# Complex Analysis and Operator Theory

## Classes of idempotents in Hilbert space

--Manuscript Draft--

| | |
|---|---|
| **Manuscript Number:** | |
| **Full Title:** | Classes of idempotents in Hilbert space |
| **Article Type:** | Original Research Paper |
| **Keywords:** | Projection; Idempotent operator |
| **Corresponding Author:** | Esteban Andruchow<br>Universidad Nacional de General Sarmiento<br>Los Polvorines, Buenos Aires ARGENTINA |
| **Corresponding Author Secondary Information:** | |
| **Corresponding Author's Institution:** | Universidad Nacional de General Sarmiento |
| **Corresponding Author's Secondary Institution:** | |
| **First Author:** | Esteban Andruchow |
| **First Author Secondary Information:** | |
| **Order of Authors:** | Esteban Andruchow |
| **Order of Authors Secondary Information:** | |
| **Funding Information:** | CONICET (PIP 0757) — Mr. Esteban Andruchow |
| **Abstract:** | An idempotent operator E in a Hilbert space H (E^2 = 1) is written as a 2 x 2 matrix in terms of the orthogonal decomposition<br>H = R(E) + R(E)^\perp<br>(R(E) is the range of E)<br>We study the sets of idempotents that one obtains when the entry E_{1,2} is a special type of operator: compact, Fredholm and injective with dense range, among others. |

# Classes of Idempotents in Hilbert space

Esteban Andruchow

December 15, 2015

**Abstract**

An idempotent operator $E$ in a Hilbert space $\mathcal{H}$ ($E^2 = 1$) is written as a $2 \times 2$ matrix in terms of the orthogonal decomposition

$$\mathcal{H} = R(E) \oplus R(E)^\perp$$

($R(E)$ is the range of $E$) as

$$E = \begin{pmatrix} 1_{R(E)} & E_{1,2} \\ 0 & 0 \end{pmatrix}.$$

We study the sets of idempotents that one obtains when $E_{1,2} : R(E)^\perp \to R(E)$ is a special type of operator: compact, Fredholm and injective with dense range, among others.

**2010 MSC:** 47A, 47B, 47B07.
**Keywords:** Projections, Idempotent operators.

## 1 Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators in $\mathcal{H}$, $\mathcal{Q}$ the set of idempotent operators, i.e. operators $E$ such that $E^2 = E$, and $\mathcal{P}$ the set of orthogonal projections in $\mathcal{H}$ (selfadjoint elements in $\mathcal{Q}$). Given an operator $A$ with closed range, $P_{R(A)}$ and $P_{N(A)}$ will denote the orthogonal projections onto the range $R(A)$ and the nullspace $N(A)$ of $A$, respectively. Given an orthogonal projection $P$, operators can be written as $2\times2$ in terms of the decomposition $\mathcal{H} = R(P) \oplus N(P)$. In particular if $E \in \mathcal{Q}$, in terms of $P_{R(E)}$,

$$E = \begin{pmatrix} 1 & E_{1,2} \\ 0 & 0 \end{pmatrix}.$$

An idempotent $E$ determines, and is determined by, the (non orthogonal) decomposition $\mathcal{H} = R(E)\dot{+}N(E)$ (we shall reserve the symbol $\oplus$ for orthogonal sums, and the symbol $\dot{+}$ for direct sums). There are well known formulas highlighting this correspondence, for instance [2]

$$P_{R(E)} = E(E + E^* - 1)^{-1}, \quad P_{N(E)} = (1 - E)(1 - E - E^*)^{-1} \tag{1}$$

and [7]

$$E = P_{R(E)}(P_{R(E)} - P_{N(E)})^{-1}. \tag{2}$$

Implicit in these formulas are the facts that $E+E^*-1$ and $P_{R(E)}-P_{N(E)}$ are invertible operators for any given $E \in \mathcal{Q}$.

In this paper we study the following subsets of $\mathcal{Q}$:

1. The set $\mathcal{Q}_d$ of idempotents $E$ such that $E^*E$ is diagonalizable (we say the $A$ is diagonalizable if there exists an orthonormal system $\{f_n\}_{n\geq 1}$ and complex numbers $\alpha_n$ such that $A\xi = \sum_{n\geq 1} \alpha_n \langle \xi, f_n\rangle f_n$, for any $\xi \in \mathcal{H}$).

2. The set $\mathcal{Q}_k$ of idempotents $E$ such that in the matrix form above, $E_{1,2}$ is compact.

3. The set $\mathcal{Q}_g$ of idempotents $E$ such that $R(E)$ and $N(E)$ are in generic position. Two subspaces $\mathcal{S}, \mathcal{T} \subset \mathcal{H}$ are in generic position [13] if

$$\mathcal{S} \cap \mathcal{T} = \mathcal{S}^\perp \cap \mathcal{T} = \mathcal{S} \cap \mathcal{T}^\perp = \mathcal{S}^\perp \cap \mathcal{T}^\perp = \{0\}.$$

4. The set $\mathcal{Q}_f$ of idempotents $E$ such that the pair $(P_{R(E)}, P_{N(E)})$ is a Fredholm pair of projections [5], [1]. A pair of projections $(P, Q)$ is a Fredholm pair if

$$PQ|_{R(Q)} : R(Q) \to R(P)$$

   is a Fredholm operator in $\mathcal{B}(R(Q), R(P))$. The index of this operator is the index of the pair, and is the integer

$$ind(P, Q) = \dim(R(P) \cap N(Q)) - \dim(N(P) \cap R(Q)).$$

5. The set $\mathcal{Q}_c$ of idempotents $E$ such that the selfadjoint contraction $A = P_{R(E)} - P_{N(E)}$ has a cyclic vector in $\mathcal{H}$.

The contents of the paper are the following. In Section 2 we recall some preliminary facts, concerning the Halmos' decomposition of $\mathcal{H}$ induced by a pair of projections. In Section 3 we study the set $\mathcal{Q}_d$, we give characterizarions and compute its connected components. $\mathcal{Q}_d$ is shown to be dense in $\mathcal{Q}$. In Section 4 we study the set $\mathcal{Q}_k$, also here we compute the connected components. These are closed submanifolds of $\mathcal{B}(\mathcal{H})$, not necesarilly complemented. Moreover, it is shown that $\mathcal{Q}_k$ admits the action of the linear Fredholm group

$$Gl_\infty(\mathcal{H}) = \{G \in \mathcal{B}(\mathcal{H}) : G \text{ is invertible and } G - 1 \text{ is compact}\}.$$

The connected components of $\mathcal{Q}_k$ are the orbits of this action. In Section 5 we study the set $\mathcal{Q}_g$. Elements $E \in \mathcal{Q}_g$ are characterized by the property that there exists a unique minimal geodesic of $\mathcal{P}$ joining $P_{R(E)}$ and $P_{N(E)}$. $\mathcal{Q}_g$ is connected. In Section 6 we study $\mathcal{Q}_f$. Elements in $\mathcal{Q}_f$ have naturally an index. It is shown that the connected components of $\mathcal{Q}_f$ are open in $\mathcal{Q}$, and are parametrized by the index. In Section 7 we introduce three symmetries (=selfadjoint unitaries in $\mathcal{H}$) with remarkable properties with respect to the classes considered. In Section 8 we study $\mathcal{Q}_c$.

The author wishes to thank Gustavo Corach for many helpful comments.

## 2 preliminary facts

Let us recall the following facts concerning the theory of two projections (see for instance [13] or [1] or [6]). Let $P_1, P_0 \in \mathcal{P}$. We shall consider the special case $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, for some $E \in \mathcal{Q}$, which corresponds with the property $P_1 - P_0$ invertible, due to the formulas above. For arbitrary $P_1, P_0$ denote

$$\mathcal{H}_{11} = R(P_1) \cap R(P_0)\ ,\ \ \mathcal{H}_{00} = N(P_1) \cap N(P_0)\ ,\ \mathcal{H}_{10} = R(P_1) \cap N(P_0)\ ,\ \ \mathcal{H}_{01} = N(P_1) \cap R(P_0)$$

and $\mathcal{H}_0$ the orthogonal complement of the sum of the above. This last subspace is usually called the generic part of the pair $P_1, P_0$. Note also that

$$N(P_1 - P_0) = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \ , \quad N(P_1 - P_0 - 1) = \mathcal{H}_{10} \ \text{ and } N(P_1 - P_0 + 1) = \mathcal{H}_{01},$$

so that the generic part depends in fact of the difference $P_1 - P_0$. In the case $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, $\mathcal{H}_{11} = \mathcal{H}_{00} = \{0\}$, therefore Halmos' decomposition consists of three subspaces. We shall refer it as the *three space decomposition* induced by $E$

Halmos proved that there is an isometric isomorphism between $\mathcal{H}_0$ and a product Hilbert space $\mathcal{L} \times \mathcal{L}$ such that in the above decomposition (putting $\mathcal{L} \times \mathcal{L}$ in place of $\mathcal{H}_0$), the *generic parts* of the projections $P_1$ and $P_0$ are, respectively

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = cos(X)$ and $S = sin(X)$ for some operator $0 < X \leq \pi/2$ in $\mathcal{L}$ with trivial nullspace. Therefore, in our case $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, one has (in the three space decomposition $\mathcal{H} = \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_0$)

$$P_1 = 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_0 = 0 \oplus 1 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}.$$

In particular,

$$(P_1 - P_0)^2 = 1 \oplus 1 \oplus \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix},$$

so that in this case ($P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$) $S$ and $X$ are invertible in $\mathcal{L}$. In the three space decomposition of $\mathcal{H}$, $E$ is of the form

$$E = 1 \oplus 0 \oplus \begin{pmatrix} 1 & -S^{-1}C \\ 0 & 0 \end{pmatrix}.$$

This follows after straightforward matrix computations, using formula (2).

The following lemma applies in any of the subsets of $\mathcal{Q}$ studied here, and will be useful in the study of their connected componentes.

**Lemma 2.1.** *Suppose that $E$ and $F$ are in the same connected component of $\mathcal{Q}$, and in the same class $\mathcal{Q}_x$ ($x = d, k, g, f$ or $c$). Then there exists a unitary operator $U$ in $\mathcal{H}$ such that $E$ and $UFU$ lie again in the same component of $\mathcal{Q}$, the same class $\mathcal{Q}_x$, and have the same range.*

*Proof.* The first two assertions are true for any unitary operator: $F$ and $UFU^*$ are in the same componwent of $\mathcal{Q}$ (the unitary group of $\mathcal{H}$ is connected), and in the same class $\mathcal{Q}_x$ (unitary conjugation trivially preserves these classes). Then it only remains to find a unitary operator $U$ such that $R(E) = R(UFU^*)$. Since $E$ and $F$ are in the same component of $\mathcal{Q}$, and the map $E \mapsto P_{R(E)}$ is continuous in $\mathcal{Q}$ (using the first of the formulas in (1)). Then $P_{R(E)}$ and $P_{R(F)}$ lie in the same connected component of $\mathcal{P}$. It is known that the connected components of $\mathcal{P}$ coincide with the orbits of the unitary conjugation. Then there exists a unitary operator $U$ such that

$$UP_{R(E)}U^* = P_{R(F)}.$$

The proof follows noting that $UP_{R(E)}U^* = P_{R(UEU^*)}$. ☐

# 3 Diagonalizable idempotents

In this section we study the set

$$\mathcal{Q}_d = \{E \in \mathcal{Q} : E^*E \text{ is diagonalizable } \}.$$

**Remark 3.1.** If $E \in \mathcal{Q}_d$, then there exist orthonormal systems $\{v_n\}_{n\geq 1}$ and $\{w_n\}_{n\geq 1}$ and real numbers $s_n \geq 1$ such that

$$E\xi = \sum_{n\geq 1} s_n \langle \xi, v_n\rangle w_n,$$

where $\langle w_i, v_j\rangle = \frac{1}{s_i}\delta_{ij}$. Moreover, $s_i = 1$ if and only if $v_i = w_i$.

Indeed, this follows from the polar decomposition of $E$, $E = V(E^*E)^{1/2}$. Since $E^*E$ is diagonalizable, there exists an orthonormal system $\{v_n\}$, and $s_n \geq 0$ (the singular values of $E$) such that

$$(E^*E)^{1/2}\xi = \sum_{n\geq 1} s_n \langle \xi, v_n\rangle v_n.$$

Then $E\xi = \sum_{n\geq 1} s_n \langle \xi, v_n\rangle V v_n$. Clearly $w_n = V v_n$ form an orthonormal system. Also, since $w_j \in R(E)$,

$$w_j = E(w_j) = \sum_{n\geq 1} s_n \langle w_j, v_n\rangle w_n,$$

and thus $s_n\langle w_j, v_n\rangle = \delta_{jn}$. Note that

$$1 = \|w_j\| = s_j\langle w_j, v_j\rangle,$$

and $0 \leq \langle w_j, v_j\rangle \leq 1$. Equality occurs in and only if $v_j$ is a multiple of $w_j$, and thus they are equal. Apparently, any operator $E$ of this form is an idempotent in $\mathcal{Q}_d$.

**Remark 3.2.** The expression obtained above implies that $E \in \mathcal{Q}_d$ if and only if $E^* \in \mathcal{Q}_d$. Indeed, if $E \in \mathcal{Q}_d$, using the usual notation $w \otimes v$ for the rank one operator $w \otimes v(\xi) = \langle \xi, v\rangle w$, one has

$$E = \sum_{n\geq 1} s_n w_n \otimes v_n,$$

(the series considereed in the strong operator topology) with $\{v_n\}, \{w_n\}$ orthonormal system satisfying $\langle w_i, v_j\rangle = \frac{1}{s_i}\delta_{ij}$. Then

$$E^* = \sum_{n\geq 1} s_n v_n \otimes w_n$$

is an idempotent operator of the same type.

Note the following elementary fact:

**Lemma 3.3.** *Let $A \in \mathcal{B}(\mathcal{H})$ be selfadjoint. Then $A$ is diagonalizable if and only if $A^2$ is diagonalizable.*

*Proof.* $A$ diagonalizable implies $A^2$ diagonalizable (with the same basis). Suppose $A^2$ diagonalizable. Then

$$A^2 = \sum_{n\geq 1} \lambda_n P_n,$$

with $\lambda_n > 0$ ($\lambda_n \neq \lambda_m$ if $n \neq m$) and $\{P_n\}_{n\geq 1}$ pairwise orthogonal. Since $A$ commutes with $A^2$, it commutes with the spectral projections $P_n$ of $A^2$. Then

$$(P_n A)^2 = \lambda_n P_n.$$

Thus if we regard $P_n A$ as an operator in $R(P_n)$, it is of the form

$$P_n A = \sqrt{\lambda_n} P_n^+ - \sqrt{\lambda_n} P_n^-,$$

with $P_n^+ + P_n^- = P_n$, $P_n^+ P_n^- = 0$. Tnen

$$A = \sum_{n\geq 1} P_n A = \sum_{n\geq 1} \sqrt{\lambda_n} P_n^+ - \sum_{n\geq 1} \sqrt{\lambda_n} P_n^-.$$

□

With the current notations we have:

**Proposition 3.4.** *The following are equivalent*

1. $E \in \mathcal{Q}_d$.
2. $E_{12}E_{12}^*$ *is diagonalizable in* $R(E)$.
3. $P_{R(E)} - P_{N(E)}$ *is diagonalizable in* $\mathcal{H}$.
4. $X$ *is diagonalizable in* $\mathcal{L}$.

*Proof.* In matrix form

$$EE^* = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E_{12}^* & 0 \end{pmatrix} = \begin{pmatrix} 1 + E_{12}E_{12}^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus apparently $EE^*$ is diagonalizable if and only if $E_{12}E_{12}^*$ is diagonalizable.

Denote $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Using formula (2),

$$EE^* = P_1(P_1 - P_0)^{-2} P_1.$$

Using the (three space) decomposition $\mathcal{H} = \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus (\mathcal{L} \times \mathcal{L})$,

$$(P_1 - P_0)^2 = 1 \oplus 1 \oplus \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}$$

and thus

$$EE^* = 1 \oplus 0 \oplus \begin{pmatrix} S^{-2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Apparently $EE^*$ is diagonalizable if and only if $S^{-2}$ is diagonalizable in $\mathcal{L}$, which is equivalent both to $S$ and $X$ being diagonalizable in $\mathcal{L}$. If $S^2$ is diagonalizable, then clearly $(P_1 - P_0)^2$ and $P_1 - P_0$ are diagonalizable in $\mathcal{H}$.

Conversely, if $(P_1 - P_0)^2$ is diagonalizable, the matrix

$$\begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}$$

is diagonalizable. Any eigenvector $(\xi_n, \eta_n)$ of this matrix with eigenvalue $s_n$ consists of a pair of eigenvectors of $S^2$ with the same eigenvalue. On the other hand, any pair of $s_n$-eigenvectors of $S^2$ is an eigenvector of this matrix. We must show that the linear span of the set of eigenvectors of $S^2$ is dense in $\mathcal{L}$. Suppose that $\xi_0$ is orthogonal to all the eigenvectors of $S^2$. Then the pair $(\xi_0, \xi_0)$ is orthogonal to all pairs of eigenvectors of $S^2$, i.e. all eigenvectors ot the matrix. Then $\xi_0 = 0$. Thus $S^2$ and $S$ are diagonalizable. $\square$

Using Lemma (2.1), one can characterize the connected components of $\mathcal{Q}_d$ (with the relative topology given by the norm of $\mathcal{B}(\mathcal{H})$). Recall the elementary fact that two orthogonal projections lie in the same connected component of $\mathcal{P}$ (or are unitarilly equivalent) if and only if they have the same rank and nullity.

**Proposition 3.5.** *Let $E, F \in \mathcal{Q}_d$. Then they lie in the same connected component if and only if*

$$\dim(R(E)) = \dim(R(F)) \textit{ and } \dim(N(E)) = \dim(N(F)).$$

*Proof.* Using Lemma (2.1), we may reduce to the case $R(E) = R(F)$. Indeed, the dimension conditions above occur if and only if $P_{R(E)}$ and $P_{R(F)}$ lie in the same connected component of $\mathcal{P}$.

Then

$$E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & F_{12} \\ 0 & 0 \end{pmatrix}$$

in the same decomposition. Let

$$E(t) = \begin{pmatrix} 1 & tE_{12} \\ 0 & 0 \end{pmatrix}.$$

Clearly $t \mapsto E(t)$ is a continuous path with values in $\mathcal{Q}_d$ ($E_{12}(t)E_{12}^*(t) = t^2 E_{12}E_{12}^*$ is diagonalizable), which connects $E$ to $P_{R(E)}$. There is a similar path $F(t)$ connecting $F$ to $P_{R(F)} = P_{R(E)}$. Thus $E$ and $F$ lie in the same connected component of $\mathcal{Q}_d$. $\square$

The following is a straightforward consequence of the Theorem of Weyl and von Neuman:

**Proposition 3.6.** *$\mathcal{Q}_d$ is dense in $\mathcal{Q}$.*

*Proof.* Pick $E \in \mathcal{Q}$. Using the three space decomposition, we can suppose that $E$ is of the form

$$1 \oplus 0 \oplus \begin{pmatrix} 1 & -S^{-1}C \\ 0 & 0 \end{pmatrix}.$$

Note that $-S^{-1}C$ is selfadjoint ($S$ and $C$ commute). Then, by the Theorem of Weyl and von Neumann, for any $\epsilon > 0$ there exists a selfadjoint operator $B_\epsilon$ acting in $\mathcal{L}$, which is diagonalizable, such that $\| - S^{-1}C - B_\epsilon\| < \epsilon$. Let $E_\epsilon$ be

$$E_\epsilon = 1 \oplus 0 \oplus \begin{pmatrix} 1 & B_\epsilon \\ 0 & 0 \end{pmatrix}.$$

Apparently, $\|E - E_\epsilon\| = \| - S^{-1}C - B_\epsilon\| < \epsilon$. Clearly $E_\epsilon \in \mathcal{Q}_d$: $B_\epsilon^2$ is diagonalizable. $\square$

## 4 Idempotents with compact off diagonal entry

In this section we study the set

$$\mathcal{Q}_k = \{E \in \mathcal{Q} : E_{12} \text{ is compact } \}$$

of idempotents with compact off-diagonal entry, or shortly, off-diagonal compact idempotents.

**Proposition 4.1.** *Let $E \in \mathcal{Q}$. The following are equivalent:*

1. $E \in \mathcal{Q}_k$.
2. $E - E^*$ *is compact.*
3. $P_{R(E)} + P_{N(E)} - 1$ *is compact.*
4. $C$ *is compact in* $\mathcal{L}$.
5. $P_{R(E)} P_{N(E)}$ *is compact.*

*Proof.* In matrix form

$$E - E^* = \begin{pmatrix} 0 & E_{12} \\ -E_{12}^* & 0 \end{pmatrix}.$$

Apparently $E - E^*$ is compact if and only if $E_{12}$ is compact. As before, denote $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Using the formulas (1),

$$P_1 - P_0 - 1 = E(E + E^* - 1)^{-1} + (1 - E)(1 - E - E^*)^{-1} - 1 = (E - E^*)\{E + E^* - 1\}^{-1},$$

it follows that $E - E^*$ is compact if and only if $P_1 + P_0 - 1$ is compact.

In the three space decomposition

$$E - E^* = 0 \oplus 0 \oplus \begin{pmatrix} 0 & -S^{-1}C \\ -S^{-1}C & 0 \end{pmatrix}.$$

Thus it is compact if and only if $C$ is compact (recall that $S$ in invertible in $\mathcal{L}$).

Finally, note that in this decomposition,

$$P_1 P_0 = 0 \oplus 0 \oplus \begin{pmatrix} C^2 & CS \\ 0 & 0 \end{pmatrix},$$

which is compact in $\mathcal{H}$ if and only if $C$ is compact in $\mathcal{L}$. □

In particular, $E \in \mathcal{Q}_k$ if and only if $E^* \in \mathcal{Q}_k$.

**Remark 4.2.** If $E \in \mathcal{Q}_k$ is non orthognal, since the operator $C = cos(X)$ has non trivial kernel, it follows that

$$X = \sum_{n \geq 1} x_n P_n,$$

with $x_n$ a strictly increasing sequence converging to $\pi/2$, and $P_n$ pairwise ortohogonal of finite rank, with $\sum_{n\geq 1} P_n = 1_{\mathcal{L}}$.

Note that $\mathcal{Q}_k \subset \mathcal{Q}_d$.

**Proposition 4.3.** *Let $E, F \in \mathcal{Q}_k$. Then $E$ and $F$ lie in the same connected component of $\mathcal{Q}_k$ if and only if*

$$\dim(R(E)) = \dim(R(F)) \text{ and } \dim(N(E)) = \dim(N(F)).$$

*Proof.* Using the same argument as in the analogous result in the previous section, based on Lemma 2.1, we can suppose that $E$ and $F$ are of the form

$$E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & F_{12} \\ 0 & 0 \end{pmatrix}$$

in the same decomposition (i.e. $R(E) = R(F)$). Both idempotents can be connected within $\mathcal{Q}_k$ by means of the line segment

$$E(t) = \begin{pmatrix} 1 & tE_{12} + (1-t)F_{12} \\ 0 & 0 \end{pmatrix}.$$

□

We shall see that $\mathcal{Q}_k$ is a differentiable submanifold of $\mathcal{B}(\mathcal{H})$. It lies inside $\mathcal{Q}$, which is a complemented submanifold of $\mathcal{B}(\mathcal{H})$ [9]. However, $\mathcal{Q}_k$ is not necessarily a *complemented* submanifold. These fact is based on the following result:

**Lemma 4.4.** *Fix an orthogonal projection $P$ in $\mathcal{B}(\mathcal{H})$. Then the set*

$$\mathcal{P}_P = \{Q \in \mathcal{P} : [Q, P] \textit{ is compact } \}$$

*is a closed $C^\infty$ submanifold of $\mathcal{B}(\mathcal{H})$.*

*Proof.* Apparently $\mathcal{P}_p$ is a closed subset of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{B}_P$ be

$$\mathcal{B}_P = \{A \in \mathcal{B}(\mathcal{H}) : [A, P] \text{ is compact } \}.$$

Then $\mathcal{B}_P$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Indeed, if

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

is the quotient map onto de Calkin algebra ($\mathcal{K}(\mathcal{H})$ is the ideal of compact operators), then

$$\mathcal{B}_P = \pi^{-1}(\{a \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) : [a, \pi(P)] = 0\}).$$

Then $\mathcal{B}_P$ is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$, being the pre-image of a $C^*$-algebra by a $*$-homomorphism. The space $\mathcal{P}_P$ is the space of selfadjoint projections of $\mathcal{B}_P$. In [9] it was proven the the space of selfadjoint projections of an arbitrary $C^*$-algebra is a complemented submanifold of the algebra. Thus $\mathcal{P}_P$ is a submanifold of $\mathcal{B}(\mathcal{H})$, which may not be complemented, since $\mathcal{B}_p$ may not be a complemented subalgebra of $\mathcal{B}(\mathcal{H})$. □

**Remark 4.5.** $\mathcal{B}_P$ is complemented in $\mathcal{B}(\mathcal{H})$ only if $P$ has finite or cofinite rank, in which case $\mathcal{B}_p = \mathcal{B}(\mathcal{H})$. Indeed, if we fix $P \in \mathcal{P}$ and write the elements of $\mathcal{B}(\mathcal{H})$ as $2 \times 2$ matrices in terms of $P$, a simple comuputation shows that

$$\mathcal{B}_P = \{A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : A_{12}, A_{21} \text{ are compact}\}.$$

Note that the subspace

$$\mathcal{S}_{12} = \{B = \begin{pmatrix} 0 & B_{12} \\ 0 & 0 \end{pmatrix} : B_{12} \text{ is compact}\}$$

is apparently complemented in $\mathcal{B}_p$. Thus, if $\mathcal{B}_P$ were complemented in $\mathcal{B}(\mathcal{H})$, then also $\mathcal{S}_{12}$ would be complemented in $\mathcal{B}(\mathcal{H})$: $\mathcal{S}_{12} \oplus \mathcal{R} = \mathcal{B}(\mathcal{H})$. Pick any operator $T \in \mathcal{B}(N(P), R(P))$, consider $T'$

$$T' = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

Then there exist unique $R' \in \mathcal{R}$ and $S \in \mathcal{S}_{12}$ such that $T' = S + R'$. Apparently, $R'$ is of the form

$$R' = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix},$$

for some $R \in \mathcal{B}(N(P), R(P))$. This would imply that the space of compact operators in $\mathcal{B}(N(P), R(P))$ would be complemented in $\mathcal{B}(N(P), R(P))$, which means that either $N(P)$ or $R(P)$ is finite dimensional.

Let us recall the following fact concerning the geometry of $\mathcal{P}$ [9]:

**Remark 4.6.** Let $P, Q \in \mathcal{P}$ such that $\|P - Q\| < 1$. Then there exists a unique selfadjoint operator $X$ which astisfies:

1. $e^{iX} P e^{-iX} = Q$.
2. $\|X\| < \pi/2$.
3. $X$ is $P$-codiagonal: $PXP = (1-P)X(1-P) = 0$.
4. $X$ is a $C^\infty$ map in the arguments $P, Q$.

This operator $X$ provides the exponent of the unique (minimal) geodesic of $\mathcal{P}$ joining $P$ and $Q$, according to the linear connection and the Finsler metric in $\mathcal{P}$, introduced by Corach, Porta and Recht in [9]. The geodesic is

$$\delta(t) = e^{itX} P e^{-itX}.$$

**Theorem 4.7.** *$\mathcal{Q}_k$ is a closed differentiable manifold of $\mathcal{Q}$ (and therefore also of $\mathcal{B}(\mathcal{H})$).*

*Proof.* It is apparent $\mathcal{Q}_k$ is closed in $\mathcal{Q}$, for instance using the characterization that $E \in \mathcal{Q}$ belongs to $\mathcal{Q}_k$ if and only if $E - E^* \in \mathcal{K}(\mathcal{H})$ (which is closed in norm).

Fix $E_0 \in \mathcal{Q}_k$, let us construct a local chart for $E_0$. Denote by $P_1 = P_{R(E_0)}$ and $P_0 = P_{N(E_0)}$. It is a known fact that two orthogonal projections $P, Q$ such that $\|P - Q\| < 1$ are unitarily equivalent, with a unitary operator $U = U(P,Q)$ which is a smooth (and explicit) formula in terms of $P$ and $Q$. By (1), the map $E \mapsto P_{R(E)}$ is continuous (in fact smooth). Thus the set

$$\mathcal{V}_{E_0} = \{E \in \mathcal{Q}_k : \|P_{R(E)} - P_1\| < r_{E_0} \leq 1\}$$

is an open neighbourhood of $E_0$ in $\mathcal{Q}_k$. Moreover, there exists a smooth map

$$\mu : \{Q \in \mathcal{P} : \|Q - P_1\| < 1\} \to \mathcal{U}(\mathcal{H}),$$

such that $\mu(E)P_1\mu(E)^* = P_{R(E)}$, and $\mu(E_0) = 1$ ($\mu$ is the unitary operator mentioned above). By the facts collected in Remark 4.6 above, $\mu(E) = e^{iX(E)}$, where $X(E)$ is a selfadjoint operator with $\|X(E)\| < \pi/2$, which is codiagonal with respect to $P_1$. Moreover, the map $E \mapsto X(E)$ defined in $\mathcal{V}_{E_0}$ is smooth.

Note that

$$P_{R(E)} + P_{N(E)} - 1 = \mu(E)\{P_1 + \mu(E)^* P_{N(E)}\mu(E) - 1\}\mu(E)^*$$

is compact, thus $P_1 + \mu(E)^* P_{N(E)}\mu(E) - 1$ is compact, or equivalently,

$$\mu(E)^* P_{N(E)}\mu(E)P_1 \text{ is compact.}$$

We can further shrink $r_{E_0}$ in the definition of $\mathcal{V}_{E_0}$ (which would make $\mu(E)$ closer to 1 and $P_{N(E)}$ closer to $P_0$), in order that $\mu(E)^* P_{N(E)}\mu(E)$ lies in a coordinate neighbourhood $\mathcal{W}_{P_0}$ of $P_0$ in the manifold $\mathcal{P}_{P_0}$ [9],

$$\varphi_{P_0} : \mathcal{W}_{P_0} \to \mathcal{Z}_{P_0} = \{Z \in \mathcal{B}_{P_0} : Z^* = Z \text{ is } P_0 - \text{codiagonal}, \|Z\| < \pi/2\}.$$

Then we can define

$$\theta_{E_0} : \mathcal{V}_{E_0} \to \{X \in \mathcal{B}(\mathcal{H}) : X^* = X, \|X\| < \pi/2, X \text{ is } P_0 - \text{codiagonal}\} \times \mathcal{Z}_{P_0},$$

$$\theta_{E_0}(E) = (X(E), \varphi_{P_0}(\mu(E)^* P_{N(E)}\mu(E))).$$

Clearly $\theta$ is a smooth map whose inverse is $\theta_{E_0}^{-1}(X,Z) = F$, where $F$ is determined by

$$P_{R(F)} = e^{iX}P_1e^{-iX} \text{ and } P_{N(F)} = e^{iX}(\varphi_{P_0}^{-1}(e^{iZ}P_0e^{-iZ}))e^{-iX}.$$

□

Let $Gl_\infty(\mathcal{H})$ be the Linear Fredholm group of $\mathcal{H}$, namely,

$$Gl_\infty(\mathcal{H}) = \{G \in \mathcal{B}(\mathcal{H}) : G \text{ is invertible and } G - 1 \text{ is compact}\}.$$

This group is an analytic Banach Lie group, whose Banach lie algebra identifies with the ideal $\mathcal{K}(\mathcal{H})$ of compact operators. Note that $Gl_\infty(\mathcal{H})$ acts in $\mathcal{Q}_k$. If $G = 1 + K \in Gl_\infty(\mathcal{H})$ with $G^{-1} = 1 + K'$, for $K, K' \in \mathcal{K}(\mathcal{H})$, then

$$GEG^{-1} - (GEG^{-1})^* = (1+K)E(1+K') - (1+K'^*)E^*(1+K^*) = E - E^* + K'',$$

for some $K'' \in \mathcal{K}(\mathcal{H})$. Thus $GEG^{-1} - (GEG^{-1})^*$ is compact.

**Proposition 4.8.** *Let $E \in \mathcal{Q}$. Then $E \in \mathcal{Q}_k$ if and only if there exists $G \in Gl_\infty(\mathcal{H})$ such that $E = GP_{R(E)}G^{-1}$.*

*Proof.* Clearly the selfadjoint projection $P_{R(E)} \in \mathcal{Q}_k$, thus for any $G \in Gl_\infty(\mathcal{H})$, $GP_{R(E)}G^{-1} \in \mathcal{Q}_k$.

Conversely, suppose that $E \in \mathcal{Q}_k$. In the three space decompostion induced by $E$, consider the operator

$$G = 1 \oplus 1 \oplus \begin{pmatrix} 1 & S^{-1}C \\ 0 & 1 \end{pmatrix}.$$

Apparently $G$ is invertible, is of the form 1 plus compact, and satisfies $GP_{R(E)} = EG$. □

Let us characterize the orbits of this action. First note that the group $Gl_\infty(\mathcal{H})$ is connected (it is an exponential group: any $G \in Gl_\infty(\mathcal{H})$ is of the form $G = e^K$, for some compact operator $K$, by a straightforward argument using the holomorphic functional calculus in the Banach algebra $\mathcal{B}(\mathcal{H})$). Therefore any pair of elements $E, F$ in the same orbit must lie in the same connected component: $\dim(N(E) = \dim(N(F))$, $\dim(R(E)) = \dim(R(F))$.

Let $P, Q \in \mathcal{P}$. Recall [15] that a projection $Q$ belongs to the *restricted Grassmannian* $G_{res}(P)$ induced by $P$ if

$$PQ|_{R(Q)} : R(Q) \to R(P)$$

is a Fredholm operator. The index of this operator parametrizes the connected components of $G_{res}(P)$: two projections $Q, Q'$ in $G_{res}(P)$ belong to the same component if and only if they have the same index. In [8], A.L. Carey and D.E. Evans proved that the components coincide with the orbits of the action of the *unitary* Fredholm group $\mathcal{U}_\infty(\mathcal{H})$,

$$\mathcal{U}_\infty(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) : U \text{ is unitary and } U - 1 \text{ is compact}\}.$$

Namely, $Q, Q'$ in $G_{res}(P)$ have the same index if and only if there exists $U \in \mathcal{U}_\infty(\mathcal{H})$ such that $Q' = UQU^*$. In order to characterize the $Gl_\infty(\mathcal{H})$ orbits of elements $E \in \mathcal{Q}_k$, the following elementary fact will be useful:

**Lemma 4.9.** *Let $G$ in $Gl_\infty(\mathcal{H})$. Then the unitary part $U$ in the polar decomposition of $G$,*

$$G = U|G|,$$

*belongs to $\mathcal{U}_\infty(\mathcal{H})$.*

*Proof.* Since $G = 1 + K$, $|G|^2 = G^*G = 1 + K^*K + K + K^*$ is of the form 1 plus compact, and selfadoint. By the diagonalization theorem of compact selfadjoint operators, it follows that $|G| \in Gl_\infty(\mathcal{H})$. Then

$$U = G|G|^{-1} \in Gl_\infty(\mathcal{H}).$$

□

**Proposition 4.10.** *Let $E, F \in \mathcal{Q}_k$. Then they lie in the same orbit of the action of $Gl_\infty(\mathcal{H})$ if and only if $P_{R(F)}$ belongs to the connected component of $P_{R(E)}$ in $G_{res}(P_{R(E)})$, i.e. the zero index component of $G_{res}(P_{R(E)})$. Or equivalently*

$$P_{R(E)}P_{R(F)}|_{R(F)} : R(F) \to R(E)$$

*is a zero-index Fredholm operator.*

*Proof.* Suppose that $E$ and $F$ lie in the same $G_\infty(\mathcal{H})$ orbit. By the above Proposition, this implies that there exists $G \in G_\infty(\mathcal{H})$ such that $GP_{R(E)}G^{-1} = P_{R(F)}$. It is well known (and an elementary fact, see for instance [9]), that this implies that the unitary part $U$ in the polar decompositon of $G$ also satisfies $UP_{R(E)}U^* = P_{R(F)}$. Therefore, by the above Lemma and remarks on the structure of the connected components of the restricted Grassmannian, it follows that $P_{R(F)}$ belongs to the zero index component of $G_{res}(P_{R(E)})$.

Conversely, suppose $UP_{R(E)}U^* = P_{R(F)}$ for some $U \in U_\infty(\mathcal{H})$. By Proposition (4.8), there exist $G, G' \in Gl_\infty(\mathcal{H})$ such that

$$E = GP_{R(E)}G^{-1} \text{ and } F = G'P_{R(F)}G'^{-1}.$$

Then

$$F = G'U^*G^{-1}E(G'U^*G^{-1})^{-1},$$

with $G'U^*G^{-1} \in Gl_\infty(\mathcal{H})$. □

Using this results, one obtains that

**Theorem 4.11.** *The orbits of the action of $Gl_\infty(\mathcal{H})$ on $\mathcal{Q}_k$ coincide with the connected components of $\mathcal{Q}_k$.*

*Proof.* Fix $E \in \mathcal{Q}_k$. We claim that the set

$$\{F \in \mathcal{Q}_k : P_{R(E)}P_{R(F)}|_{R(F)} \in \mathcal{B}(R(F), R(E)) \text{ is a zero index Fredholm operator}\},$$

is an open subset of $\mathcal{Q}_k$. Note that by the above Proposition, this set coincides with the $Gl_\infty(\mathcal{H})$-orbit of $E$. Indeed, by the first of the formulas in 1, the map

$$\mathcal{Q}_k \to \mathcal{P} \times \mathcal{P} \ , \quad F \mapsto (P_{R(E)}, P_{R(F)})$$

is continuous. Thus it suffices to show that the set

$$\{(P, Q) \in \mathcal{P} \times \mathcal{P} : PQ|_{R(Q)} : R(Q) \to R(P) \text{ is a zero index Fredholm operator}\}$$

is open in $\mathcal{P} \times \mathcal{P}$. The proof of this fact is fairly straightforward ([3]). We include a proof of this fact in the Section treating Fredholm idempotents (Section 5).

Therefore the $Gl_\infty(\mathcal{H})$-orbits $\mathcal{O}_E$ of elements $E$ in $\mathcal{Q}_k$ are open. Therefore they are also closed:

$$\mathcal{Q}_k \setminus \mathcal{O}_E = \cup_{\mathcal{O}_F \neq \mathcal{O}_E} \mathcal{O}_F$$

is open in $\mathcal{Q}_k$. It follows that the orbits coincide with the connected components. □

Thus we have:

**Corollary 4.12.** *Let $E, F \in \mathcal{Q}_k$. Then*

$$P_{R(E)}P_{R(F)}|_{R(F)} : R(F) \to R(E) \quad \textit{is a zero index Fredholm operator}$$

*if and only if*

$$\dim(R(E)) = \dim(R(F)) \quad \textit{and} \quad \dim(N(E)) = \dim(N(F)).$$

## 5 Idempotents in generic position

In this section we study the set $\mathcal{Q}_g$,

$$\mathcal{Q}_g = \{E \in \mathcal{Q} : R(E) \text{ and } N(E) \text{ are in generic position}\}.$$

This means that $R(E) \cap N(E)^\perp = N(E) \cap R(E)^\perp = \{0\}$. Given $E \in \mathcal{Q}_g$, putting $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$, in [3] it was proven that these comditions imply that there exists a unique (minimal) geodesic in $\mathcal{P}$ joining $P_1$ and $P_0$:

$$P_0 = e^{iZ} P_1 e^{-iZ}$$

for a uniquely determined selfadjoint operator $Z$ which is $P_1$ and $P_0$ codiagonal and satisfies $\|Z\| \leq \pi/2$. In terms of the operator $X$ acting in $\mathcal{L}$ (in Halmos' model), $C = \cos(X)$, $S = \sin(X)$, $e^{iZ}$ and $Z$ are given by

$$e^{iZ} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}.$$

Chandler Davis in [10] proved that to any decomposition $A = P_1 - P_0$ of an operator as a difference of projections in generic position, there corresponds a unique symmetry $V = V(P_1, P_0)$, $V^* = V = V^{-1}$, which anti-commutes with $A$: $VA = -AV$. Explicitly

$$P_1 = \frac{1}{2}\{1 + A + V(1 - A^2)^{1/2}\} \text{ and } P_0 = \frac{1}{2}\{1 - A + V(1 - A^2)^{1/2}\}.$$

Note that this symmetry $V$ satisfies $VP_1V = P_0$ and therefore

$$VEV = 1 - E.$$

The symmetry $V$ and the unique geodesic joining $P_1$ and $P_0$ are related by the formula [4]

$$V = e^{iZ}(2P_1 - 1) = (2P_0 - 1)e^{-iZ}.$$

**Proposition 5.1.** *Let $E \in \mathcal{Q}$. The following are equivalent:*

1. *$E \in \mathcal{Q}_g$.*
2. *$N(E + E^* - 2) = N(E + E^*) = \{0\}$.*
3. *$E_{12}$ has trivial nullspace and dense range.*
4. *There exists a unique minimal geodesic of $\mathcal{P}$ joining $P_1$ and $P_0$.*

*Proof.* As usual $P_1 = P_{R(E)}$ and $P_0 = P_{N(E))}$. As remarked above, $\mathcal{H}_{10} = N(P_1 - P_0 - 1)$ and $\mathcal{H}_{01} = N(P_1 - P_0 + 1)$. Note that

$$P_1 - P_0 - 1 = (E + E^* - 1)^{-1} - 1 = (E + E^* - 1)^{-1}\{2 - E - E^*\},$$

And thus $\mathcal{H}_{10} = N(E + E^* - 2)$. Similarly $\mathcal{H}_{01} = N(E + E^*)$. This proves that the first two conditions are equivalent.

In matrix form

$$E + E^* - 2 = \begin{pmatrix} 0 & E_{12} \\ E_{12}^* & -2 \end{pmatrix}.$$

Then $(\xi_1, \xi_2) \in N(E + E^* - 2)$ if and only if $E_{12}\xi_2 = 0$ and $E_{12}^*\xi_1 = 2\xi_2$. Then

$$E_{12}E_{12}^*\xi_1 = 2E_{12}\xi_2 = 0,$$

which implies $E_{12}^*\xi_1 = 0$, and thus also $\xi_2 = 0$. Conversely, clearly a pair $(\xi_1, \xi_2) \in N(E_{12}^*) \oplus \{0\}$ lies in the nullspace of $E + E^* - 2$. Then

$$N(E + E^* - 2) = N(E_{12}^*) \oplus \{0\}.$$

Similarly

$$N(E + E^*) = \{0\} \oplus N(E_{12}).$$

Thus $E \in \mathcal{Q}_g$ if and only if $N(E_{12}) = N(E_{12}^*) = \{0\}$, i.e. $E_{12}$ has trivial nullspace and dense range.

The equivalence with the last condition was stated above. □

In particular, $E \in \mathcal{Q}_g$ if and only if $E^* \in \mathcal{Q}_g$

Note that if $E \in \mathcal{Q}_g$, the unitary part in the polar decomposition of $E_{12} : N(E) \to R(E)$ is an onto isometry between $N(E)$ and $R(E)$.

**Theorem 5.2.** *$\mathcal{Q}_g$ is arcwise connected.*

*Proof.* The last sentence above implies that if $E \in \mathcal{Q}_g$, both $N(E)$ and $R(E)$ are infinite dimensional, thus any pair $E, F \in \mathcal{Q}_g$ belong to the same connected component in $\mathcal{Q}$. Thus we may use again Lemma 2.1, and reduce to the case when $R(E) = R(F)$. Also $\mathcal{H} = \mathcal{H}_0$ can be replaced by the space $\mathcal{L} \times \mathcal{L}$. In matrix form

$$E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 1 & F_{12} \\ 0 & 0 \end{pmatrix}.$$

Let $E_{12} = U_E|E_{12}|$ and $F_{12} = U_F|F_{12}|$, where $U_E$ and $U_F$ are unitary operators in $\mathcal{L}$. Since the unitary group of $\mathcal{L}$ is connected, there are continuous paths $U_E(t)$ and $U_F(t)$ of unitaries in $\mathcal{L}$ connecting $U_E(0) = U_E$ with $U_E(1) = 1$ and $U_F(0) = U_F$ with $U_F(1) = 1$. The continuous path

$$\begin{pmatrix} 1 & U_E(t)|E_{12}| \\ 0 & 0 \end{pmatrix}$$

connects $E$ with

$$\begin{pmatrix} 1 & |E_{12}| \\ 0 & 0 \end{pmatrix}$$

inside $\mathcal{Q}_g$. Similarly for $F$. Thus it remains to see that

$$\begin{pmatrix} 1 & |E_{12}| \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & |F_{12}| \\ 0 & 0 \end{pmatrix}$$

can be connected inside $\mathcal{Q}_g$. Or equivalently, that two positive operators $|E_{12}|, |F_{12}|$ with trivial nullspace (and therefore dense range) can be connected with a continuous path of positive operators with trivial nullspace. It is easy to see that the set of positive operators with trivial nullspace is convex, and the proof follows. □

# 6 Fredholm idempotents

In this section we study the set $\mathcal{Q}_f$ of Fredholm idempotents,

$$\mathcal{Q}_f = \{E \in \mathcal{Q} : (P_{R(E)}, P_{N(E)}) \text{ is a Fredholm pair}\}.$$

In other words, $E \in \mathcal{Q}_f$ if [5], [1] if and only if

$$P_{N(E)} P_{R(E)}|_{R(E)} : R(E) \to N(E)$$

is a Fredholm operator. The index of this operator (usually called the index of the pair), which we shall call here $i(E)$, the index of $E$, is

$$i(E) = i(P_{R(E)}, P_{N(E)}) = \dim(R(E) \cap N(E)^{\perp}) - \dim(N(E) \cap R(E)^{\perp}).$$

By the computations in the previous section, this index is also

$$i(E) = \dim(N(E + E^* - 2)) - \dim(N(E + E^*)).$$

These pairs can also be described as those such that $P_{N(E)}$ belongs to the restricted Grassmannian $G_{res}(P_{R(E)})$ (as in Section 3).

**Remark 6.1.** In [5] it was proven that $(P, Q)$ is a Fredholm pair if and only if $\pm 1$ are isolated eigenvalues of finite multiplicity. or do not belong in the spectrum of $P - Q$. Let us abreviate this condition by saying that $\pm 1$ are eigenvalues with zero or finite multiplicity.

The following characterization follows:

**Proposition 6.2.** *Let $E \in \mathcal{Q}$. The following are aquivalent:*

1. *$E \in \mathcal{Q}_f$.*
2. *$0, 2$ are isolated eigenvalues of $E + E^*$, with zero or finite multiplicity.*
3. *$E_{12} : R(E)^{\perp} \to R(E)$ is a Fredholm operator.*

*In this case, $i(E) = index(E_{12})$.*

*Proof.* The equivalence of the first two conditions follows form the above remark and the computations in the previous section. Recall also that (in terms of the decomposition $\mathcal{H} = R(E) \oplus R(E)^{\perp}$) :

$$N(E + E^* - 2) = N(E_{12}^*) \oplus \{0\} \text{ and } N(E + E^*) = \{0\} \oplus N(E_{12}).$$

Thus $E \in \mathcal{Q}_f$ if and only if $N(E_{12})$ and $N(E_{12})^*$ are finite dimensional and $0, 2$ are isolated eigenvalues of $E + E^*$ with zero or finite multiplicty. Let us examine this latter condition. It is equivalent to $\pm 1$ being isolated in the spectrum of $P_{R(E)} - P_{N(E)}$, or equivalently, that 1 is isolated in the spectrum of $(P_{R(E)} - P_{N(E)})^2$. In matrix form

$$(P_{R(E)} - P_{N(E)})^2 = (E + E^* - 1)^2 = \begin{pmatrix} 1 & E_{12} \\ E_{12}^* & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 + E_{12} E_{12}^* & 0 \\ 0 & 1 + E_{12}^* E_{12} \end{pmatrix}.$$

Then 1 is isolated in the spectrum of $(P_{R(E)} - P_{N(E)})^2$ if and only if 0 is isolated in the spectrum of $E_{12}E_{12}^*$ (the other follows). This is equivalent to the fact that $E_{12}$ has closed range. It follows that $E \in \mathcal{Q}_f$ if and only if $E_{12} : R(E)^\perp \to R(E)$ is a Fredholm operator. Apparently

$$index(E_{12}) = \dim(N(E_{12}^*)) - \dim(N(E_{12})) = \dim(N(E + E^* - 2)) - \dim(N(E + E^*)) = i(E).$$

□

In particular, $E \in \mathcal{Q}_f$ if and only if $E^* \in \mathcal{Q}_f$. Also note that in this case, $i(E) = i(E^*)$.

**Theorem 6.3.** *Let $E, F \in \mathcal{Q}_f$. Then they lie in the same connected component of $\mathcal{Q}_f$ if and only if*

$$i(E) = i(F).$$

*Proof.* First note the fact that $E \in \mathcal{Q}_f$ implies that both $R(E)$ and $N(E)$ are infinite dimensional ($E_{12}$ is a Fredholm operator between these spaces). It follows that $E$ and $F$ lie in the same connected component in $\mathcal{Q}$. Lemma 2.1 applies again here, and we may suppose that $R(E) = R(F)$. It follows that $E_{12}, F_{12}$ are Fredholm operators in $\mathcal{B}(R(E)^\perp, R(E))$. It is a well known fact that they lie in the same connected component of the set of Fredholm operators between $R(E)^\perp$ and R$(E)$ if and only if they have the same index. A continuous path $E_{12}(t)$ between $E_{12}$ and $F_{12}$ provides a continuous path between $E$ and $F$ inside $\mathcal{Q}_f$:

$$E(t) = \begin{pmatrix} 1 & E_{12}(t) \\ 0 & 0 \end{pmatrix}.$$

□

**Proposition 6.4.** *$\mathcal{Q}_f$ is open in $\mathcal{Q}$.*

*Proof.* By the continuity of the range projection map $F \mapsto P_{R(F)}$ in $\mathcal{Q}$, given a fixed $E \in \mathcal{Q}_f$, there exists a positive radius $d = d_E$ such that if $F \in \mathcal{Q}$ satisfies $\|F - E\| < d$ then $\|P_{R(F)} - P_{R(E)}\| < 1$. Then there exists a unitary operator $\mu(F)$ in $\mathcal{H}$ (a continuous map in the parameter $F$, with $\mu(E) = 1$) such that $\mu(F)P_{R(E)}\mu^*(F) = P_{R(F)}$. Thus $\mu^*(F)F\mu(F)$ and $E$ have the same range. In matrix form in terms of $\mathcal{H} = R(E) \oplus R(E)^\perp$,

$$\mu^*(F)F\mu(F) = \begin{pmatrix} 1 & F'_{12} \\ 0 & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & E_{12} \\ 0 & 0 \end{pmatrix}.$$

Note that if one shrinks $d = d_E$, then $\|\mu^*(F)F\mu(F) - E\| = \|F'_{12} - E_{12}\|$ tends to zero. Since the set of Fredholm operators between $R(E)^\perp$ and $R(E)$ is open, it follows that there exists $d$ such that $\|F - E\| < d$ implies $F'_{12}$ is a Fredholm operator in $\mathcal{B}(R(E)^\perp, R(R))$. Note that $\mu(E)$ maps $R(E)$ onto $R(F)$ (and thus also their orthogonal supplements). It follows that $\|E - F\| < d$ implies that

$$\mu(E)F'_{12}\mu^*(E) = P_{R(F)}FP_{R(F)^\perp} = F_{12}$$

is a Fredholm operator between $R(F)^\perp$ and $R(F)$, i.e. $F \in \mathcal{Q}_f$. □

## 7 Three symmetries in $\mathcal{Q}$

Given $E \in \mathcal{Q}$, there are several symmetries induced by $E$. Among these, we shall focus on the following. The first was considered by Corach, Porta and Recht in [9]:

Consider the polar decomposition

$$2E - 1 = \rho_E|2E - 1|.$$

Then $\rho_E$ is a selfadjoint unitary operator (a symmetry), which satisfies $\rho_E|2E-1| = |2E-1|^{-1}\rho_E$. In particular this implies that $\rho_E(2E-1) = (2E^*-1)\rho_E$, or equivalently,

$$\rho_E E \rho_E = E^*.$$

The second symmetry is obtained from the polar decomposition of $P_{R(E)} - P_{N(E)}$. Since this operator is invertible and selfadjoint, the unitary part $s_E$ in the (commuting) factorization

$$P_{R(E)} - P_{N(E)} = s_E|P_{R(E)} - P_{N(E)}| = |P_{R(E)} - P_{N(E)}|s_E$$

is a selfadjoint unitary operator.

**Proposition 7.1.** *With the above notations,*

$$s_E E s_E = E^*.$$

*Proof.* Recall that $P_{R(E)} - P_{N(E)} = (E + E^* - 1)^{-1}$. In matrix form, as seen above

$$(E + E^+ - 1)^2 = \begin{pmatrix} 1 + E_{12}E_{12}^* & 0 \\ 0 & 1 + E_{12}^*E_{12} \end{pmatrix},$$

and thus

$$s_E = (E + E^* - 1)|E + E^* - 1|^{-1} = \begin{pmatrix} (1 + E_{12}E_{12}^*)^{-1/2} & E_{12}(1 + E_{12}^*E_{12})^{-1/2} \\ E_{12}^*(1 + E_{12}E_{12}^*)^{-1/2} & -(1 + E_{12}^*E_{12})^{-1/2} \end{pmatrix}.$$

After straightforward computations

$$s_E E s_E = \begin{pmatrix} 1 & 0 \\ E_{12}^* & 0 \end{pmatrix} = E^*.$$

□

**Remark 7.2.** Both symmetries $\rho_E$ and $s_E$ conjugate $E$ with $E^*$. They can be computed in the three space decomposition. Namely, recall that $S \geq 0$, and then

$$(P_1 - P_0)^2 = 1 \oplus 1 \oplus \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix} \quad \text{so that} \quad |P_1 - P_0| = 1 \oplus 1 \oplus \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}.$$

Thus

$$s_E = (P_1 - P_0)|P_1 - P_0|^{-1} = 1 \oplus -1 \oplus \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix}.$$

For the computation of $\rho_E$, put $\Gamma = S^{-1}C$ (the cotangent of $X$). Note that

$$2E - 1 = 1 \oplus -1 \oplus \begin{pmatrix} 1 & -\Gamma \\ 0 & -1 \end{pmatrix} \text{ and } |2E-1|^2 = 1 \oplus 1 \oplus \begin{pmatrix} 1 & -\Gamma \\ -\Gamma & 1+\Gamma^2 \end{pmatrix}.$$

Straightforward computations show that the square root of this operator is

$$|2E-1| = 1 \oplus 1 \oplus \begin{pmatrix} 2(4+\Gamma^2)^{-1/2} & -\Gamma(4+\Gamma^2)^{-1/2} \\ -\Gamma(4+\Gamma^2)^{-1/2} & (\Gamma^2+2)(4+\Gamma^2)^{-1/2} \end{pmatrix},$$

and thus

$$\rho_E = |2E-1|(2E-1) = 1 \oplus -1 \oplus \begin{pmatrix} 2(4+\Gamma^2)^{-1/2} & -\Gamma(4+\Gamma^2)^{-1/2} \\ -\Gamma(4+\Gamma^2)^{-1/2} & -2(4+\Gamma^2)^{-1/2} \end{pmatrix}.$$

The fact that both $s_E$ and $\rho_E$ intertwine $E$ and $E^*$ imply that the products

$$\rho_E s_E \text{ and } s_E \rho_E$$

commute with $E$ and $E^*$.

The third symmetry was introduced in Section 4. It is the symmetry $V = V_E$, obtained by Davis [10], which is defined only for $E \in \mathcal{Q}_g$, and satisfies

$$V_E E V_E = 1 - E.$$

Note that this symmetry could not be defined in the other classes of $\mathcal{Q}$, which are not invariant for the map $E \mapsto 1 - E$. In terms of $C$ and $S$ in Halmos' model,

$$V = \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

The symmetry $V$ has the following geometric characterization:

**Theorem 7.3.** *Let $E \in \mathcal{Q}_g$. Then the projection $\frac{1}{2}(1+V)$ onto the 1 eigenspace of $V$, is the middlepoint of the unique geodesic joinming $P_{R(E)}$ and $P_{N(E)}$*

*Proof.* As before, put $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Let $\delta(t) = e^{itZ} P_1 e^{-itZ}$ be the unique geodesic joining $P_1$ and $P_0$. Recall from Section 4 that $V = e^{iZ}(2P_1 - 1)$. Since $Z$ anti-commutes with $V$, one has that

$$V = e^{\frac{i}{2}Z}(2P_1 - 1)e^{-\frac{i}{2}Z},$$

and thus

$$\frac{1}{2}(1+V) = e^{\frac{i}{2}Z} P_1 e^{-\frac{i}{2}Z} = \delta(\frac{1}{2}).$$

□

**Remark 7.4.** Suppose that $E \in \mathcal{Q}_k$. In Proposition 4.10 it was shown that $E = GP_{R(E)}G^{-1}$ for some $G \in Gl_\infty(\mathcal{H})$. In the polar decomposition of $2E - 1 = \rho_E|2e-1|$ above, Corach, Porta and Recht [9] noted that

$$2E - 1 = \rho_E|2E-1| = |2E-1|^{-1}\rho_E.$$

Thus $2E-1 = |2E-1|^{-1/2}\rho_E|2E-1|^{1/2}$, and therefore

$$E = |2E-1|^{-1/2}\frac{1}{2}\{\rho_E + 1\}|2E-1|^{1/2},$$

where $\frac{1}{2}\{\rho_E+1\}$ is the orthogonal projection onto the 1-eigenspace of the symmetry $\rho_E$. Note that $|2E-1| \in Gl_\infty(\mathcal{H})$. Indeed, in the three space decompositon of $|2E-1|$, $\Gamma = S^{-1}C$ is a compact operator in $\mathcal{L}$. Then also $|2E-1|^{1/2} \in Gl_\infty(\mathcal{H})$. It follows that $P_{R(E)}$ and $\frac{1}{2}\{\rho_E+1\}$ are orthogonal projections for which there exists $G_0 \in Gl_\infty(\mathcal{H})$ such that $G_0 P_{R(E)} G_0^{-1} = \frac{1}{2}\{\rho_E+1\}$. Then, the unitary $U_0$ in the polar decompositioon of $G_0$ verifies

$$U_0 P_{R(E)} U_0^* = \frac{1}{2}\{\rho_E + 1\},$$

and by Lemma 4.9, $U_0 \in \mathcal{U}_\infty(\mathcal{H})$.

# 8 Cyclic idempotents

In this section we study the set $\mathcal{Q}_c$ of cyclic idempotents

$$\mathcal{Q}_c = \{E \in \mathcal{Q} : P_{R(E)} - P_{N(E)} \text{ is a cyclic operator in } \mathcal{H}\}.$$

In other words, the commutative $C^*$-algebra $C^*(P_{R(E)} - P_{N(E)})$ has a cyclic vector. Apparently, this implies that the $C^*$-algebra $C^*(P_{R(E)}, P_{N(E)}) = C^*(E)$ generated by the two projections (or equivalently by $E$) has a cyclic vector in $\mathcal{H}$. It is clearly a weaker condition.

The equality $P_{R(E)} - P_{N(E)} = (E + E^* - 1)^{-1}$ clearly implies the following:

**Proposition 8.1.** *$E \in \mathcal{Q}_c$ if and only if $E+E^*$ (or equivalently $E+E^*-1$) is a cyclic operator in $\mathcal{H}$.*

Also it is apparent that for any unitary operator $U$, $E \in \mathcal{Q}_c$ implies that $UEU^* \in \mathcal{Q}_c$. In particular, $E^* \in \mathcal{Q}_c$.

**Remark 8.2.** In the three space decomposition $\mathcal{H} = \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_0$, recall that

$$\mathcal{H}_{10} = N(E + E^* - 2) \quad \text{and} \quad \mathcal{H}_{01} = N(E + E^*).$$

If $E \in \mathcal{Q}_c$, this implies that

$$\dim \mathcal{H}_{10} \le 1 \quad \text{and} \quad \dim \mathcal{H}_{01} \le 1.$$

Indeed, the fact that $E + E^*$ is cyclic implies that any eigenvalue must have multiplicity less or equal than 1.

In terms of the Halmos' model:

**Theorem 8.3.** *$E \in \mathcal{Q}_c$ if and only if*

$$\dim \mathcal{H}_{10} \le 1\ , \quad \dim \mathcal{H}_{01} \le 1$$

*and the operator $Z$ acting in the generic part $\mathcal{H}_0$,*

$$Z = \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}$$

*is cyclic in $\mathcal{H}_0$.*

*This operator $Z$ is the exponent of the unique geodesic joining the generic parts of $P_{R(E)}$ and $P_{N(E)}$.*

*Proof.* As usual, denote $P_1 = P_{R(E)}$ and $P_0 = P_{N(E)}$. Suppose first that $E \in \mathcal{Q}_c$. As seen above this implies the bounds for the dimensions of $\mathcal{H}_{10}$ and $\mathcal{H}_{01}$. Let $A_0$ be the generic part of $P_1 - P_0$. Identifying $\mathcal{H}_0$ and $\mathcal{L} \times \mathcal{L}$, we have

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} = \begin{pmatrix} S^2 & -CS \\ -CS & -S^2 \end{pmatrix}.$$

The symmetry defined by Davis, induced by this decomposition of $A_0$ is

$$V = \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

Clearly, the assumption that $A = P_1 - P_0$ is cyclic in $\mathcal{H}$ implies that $A_0$ is cyclic in $\mathcal{H}_0$. Consider

$$B_0 = VA_0.$$

Clearly $B_0$ also anti-commutes with $V$. In [4] it was shown that if $A_0$ is cyclic, then one can find a cyclic vector $\xi_0$ such that $V\xi_0 = \xi_0$. Then

$$B_0^n \xi_0 = (VA_0)^n \xi_0 = (-1)^n A_0 V \xi_0 = (-1)^n A_0^n \xi_0.$$

It follows that $B_0$ is also cyclic (with the same cyclic vector $\xi_0$). Note that in matrix form

$$B_0 = VA_0 = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \begin{pmatrix} S^2 & -CS \\ -CS & -S^2 \end{pmatrix} = \begin{pmatrix} 0 & -S \\ S & 0 \end{pmatrix}.$$

It follows that $iB_0$ is selfadjoint and cyclic. We claim that $iB_0 = \sin(Z)$ and that $Z$ is also cyclic (with the same cyclic vector $\xi_0$). Indeed, the first claim follows from a straightforward matrix computation. In our case, $S$ is invertible in $\mathcal{L}$. Clearly $Z$ is an analytic function in terms of $iB_0$, $Z = f(iB_0)$, with $f(0) = 0$. In particular, any vector in $\mathcal{H}_0$ which is orthogonal to $Z^n\xi_0$, for all $n \geq 1$, is also orthogonal to $(iB_0)^n\xi_0$ for all $n \geq 1$, and thus trivial. Then $Z$ is cyclic with cyclic vector $\xi_0$.

The fact that $e^{iZ}P_1e^{-iZ} = P_0$ was shown in Section 4.

Conversely, assuming $\dim \mathcal{H}_{10} \leq 1$ and $\dim \mathcal{H}_{01} \leq 1$, it remains to prove that $A_0$ is cyclic in $\mathcal{H}_0$. The same argument above shows that if $Z$ cyclic with cyclic vector $\xi_0$, then $\sin(Z) = iB_0$ is cyclic, and therefore $A_0 = VB_0$, by the same computation above. □

With respect to the off-diagonal entry $E_{12}$, we have sufficient conditions:

**Proposition 8.4.** *Let $E \in \mathcal{Q}$ such that $N(E_{12}) = \{0\}$, $N(E_{12}E_{12}^* - 1) = \{0\}$, and $E_{12}E_{12}^*$ is cyclic in $R(E)$, with cyclic vector $\xi_1 \in R(E)$. Then $E \in \mathcal{Q}_c$, with $\xi_0 = \xi_1 + E_{12}^*\xi_1$ cyclic for $E + E^* - 1$.*

*Proof.* First let us compute the powers of $E + E^* - 1$. After straightforward computations, if $n = 2k$ is even,

$$(E + E^* - 1)^n = \begin{pmatrix} (1 + E_{12}E_{12}^*)^k & 0 \\ 0 & (1 + E_{12}^*E_{12})^k \end{pmatrix}.$$

If $n = 2k + 1$ is odd

$$(E + E^* - 1)^n = \begin{pmatrix} (1 + E_{12}E_{12}^*)^k & (1 + E_{12}E_{12}^*)^k E_{12} \\ (1 + E_{12}^*E_{12})^k E_{12}^* & (1 + E_{12}^*E_{12})^k \end{pmatrix}.$$

Let $\eta = \eta_1 + \eta_2 \in \mathcal{H}$, $\eta_1 \in R(E)$, $\eta_2 \in R(E)^\perp$, such that $\eta \perp (E + E^* - 1)(\xi_1 + E_{12}^*\xi_1)$ for all $n \geq 0$. Then if $n = 2k$

$$\langle \eta_1, (1 + E_{12}E_{12}^*)^k \xi_1 \rangle + \langle \eta_2, (1 + E_{12}^*E_{12})^k E_{12}^*\xi_1 \rangle = 0 \tag{3}$$

for all $k \geq 0$. If $n = 2j + 1$,

$$\langle \eta_1, (1 + E_{12}E_{12}^*)^j \xi_1 + (1 + E_{12}E_{12}^*)^j E_{12}E_{12}^*\xi_1 \rangle + \langle \eta_2, 2(1 + E_{12}^*E_{12})^j E_{12}^*\xi_1 \rangle = 0$$

for all $j \geq 0$. This term equals

$$\langle \eta_1, (1 + E_{12}E_{12}^*)^{j+1} \xi_1 \rangle + 2\langle \eta_2, (1 + E_{12}^*E_{12})^j E_{12}^*\xi_1 \rangle = 0. \tag{4}$$

Putting $j = k \geq 0$, multiplying equation (3) by 2 and substracting from it equation (4), one obtains

$$\langle \eta_1, (1 - E_{12}E_{12}^*)(1 + E_{12}E_{12}^*)^k \xi_1 \rangle = 0$$

Apparently, the fact that the set of vectors $\{(E_{12}E_{12}^*)^k \xi_1 : k \geq 0\}$ spans a dense subspace of $R(E)$, implies that also the set $\{(1 + E_{12}E_{12}^*)^k \xi_1 : k \geq 0\}$ spans a dense subspace of $R(E)$. By hypothesis, $1 - E_{12}E_{12}^*$ has dense range in $R(E)$, it follows that the set

$$\{(1 - E_{12}E_{12}^*)(1 + E_{12}E_{12}^*)^k \xi_1 : k \geq 0\}$$

spans a dense subset of $R(E)$. It follows that $\eta_1 = 0$. Similarly, putting $j + 1 = k$ for $j \geq 0$, and substracting equation (3) from equation (4), one obtains

$$0 = \langle \eta_2, (1 - E_{12}^*E_{12})(1 + E_{12}^*E_{12})^j E_{12}^*\xi_1 \rangle = \langle \eta_2, E_{12}^*(1 - E_{12}E_{12}^*)(1 + E_{12}E_{12}^*)^j \xi_1 \rangle$$

for all $j \geq 0$. The hypothesis $N(E_{12}) = \{0\}$ implies that $E_{12}^* : R(E) \to R(E)^\perp$ has dense range. Thus similarly as above, $\eta_2 = 0$, and therefore $\xi_1 + E_{12}^*\xi_1$ is a cyclic vector for $E + E^* - 1$ in $\mathcal{H}$. □

**Remark 8.5.** Analogously, one can prove that if $N(E_{12}^*) = N(1 - E_{12}^*E_{12}) = \{0\}$ and $E_{12}^*E_{12}$ is cyclic in $R(E)^\perp$ with cyclic vector $\xi_2$, then $E + E^* - 1$ is cyclic in $\mathcal{H}$, with cyclic vector $\xi_2 + E_{12}\xi_2$.

**Remark 8.6.**

1. In the above Proposition, the condition $N(E_{12}) = \{0\}$ could be replaced by the condition $\mathcal{H}_{01} = \{0\}$. Indeed, recall from Section 4 that

$$\mathcal{H}_{10} = N(E + E^*) = \{0\} \oplus N(E_{12}).$$

Also note that if $E$ is cyclic, one has $\dim \mathcal{H}_{01} \leq 1$, so that $E_{12}$ is not far from having trivial nullspace. However it appears not to be a necessary condition.

2. Something similar happens with the other condition, $N(E_{12}E_{12}^* - 1) = \{0\}$. If one asks that $E_{12}E_{12}^*$ be cyclic in $R(E)$, then all eventual eigenvalues must have multiplicity at most 1, i.e. $\dim N(E_{12}E_{12}^* - 1) \leq 1$.

With reference to this last condition, let us point out that in Halmos' model for the generic part of $E$, this last condition is automatically fulfilled:

**Lemma 8.7.** *Let $E_0$ be the generic part of $E$ acting in $\mathcal{H}_0 = \mathcal{L} \times \mathcal{L}$:*

$$E_0 = \begin{pmatrix} 1 & -S^{-1}C \\ 0 & 0 \end{pmatrix}.$$

*Then $N((S^{-1}C)^2 - 1) = \{0\}$.*

*Proof.* Suppose that there exists a vector $\xi \in \mathcal{L}$ such that $(S^{-1}C)^2\xi = \xi$. Since $S^{-1}C$ is a positive operator in $\mathcal{L}$ ($C$ and $S$ commute), this implies that $S^{-1}C\xi = \xi$. Recall that there exist $0 \leq X \leq \pi/2$ such that $C = \cos(X)$ and $S = \sin(X)$. The fact that $S$ is invertible implies further that $0 < r \leq X \leq \pi/2$. Therefore the continuous function $cotg : [r, \pi/2] \to [0, cotg(r)]$, $cotg(t) = \frac{\cos(t)}{\sin(t)}$ has a continuous inverse $cotg^{-1}$. Note that $cotg(X) = S^{-1}C$ and thus $cotg^{-1}(S^{-1}C) = X$. The function $cotg^{-1}$ is a uniform limit of polynomials in the interval $[0, cotg(r)]$,

$$cotg^{-1}(t) = \lim_{n\to\infty} p_n(t).$$

Since $S^{-1}C\xi = \xi$, it follows that $p_n(S^{-1}C)\xi = p_n(1)\xi$. Taking limits,

$$X\xi = cotg^{-1}(X)\xi = \lim_{n\to\infty} p_n(X)\xi = \lim_{n\to\infty} p_n(1)X\xi = cotg^{-1}(1)\xi = \frac{\pi}{4}\xi.$$

Therefore $S\xi = C\xi = \frac{1}{\sqrt{2}}\xi$. Consider the vector $\bar{\xi} = (\xi, 0) \in \mathcal{L} \times \mathcal{L}$. Then

$$P_{R(E_0)}\bar{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \bar{\xi}$$

and

$$P_{N(E_0)}\bar{\xi} = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \bar{\xi},$$

i.e. $\xi = 0$, a contradiction. □

The folowing result holds:

**Corollary 8.8.** *Suppose that $E \in \mathcal{Q}_g$ (the set of idempotents in generic position). With the above notations, if $X$ (or equivalently, $CS^{-1}$) is cyclic in $\mathcal{L}$, then $E \in \mathcal{Q}_c$.*

*Proof.* By Lemma 8.7, in this case the sufficient conditions in Proposition 8.4 applied to the Halmos model reduce to $CS^{-1}$ being cyclic in $\mathcal{L}$. By the computation in Lemma 8.7, $CS^{-1} = cotg(X)$ is cyclic in $\mathcal{L}$ if and only if $X$ is cyclic in $\mathcal{L}$ □

**Remark 8.9.** This result means that the conditions in Proposition 8.4 are not necessary for $E$ to belong to $\mathcal{Q}_c$. Indeed, the class $\mathcal{Q}_c$ is unitarily invariant. Whereas for an arbitrary idempotent $E$ in generic position (which is unitarily equivalent to a Halmos model), the off diagonal entry $E_{12}$ (with trivial nullspace and dense range) need not verify $N(E_{12}E_{12}^* - 1) = \{0\}$. In other words, this last condition is not unitarily invariant.

## References

[1] Amrein, W. O.; Sinha, K. B. On pairs of projections in a Hilbert space. Linear Algebra Appl. 208/209 (1994), 425–435.

[2] T. Ando. Unbounded or bounded idempotent operators in Hilbert space, Linear Algebra Appl. 438 (2013), no. 10, 3769–3775.

[3] Andruchow, E. Pairs of projections: Fredholm and compact pairs, Complex Anal. Oper. Theory 8 (2014), no. 7, 1435–1453.

[4] Andruchow, E. Operators which are the difference of two projections. J. Math. Anal. Appl. 420 (2014), no. 2, 1634–1653.

[5] Avron, J.; Seiler, R.; Simon, B. The index of a pair of projections. J. Funct. Anal. 120 (1994), no. 1, 220–237.

[6] Böttcher, A.; Spitkovsky, I. M. A gentle guide to the basics of two projections theory. Linear Algebra Appl. 432 (2010), no. 6, 1412–1459.

[7] Buckholtz, D. Hilbert space idempotents and involutions, Proc. Amer. Math. Soc. 128 (2000), no. 5, 1415–1418.

[8] Carey, A.L.; Evans, D.E. Algebras almost commuting with Clifford algebras, J. Funct. Anal. 88 (1990), no. 2, 279–298.

[9] Corach, G.; Porta, H.; Recht, L. The geometry of spaces of projections in $C^*$-algebras. Adv. Math. 101 (1993), no. 1, 59–77.

[10] Davis, C. Separation of two linear subspaces. Acta Sci. Math. Szeged 19 (1958) 172–187.

[11] Dixmier, J. Position relative de deux variétés linéaires fermées dans un espace de Hilbert. (French) Revue Sci. 86, (1948). 387–399.

[12] Koliha, J. J.; Rakocevic, V. Fredholm properties of the difference of orthogonal projections in a Hilbert space. Integral Equations Operator Theory 52 (2005), no. 1, 125–134.

[13] Halmos, P. R. Two subspaces. Trans. Amer. Math. Soc. 144 1969 381–389.

[14] Porta, H.; Recht, L. Minimality of geodesics in Grassmann manifolds. Proc. Amer. Math. Soc. 100 (1987), no. 3, 464–466.

[15] Segal, G.; Wilson, G. Loop groups and equations of KdV type. Inst. Hautes tudes Sci. Publ. Math. No. 61 (1985), 5–65.

(Esteban Andruchow) Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina and Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.