mathbb{Z}}\sum_{j=1}^{n} \left|\mathbb{E}(X_k^j\overline{Y})\right|^2 = \sum_{k\in\mathbb{Z}}\sum_{j=1}^{n} \left|\int_{[0,1)} e^{i2\pi\lambda k} \sum_{j=1}^{n} \overline{f}_j \frac{d\mu_{rj}}{d\nu} d\nu\right|^2 \leq B\mathbb{E}|Y|^2 < \infty$$

Hence, (7) gives the Fourier coefficients of a function belonging to $L^2[0,1)$ (thus in $L^1[0,1)$). In particular, if we take $f = e_r = (0,\ldots,0,1,0\ldots)$, the vector which is zero in all its coordinates but for the rth. coordinate, we get that $(\hat{\mu}_{rr}(k))_k \in l^2(\mathbb{Z})$ are the Fourier coefficients of an integrable function. Then by the uniqueness theorem of the Fourier transform of measures, we have that $\mu_{rr} << \mathcal{L}$ and then $\nu = \sum_{k=1}^{n} \mu_{kk} << \mathcal{L}$. From this: $\mu_{rj} << \mathcal{L}$, so by the Radon–Nykodym theorem there exist derivativeS(spectral densities) $\frac{d\mu_{rj}}{d\mathcal{L}} \in L^1[0,1)$. So if $D_{ij}(\lambda) = \frac{d\mu_{ij}}{d\mathcal{L}}(\lambda)$ is the matrix of spectral densities, we have that

$$\mathbb{E}\left(YX_k^j\right) = \int_{[0,1)} e^{i2\pi\lambda k} e_j D f^* d\lambda \text{ and } \mathbb{E}|Y|^2 = \int_{[0,1)} fDf^* d\lambda.$$

Then

$$A\int_{[0,1)} fDf^* d\lambda \leq \sum_{k\in\mathbb{Z}}\sum_{j=1}^{n} \left|\int_{[0,1)} e^{i2\pi\lambda k} e_j D f^* d\lambda\right|^2 \leq B\int_{[0,1)} fDf^* d\lambda$$

and from Parseval's identity we have

$$\sum_{k\in\mathbb{Z}}\sum_{j=1}^{n}\left|\int_{[0,1)} e^{i2\pi\lambda k}e_jDf^*d\lambda\right|^2 = \int_{[0,1)} \|Df^*\|^2\, d\lambda.$$

Hence we can write

$$A\int_{[0,1)} fDf^*d\lambda \le \int_{[0,1)} \|Df^*\|^2 d\lambda \le B\int_{[0,1)} fDf^*d\lambda. \quad (8)$$

Now, given $x\in\mathbb{C}^n$ we can take $\epsilon>0$, $\lambda_0\in[0,1]$ and then if we replace in (8)$f = x\mathbf{1}_{B(\lambda_0,\epsilon)}$, by Lebesgue's differentiation theorem, we have that $\forall\, x\in\mathbb{C}^n$ there exists a measurable set F_x such that $\mathcal{L}(F_x^c)=0$ and

$$AxD(\lambda)x^* \le \|D(\lambda)x^*\|^2 \le BxD(\lambda)x^* \quad \forall\,\lambda\in F_x. \quad (9)$$

Take $\mathcal{D}$ a countable dense subset of $\mathbb{C}^n$, and define $F=\bigcap_{x\in\mathcal{D}} F_x$, then clearly $\mathcal{L}(F^c)=0$ and given any $\lambda\in F$ inequality (9) holds for every $x\in\mathcal{D}$. On the other hand, for each $\lambda\in F$, $h(x) = CxDx^* - \|Dx^*\|^2$ is continuous, so given $\lambda\in F$ the inequality must hold for every $x\in\mathbb{C}^n$. But since D is self adjoint, then $Nul(D) = Col(D)^{\perp}$, so $Col(D)\oplus Nul(D) = \mathbb{C}^n$, a.e. Now, if $v^*\notin Nul(D)$ is an eigenvector associated to $z\in\sigma(D)\setminus\{0\}$, $\|Dv^*\|^2 = z^2\|v^*\|^2$ and $vDv^* = z\|v\|^2$, then as we have seen, (9) holds for every $\lambda\in F$ and $x\in\mathbb{C}^n$, thus, $A\le z\le B$.

($\Leftarrow$) It is easy to see that assuming (9), from (9) and (6) one can reverse the argument.

Part b) ($\Rightarrow$) If $\mathcal{X}$ is a Riesz basis then it is a frame (Theorem 2.1). From part a) we have that $\mu_{ij} << \mathcal{L}$ and $\sigma(D)\subseteq\{0\}\cup[A,B]$ a.e.. Let us check that $0\notin\sigma(D)$ a.e., if this is not the case, by Lemma 2.1 we can take a column $g^* = (P_{Nul(D)})_j \ne 0$ over a set of positive $\mathcal{L}$ measure, moreover we can suppose $\|g\|=1$ over some A of positive measure. If $c_k^r(g) = \int_{[0,1)} e^{-i\lambda 2\pi k}g_r(\lambda)d\lambda$ then there exists k,r such that $c_k^r(g)\ne 0$, and on the other hand $(c_k^r(g))_k\in l^2(\mathbb{Z})$. Now we can define $Y\in H(\mathcal{X})$ as $Y=\sum_{j=1}^{n}\int_{[0,1)} g_j d\Phi_j$, clearly $\mathbb{E}|Y|^2 = \int_{[0,1)} gDg^*d\lambda = 0$. If we show that $Y = \sum_{k\in\mathbb{Z}}\sum_{r=1}^{n} c_k^r(g)X_k^r = 0$ we are done, since from Theorem 2.2 then $\{X_k^r\}_{k\in\mathbb{Z}}^r$ can't be a frame. Let us define for $r=1,\dots,n$, $g_{N\,r} = \sum_{|k|\le N} c_k^r(g)e^{i2\pi\lambda k}$, $g_N = (g_{N1},\dots,g_{Nn})$, and finally $Y_N = \sum_{j=1}^{n}\int_{[0,1)} g_{Nj}d\Phi_j = \sum_{|k|\le N}\sum_{r=1}^{n} c_k^r(g)X_k^r$. The result will follow if we show that $\mathbb{E}|Y_N - Y|^2 \underset{N\to\infty}{\to} 0$. This is true since $M(\lambda) = \sup_{\|x\|=1} xD(\lambda)x^*$ is measurable and $M\le B$ a.e. and on the other hand $g_{N,r}\to g_r\in L^2[0,1)$ converges in $L^2[0,1)$ and a.e. from which we have $\mathbb{E}|Y_N - Y|^2$

$$= \int_{[0,1)} (g - g_N)D(g-g_N)^*d\lambda \le \int_{[0,1)} \|g_N - g\|^2 M d\lambda$$

$$\le B\int_{[0,1)} \|g_N - g\|^2 d\lambda \underset{N\to\infty}{\to} 0.$$

($\Leftarrow$) Under these conditions, again from part a), we know that $\mathcal{X}$ is a frame. On the other hand $\sigma(D)\subseteq[A,B]$ a.e. implies that $\exists D^{-1}$ a.e.. Moreover $\sigma(D^{-1})\subseteq[B^{-1},A^{-1}]$. Then if we call Λ_i $i=1\dots n$ to the eigenvalues of D taking in account their multiplicity, we have that

$$\frac{1}{B}\le tr(D^{-1}) = \sum_{i=1}^{n}\Lambda_i^{-1} \le \frac{n}{A} \text{ a.e. in}[0,1)[\mathcal{L}]$$
$$\Longrightarrow \int_{[0,1)} tr(D^{-1})d\lambda < \infty.$$

This proves the result, since by Theorem 2.3 this implies that $\mathcal{X}$ is minimal, and then a Riesz basis, by Theorem 2.2. ∎

1) Sampling: Given a continuous time stationary process we can give conditions for samples taken at an uniform rate to form a frame. This is related to reconstructing a continuous parameter process from its samples. Recall in Section II we noted that the spectral measure of a sampled process is $(\mu_{ij})_{\Pi}$, where μ_{ij} is the spectral measure of the original continuous time process. Let us prove the following result.

Lemma 3.1: Given the spectral measures μ_{ij} over $(\mathbb{R},\mathcal{B}(\mathbb{R}))$: $\mu_{ij} << \mathcal{L} \iff (\mu_{ij})_{\Pi} << \mathcal{L}$.

Proof: In general, if $f:(X,\Sigma)\longrightarrow(Y,\Sigma')$ is a map between measurable spaces we have that given a (positive) measure ν defined over Σ then $\nu\equiv 0 \iff \nu_f\equiv 0$, where ν_f is the induced measure by f. Since given $A'\in\Sigma'$ then $\nu_f(A') = \nu(f^{-1}(A'))$ and on the other hand given $A\in\Sigma$ then $\nu(A)\le\nu(f^{-1}(f(A))) = \nu_f(f(A))$. In our case, taking $f=\Pi$, and $\nu=\sum_j \mu_{jj}$ as in def. 1, we have that $(\nu_s)_{\Pi}\equiv 0 \iff \nu_s\equiv 0$, but by prop. 2.1$(\nu_s)_{\Pi} = (\nu_{\Pi})_s$ then $(\nu_{\Pi})_s\equiv 0\iff\nu_s\equiv 0$ which is equivalent to the result, since $\mu_{ij} << \nu$ and $(\mu_{ij})_{\Pi} << \nu_{\Pi}$. ∎

Theorem 3.2: Let $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}^{r=1,\dots,n}$ be a continuous time stationary process, and let $\mathcal{Y} = \{X_k^r\}_{k\in\mathbb{Z}}^{r=1,\dots,n}$ then the following are equivalent.

a) $\mathcal{Y}$ is a frame of its span $H(\mathcal{Y})$.

b) $\mu_{ij} << \mathcal{L}$, where μ_{ij} are the spectral measures of $\mathcal{X}$, and there exists $A,B>0$ such that D_{Π} the matrix of periodized spectral densities verifies $\sigma(D_{\Pi})\subseteq\{0\}\cup[A,B]$ a.e $[\mathcal{L}]$.

Proof: This is immediate from the previous Lemma 3.1 and Theorem 3.1. ∎

Let us study under which conditions, given a w.s.s. process $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}^{r=1\dots n}$, there exists another stationary procesS(indexed by $k\in\mathbb{Z}$) $\mathcal{W} = \{W_k^r\}_{k\in\mathbb{Z}}^{r=1\dots m}$ such that $\overline{\text{span}}\{W_k^r\}_{k\in\mathbb{Z}}^{r=1\dots m} = \overline{\text{span}}\{X_k^r\}_{k\in\mathbb{Z}}^{r=1\dots n}$. Let us introduce some notation. If $\Lambda\in\mathbb{R}^{n\times n}$ is a diagonal non negative matrix, for $\alpha\in\mathbb{R}$ we define $\Lambda^{(\alpha)}$ as $\Lambda_{tl}^{(\alpha)} = \Lambda_l^{\alpha}$ if $t=l$ and $\Lambda_l\ne 0$, or $\Lambda_{tl}^{(\alpha)} = 0$ otherwise. If $A\in\mathbb{R}^{n\times n}$ is a (symmetric) non negative definite matrix, A admits a diagonal decomposition $A = P\Lambda P^*$, from this, we define $A^{(\alpha)} = P\Lambda^{(\alpha)}P^*$.

Lemma 3.2: Let $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}^{r=1\dots n}$ be a w.s.s. process. $\mathcal{X}$ contains a stationary sequence which is a frame of the closed linear span of $\mathcal{Y} = \{X_k^r\}_{k\in\mathbb{Z}}^{r=1\dots n}$, $H(\mathcal{Y}) \iff$ the spectral measures verify, for all ij: $\mu_{ij} << \mathcal{L}$.

Proof: ($\Rightarrow$) We denote $\mathcal{Y} = \{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$. If $\mathcal{Y}$ contains a stationary sequence, say $\mathcal{W}$, which is a frame of its span, then by the previous results, Theorem 3.1, the (matrix) spectral measure associated to $\mathcal{W}$ must be absolutely continuous, so there exists a spectral density matrix $D^{\mathcal{W}}$, defined over $[0,1)$. On the other hand if $\mathcal{Y}$ is obtainable by a linear transformation from $\mathcal{W}$, say a $n \times m$ measurable matrix function $\varphi^{\mathcal{Y}\mathcal{W}}$, then the spectral measures of $\mathcal{Y}$ are given by [20]

$$\forall\, A \in \mathcal{B}[0,1) : \mu'_{ij}(A) = \int_A \varphi_i^{\mathcal{Y}\mathcal{W}} D^{\mathcal{W}} (\varphi_j^{\mathcal{Y}\mathcal{W}})^* d\lambda$$

so $\mu'_{ij} << \mathcal{L}$, but since $\mu'_{ij} = (\mu_{ij})_{\Pi}$ then by the claim 3.1: $\mu_{ij} << \mathcal{L}$.

($\Leftarrow$) The spectral density matrix of $\mathcal{Y}$, $D_{\Pi}^{\mathcal{X}}$ exists since $\mu_{ij} << \mathcal{L}$ is diagonalizable in a measurable form, i.e., there exists a measurable [18] P orthogonal matrix and a diagonal matrix of eigenvalues Λ such that $\Lambda = P^* D_{\Pi}^{\mathcal{X}} P$. Take $\mathcal{W}$ the process obtained from $\mathcal{Y}$ by the linear invertible transformation[1] [20] induced by $G^{(-\frac{1}{2})} = P\Lambda^{(-\frac{1}{2})}P^*$. Then $G^{(-\frac{1}{2})} D_{\Pi}^{\mathcal{X}} (G^{(-\frac{1}{2})})^* = P\Lambda^{(-\frac{1}{2})}\Lambda\Lambda^{(-\frac{1}{2})}P^* = B$, where B is a diagonal matrix which takes the values 0 or 1 in the diagonal. From this we have that this linear operation is well defined since each column of $G^{(-\frac{1}{2})}$ is in $\mathbf{L}^2[0,1)$. On the other hand by the result [20] which relates a linear transformation and the spectral measure: $D^{\mathcal{W}} = B \Rightarrow \sigma(D^{\mathcal{W}}) \subseteq \{0,1\}$ a.e., then, by Theorem 3.1, $\mathcal{W}$ is a frame of $H(\mathcal{Y})$. ■

In previous works, e.g.,[20], [26], a common condition for the existence of orthogonal or minimal stationary sequences, is that the eigenvalues of the spectral density be not null a.e. In Theorem 3.1, we proved that the spectral density matrix of a stationary sequence which is a Riesz basis must have all its eigenvalues inside a positive bounded interval $[A,B]$. However there are some limitations on the supports of the spectral densities.

Lemma 3.3: Let $\mathcal{X} = \{X_k\}_{k\in\mathbb{Z}}$ be a w.s.s. sequence, with absolutely continuous spectral measure, and let $\mathcal{Z} = \{Z_k\}_{k\in\mathbb{Z}}$ be another w.s.s. sequence such that $H(\mathcal{X}) = H(\mathcal{Z})$, then $\mathcal{Z}$ has an absolutely continuous spectral measure, and if $\phi^{\mathcal{X}}$ and $\phi^{\mathcal{Z}}$ are the spectral densities of $\mathcal{X}$ and $\mathcal{Z}$ respectively, then

$$\mathcal{L}(\mathrm{supp}(\phi^{\mathcal{X}}) \triangle \mathrm{supp}(\phi^{\mathcal{Z}})) = 0.$$

Proof: From 3.2 there exists $\mathcal{Y} = \{Y_k\}_{k\in\mathbb{Z}}$ a stationary sequence which is a frame of $H(\mathcal{Z})$. Let us call $\Phi_{\mathcal{Z}}$ and $\Phi_{\mathcal{X}}$ the random spectral measures of $\mathcal{Z}$ and $\mathcal{X}$ respectively. Then as we have seen in Lemma 3.2:

$$Y_k = \int_{[0,1)} (\phi^{\mathcal{Z}})^{-\frac{1}{2}} \mathbf{1}_{\mathrm{supp}(\phi^{\mathcal{Z}})} e^{i2\pi\lambda k} d\Phi_{\mathcal{Z}}.$$

On the other hand, as $Z_k \in H(\mathcal{X})$, Z_k admits the following representation:

$$Y_k = \int_{[0,1)} f e^{i2\pi\lambda k} d\Phi_{\mathcal{X}}$$

for some $f \in L^2(\mathbb{R}, \phi^{\mathcal{X}} d\lambda)$. Then $\phi^{\mathcal{Y}} = \mathbf{1}_{\mathrm{supp}(\phi^{\mathcal{Z}})}$ a.e. $[\mathcal{L}]$. But [20] $\phi^{\mathcal{Z}} = |f|^2 \phi^{\mathcal{X}}$ a.e. (in particular $\mathcal{Z}$ has an absolutely continuous spectral measure), then $\phi^{\mathcal{Y}} = \frac{|f|^2}{\phi^{\mathcal{Z}}} \mathbf{1}_{\mathrm{supp}(\phi^{\mathcal{Z}})} \phi^{\mathcal{X}} = \mathbf{1}_{\mathrm{supp}(\phi^{\mathcal{Z}})}$ a.e. From this $\mathcal{L}(\mathrm{supp}(\phi^{\mathcal{Z}}) \setminus \mathrm{supp}(\phi^{\mathcal{X}})) = 0$. Interchanging the roles of $\mathcal{Z}$ and $\mathcal{X}$ we get $\mathcal{L}(\mathrm{supp}(\phi^{\mathcal{X}}) \triangle \mathrm{supp}(\phi^{\mathcal{Z}})) = 0$. ■

[1]Note that the operation is always defined over the closed linear span of the process.

We have seen that the closed linear span of a stationary sequence always contains a stationary sequence which is a frame. An arbitrary stationary sequence may not contain any *stationary* Riesz basis. Moreover, from Lemmas 3.3, 3.2, and Theorem 3.1, it is immediate that a necessary and sufficient condition for a 1-D stationary sequence to contain a (stationary) Riesz basis of its span, is to have a spectral measure equivalent to $\mathcal{L}$.

2) Fundamental Frame: We give conditions for a sequence of samples to be a fundamental frame, i.e., to be a frame and to be complete with respect to the closed linear span of the continuous time process, i.e.

$$\overline{\mathrm{span}}\,\{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n} = \overline{\mathrm{span}}\,\{X_t^r\}_{t\in\mathbb{R}}^{r=1\ldots n} = H(\mathcal{X}). \tag{10}$$

This is related to reconstructing a signal from its samples, since the sampling problem can be formulated in terms of finding conditions for which (10) holds [14], [13]. The following Theorem 3.3 was proved by Lloyd [14], and it is an analogue result of the result of [1] for $L^2(\mathbb{R})$ functions.

Theorem 3.3 [14]: Let $\mathcal{X} = \{X_t\}_{t\in\mathbb{R}}$ be a w.s.s. process with spectral measure μ. The following are equivalent.

i) $\overline{\mathrm{span}}\{X_k\}_{k\in\mathbb{Z}} = \overline{\mathrm{span}}\{X_t\}_{t\in\mathbb{R}}$, (in $L^2(\Omega, \mathcal{F}, \mathbf{P})$).
ii) $\overline{\mathrm{span}}\{e^{i2\pi kx}\}_{k\in\mathbb{Z}} = \{f \in L^2(\mu) : f \; is \; 1-periodic\} = L^2(\mu)$.
iii) There exists $A \in \mathcal{B}(\mathbb{R})$ such that $\mu(A^c) = 0$ and $A \cap A + k = \emptyset$, $k \in \mathbb{Z} \setminus \{0\}$.

The following is an extension of Theorem 3.3.

Lemma 3.4: Let $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}^{r=1\ldots n}$ be w.s.s. process and let $\mathcal{Y} = \{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$ be the discrete time "sampled" process. Then

a) $H(\mathcal{X}) = H(\mathcal{Y}) \iff$ for each eigenvalue Λ_j of $\frac{d\mathbf{M}}{d\nu}$ there exists $A_j \in \mathcal{B}(\mathbb{R})$, such that $\Lambda_j = 0$ a.e. $[\nu]$ on A_j^c and $A_j \cap A_j + k = \emptyset$ for every $k \in \mathbb{Z} \setminus \{0\}$.
b) Let $\mathcal{X}$ be such that the spectral measures $\mu_{ij} << \mathcal{L}$, then $H(\mathcal{X}) = H(\mathcal{Y}) \iff$ for each eigenvalue Λ_j of D there exists $A_j \in \mathcal{B}(\mathbb{R})$, such that $\Lambda_j = 0$ a.e. $[\mathcal{L}]$ on A_j^c and $A_j \cap A_j + k = \emptyset$ for every $k \in \mathbb{Z} \setminus \{0\}$.

Remark: Recall that $\nu = tr(\mathbf{M})$.

Proof: a) Since $\frac{d\mathbf{M}}{d\nu}$ is non-negative definite and self adjoint, there exists a measurable [18] P orthogonal matrix and a diagonal matrix of eigenvalues Λ such that $\Lambda = P^* \frac{d\mathbf{M}}{d\nu} P$. Let us introduce the process $\mathcal{Z} = \{Z_t^r\}_{t\in\mathbb{R}}^{r=1\ldots n}$ defined by the linear operation on $\mathcal{X}$ [20, Ch. 1, Sec. 8]

$$Z_t^r = \sum_{j=1}^n \int_{\mathbb{R}} e^{i\lambda 2\pi t} P_{rj}^* d\Phi_j. \tag{11}$$

This operation is well defined since P_i, the ith column of P, is in $\mathbf{L}^2(\mathbb{R})$:

$$\int_{\mathbb{R}} P_i^* \frac{d\mathbf{M}}{d\nu} P_j d\nu \le \int_{\mathbb{R}} \|P_i\|^2 \left\| \left(\frac{d\mathbf{M}}{d\nu}\right)^{\frac{1}{2}} \right\|_{op}^2 d\nu$$
$$\le \int_{\mathbb{R}} tr\left(\frac{d\mathbf{M}}{d\nu}\right) d\nu = \nu(\mathbb{R}) < \infty,$$

and if we denote μ'_{ij} the spectral measures of the process $\mathcal{Z}$, then by the result [20] which relates a linear transformation and the spectral measure:

$$\forall\, A \in \mathcal{B}(\mathbb{R}) : \mu'_{ij}(A) = \int_A P_i^* \frac{d\mathbf{M}}{d\nu} P_j d\nu = \int_A \Lambda_{ij} d\nu. \quad (12)$$

The linear operation induced by P over $H(\mathcal{X})$ in (11) is invertible, since P^{-1} exists for almost all λ, then $H(\mathcal{X}) = H(\mathcal{Z})$. Equation (11) also implies that for each $k \in \mathbb{Z}$, the X_k^r are obtainable from the Z_k^r, for every $r = 1\ldots n$ and reciprocally, since P^{-1} exists. If we introduce another process of samples from $\mathcal{Z}$: $\mathcal{S} = \{Z_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$, from the latter discussion we have that $H(\mathcal{X}) = H(\mathcal{Y}) \iff H(\mathcal{Z}) = H(\mathcal{S})$. But (12) means that $\mu_{ij} \equiv 0$ for $i \neq j$, or equivalently $\mathbb{E}\left(Z_{t'}^i Z_t^j\right) = 0$ for $i \neq j$, so that we have the orthogonal sum $H(\mathcal{Z}) = \overset{n}{\underset{j=1}{\oplus}} \overline{\mathrm{span}}\{Z_t^j\}_{t\in\mathbb{R}}$. Hence it will suffice to study when $\overline{\mathrm{span}}\{Z_t^j\}_{t\in\mathbb{R}} = \overline{\mathrm{span}}\{Z_k^j\}_{k\in\mathbb{Z}}$, for each j. This holds if and only if (by Theorem 3.3) there exists $A_j \in \mathcal{B}(\mathbb{R})$ such that $\mu'_{jj}(A^c) = 0$ and $A_j \cap A_j + k = \emptyset$ for integer $k \neq 0$. But from (12)$\mu'_{jj}(A^c) = 0$ is equivalent to the condition on the eigenvalue $\Lambda_j = \Lambda_{jj} = 0$ a.e. $[\nu]$ on A_j^c.

b) Is an immediate consequence of a). ■

Now, we can characterize fundamental frames of uniform samples, in other words, given $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}^{r=1\ldots n}$ a w.s.s. process and $\mathcal{Y} = \{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$ the discrete time 'sampled' process, we want to give conditions when $\mathcal{Y}$ is a fundamental frame for $H(\mathcal{X})$ in terms of the spectral measure of the process. For this purpose, in an analogue way to definition 1 for each $\lambda \in [0,1)$ we introduce the following sequence space:

$$l^2(D) = \left\{ x : \mathbb{Z} \to \mathbb{C}^n, \sum_{k\in\mathbb{Z}} x_k D(\lambda + k) x_k^* < \infty \right\}.$$

In a similar manner to that of definition 1, $l^2(D)$ can be identified with a Hilbert space with norm $\|x\|_{l^2(D)}^2 = \sum_{k\in\mathbb{Z}} x_k D(\lambda + k)x_k^*$. We give a condition in terms of $l^2(D)$ and an alternative condition combining the preceding results.

Theorem 3.4: Let $\mathcal{X} = \{X_t^r\}_{t\in\mathbb{R}}^{r=1\ldots n}$ be a w.s.s. process, with spectral measure μ_{ij}, and let $\mathcal{Y} = \{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$ be the discrete time 'sampled' process. Then:

a) $\mathcal{Y}$ is a frame with constants $A,\ B > 0$ for $H(\mathcal{X}) \iff \mu_{ij} << \mathcal{L}$ and for almost all $\lambda \in [0,1)$:

$$A\|x\|_{l^2(D)}^2 \leq \left\| \sum_{k\in\mathbb{Z}} D(\lambda + k) x_k^* \right\|_{\mathbb{C}^n}^2 \leq B\|x\|_{l^2(D)}^2$$

$\forall\, x \in l^2(D)$.

b) $\mathcal{Y}$ is a frame with constants $A, B > 0$ for $H(\mathcal{X}) \iff$ The following conditions hold simultaneously: i) $\mu_{ij} << \mathcal{L}$ and there exists $A, B > 0$ such that D_Π the matrix of periodized spectral densities verifies $\sigma(D_\Pi) \subseteq \{0\} \cup [A, B]$ a.e $[\mathcal{L}]$. ii) For each eigenvalue Λ_j of D there exists $A_j \in \mathcal{B}(\mathbb{R})$, such that $\Lambda_j = 0$ a.e. $[\mathcal{L}]$ on A_j^c and $A_j \cap A_j + k = \emptyset$ for every $k \in \mathbb{Z} \setminus \{0\}$.

Proof: (Part a) ($\Rightarrow$) First note that by Theorem 3.2, $\mu_{ij} << \mathcal{L}$ and then there exists D the spectral density matrix. Recall (6), if $Y \in H(\mathcal{X})$ then there exists $f \in \mathbf{L}^2(\mathbb{R})$ such that $Y = \sum_{j=1}^n \int_{\mathbb{R}} f_j d\Phi_j$. Then for such Y and f we have

$$\sum_{k\in\mathbb{Z}} \sum_{j=1}^n |\mathbb{E}(X_k^j \overline{Y})|^2 = \int_{[0,1)} \left\| \sum_{k\in\mathbb{Z}} D(\lambda + k) f^*(\lambda + k) \right\|_{\mathbb{C}^n}^2 d\lambda$$

on the other hand

$$\mathbb{E}|Y|^2 = \int_{[0,1)} \sum_{k\in\mathbb{Z}} f(\lambda + k) D(\lambda + k) f^*(\lambda + k) d\lambda.$$

Calling the sequence $\mathbf{f}_k = f(\cdot + k)$, then by a similar argument to that of the proof of Theorem 3.1 we can rewrite condition (6) as

$$A \int_{[0,1)} \|\mathbf{f}\|_{l^2(D)}^2 d\lambda \leq \int_{[0,1)} \left\| \sum_{k\in\mathbb{Z}} D(\lambda + k) f^*(\lambda + k) \right\|_{\mathbb{C}^n}^2 d\lambda \leq B \int_{[0,1)} \|\mathbf{f}\|_{l^2(D)}^2 d\lambda. \quad (13)$$

Let us call $\mathcal{D}$ the set of all sequences in $(\mathbb{C}^n)^{\mathbb{Z}}$ such that x_k^j belongs to $\mathcal{D}$ if $x_k^j = 0$ but for finitely many jk's. $\mathcal{D}$ is a countable set which is dense in $l^2(D)$ for every λ. Given $x \in \mathcal{D}$, $\epsilon > 0$ and $\lambda_0 \in [0,1)$ set $\mathbf{f}(\lambda) = \sum_{k\in\mathbb{Z}} x_k \mathbf{1}_{B(\lambda_0,\epsilon)}(\lambda - k)\mathbf{1}_{[k,k+1)}$. In this case (13) becomes

$$A \int_{B(\lambda_0,\epsilon)\cap[0,1)} \|x\|_{l^2(D)}^2 d\lambda \leq \int_{B(\lambda_0,\epsilon)\cap[0,1)} \left\| \sum_{k\in\mathbb{Z}} D(. + k) x_k^* \right\|_{\mathbb{C}^n}^2 d\lambda \leq B \int_{B(\lambda_0,\epsilon)\cap[0,1)} \|x\|_{l^2(D)}^2 d\lambda.$$

Then for each $x \in \mathcal{D}$, there exists F_x, such that $\mathcal{L}(F_x^c) = 0$ and $A\|x\|_{l^2(D)}^2 \leq \left\| \sum_{k\in\mathbb{Z}} D(\lambda + k) x_k^* \right\|_{\mathbb{C}^n}^2 \leq B\|x\|_{l^2(D)}^2$, for all $\lambda \in F_x$. Then taking $F = \bigcap_{x\in\mathcal{D}} F_x$, we have $\mathcal{L}(F^c) = 0$ and that for all $\lambda \in F$ it is

$$A\|x\|_{l^2(D)}^2 \leq \left\| \sum_{k\in\mathbb{Z}} D(\lambda + k) x_k^* \right\|_{\mathbb{C}^n}^2 \leq B\|x\|_{l^2(D)}^2 \quad \forall\, x \in \mathcal{D}. \quad (14)$$

We want to prove that (14) holds for every $x \in l^2(D)$. For such x let us define $y_k = D^{\frac{1}{2}}(\lambda + k)x_k^*$, then $\sum_{k=N+1}^{M} D(\lambda + k)x_k^* = \sum_{k=N+1}^{M} D^{\frac{1}{2}}(\lambda + k) y_k$. First we shall see that given $x \in l^2(D)$: $\lim_{N,M\to\infty} \left\| \sum_{k=N+1}^{M} D(\lambda + k)x_k^* \right\| = 0$, for $j = 1\ldots n$. Denoting $D_j^{\frac{1}{2}}(\lambda + k)$ the jth row of $D^{\frac{1}{2}}(\lambda + k)$, we have: $\left\| \sum_{k=N+1}^{M} D^{\frac{1}{2}}(\lambda + k) y_k \right\|^2 = \sum_{j=1}^{n} \left| \sum_{k=N+1}^{M} D_j^{\frac{1}{2}}(\lambda + k) y_k \right|^2 \leq$

$\sum_{j=1}^{n}\left(\sum_{k=N+1}^{M}\left\|D_j^{\frac{1}{2}}(\lambda+k)\right\|_{\mathbb{C}^n}\|y_k\|_{\mathbb{C}^n}\right)^2$. Applying again the Cauchy–Schwartz inequality, we get

$$\left\|\sum_{k=N+1}^{M} D^{\frac{1}{2}}(\lambda+k)y_k\right\|^2 \leq \sum_{j=1}^{n}\left(\sum_{k=N+1}^{M}\left\|D_j^{\frac{1}{2}}(\lambda+k)\right\|_{\mathbb{C}^n}^2\right)\left(\sum_{k=N+1}^{M}\|y_k\|_{\mathbb{C}^n}^2\right).$$

If we denote $\|\cdot\|_{\mathbb{C}^{n\times n}}$ the Froebenius/euclidean norm, denoting $x_N(k)=x_k\mathbf{1}_{(-\infty,N]}(k)$ and recalling the definition of y_k we can rewrite the last equation as

$$\begin{aligned}\sum_{k=N+1}^{M}&\left\|D^{\frac{1}{2}}(\lambda+k)\right\|_{\mathbb{C}^{n\times n}}^2\|x_N-x_M\|_{l^2(D)}^2\\ &=\sum_{k=N+1}^{M} tr(D^{\frac{1}{2}}(\lambda+k)D^{\frac{1}{2}}(\lambda+k))\|x_N-x_M\|_{l^2(D)}^2\\ &=tr\left(\sum_{k=N+1}^{M}D(\lambda+k))\right)\|x_N-x_M\|_{l^2(D)}^2.\end{aligned}$$

But by Theorem 3.1, $tr\left(\sum_{k=N+1}^{M}D(\lambda+k))\right)\leq tr(D_{\Pi}(\lambda))\leq nB$ a.e. in $[0,1)$. Equivalently this holds for all $\lambda\in F'$ with $\mathcal{L}((F')^c)=0$, so we can take $G=F\cap F'$. Then, the result follows, since for $\lambda\in G$ and $x\in l^2(D)$: $\lim_{M\to\infty}\left\|\sum_{k\in\mathbb{Z}}D(\lambda+k)x_M^*(k)\right\| = \left\|\sum_{k\in\mathbb{Z}}D(\lambda+k)x_k^*\right\|$ and $\lim_{M\to\infty}\|x_M\|_{l^2(D)}=\|x\|_{l^2(D)}$.

($\Leftarrow$) It is easy to reverse the proof using the condition of the hypothesis and (13).

(*Part b.*) It follows from combining Theorem 3.2 and Lemma 3.4. ∎

In particular we have that if $\mathcal{Y}$ is a fundamental frame, then for almost all λ $l^2(D)$ must be isomorphic to a finite dimensional space and that $\mathcal{L}\left(\bigcap_{n\in\mathbb{Z}}\bigcup_{k\geq n}\{\lambda: D(\lambda+k)\neq 0\}\right)=0$, since if we take $\lambda\in\bigcap_{n\in\mathbb{Z}}\bigcup_{k\geq n}\{\lambda : D(\lambda+k)\neq 0\}$ for infinitely many k's there exists $v_k\neq 0$ in $Col(D(\lambda+k))$, and setting $v_k=0$ if $D(\lambda+k)=0$ we can define the sequence/set of vectors $\mathcal{U}=\{w_k\}_{k\in\mathbb{Z}}$ as $w_k(n)=v_k\delta_{k\,n}$. Then $\mathcal{U}$ is linearly independent and then $\dim(l^2(D))=\infty$.

IV. Canonical Dual Frame and A.S. Convergence

A. Canonical Dual Frame

In the case of shift invariant subspaces of $L^2(\mathbb{R})$ some useful formulations for the dual frame are obtained in terms of the Fourier transforms of the generators. In the following we consider a similar problem for the frame formed by the stationary sequence: $\{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$. In this case is possible to give conditions in terms of the spectral density. Recall that for each $t\in\mathbb{R}$ or $\mathbb{Z}$, X_t^r can be written as the result of the action of the (unitary) time shift operator T on X_0^r [20]. In our case, the frame operator $S: H(\mathcal{X})\longrightarrow H(\mathcal{X})$ is given by

$$Y\longmapsto S(Y)=\sum_{r=1}^{n}\sum_{k\in\mathbb{Z}}\mathbb{E}(\overline{Y}X_k^r)X_k^r.$$

Recall that S has a bounded inverse and on the other hand each $Y\in H(\mathcal{X})$ admits the following representation:

$$Y=\sum_{r=1}^{n}\sum_{k\in\mathbb{Z}}\mathbb{E}(\overline{Y}S^{-1}X_k^r)X_k^r.$$

Taking into account (1) and if we suppose that T commutes with S^{-1} we have

$$\begin{aligned}Y&=\sum_{r=1}^{n}\sum_{k\in\mathbb{Z}}\mathbb{E}(\overline{Y}S^{-1}T^kX_0^r)X_k^r\\ &=\sum_{r=1}^{n}\sum_{k\in\mathbb{Z}}\mathbb{E}(\overline{Y}T^kS^{-1}X_0^r)X_k^r\end{aligned}$$

so, in this case the canonical dual frame is a new stationary sequence given by $W_k^r=T^k\left(S^{-1}X_0^r\right)$, $k\in\mathbb{Z}$, and it would suffice to show that $W_0^r=S^{-1}X_0^r$. We need the following lemma to solve this problem.

Lemma 4.1: Let $\mathcal{X}=\{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$ be a stationary sequence which is a frame of its span $H(\mathcal{X})$. Then, for all $Y\in H(\mathcal{X})$ the following holds: $ST^kY=T^kSY$ and $S^{-1}T^kY=T^kS^{-1}Y$.

Proposition 4.1: Let $\mathcal{X}=\{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$ be a stationary sequence which is a frame of its span $H(\mathcal{X})$. Then the canonical dual frame $\{W_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$ is given by

$$W_k^r=\sum_{j=1}^{n}\int_{[0,1)}e^{i\lambda 2\pi k}(D^{\sharp}e_r^*)_j d\Phi_j. \tag{15}$$

Proof: Let us define $Z_0^m=\sum_{j=1}^{n}\int_{[0,1)}(D^{\sharp}e_m^*)_j d\Phi_j$. Where $D^{\sharp}$ is measurable by Lemma 2.1. If we show that $SZ_0^m=X_0^m$ we are done, since S is invertible. Define $M_N=\sum_{r=1}^{n}\sum_{|k|\leq N}\mathbb{E}(\overline{Z_0^m}X_k^r)X_k^r$ and $h_{N,r}(\lambda)=\sum_{|k|\leq N}\mathbb{E}(\overline{Z_0^m}X_k^r)e^{i\lambda 2\pi k}$, and recall that $SZ_0^m=\sum_{r=1}^{n}\sum_{k\in\mathbb{Z}}\mathbb{E}(\overline{Z_0^m}X_k^r)X_k^r$. By Lemma 2.1 we have that $X_0^m=\sum_{j=1}^{n}\int_{[0,1)}(e_m)_j d\Phi_j=\sum_{j=1}^{n}\int_{[0,1)}(P_{Col(D)}e_m^*)_j d\Phi_j$ almost surely. On the other hand:

$$\begin{aligned}\mathbb{E}(\overline{Z_0^m}X_k^r)&=\int_{[0,1)}e^{i\lambda 2\pi k}e_rDD^{\sharp}e_m^*d\lambda\\ &=\int_{[0,1)}e^{i\lambda 2\pi k}e_rP_{Col(D)}e_m^*d\lambda.\end{aligned}$$

So $h_{N,r} \longrightarrow e_r P_{Col(D)} e_m^*$ in $L^2[0,1)$ and a.e. But

$$\begin{aligned}\mathbb{E}|X_0^m - M_N|^2 &= \int_{[0,1)} (h_N - P_{Col(D)} e_m^*)^* D (h_N - P_{Col(D)} e_m^*) d\lambda \\ &\leq B \int_{[0,1)} \left\| h_N - P_{Col(D)} e_m^* \right\|^2 d\lambda\end{aligned}$$

where again, $M(\lambda) = \sup_{\|x\|=1} x D(\lambda) x^*$ is measurable and $M \leq B$ a.e. which proves the result. ∎

1) On the Expansion Coefficients: Observing (15) we expect that the coefficients of the expansion may be rewritten in terms of ordinary Fourier transforms. Let us discuss the case for fundamental frames and when $n = 1$. In this case, given a stationary process $Y_t \in H(\mathcal{X})$, we know that $Y_t = \int_{\mathbb{R}} f\, e^{i\lambda 2\pi t} d\Phi$ for some $f \in L^2(\mathbb{R}, \phi d\lambda)$, where ϕ is the spectral density. On the other hand, (15) becomes $W_k = \int_{\mathbb{R}} (\phi)^{-1} \mathbf{1}_S e^{i\lambda 2\pi k} d\Phi$, where $S = \{\lambda : \phi(\lambda) \neq 0\}$, then $\mathbb{E}(Y_t \overline{W_k}) = \int_S f(\lambda) e^{-i2\pi\lambda(t-k)} d\lambda = \widehat{f}(t-k)$. Note that we can take f such that $f = 0$ outside S, and moreover $f \in L^2(\mathbb{R})$ since $B \int_{\mathbb{R}} |f|^2 d\lambda \leq \int_{\mathbb{R}} |f|^2 \phi d\lambda < \infty$. Then $Y_t \in H(\mathcal{X})$, provided that $\mathcal{Y} = \{X_k\}_{k\in\mathbb{Z}}$ is a fundamental frame, admits the following representation:

$$Y_t = \sum_{k\in\mathbb{Z}} \widehat{f}(t-k) X_k.$$

Let us discuss the case when another dual frame is used. First we need an auxiliary lemma.

Lemma 4.2: Let $f, g \in L^2(\mathbb{R})$ and let $\widehat{f}, \widehat{g}$ be their Fourier transforms, then

$$\int_{\mathbb{R}} |f * g|^2 dt = \int_{\mathbb{R}} |\widehat{f}\widehat{g}|^2 d\lambda.$$

When one side of the above equation is finite, then $\widehat{f * g} = \widehat{f}\widehat{g}$ a.e.

Proof: In the Appendix.

In the general case, let $\{W_k\}_{k\in\mathbb{Z}}$ be the dual frame. If $g \in L^2(\mathbb{R}, \phi d\lambda)$ is such that $W_k = \int_{\mathbb{R}} g e^{i2\pi\lambda k} d\Phi$, supposing that f and Y_t are as in the previous discussion, then $\mathbb{E}(\overline{Y_t} W_k) = \int_{\mathbb{R}} \overline{f(\lambda)} g(\lambda) \phi(\lambda) e^{-i2\pi\lambda(t-k)} d\lambda$. First, we have that $fg\phi \in L^1(\mathbb{R})$. And, again as in the previous case $f, g \in L^2(\mathbb{R})$. We also have that $\|g\phi\|^2_{L^2(\mathbb{R})} \leq A \int_{\mathbb{R}} |g|^2 \phi d\lambda < \infty$, since $g \in L^2(\mathbb{R}, \phi d\lambda)$. Now by Lemma 4.2 $\widehat{h} := \widehat{g\phi} = \widehat{g} * \widehat{\phi}$ and $\widehat{h} = \widehat{g} * \widehat{\phi} \in L^2(\mathbb{R})$. Then, $\widehat{h} * \widehat{f}$ is well defined and then $\widehat{hf} = \widehat{h} * \widehat{f}$ (in $\mathcal{S}'(\mathbb{R})$) but $hf \in L^1(\mathbb{R})$ so $\widehat{h} * \widehat{f}$ is well defined a.e. and then $\mathbb{E}(Y_t W_k) = \widehat{g} * \widehat{\phi} * \widehat{f}$. Finally, from this

$$Y_t = \sum_{k\in\mathbb{Z}} (\widehat{g} * \widehat{\phi} * \widehat{f})(t-k) X_k.$$

B. Almost Sure Convergence

The representations given above converge in norm. Let us discuss briefly the problem of almost sure convergence for these representations. Not note that point-wise convergence strongly depends on the summation method. First let us examine what happens in this context of the frame algorithm described before in Section II. Here, given $Y \in H(\mathcal{X})$, we can write $Y_0 = 0$ and define $Y_{n+1} = Y_n + \lambda S(Y - Y_n)$, with λ and δ defined as before, then $\mathbb{E}|Y - Y_n|^2 \leq \delta^{2n} \mathbb{E}|Y|^2$. But given $\epsilon > 0$ by Chevyshev's inequality:

$$\mathbf{P}(|Y - Y_n| > \epsilon) \leq \frac{\mathbb{E}|Y - Y_n|^2}{\epsilon^2} \leq \frac{\delta^{2n} \mathbb{E}|Y|^2}{\epsilon^2}$$

then $\sum_n \mathbf{P}(|Y - Y_n| > \epsilon) < \infty$, and then by the first Borel–Cantelli Lemma $Y_n \underset{n\to\infty}{\longrightarrow} Y$ a.s. The Menchoff–Rademacher theorem [15] gives a sufficient condition for the a.s. convergence of orthogonal expansions, a similar result holds for sequences which form a frame. This result could be obtained in a similar manner to [10] adapting the classical proof [15] of the original theorem. We include in the Appendix a sketch of a shorter argument, similar to that of [25] involving absolutely summing operators. To prove such a result one would like to bound the maximal operator $\sup_{N\leq s} \left| \sum_{r=1}^{n} \sum_{|k|\leq N} (\log(k+1))^{-1} c_k^r X_k^r \right|$. Note that this also can be written in terms of norms of sequences, and combining props. 33 and 21 from [25] it is possible to summarize these results in the following.

Lemma 4.3: Let $(c_k)_k \in l^2(\mathbb{Z})$ be such that $\sum_k c_k^2 \log^2(k+1) < \infty$, if we define $T : l^2 \longrightarrow l^\infty$ as $(T\xi)_j = \sum_{|k|\leq j} c_k \xi_k$ and given a random vector $\Theta = (X_{-s}, \ldots, X_0, \ldots, X_s)$ we have $\mathbb{E}\|T\Theta\|^2_{l^\infty} \leq C \sum_k c_k^2 \log^2(k+1) \sup_{x\in l^2 \|x\|=1} \mathbb{E}|\Theta x^*|^2$.

This lemma contains all we need to prove the following.

Proposition 4.2: $\mathcal{X} = \{X_k^r\}_{k\in\mathbb{Z}}^{r=1\ldots n}$ be a stationary sequence which is a frame of its span $H(\mathcal{X})$. If $(c_k^r)_{k\,r} \in l^2$ then $\sum_{r=1}^{n} \sum_{|k|\leq N} (\log(k+1))^{-1} c_k^r X_k^r$ converges almost surely as $N \longrightarrow \infty$.

Proof: See the Appendix.

V. Application: Sampling in an Union of Subspaces

Here we describe briefly the problem of sampling a random signal from an union of subspaces. Using the previous results we can give similar conditions to the work of Lu and Do for deterministic signals in $L^2(\mathbb{R})$ [16].

$A : \chi \longrightarrow l^2(\Lambda)$ is a sampling operator characterized by a stationary sequence generated by $\mathcal{W} = \{W_0^r\}^{r=1\ldots M}$ and let χ be an union of subspaces:

$$\chi = \bigcup_{\gamma\in\Gamma} H(\mathcal{X}_\gamma).$$

We can give conditions in terms of the spectral densities for which this sampling operator is invertible and stable. Recall that every $Y \in H(\mathcal{X})$ admits the following representation $Y = \sum_j \int_{[0,1)} f_j d\Phi_j$ where $H(\mathcal{X}) = \overline{\mathrm{span}}\{T_k X_0^r\}_{k\in\mathbb{Z}}^r$. Let g_r be such that $W_0^r = \sum_j \int_{[0,1)} (g_r)_j d\Phi_j$. If we suppose that $\mathcal{X}$ is a Riesz basis, by Theorem 3.1, the spectral density matrix $D_{\mathcal{X}}$ exists. So we can write,

$$d_{mr} = \int_{[0,1)} f D_{\mathcal{X}} g_r^* e^{i2m\pi\lambda} d\lambda.$$

Define

$$\hat{d}_r(\lambda) = \sum_{m\in\mathbb{Z}} d_{mr} e^{i2m\pi\lambda} = \left(f D_{\mathcal{X}} g_r^*\right)(\lambda).$$

So, if $G_{\mathcal{W}} = (g_1^* \dots g_M^*)$ we can write

$$\hat{d} = f\Gamma_{\mathcal{X}\mathcal{W}} \text{ where } \Gamma_{\mathcal{X}\mathcal{W}} = D_{\mathcal{X}} G_{\mathcal{W}}. \tag{16}$$

Under suitable conditions $H'(\mathcal{X}_{\gamma\theta}) := H(\mathcal{X}_\gamma) + H(\mathcal{X}_\theta)$ is closed. If $Len(H) := \min\left\{K : H \text{ can be generated by } \{X_0^r\}_{r=1}^K\right\}$. Provided that $\mathcal{X}_{\gamma\theta}$ forms a Riesz basis for $H'(\mathcal{X}_{\gamma\theta})$ and if $\mathcal{U}_{\gamma\theta}, \mathcal{U}_\theta, \mathcal{U}_\gamma$ are the (finite) generators of $H'(\mathcal{X}_{\gamma\theta})$, $H(\mathcal{X}_\theta)$, and $H(\mathcal{C}_\gamma)$ respectively, then we have

$$\sharp(\mathcal{U}_{\gamma\theta}) \le \sharp(\mathcal{U}_\gamma) + \sharp(\mathcal{U}_\theta).$$

If $Y \in H'(\mathcal{X}_{\gamma\theta})$, AY can be expressed as the following frequency domain relationship between the sampling data and the signal:

$$f\Gamma_{\mathcal{X}_{\gamma\theta}\mathcal{W}}, \Gamma_{\mathcal{X}_{\gamma\theta}\mathcal{W}} \in \mathbb{C}^{Len(\mathcal{X}_{\gamma\theta})\times M}.$$

Then, as in [16] one can prove the following results. We omit the proofs, since from the previous discussion is clear that the proofs of these results are very similar to their $L^2(\mathbb{R})$ counterparts. The first result provides an easy way to compute minimum sampling requirement M_{min}, interpreted as the minimum number of channels in a multi channel sampling.

Proposition 5.1: If $A : Y \longmapsto \left\{\mathbb{E}\left(\overline{Y} W_k^r\right)\right\}_k^r$ is an invertible sampling operator for $\chi = \bigcup_{\gamma\in\Gamma} H(\mathcal{X}_\gamma)$ then $M \ge M_{min} = \sup_{(\gamma,\theta)\in\Gamma\times\Gamma} Len(H'(\mathcal{X}_{\gamma\theta}))$.

The following gives a condition for stable sampling.

Proposition 5.2: Let $\mathcal{W} = \{W_0^r\}^{r=1\dots M}$ be a set of sampling functions (random variables) and $\mathcal{X}_{\gamma\theta}$ be a set of generators of Riesz basis for $H'(\mathcal{X}_{\gamma\theta})$. Then $\{W_k^r\}_{k\in\mathbb{Z}}^{r=1\dots M}$ provides a stable sampling operator for χ if and only if

$$\begin{aligned} 0 &< \operatorname*{essinf}_{(\gamma\theta)\in\Gamma\times\Gamma,\lambda\in[0,1]} \sigma^2(\Gamma_{\mathcal{X}_{\gamma\theta}\mathcal{W}}(\lambda)) \\ &\le \operatorname*{esssup}_{(\gamma\theta)\in\Gamma\times\Gamma,\lambda\in[0,1]} \sigma^2(\Gamma_{\mathcal{X}_{\gamma\theta}\mathcal{W}}(\lambda)) < \infty. \end{aligned}$$

Appendix
Proofs of Some Auxiliary Results

1) Proof of Lemma 2.1:

a) In [18] it is proved that every non negative self adjoint matrix is diagonalizable in a measurable form $D = P\Lambda P^*$, then $D^\sharp = D^{(-1)}$ and $DD^\sharp = P_{Col(D)}$ are measurable.

b) From the previous $I - P_{Col(D)} = P_{Nul(D)}$ is measurable. Now if $\mathcal{L}(\{\lambda : \mathrm{rg}(D)(\lambda) < n\}) > 0$ and if $A_j = \{\lambda : (P_{Nul(D)})_j(\lambda) \neq 0\}$, then $\{\lambda : \mathrm{rg}(D)(\lambda) < n\} = \bigcup_{j=1}^n A_j$. From this, there exists a column $(P_{Nul(D)})_j(\lambda) \neq 0$ for every λ in some measurable A_j, with $\mathcal{L}(A_j) > 0$.

Part c) For such g, put $Z = \sum_{j=1}^n \int_{[0,1)} (f_j + g_j) d\Phi_j$, then $\mathbb{E}|Y - Z|^2 = \int_{[0,1)} gDg^* d\lambda = 0$ since $g \in Nul(D)$ a.e. so $Y = Z$ a.s. ■

2) Proof of Lemma 4.1: Given $Y \in H(\mathcal{X})$, $k \in \mathbb{Z}$: $S(T^jY) = \sum_{r=1}^n \sum_{k\in\mathbb{Z}} \mathbb{E}(T^jY\overline{T^kX_0^r})T^kX_0^r$. Recall that T is unitary so $T^* = T^{-1}$ and then $S(T^jY) = \sum_{r=1}^n \sum_{k\in\mathbb{Z}} \mathbb{E}\left(Y\overline{T^{k-j}X_0^r}\right)T^kX_0^r$. Making a change of variables $k - j =: m$ we have

$$S(T^jY) = \sum_{r=1}^n \sum_{m\in\mathbb{Z}} \mathbb{E}\left(Y\overline{T^mX_0^r}\right)T^{m+j}X_0^r = T^j(SY).$$

Finally, $S^{-1}T^kSS^{-1}Y = (S^{-1}S)T^kSY = T^kS^{-1}Y$. ■

3) Proof of Lemma 4.2: Note that $(f * g)(t)$ is well defined for all t and is a bounded function as a consequence of the Cauchy Schwartz inequality. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in the Schwartz space $\mathcal{S}(\mathbb{R})$ such that $\varphi_n \longrightarrow g$ in $L^2(\mathbb{R})$ as $n \longrightarrow \infty$, then by Cauchy–Schwartz: $|f*g(t) - f*\varphi_n(t)| \le \|f\|_{L^2(\mathbb{R})}\|g - \varphi_n\|_{L^2(\mathbb{R})}$ so that $f * \varphi_n \underset{n\to\infty}{\longrightarrow} f * g$ uniformly. Taking $\psi \in \mathcal{S}(\mathbb{R})$ we have that $|f * (g - \varphi_n)(t)||\widehat{\psi}(t)| \le \|f\|_{L^2(\mathbb{R})}M|\widehat{\psi}(t)| \in L^1(\mathbb{R})$ then by the dominated convergence theorem:

$$\lim_{n\to\infty} \int_{\mathbb{R}} (f*\varphi_n)(t)\widehat{\psi}(t)dt = \int_{\mathbb{R}} (f*g)(t)\widehat{\psi}(t)dt. \tag{17}$$

But we also have that

$$\forall\, n : \int_{\mathbb{R}} (f*\varphi_n)(t)\widehat{\psi}(t)dt = \int_{\mathbb{R}} (\widehat{f}\widehat{\varphi}_n)(t)\psi(t)dt, \tag{18}$$

and again by the Cauchy–Schwartz inequality:

$$\left|\int_{\mathbb{R}} \widehat{f}\psi(\widehat{\varphi}_n - \widehat{g})dt\right| \le \|\widehat{f}\psi\|_{L^2(\mathbb{R})}\|\widehat{g} - \widehat{\varphi}_n\|_{L^2(\mathbb{R})} \underset{n\to\infty}{\longrightarrow} 0. \tag{19}$$

Combining (17), (18) and (19) we have that

$$\begin{aligned} \langle f*g, \widehat{\psi}\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}} (f*\varphi_n)(t)\widehat{\psi}(t)dt \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} (\widehat{f}\widehat{\varphi}_n)(t)\psi(t)dt = \langle \widehat{f}\widehat{g}, \psi\rangle. \end{aligned}$$

On the other hand

$$\begin{aligned} \|\widehat{f}\widehat{g}\| &= \sup_{\|\psi\|=1 \psi\in\mathcal{S}(\mathbb{R})} \int_{\mathbb{R}} \widehat{f}\widehat{g}\psi dt \\ &= \sup_{\|\psi\|=1 \psi\in\mathcal{S}(\mathbb{R})} \int_{\mathbb{R}} (f*g)\widehat{\psi} dt = \|f*g\|. \end{aligned}$$

■

4) Proof of Proposition 4.2: The result follows if we bound the expected value of the square of the maximal function

$$\sup_{N\leq s}\left|\sum_{r=1}^{n}\sum_{|k|\leq N}(\log(k+1))^{-1}c_k^rX_k^r\right| \leq \sum_{r=1}^{n}\sup_{N\leq s}\left|\sum_{|k|\leq N}(\log(k+1))^{-1}c_k^rX_k^r\right|$$

then

$$\left(\mathbb{E}\sup_{N\leq s}\left|\sum_{r=1}^{n}\sum_{|k|\leq N}(\log(k+1))^{-1}c_k^rX_k^r\right|^2\right)^{\frac{1}{2}} \leq \sum_{r=1}^{n}\left(\mathbb{E}\sup_{N\leq s}\left|\sum_{|k|\leq N}(\log(k+1))^{-1}c_k^rX_k^r\right|^2\right)^{\frac{1}{2}}.$$

Now, from Lemma 4.3, for each r:

$$\mathbb{E}\sup_{N\leq s}\left|\sum_{|k|\leq N}(\log(k+1))^{-1}c_k^rX_k^r\right|^2 \leq C_r\|c^r\|_{l^2}\sup_{a\in l^2(\mathbb{Z})\|a\|=1}\mathbb{E}\left|\sum_{|k|\leq s}a_kX_k^r\right|^2.$$

Since the sequence $\mathcal{X}$ is Besselian $\mathbb{E}\left|\sum_{|k|\leq s}a_kX_k^r\right|^2 \leq B\|a\|^2_{l^2(\mathbb{Z})} \leq B$. From this: $\mathbb{E}\sup_{N\leq s}\left|\sum_{|k|\leq N}(\log(k+1))^{-1}c_k^rX_k^r\right|^2 \leq C_rB$. Then using the Cauchy–Schwartz inequality:

$$\mathbb{E}\sup_{N\leq s}\left|\sum_{r=1}^{n}\sum_{|k|\leq N}(\log(k+1))^{-1}c_k^rX_k^r\right|^2 \leq \left(\sum_{r=1}^{n}C_rB\|c^r\|_{l^2(\mathbb{Z})}\right)^2 \leq C'\sum_{r=1}^{n}\sum_{k\in\mathbb{Z}}|c_k^r|^2.$$

■

ACKNOWLEDGMENT

The authors would to like to thank the anonymous reviewers for their very helpful comments.

REFERENCES

[1] M. G. Beaty and M. M. Dodson, "The WKS theorem, spectral translates and Plancherel's formula," *J. Fourier Anal. Appl.*, vol. 10, no. 2, pp. 179–199, 2004.

[2] J. J. Benedetto and L. Shidong, "The theory of multiresolution analysis frames and applications to filter banks," *Appl. Comput. Harmon. Anal.*, vol. 5, pp. 389–427, 1998.

[3] M. Bownik, "The structure of shift invariant subspaces of $L^2(\mathbb{R})$," *J. Funct. Anal.*, vol. 177, no. 2, pp. 282–309, 2000.

[4] C. De Boor, R. DeVore, and A. Ron, "The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$," *J. Funct. Anal.*, vol. 119, no. 1, pp. 37–78, 1994.

[5] S. Cambanis and E. Masry, "Wavelet approximation of determinstic and random signals: Convergence properties and rates," *IEEE Trans. Inf. Theory*, vol. 40, pp. 1013–1029, 1994.

[6] O. Christensen, "Frames and bases," in *Applied and Numerical Harmonic Analysis Series*. Boston, MA: Birkhäuser, 2008.

[7] H. Dym and H. P. McKean, *Gaussian Processes, Function Theory and The Inverse Spectral Problem*. New York: Academic, 1976.

[8] K. Gröchenig, *Time Frequency Analysis*. Birkhäuser: , 2001.

[9] C. Houdré, "Wavelets, Probability and Statistics, Some Bridges," in *Chapter 9 in Wavelets: Mathematics and Applications*, J. Benedetto and M. Frazier, Eds. Boca Raton, FL: CRC, 1994, vol. 52.

[10] C. Houdré, "On the almost sure convergence of series of stationary and related non stationary random variables," *Ann. Probabil.*., vol. 23, no. 3, pp. 1204–1218, 1995.

[11] I. A. Ibragimov and Y. A. Rozanov, *Gaussian Random Processes*. Berlin, Germany: Springer Verlag, 1978.

[12] S. Kwapien and W. A. Woyczynski, *Random Series and Stochastic Integrals Single and Multiple*. Boston, MA: Birkhäuser, 1992.

[13] A. J. Lee, "Sampling theorems for non stationary random processes," *Trans. Amer. Math. Soc.*, vol. 242, pp. 225–241, 1978.

[14] S. P. Lloyd, "A sampling theorem for stationary (wide sense) stochastic processes," *Trans. Amer. Math. Soc*, vol. 92, no. 1, pp. 1–12, 1959.

[15] M. Loéve, *Probability Theory*. Berlin, Germany: Springer Verlag, 1977, vol. II.

[16] Y. M. Lu and M. N. Do, "A theory for sampling signals from a union of subspaces," *IEEE Trans. Signal Process.*, vol. 56, no. 6, pp. 2334–2345, 2008.

[17] P. Masani, "Shift invariant spaces and prediction theory," *Acta Math.*, vol. 107, pp. 275–290, 1962.

[18] A. Ron and Z. Shen, "Frames and stable bases for shift -invariant subspaces of $L^2(\mathbb{R}^d)$," *Can. J. Math.*, vol. 47, pp. 1051–1094, 1995.

[19] M. Rosenblatt, *Stationary Sequences and Random Fields*. Boston, MA: Birkhäuser, 1985.

[20] Y. A. Rozanov, *Stationary Random Processes*. San Francisco, CA: Holden-Day, 1967.

[21] Y. A. Rozanov, *On Stationary Sequences Forming a Basis*. Moscow, Russia: Soviet Math.-Doklady, 1960, vol. 1, pp. 91–93.

[22] A. Weron, "On characterizations of interpolable and minimal stationary processes," *Studia Math.*, vol. 49, pp. 165–183, 1974.

[23] A. Makagon and A. Weron, "Q-variate minimal stationary processes," *Studia Math.*, vol. 59, pp. 41–52, 1976.

[24] N. Wiener and P. Masani, "The prediction theory of multivariate stochastic processes, II: The linear predictor," *Acta Math.*, vol. 99, pp. 94–137, 1959.

[25] P. Wojtaszczyc, *Banach Spaces for Analysts*. Cambridge, U.K.: Cambridge Univ. Press, 1991.

[26] P. Zhao, G. Lin, and C. Zhao, "A matrix-valued wavelet KL-like expansion for wide sense stationary processes," *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 914–920, 2004.

Juan Miguel Medina (M'11) was born in Buenos Aires, Argentina, in 1977. He received the Licenciado en Ciencias Matemáticas degree in pure mathematics from the Universidad de Buenos Aires, Argentina. He received the Ph.D. degree in mathematics from the Universidad de Buenos Aires in 2011.

He is currently with the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) at the Instituo Argentino de Matemática "A.P. Calderòn," Buenos Aires.

Bruno Cernuschi-Frías (S'77–M'78–SM'01) was born in 1952 in Montevideo, Uruguay. He received the Ph.D. degree in electrical engineering from Brown University, Providence, RI, in 1984.

He is currently a full tenured Professor with the Department of Electronics, Faculty of Engineering, University of Buenos Aires, Argentina, and Principal Researcher at the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET).