Representations of the weighted WG inverse and a rank equation's solution

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Abstract

In this paper, we present several representations of the W-weighted WG inverse. These representations are expressed in terms of matrix powers as well as in terms of matrix products involving only the Moore-Penrose inverse. In addition, a new characterization of the W-weighted WG inverse is presented by using a rank equation.

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1 Introduction

The theory of generalized inverses seems to be maturing very fastly over the last century. It all started with the Moore-Penrose inverse and grew hand in hand of several contributors. In fact, recently several new generalized inverses were introduced [1, 2, 3, 4]. Roughly speaking, they are defined either by using the Moore-Penrose inverse and/or Drazin inverse, or by using projectors. From the viewpoint of the applications, generalized inverses appear as a useful tool in areas such as Markov chains [5, 6], Chemical equations [7], Robotics [8], Coding theory [9], etc.

While the Moore-Penrose inverse was introduced for rectangular matrices, Drazin inverse was firstly considered for square matrices. In 1980, Cline and Greville [10] extended the Drazin inverse to rectangular matrices and it was called the W-weighted Drazin inverse. This weighted generalized inverse has attracted great interest for mathematician researchers in the area of generalized inverse theory [11, 12, 13]. The W-weighted Drazin inverse is useful in various applications (for instance, in singular equations [14], numerical analysis [15], neural computing [16], partial orders [17, 18], etc.).

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Recently, the core-EP inverse has caught the attention of many authors. The core-EP inverse of a square matrix was defined in [3], and generalized to a rectangular matrix in [19]. Recently, several weighted generalized inverses such as weighted DMP inverses [20], weighted CMP inverses [21, 22], and weighted WG inverses [23] have been introduced as well.

We denote by $\mathbb{C}^{m\times n}$ the set of all $m \times n$ complex matrices. For $A \in \mathbb{C}^{m\times n}$, the symbols A^*, A^{-1}, $\mathrm{rk}(A)$, $\mathcal{N}(A)$, and $\mathcal{R}(A)$ will denote the conjugate transpose, the inverse (whenever it exists), the rank, the kernel, and the range space of A, respectively. Moreover, I_n will refer to the $n \times n$ identity matrix.

Let $A \in \mathbb{C}^{m\times n}$. The Moore-Penrose inverse of A is the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ satisfying the following four equations [5]

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse is used to represent the orthogonal projectors $P_A := AA^\dagger$ and $Q_A := A^\dagger A$ onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

For a given complex square matrix A, the index of A, denoted by $\mathrm{Ind}(A)$, is the smallest nonnegative integer k such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$.

Let $W \in \mathbb{C}^{n\times m}$ be a fixed nonzero matrix. We recall that the W-weighted Drazin inverse of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $A^{d,W} \in \mathbb{C}^{m\times n}$ satisfying the three equations [10]

$$A^{d,W}WAWA^{d,W} = A^{d,W}, \quad AWA^{d,W} = A^{d,W}WA, \quad A^{d,W}W(AW)^{k+1} = (AW)^k,$$

where $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$.

For the particular $k = 1$ case, the W-weighted Drazin inverse of A is called the weighted group inverse of A and is denoted by $A^{\#,W}$. When $m = n$ and $W = I_n$, we recover the Drazin inverse, that is, $A^{d,W} = A^d$. Moreover, if $\mathrm{Ind}(A) = 1$, then the Drazin inverse is called the group inverse of A and denoted by $A^\#$.

Several representations and properties of the W-weighted Drazin inverse can be found in [10, 12, 13, 15]. The W-weighted Drazin inverse satisfies the following two dual representations

$$A^{d,W} = A[(WA)^d]^2 = [(AW)^d]^2 A, \tag{1}$$

and the following two important properties

$$A^{d,W}W = (AW)^d, \quad WA^{d,W} = (WA)^d. \tag{2}$$

The core inverse was introduced by O. Baksalary and G. Trenkler in [1]. For a given matrix $A \in \mathbb{C}^{n\times n}$, the core inverse of A is the unique matrix $A^{\textcircled{\#}} \in \mathbb{C}^{n\times n}$ defined by the conditions

$$AA^{\textcircled{\#}} = P_A, \quad \mathcal{R}(A^{\textcircled{\#}}) \subseteq \mathcal{R}(A).$$

It is well known that A is core invertible if and only if $\mathrm{Ind}(A) \leq 1$. Some more characterizations were given in [24] and numerical aspects were investigated in [25].

K. Manjunatha Prasad and K.S. Mohana extended this concept for $n \times n$ complex matrices of arbitrary index [3]. They defined the core EP inverse as the (unique) matrix $A^{\textcircled{\dagger}} = A^k((A^*)^k A^{k+1})^\dagger (A^*)^k$, where $k = \text{Ind}(A)$.

Later, the core EP inverse was extended from square matrices to rectangular matrices in [19] and was called the weighted core EP inverse and denoted by $A^{\textcircled{\dagger},W}$. We recall that it is given by $A^{\textcircled{\dagger},W} = (WAWP_{(AW)^k})^\dagger$.

H. Wang and J. Chen [4] defined other generalized inverse for square matrices by using the core EP inverse, given by the matrix $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$, and called the weak group inverse of A. Recently, in [23] the authors extended the weak group inverse from square to rectangular matrices and it is known as the W-weighted WG inverse. For $A \in \mathbb{C}^{m\times n}$, it is given by the unique matrix $A^{\textcircled{w},W} \in \mathbb{C}^{m\times n}$ satisfying the two conditions

$$AWA^{\textcircled{w},W}WA^{\textcircled{w},W} = A^{\textcircled{w},W}, \qquad AWA^{\textcircled{w},W} = A^{\textcircled{\dagger},W}WA. \tag{3}$$

Moreover, this new weighted inverse admits the following representation in terms of the weighted core EP inverse: $A^{\textcircled{w},W} = A^{\textcircled{\dagger},W}WA^{\textcircled{\dagger},W}WA = [A^{\textcircled{\dagger},W}W]^2 A$.

Another generalized inverse, named the CMP inverse and considered for rectangular matrices, was investigated by D. Mosić in [21] and generalized to invertible bounded linear operator between two Hilbert spaces in [22].

The main aim of this paper is to present several new representations of the W-weighted WG inverse. These representations are expressed in terms of different matrix powers as well as in terms of matrix products involving only the Moore-Penrose inverse. The importance of these representations is that Moore-Penrose inverse can be automatically computed in different computational packages. In addition, a new characterization of the W-weighted WG inverse is introduced by using a rank equation.

The paper is organized as follows. Section 2 presents some preliminaries. Section 3 provides some representations for the W-weighted WG inverse in terms of purely Moore-Penrose inverses and other by means of only weighted WG inverse of square matrices. Section 4 gives a new characterization for W-weighted WG inverses by studying an adequate rank equation and some consequences are derived. full-rank decompositions are investigated for computing weighted core EP inverses and weighted WG inverses. Finally, Section 5 derives an additional representation for W-weighted WG inverses by using full-rank decompositions.

2 Preliminary results

In [26], H. Wang introduced the core EP decomposition. It was proved that for every nonzero matrix $A \in \mathbb{C}^{n\times n}$ with $\text{Ind}(A) = k$, there exist unique matrices $A_1, A_2 \in \mathbb{C}^{n\times n}$ such that $A = A_1 + A_2$ satifying $\text{Ind}(A_1) \leq 1$, $A_2^k = 0$, and $A_1^* A_2 = A_2 A_1 = 0$ ([26, Theorem 2.1, Theorem 2.4]). Moreover, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that A can be represented as the sum of

$$A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* \qquad \text{and} \qquad A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \tag{4}$$

where T is nonsingular, $\mathrm{rk}(T) = \mathrm{rk}(A^k)$, and N is nilpotent of index k. This representation of A is called the core EP decomposition of A.

Based on decomposition (4) for A, H. Wang proved that the core EP inverse of A has the form

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{5}$$

Similarly, in [4] it was proved that the weak group inverse can be factorized as

$$A^{\textcircled{w}} = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*, \tag{6}$$

provided that $A = A_1 + A_2$ be written as in (4).

Throughout this paper, a nonzero matrix $W \in \mathbb{C}^{n\times m}$ will be fixed and used as a weight. In what follows, this weight matrix W will be not explicitly mentioned. For $A \in \mathbb{C}^{m\times n}$, we notice that $AW \in \mathbb{C}^{m\times m}$ and $WA \in \mathbb{C}^{n\times n}$ are both square matrices.

In [19] the authors introduced a new decomposition, called weighted core EP decomposition, extending the core EP decomposition from square to rectangular matrices. This result establishes a simultaneous unitary block upper triangularization of a pair of rectangular matrices.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then there exist two unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$, two nonsingular matrices $A_1, W_1 \in \mathbb{C}^{t\times t}$, and two matrices $A_2 \in \mathbb{C}^{(m-t)\times(n-t)}$ and $W_2 \in \mathbb{C}^{(n-t)\times(m-t)}$ such that A_2W_2 and W_2A_2 are nilpotent of indices* $\mathrm{Ind}(AW)$ *and* $\mathrm{Ind}(WA)$*, respectively, with*

$$A = U \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} V^* \quad \text{and} \quad W = V \begin{bmatrix} W_1 & W_{12} \\ 0 & W_2 \end{bmatrix} U^*. \tag{7}$$

The expressions for A and W provided in Theorem 2.1 give the so called weighted core EP decomposition of the pair $\{A, W\}$.

The weighted core EP inverse of a rectangular matrix can be represented by using the weighted core EP decomposition [19, Theorem 5.2]. More precisely, the weighted core EP inverse of $A \in \mathbb{C}^{m\times n}$ has the form

$$A^{\textcircled{\dagger},W} = U \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*. \tag{8}$$

In the same paper, the authors also gave the following useful representations:

$$(AW)^{\textcircled{\dagger}} = U \begin{bmatrix} (A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad (WA)^{\textcircled{\dagger}} = V \begin{bmatrix} (W_1A_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*. \tag{9}$$

Remark 2.2. When $m = n$ and $W = I_n$, from the representations given in (5) and (8), it is easy to verify that the weighted core EP inverse and the core EP inverse coincide.

In [23], the authors introduced a new canonical form for the W-weighted Drazin inverse of a rectangular matrix by using the weighted core EP decomposition of the pair $\{A, W\}$.

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$, with $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$, be written as in (7). Then*

$$A^{d,W} = U \begin{bmatrix} (W_1A_1W_1)^{-1} & A_1R_{WA} \\ 0 & 0 \end{bmatrix} V^*, \tag{10}$$

where

$$R_{WA} = \sum_{j=0}^{k-1} (W_1A_1)^{j-k-2}(W_1A_{12} + W_{12}A_2)(W_2A_2)^{k-1-j}.$$

In particular, if $k = 1$ we have

$$A^{\#,W} = U \begin{bmatrix} (W_1A_1W_1)^{-1} & (A_1W_1)^{-2}(A_{12} + W_1^{-1}W_{12}A_2) \\ 0 & 0 \end{bmatrix} V^*. \tag{11}$$

Based on the weighted core-EP decomposition (7), the weighted weak group inverse $A^{ⓦ,W}$ is expressed by [23]

$$A^{ⓦ,W} = U \begin{bmatrix} (W_1A_1W_1)^{-1} & (A_1W_1)^{-2}(A_{12} + W_1^{-1}W_{12}A_2) \\ 0 & 0 \end{bmatrix} V^*. \tag{12}$$

Remark 2.4. When $k = 1$, it is easy to verify that the W-weighted Drazin (group) inverse and the weighted weak group inverse coincide, i.e., $A^{ⓦ,W} = A^{\#,W}$.

We finish this section by presenting two propositions that will be useful in the rest of the paper.

Proposition 2.5. [5] *Let $A \in \mathbb{C}^{n\times n}$ with Ind$(A) = k$. Then for each integer $\ell \geq k$ we have,*

$$A^d = A^\ell \left(A^{2\ell+1}\right)^\dagger A^\ell. \tag{13}$$

Proposition 2.6. [19] *Let $A \in \mathbb{C}^{n\times n}$ be written as in (4) such that Ind$(A) = k$. Then, for each integer $\ell \geq k$,*

$$P_{A^\ell} = U \begin{bmatrix} I_{rk(A^k)} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{14}$$

Proposition 2.7. [27] *Let $A \in \mathbb{C}^{n\times n}$ be written as in (4) such that Ind$(A) = k$. Then, for each integer $\ell \geq k$,*

$$A^{\textcircled{\dagger}} = A^d P_{A^\ell}. \tag{15}$$

3 Representations of the W-weighted WG inverse

As we mentioned in the introduction, in [23, Theorem 6] the authors gave the following representation for the W-weighted WG inverse of $A \in \mathbb{C}^{m\times n}$:

$$A^{ⓦ,W} = [A^{\dagger,W} W]^2 A. \tag{16}$$

On the other hand, in [19] the authors gave the following representation for the weighted core EP inverse of $A \in \mathbb{C}^{m\times n}$:

$$A^{\dagger,W} = \left(WAWP_{(AW)^k}\right)^{\dagger} = \left[W(AW)^{k+1}\left((AW)^k\right)^{\dagger}\right]^{\dagger}, \tag{17}$$

where $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. We can use the expression in (17) to obtain a new representation of the inverse W-weighted WG inverse, that is,

$$A^{ⓦ,W} = \left[\left[W(AW)^{k+1}\left((AW)^k\right)^{\dagger}\right]^{\dagger} W\right]^2 A. \tag{18}$$

A computational disadvantage of the representation (18) arises from the need of computing the Moore-Penrose inverse of two different matrices. In [11], the authors obtained some representations for the weighted core EP inverse which involve only one Moore-Penrose inverse. In the same way, the following results give new representations for the W-weighted WG inverse involving only one Moore-Penrose inverse.

Firstly, we recall that the weighted core EP inverse can be represented as $A^{\dagger,W} = A^{d,W}P_{(WA)^k}$ [11, Theorem 4.1]. By using Proposition 2.6, it immediately follows the following theorem.

Theorem 3.1. *If $A \in \mathbb{C}^{m\times n}$ with $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$ then, for each integer $\ell \geq k$,*

$$A^{\dagger,W} = A^{d,W}P_{(WA)^\ell} = A^{d,W}(WA)^\ell((WA)^\ell)^\dagger. \tag{19}$$

By applying above theorem and some properties of the core EP inverse of a square matrix we obtain the following interesting representation of the W-weighted WG inverse in terms of the Drazin inverse and the core EP inverse of a square matrix.

Theorem 3.2. *If $A \in \mathbb{C}^{m\times n}$ with $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$ then, for each integer $\ell \geq k$,*

$$A^{ⓦ,W} = A[(WA)^d]^2(WA)^{\dagger}WA. \tag{20}$$

Proof. From (16), (19), (2), and Theorem 2.7, respectively, we have

$$\begin{aligned} A^{ⓦ,W} &= [A^{\dagger,W}W]^2A \\ &= A^{d,W}P_{(WA)^\ell}(WA^{d,W})P_{(WA)^\ell}WA \\ &= A^{d,W}P_{(WA)^\ell}[(WA)^dP_{(WA)^\ell}]WA \\ &= A^{d,W}P_{(WA)^\ell}(WA)^{\dagger}WA \\ &= A^{d,W}(WA)^{\dagger}WA, \end{aligned}$$

where the last equality is due to the fact that $\mathcal{R}((WA)^{\textcircled{\dagger}}WA) = \mathcal{R}((WA)^{\textcircled{\dagger}}) = \mathcal{R}((WA)^{\ell}))$. Now, (20) follows directly from (1). $\square$

Corollary 3.3. *If $A \in \mathbb{C}^{m\times n}$ with $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$ then, for each integer $\ell \geq k$,*

$$A^{\textcircled{w},W} = A[(WA)^{d}]^{3} P_{(WA)^{\ell}} WA. \tag{21}$$

Proof. Follows from Theorem 3.2 and Proposition 2.7. $\square$

Corollary 3.4. *If $A \in \mathbb{C}^{m\times n}$ with $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$ then, for each integer $\ell \geq k$,*

$$A^{\textcircled{w},W} = A[(WA)^{\ell}\left((WA)^{2\ell+1}\right)^{\dagger}(WA)^{\ell}]^{3}(WA)^{\ell}\left((WA)^{\ell}\right)^{\dagger} WA. \tag{22}$$

Proof. Follows from Corollary 3.3 and Proposition 2.5. $\square$

In above corollary we need to compute the Moore-Penrose inverse of two matrices. Next, we presents a more symmetrical result that requires the computation of only one Moore-Penrose inverse.

Corollary 3.5. *If $A \in \mathbb{C}^{m\times n}$ with $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$ then, for each integer $\ell \geq k$,*

$$A^{\textcircled{w},W} = A[(WA)^{\ell}\left((WA)^{2\ell+1}\right)^{\dagger}(WA)^{\ell}]^{3}(WA)^{2\ell+1}\left((WA)^{2\ell+1}\right)^{\dagger} WA. \tag{23}$$

Proof. Follows from Corollary 3.4 and Proposition 2.6. $\square$

Now, we give several new representations and properties of $A^{\textcircled{w},W}$.

Theorem 3.6. *For each integer $\ell \geq k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$, the W-weighted WG inverse of $A \in \mathbb{C}^{m\times n}$ can be represented as follows:*

(a) $A^{\textcircled{w},W} = A\left[\left[(WA)^{\ell}\left((WA)^{2\ell+1}\right)^{\dagger}(WA)^{\ell}\right]^{2}(WA)^{\ell}\left((WA)^{\ell}\right)^{\dagger}WA\right]^{2}.$

(b) $A^{\textcircled{w},W} = A\left[\left[(WA)^{\ell}\left((WA)^{2\ell+1}\right)^{\dagger}(WA)^{\ell}\right]^{2}(WA)^{2\ell+1}\left((WA)^{2\ell+1}\right)^{\dagger}WA\right]^{2}.$

Proof. (a) From (16), (19), and (1), respectively, we have

$$\begin{aligned} A^{\textcircled{w},W} &= [A^{\textcircled{\dagger},W}W]^{2}A \\ &= A^{d,W}P_{(WA)^{\ell}}WA^{d,W}P_{(WA)^{\ell}}WA \\ &= A((WA)^{d})^{2}P_{(WA)^{\ell}}WA((WA)^{d})^{2}P_{(WA)^{\ell}}WA \\ &= A\left[[(WA)^{d}]^{2}P_{(WA)^{\ell}}WA\right]^{2}. \end{aligned}$$

Now, the assertion follows directly from Proposition 2.5.
(b) It follows from part (a) and Proposition 2.6. $\square$

Some well-known representations of the weak group inverse can be derived as particular cases by setting $W = I_n$ in the above theorem.

Corollary 3.7. *For each integer $\ell \geq k = \operatorname{Ind}(A)$, the weak group inverse of $A \in \mathbb{C}^{n\times n}$ can be represented as follows:*

(a) $A^{ⓦ} = A[A^{\ell}\left(A^{2\ell+1}\right)^{\dagger}A^{\ell}]^3A^{2\ell+1}\left(A^{2\ell+1}\right)^{\dagger}A.$

(b) $A^{ⓦ} = A\left[\left[A^{\ell}\left(A^{2\ell+1}\right)^{\dagger}A^{\ell}\right]^2A^{\ell}\left(A^{\ell}\right)^{\dagger}A\right]^2.$

(c) $A^{ⓦ} = A\left[\left[A^{\ell}\left(A^{2\ell+1}\right)^{\dagger}A^{\ell}\right]^2A^{2\ell+1}\left(A^{2\ell+1}\right)^{\dagger}A\right]^2.$

Before the study of some properties of $A^{ⓦ,W}$, we present an auxiliary lemma.

Lemma 3.8. *Let $A \in \mathbb{C}^{m\times n}$ and consider the weighted core EP decomposition of the pair $\{A, W\}$ as in (7). It then results that*

(i) $(AW)^{ⓦ} = U\begin{bmatrix} (A_1W_1)^{-1} & (A_1W_1)^{-2}(A_1W_{12}+A_{12}W_2) \\ 0 & 0 \end{bmatrix}U^*.$

(ii) $(WA)^{ⓦ} = V\begin{bmatrix} (W_1A_1)^{-1} & (W_1A_1)^{-2}(W_1A_{12}+W_{12}A_2) \\ 0 & 0 \end{bmatrix}V^*.$

Proof. (i) From Theorem 2.1 we obtain

$$AW = U\begin{bmatrix} A_1W_1 & A_1W_{12}+A_{12}W_2 \\ 0 & A_2W_2 \end{bmatrix}U^*. \tag{24}$$

So, a core EP decomposition of AW is given by $AW = (AW)_1 + (AW)_2$, where

$$(AW)_1 = U\begin{bmatrix} A_1W_1 & A_1W_{12}+A_{12}W_2 \\ 0 & 0 \end{bmatrix}U^*, \quad (AW)_2 = U\begin{bmatrix} 0 & 0 \\ 0 & A_2W_2 \end{bmatrix}U^*. \tag{25}$$

Now, by applying (6) we get

$$(AW)^{ⓦ} = U\begin{bmatrix} (A_1W_1)^{-1} & (A_1W_1)^{-2}(A_1W_{12}+A_{12}W_2) \\ 0 & 0 \end{bmatrix}V^*.$$

Part (ii) can be proved in a similar way. □

Next, some new properties of $A^{ⓦ,W}$ are given.

Theorem 3.9. *Let $A \in \mathbb{C}^{m\times n}$ and consider the weighted core EP decomposition of the pair $\{A, W\}$ as in (7). It then results that*

(i) $WA^{ⓦ,W} = (WA)^{ⓦ}.$

(ii) $A^{ⓦ,W} = A[(WA)^{ⓦ}]^2.$

(iii) $A^{ⓦ,W} = (AW)^{ⓦ}A(WA)^{ⓦ}.$

(iv) $A^{ⓦ,W}WAWA^{ⓦ,W} = A^{ⓦ,W}$.

(v) $A^{ⓦ,W} = A[(WA)^{\textcircled{\dagger}}]^3WA$.

Proof. Items (i)-(v) can be easily derived from (7), (9), (12) and Lemma 3.8. $\square$

Remark 3.10. We note that parts (i) and (ii) in Theorem 3.9 give two interesting properties of the W-weighted WG inverse similar to that satisfied by the W-weighted Drazin inverse (See Eqs. (1) and (2)). However, the equalities $A^{d,W} = [(AW)^d]^2A$ and $A^{d,W}W = (AW)^d$ do not remain valid for the W-weighted WG inverse, provided that $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\} \geq 2$, as we can check with the following examples.

Example 3.11. *Let*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \textit{and} \quad W = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\} = \max\{1, 2\} = 2$.

$$A^{ⓦ,W} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad [(AW)^{ⓦ}]^2A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 3.12. *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to check that $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\} = \max\{1, 2\} = 2$.

$$A^{ⓦ,W}W = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad (AW)^{ⓦ} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4 Characterization of the W-weighted WG inverse

In this section we give a new characterization of the W-weighted WG inverse by using a rank equation.

It is well known that if A is a nonsingular matrix of size $n \times n$, then the inverse A^{-1} of A is the unique matrix X that satisfies the rank equation

$$\mathrm{rk} \begin{bmatrix} A & I_n \\ I_n & X \end{bmatrix} = \mathrm{rk}(A).$$

The following two results are needed in what follows.

Lemma 4.1. *([28, Lemma 1]) Let $A \in \mathbb{C}^{n\times n}$ and M be a $2n \times 2n$ matrix partitioned as*

$$M = \begin{bmatrix} A & AQ \\ PA & B \end{bmatrix},$$

for P, Q, and B being matrices of adequate sizes. Then $\mathrm{rk}(M) = \mathrm{rk}(A) + \mathrm{rk}(B - PAQ)$.

Now, we present the main result of this section.

Theorem 4.2. *Let $A \in \mathbb{C}^{m\times n}$ and consider the weighted core EP decomposition of the pair $\{A, W\}$ as in (7) with $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$ and $t = \mathrm{rk}(A_1) = \mathrm{rk}(W_1)$. Then there exist a unique matrix X such that*

$$X(WA)^k = 0, \quad X^2 = X, \quad ((WA)^k)^* WAX = 0, \quad \mathrm{rk}(X) = n - t, \tag{26}$$

a unique matrix Y such that

$$Y(AW)^k = 0, \quad Y^2 = Y, \quad ((WA)^k)^*(WA)^2 WY = 0, \quad \mathrm{rk}(Y) = m - t, \tag{27}$$

and a unique matrix Z such that

$$\mathrm{rk}\begin{bmatrix} WAW & I - X \\ I - Y & Z \end{bmatrix} = \mathrm{rk}(WAW). \tag{28}$$

The matrix Z is the weighted weak group inverse $A^{ⓦ,W}$ of A. Furthermore, we have

$$X = I_n - WAWA^{ⓦ,W}, \quad Y = I_m - A^{ⓦ,W}WAW. \tag{29}$$

Proof. We assume that the pair $\{A, W\}$ is written as in (7) in the weighted core EP decomposition. It is straightforward to see that

$$WA = V \begin{bmatrix} W_1A_1 & W_1A_{12} + W_{12}A_2 \\ 0 & W_2A_2 \end{bmatrix} V^* \tag{30}$$

and

$$(WA)^k = V \begin{bmatrix} (W_1A_1)^k & \widetilde{T}_{WA} \\ 0 & 0 \end{bmatrix} V^*, \tag{31}$$

where $\widetilde{T}_{WA} = \sum_{j=0}^{k-1} (W_1A_1)^{j-k-1}(W_1A_{12} + W_{12}A_2)(W_2A_2)^{k-1-j}$.
By Lemma 3.8 and Theorem 3.9, it is easy to check that

$$\begin{aligned} X &:= I_n - WAWA^{ⓦ,W} = I_n - WA(WA)^{ⓦ} \\ &= V \begin{bmatrix} 0 & -(W_1A_1)^{-1}(W_1A_{12} + W_{12}A_2) \\ 0 & I_{n-t} \end{bmatrix} V^* \end{aligned}$$

satisfies conditions $X(WA)^k = 0$, $X^2 = X$, and $((WA)^k)^*WAX = 0$. Moreover, it is clear that $\mathrm{rk}(X) = n - t$.

In order to show uniqueness, let X_0 be a matrix which satisfies (26). Let $X_1 = V^* X_0 V$, and let X_1 be partitioned as

$$X_1 = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

with E and H of sizes $t \times t$ and $(n-t) \times (n-t)$, respectively.

From $X_0(WA)^k = 0$ and the fact that W_1A_1 is nonsingular we obtain $E = 0$ and $G = 0$. Since X_0 satisfies $X_0^2 = X_0$ and $\mathrm{rk}(X_0) = n - t$, it follows that H is nonsingular, and so $H = I_{n-t}$. Therefore,

$$X_1 = \begin{bmatrix} 0 & F \\ 0 & I_{n-t} \end{bmatrix}.$$

Finally, from $((WA)^k)^* WAX_0 = 0$, we have $((W_1A_1)^k)^*(W_1A_1F + W_1A_{12} + W_{12}A_2) = 0$ which is equivalent to $F = -(W_1A_1)^{-1}(W_1A_{12} + W_{12}A_2)$. Consequently, we obtain

$$X_0 = V \begin{bmatrix} 0 & -(W_1A_1)^{-1}(W_1A_{12} + W_{12}A_2) \\ 0 & I_{n-t} \end{bmatrix} V^* = X.$$

Now, we shall prove that there exists a unique matrix Y satisfying condition (27). It is straightforward to see that

$$AW = U \begin{bmatrix} A_1W_1 & A_1W_{12} + A_{12}W_2 \\ 0 & A_2W_2 \end{bmatrix} U^* \tag{32}$$

and

$$(AW)^k = U \begin{bmatrix} (A_1W_1)^k & \widetilde{T}_{AW} \\ 0 & 0 \end{bmatrix} U^*, \tag{33}$$

where $\widetilde{T}_{AW} = \sum_{j=0}^{k-1} (A_1W_1)^j (A_1W_{12} + A_{12}W_2)(A_2W_2)^{k-1-j}$.

From Lemma 3.8 and (32), it is not difficult to check that

$$\begin{aligned} Y : &= I_m - A^{ⓦ,W} WAW \\ &= U \begin{bmatrix} 0 & * \\ 0 & I_{m-t} \end{bmatrix} U^*, \end{aligned}$$

where $*$ is a matrix which will be not necessary in what follows. According to (33), it easy to see that $Y(AW)^k = 0$, $Y^2 = Y$, and $\mathrm{rk}(Y) = m - t$. On the other hand, since $BB^{ⓦ} = B^{\textcircled{\dagger}}B$ when B is a square matrix, and from the fact that $AW(AW)^{\textcircled{\dagger}} = P_{(AW)^k} = (P_{(AW)^k})^*$ (see [19, Lemma 2.6]) we obtain

$$\begin{aligned}
((WA)^k)^*(WA)^2WY &= ((WA)^k)^*(WA)^2W(I_m - A^{ⓦ,W}WAW) \\
&= ((WA)^k)^*(WA)^2(W - WA^{ⓦ,W}WAW) \\
&= ((WA)^k)^*(WA)^2(I_m - (WA)^{ⓦ}WA)W \\
&= ((WA)^k)^*WA(WA - WA(WA)^{ⓦ}WA)W \\
&= ((WA)^k)^*WA(I_m - WA(WA)^{ⓦ})WAW \\
&= ((WA)^k)^*WA(I_m - (WA)^{\textcircled{\dagger}}WA)WAW \\
&= ((WA)^k)^*(WA - WA(WA)^{\textcircled{\dagger}}WA)WAW \\
&= ((WA)^k)^*(I_m - WA(WA)^{\textcircled{\dagger}})(WA)^2W \\
&= ((WA)^k)^*(I_m - P_{(AW)^k})(WA)^2W \\
&= \left[((WA)^k)^* - (P_{(AW)^k}(WA)^k)^*\right](WA)^2W \\
&= 0.
\end{aligned}$$

The uniqueness of such a matrix Y can be similarly proved to that of X.

Finally, let $A^{ⓦ,W}$ be the weighted weak group inverse of A. Observe that Eq. (29) holds. For these X and Y, we have

$$\begin{bmatrix} WAW & I_n - X \\ I_m - Y & Z \end{bmatrix} = \begin{bmatrix} WAW & WAWA^{ⓦ,W} \\ A^{ⓦ,W}WAW & Z \end{bmatrix}.$$

Thus, by Lemma 4.1 and the condition (28) we get

$$\mathrm{rk}(Z - A^{ⓦ,W}WAWA^{ⓦ,W}) = 0,$$

which is equivalent to $Z = A^{ⓦ,W}$ because $A^{ⓦ,W}WAWA^{ⓦ,W} = A^{ⓦ,W}$ by Theorem 3.9 (iv). This completes the proof of theorem. □

Consequently, we give a new characterization of the weighted group inverse $A^{\#,W}$ of A.

Corollary 4.3. *Let $A \in \mathbb{C}^{m\times n}$, with $\max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\} = 1$ and $t = \mathrm{rk}(A_1) = \mathrm{rk}(W_1)$, be written as in (7). Then there exist a unique matrix X such that*

$$XWA = 0, \quad X^2 = X, \quad (WA)^*WAX = 0, \quad \mathrm{rk}(X) = n - t, \tag{34}$$

a unique matrix Y such that

$$YAW = 0, \quad Y^2 = Y, \quad (WA)^*(WA)^2WY = 0, \quad \mathrm{rk}(Y) = m - t, \tag{35}$$

and a unique Z such that

$$\mathrm{rk}\begin{bmatrix} WAW & I - X \\ I - Y & Z \end{bmatrix} = \mathrm{rk}(WAW). \tag{36}$$

The matrix Z is the weighted group inverse $A^{\#,W}$ of A. Furthermore, we have

$$X = I_n - WAWA^{\#,W}, \quad Y = I_m - A^{\#,W}WAW. \tag{37}$$

Remark 4.4. From (2), we observe that (37) is equivalent to

$$X = I_n - WA(WA)^{\#}, \quad Y = I_m - (AW)^{\#}AW.$$

A well-known characterization of the group inverse [29, 30] can be derived by setting $W = I_n$ and $A \in \mathbb{C}^{n\times n}$ of index 1 in corollary above.

Corollary 4.5. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index 1 such that $t = \mathrm{rk}(A)$. Then, there exist a unique matrix Y such that*

$$YA = 0, \quad AY = 0, \quad Y^2 = Y, \quad \mathrm{rk}(Y) = n - t, \tag{38}$$

and a unique matrix X such that

$$\mathrm{rk}\begin{bmatrix} A & I_n - Y \\ I_n - Y & X \end{bmatrix} = \mathrm{rk}(A). \tag{39}$$

The matrix X is the group inverse $A^{\#}$ of A. Furthermore, we have $Y = I_n - AA^{\#}$.

5 Algorithm and numerical example

In this section, we derive one more representation for the generalized inverse $A^{ⓦ,W}$ based on the procedure of Cline [31]. In addition, we present an algorithm for computing it.

In view of the representations obtained in Section 3, if $\max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\} \geq 1$, it appears greater than one powers of WA or AW when calculating the W-weighted WG inverse of $A \in \mathbb{C}^{m\times n}$. Specifically, if WA (or AW) is ill-conditioned, the best method is probably the sequential procedure of Cline [31], which involves full-rank decomposition of matrices of successively smaller sizes until a nonsingular matrix is reached. Thus, by [5, p. 166], if we take $WA = P_1Q_1$, $Q_iP_i = P_{i+1}Q_{i+1}$ is a full-rank decomposition of Q_iP_i, $i = 1, 2, \ldots, k-1$, and Q_kP_k nonsingular, then

$$(WA)^d = P(Q_kP_k)^{-k-1}Q. \tag{40}$$

Next, by using Corollary 3.3, we derive a new representation for computing the W-weighted WG inverse by means of the sequential procedure of Cline.

Theorem 5.1. *Let $A \in \mathbb{C}^{m\times n}$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Let P_1Q_1 be a full-rank decomposition of WA, $P_{i+1}Q_{i+1}$ a full-rank decomposition of Q_iP_i, $i = 1, 2, \ldots, k-1$, and Q_kP_k nonsingular. Then the following hold:*

$$A^{ⓦ,W} = A[P(Q_kP_k)^{-k-1}Q]^3P(P^*P)^{-1}P^*P_1Q_1, \tag{41}$$

where $P = P_1P_2\cdots P_k$ and $Q = Q_k\cdots Q_2Q_1$.

Proof. As $WA = P_1Q_1$ is assumed to be a full-rank factorization, from (40) we have $(WA)^d = P(Q_kP_k)^{-k-1}Q$, where $P = P_1P_2\cdots P_k$ and $Q = Q_k\cdots Q_2Q_1$.
Assuming that $P_{i+1}Q_{i+1}$ is a full-rank decomposition of Q_iP_i, for $i = 1, 2, \ldots, k-1$, and Q_kP_k is nonsingular, we can see that PQ is a full-rank decomposition of $(WA)^k$. In fact, the equality $(WA)^k = PQ$ is clear; in particular $(WA)^2 = P_1P_2Q_2Q_1$. Since $\mathrm{rk}(WA) = \mathrm{rk}(P_1) = \mathrm{rk}(Q_1)$, we get that P_1 admits a left inverse $P_1^{(\ell)}$ and Q_1 admits a right inverse $Q_1^{(r)}$. If $P_2 \in \mathbb{C}^{n\times s}$, from $\mathrm{rk}(P_2Q_2) \geq \mathrm{rk}(P_2) + \mathrm{rk}(Q_2) - s = \mathrm{rk}(P_2) = \mathrm{rk}(Q_2)$, we get

$$\mathrm{rk}(P_2) = \mathrm{rk}(Q_2) = \mathrm{rk}(P_2Q_2) = \mathrm{rk}(P_1^{(\ell)}(WA)^2Q_1^{(r)}) \leq \mathrm{rk}((WA)^2) \leq \mathrm{rk}(P_2Q_2).$$

Following a similar argument we arrive at $\mathrm{rk}((WA)^k) = \mathrm{rk}(P) = \mathrm{rk}(Q)$. Now, for $\ell \geq k$ we have

$$P_{(WA)^\ell} = P_{(WA)^k} = (WA)^k((WA)^k)^\dagger = PQQ^*(QQ^*)^{-1}(P^*P)^{-1}P^* = P(P^*P)^{-1}P^*. \tag{42}$$

Now, expression (41) follows from Corollary 3.3, (40), and (42). □

Following the same notation as in Theorem 5.1, we derive a procedure for computing the W-weighted WG inverse inverse $A^{Ⓦ,W}$ in the following algorithm.

Algorithm
Input: $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$.
Output: $A^{Ⓦ,W}$.
Step 1
Compute $k = \max\{\mathrm{Ind}(WA), \mathrm{Ind}(AW)\}$.
Step 2 Perform elementary row operations on WA to get the full-rank decomposition P_1Q_1 of WA.
Step 3 For $i = 1$ to $k-1$ perform the product Q_iP_i and calculate the full-rank decomposition $P_{i+1}Q_{i+1}$ of Q_iP_i.
Step 4 Compute $P = P_1P_2\cdots P_k$ and $Q = Q_k\cdots Q_2Q_1$.
Step 5 Compute $A^{Ⓦ,W} = A[P(Q_kP_k)^{-k-1}Q]^3P(P^*P)^{-1}P^*P_1Q_1$.
End

Now, we give an example to demonstrate the performance of the algorithm for computing the generalized inverse $A^{Ⓦ,W}$.

Example 5.2. *Let*

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We use the above algorithm to compute the W-weighted WG inverse $A^{\textcircled{w},W}$ of the matrix A with respect to weight W.

We have

$$WA = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and $k = \max\{\mathrm{Ind}(WA), \mathrm{Ind}(AW)\} = 3$ as required in Step 1. Computing a full-rank decomposition of the product WA, we obtain $WA = P_1Q_1$, where

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

as required in Step 2. Since $k = 3$, from Step 3, we need to compute full-rank decomposition of Q_1P_1 and Q_2P_2, respectively. In fact, for $i = 1$

$$Q_1P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P_2Q_2,$$

where

$$P_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For $i = 2$, we have

$$Q_2P_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = P_3Q_3,$$

where

$$P_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Q_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

From Step 4, we obtain

$$P = P_1P_2P_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad Q = Q_3Q_2Q_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Finally, from Step 5, we conclude that

$$A^{ⓦ,W} = \begin{bmatrix} 2/3 & 1/3 & 1/3 & 0 & 0 \\ 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Conflict of interests

No potential conflict of interest was reported by the authors.

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