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Matrix-Valued Orthogonal Polynomials Related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$

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The matrix-valued spherical functions for the pair $(K \times K, K)$, $K = \mathrm{SU}(2)$, are studied. By restriction to the subgroup A, the matrix-valued spherical functions are diagonal. For suitable set of spherical functions, we take these diagonals as a matrix-valued function, which are the full spherical functions. Their orthogonality is a consequence of the Schur orthogonality relations. From the full spherical functions, we obtain matrix-valued orthogonal polynomials of arbitrary size, and they satisfy a three-term recurrence relation which follows by considering tensor product decompositions. An explicit expression for the weight and the complete block-diagonalization of the matrix-valued orthogonal polynomials is obtained. From the explicit expression, we obtain right-hand-sided differential operators of first and second order for which the matrix-valued orthogonal polynomials are eigenfunctions. We study the low-dimensional cases explicitly, and for these cases additional results, such as the Rodrigues' formula and being eigenfunctions to first-order differential–difference and second-order differential operators, are obtained.

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1 Introduction

The connection between special functions and representation theory of Lie groups is a very fruitful one, see, for example, [34, 35]. For the special case of the group SU(2), we know that the matrix elements of the irreducible finite-dimensional representations are explicitly expressible in terms of Jacobi polynomials, and in this way many of the properties of the Jacobi polynomials can be obtained from the group theoretic interpretation. In particular, the spherical functions with respect to the subgroup $S(U(1) \times U(1))$ are the Legendre polynomials, and using this interpretation one obtains product formula, addition formula, integral formula, etc. for the Legendre polynomials, see, for example, [11, 19, 20, 34, 35] for more information on spherical functions.

In the development of spherical functions for a symmetric pair (G, K) the emphasis has been on spherical functions with respect to one-dimensional representations of K, and in particular, the trivial representation of K. Godement [12] considered the case of higher-dimensional representations of K, see also [11, 33] for the general theory. Examples studied are [2, 5, 15, 23, 31]. However, the focus is usually not on obtaining explicit expressions for the matrix-valued spherical functions, see Section 2 for the definition, except for [15, 23, 32]. In [15] the matrix-valued spherical functions are studied for the case $(U, K) = (SU(3), U(2))$, and the calculations revolve around the study of the algebra of differential operators for which these matrix-valued orthogonal polynomials are eigenfunctions. The approach in this paper is different.

In our case, the paper [23] by Koornwinder is relevant. Koornwinder studies the case of the compact symmetric pair $(U, K) = (SU(2) \times SU(2), SU(2))$, where the subgroup is diagonally embedded, and he calculates explicitly vector-valued orthogonal polynomials. The goal of this paper is to study this example in more detail and to study the matrix-valued orthogonal polynomials arising from this example. The spherical functions in this case are the characters of SU(2), which are the Chebyshev polynomials of the second kind corresponding to the Weyl character formula. So the matrix-valued orthogonal polynomials can be considered as analogues of the Chebyshev polynomials. Koornwinder [23] introduces the vector-valued orthogonal polynomials which coincide with rows in the matrix of the matrix-valued orthogonal polynomials in this paper. We provide some of Koornwinder's results with new proofs. The matrix-valued spherical functions can be given explicitly in terms of the Clebsch–Gordan coefficients, or $3-j$-symbols, of SU(2). Moreover, we find many more properties of these matrix-valued orthogonal polynomials. In particular, we give an explicit expression for the weight, that is, the matrix-valued orthogonality measure, in terms of Chebyshev polynomials by using an expansion in terms of spherical functions of the matrix elements and explicit knowledge of Clebsch–Gordan coefficient. This gives some strange identities for sums of hypergeometric functions in Appendix 1. Another important result is the explicit three-term recurrence relation which is obtained by considering tensor product decompositions. Also, using the explicit expression for the weight function we can obtain differential operators for which these matrix-valued orthogonal polynomials are eigenfunctions.

Matrix-valued orthogonal polynomials arose in the work of Krein [26, 27] and have been studied from an analytic point of view by Durán and others, see [6–10, 13, 14, 16, 17] and references given there. As far as we know, the matrix-valued orthogonal polynomials that we obtain have not been considered before. Also 2×2-matrix-valued orthogonal polynomials occur in the approach of the noncommutative oscillator, see [22] for more references. A group theoretic interpretation of this oscillator in general seems to be lacking.

The results of this paper can be generalized in various ways. First of all, the approach can be generalized to pairs (U, K) with the centralizer $(U\mathfrak{g})^{\mathfrak{k}}$ abelian, but this is rather restrictive [25]. Given a pair (U, K) and a representation δ of K such that $[\pi|K : \delta] \leq 1$ for all representations π of G and $\delta|_M$ is multiplicity free, we can perform the same construction to get matrix-valued orthogonal polynomials. Needless to say, in general, it might be difficult to be able to give an explicit expression of the weight function. Another option is to generalize to $(K \times K, K)$ to obtain matrix-valued orthogonal polynomials generalizing Weyl's character formula for other root systems, see, for example, [19].

We now discuss the contents of the paper. In Section 2, we introduce the matrix-valued spherical functions for the pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ taking values in the matrices of size $(2\ell + 1) \times (2\ell + 1)$, $\ell \in \frac{1}{2}\mathbb{N}$. In Section 3, we prove the recurrence relation for the matrix-valued spherical functions using a tensor product decomposition. This result gives us the opportunity to introduce polynomials, and this coincides with results of Koornwinder [23]. In Section 4, we introduce the full spherical functions on the subgroup A_*, corresponding to the Cartan decomposition $U = K_* A_* K_*$, by putting the restriction to A_* of the matrix-valued spherical function into a suitable matrix. In Section 5, we discuss the explicit form and the symmetries of the weight. Moreover, we calculate the commutant explicitly and this gives rise to a decomposition of the full spherical functions, the matrix-valued orthogonal polynomials and the weight function in a 2×2-block diagonal matrix, which cannot be reduced further. After a brief review of generalities of matrix-valued orthogonal polynomials in Section 6, we discuss the even- and odd-dimensional cases separately. In the even-dimensional case an interesting relation between the two blocks occurs. In Section 7, we discuss the right-hand-sided differential operators, and we show that the matrix-valued orthogonal polynomials associated to the full spherical function are eigenfunctions to a first-order differential operator as well as to a second-order differential operator. Section 8 discusses explicit low-dimensional examples, and gives some additional information such as the Rodrigues' formula for these matrix-valued orthogonal polynomials and more differential operators. Finally, in the appendices we give somewhat more technical proofs of two results.

2 Spherical Functions of the Pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$

Let $K = \mathrm{SU}(2)$, $U = K \times K$ and $K_* \subset U$ the diagonal subgroup. An element in K is of the form

$$k(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha, \beta \in \mathbb{C}. \tag{2.1}$$

Let $m_t := k(\mathrm{e}^{\mathrm{i}t/2}, 0)$ and let $T \subset K$ be the subgroup consisting of the m_t. T is the (standard) maximal torus of K. The subgroup $T \times T \subset U$ is a maximal torus of U. Define

$$A_* = \{(m_t, m_{-t}) : 0 \leq t < 4\pi\} \quad \text{and} \quad M = \{(m_t, m_t) : 0 \leq t < 4\pi\}.$$

We write $a_t = (m_t, m_{-t})$ and $b_t = (m_t, m_t)$. We have $M = Z_{K_*}(A_*)$ and the decomposition $U = K_* A_* K_*$. Note that M is the standard maximal torus of K_*.

The equivalence classes of the unitary irreducible representations of K are parametrized by $\hat{K} = \frac{1}{2}\mathbb{N}$. An element $\ell \in \frac{1}{2}\mathbb{N}$ determines the space

$$H^\ell := \mathbb{C}[x, y]_{2\ell},$$

the space of homogeneous polynomials of degree 2ℓ in the variables x and y. We view this space as a subspace of the function space $C(\mathbb{C}^2, \mathbb{C})$ and as such, K acts naturally on it via

$$k : p \mapsto p \circ k^{\mathrm{t}},$$

where k^{t} is the transposed. Let

$$\psi_j^\ell : (x, y) \mapsto \binom{2\ell}{\ell - j}^{\frac{1}{2}} x^{\ell - j} y^{\ell + j}, \quad j = -\ell, -\ell + 1, \ldots, \ell - 1, \ell. \tag{2.2}$$

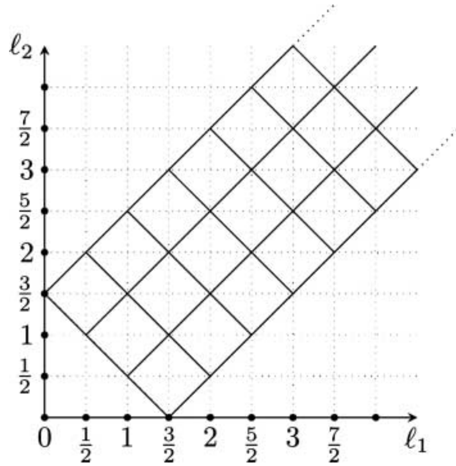

Fig. 1. Plot of the parametrization of the pairs (ℓ_1, ℓ_2) that contain ℓ upon restriction.

We stipulate that this is an orthonormal basis with respect to a Hermitian inner product that is linear in the first variable. The representation $T^\ell : K \to \mathrm{GL}(H^\ell)$ is irreducible and unitary.

The equivalence classes of the unitary irreducible representations of U are paramatrized by $\hat{U} = \hat{K} \times \hat{K} = \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$. An element $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ gives rise to the Hilbert space $H^{\ell_1,\ell_2} := H^{\ell_1} \otimes H^{\ell_2}$ and in turn to the irreducible unitary representation on this space, given by the outer tensor product

$$T^{\ell_1,\ell_2}(k_1, k_2)(\psi^{\ell_1}_{j_1} \otimes \psi^{\ell_2}_{j_2}) = T^{\ell_1}(k_1)(\psi^{\ell_1}_{j_1}) \otimes T^{\ell_2}(k_2)(\psi^{\ell_2}_{j_2}).$$

The restriction of $(T^{\ell_1,\ell_2}, H^{\ell_1,\ell_2})$ to K_* decomposes multiplicity free in summands of type $\ell \in \frac{1}{2}\mathbb{N}$ with

$$|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2 \quad \text{and} \quad \ell_1 + \ell_2 - \ell \in \mathbb{Z}. \tag{2.3}$$

Conversely, the representations of U that contain a given $\ell \in \frac{1}{2}\mathbb{N}$ are the pairs $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ that satisfy (2.3). We have pictured this parametrization in Figure 1 for $\ell = \frac{3}{2}$.

The following theorem is standard, see [24].

Theorem 2.1. The space H^{ℓ_1,ℓ_2} has a basis

$$\{\phi^{\ell_1,\ell_2}_{\ell,j} : \ell \text{ satisfies (2.3) and } |j| \le \ell\}$$

such that, for every ℓ, the map $\beta_\ell^{\ell_1,\ell_2} : H^\ell \to H^{\ell_1,\ell_2}$ defined by $\psi_j^\ell \mapsto \phi_{\ell,j}^{\ell_1,\ell_2}$ is a K-intertwiner. The base change with respect to the standard basis $\{\psi_{j_1}^{\ell_1} \otimes \psi_{j_2}^{\ell_2}\}$ of H^{ℓ_1,ℓ_2} is given by

$$\phi_{\ell,j}^{\ell_1,\ell_2} = \sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell}\, \psi_{j_1}^{\ell_1} \otimes \psi_{j_2}^{\ell_2},$$

where the $C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell}$ are the Clebsch–Gordan coefficients, normalized in the standard way. The Clebsch–Gordan coefficient satisfies $C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell} = 0$, if $j_1 + j_2 \neq j$. □

Definition 2.2 (Spherical Function)**.** Fix a K-type $\ell \in \frac{1}{2}\mathbb{N}$ and let $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ be a representation that contains ℓ upon restriction to K_*. The spherical function of type $\ell \in \frac{1}{2}\mathbb{N}$ associated to $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ is defined by

$$\Phi_{\ell_1,\ell_2}^\ell : U \to \mathrm{End}(H^\ell) : x \mapsto \left(\beta_\ell^{\ell_1,\ell_2}\right)^* \circ T^{\ell_1,\ell_2}(x) \circ \beta_\ell^{\ell_1,\ell_2}. \tag{2.4}$$

□

If $\Phi_{\ell_1,\ell_2}^\ell$ is a spherical function of type ℓ, then it satisfies the following properties:

(i) $\Phi_{\ell_1,\ell_2}^\ell(e) = I$, where e is the identity element in the group U and I is the identity transformation of H^ℓ,

(ii) $\Phi_{\ell_1,\ell_2}^\ell(k_1 x k_2) = T^\ell(k_1)\Phi_\ell(x)T^\ell(k_2)$ for all $k_1, k_2 \in K_*$ and $x \in U$,

(iii) $\Phi_{\ell_1,\ell_2}^\ell(x)\Phi_{\ell_1,\ell_2}^\ell(y) = \int_{K^*} \chi_\ell(k^{-1})\Phi_{\ell_1,\ell_2}^\ell(xky)dk$, for all $x, y \in U$. Here ξ_ℓ denotes the character of T^ℓ and $\chi_\ell = (2\ell+1)\xi_\ell$.

Remark 2.1. Definition 2.2 is not the definition of a spherical function given by Godement [12], Gangolli and Varadarajan [11], or Tirao [33], but it follows from property (iii) that it is equivalent in this situation. The point where our definition differs is essentially that we choose one space, namely $\mathrm{End}(H^\ell)$, in which all the spherical functions take their values, instead of different endomorphism rings for every U-representation. We can do this because of the multiplicity free splitting of the irreducible representations. □

Proposition 2.3. Let $\mathrm{End}_M(H^\ell)$ be the algebra of elements $Y \in \mathrm{End}(H^\ell)$ such that $T^\ell(m)Y = YT^\ell(m)$ for all $m \in M$. Then $\Phi_{\ell_1,\ell_2}^\ell(A_*) \subset \mathrm{End}_M(H^\ell)$. The restriction of $\Phi_{\ell_1,\ell_2}^\ell$ to A_* is diagonalizable. □

Proof. This is observation [23, (2.6)]. Another proof, similar to [15, Proposition 5.11], uses $ma = am$ for all $a \in A_*$ and $m \in M$ so that by (ii)

$$\Phi^{\ell}_{\ell_1,\ell_2}(a) = T^{\ell}(m)\Phi^{\ell}_{\ell_1,\ell_2}(a)T^{\ell}(m)^{-1}.$$

The second statement follows from the fact that the restriction of any irreducible representation of $K_* \cong \mathrm{SU}(2)$ to $M \cong \mathrm{U}(1)$ decomposes multiplicity free. ∎

The standard weight basis (2.2) is a weight basis in which $\Phi^{\ell}_{\ell_1\ell_2}|_{A_*}$ is diagonal. The restricted spherical functions are given by

$$(\Phi^{\ell}_{\ell_1,\ell_2}(a_t))_{j,j} = \sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} \mathrm{e}^{\mathrm{i}(j_2-j_1)t} (C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j})^2, \tag{2.5}$$

which follows from Definition 2.2 and Theorem 2.1.

3 Recurrence Relation for the Spherical Functions

A zonal spherical function is a spherical function $\Phi^{\ell}_{\ell_1,\ell_2}$ for the trivial K-type $\ell = 0$. We have a diffeomorphism $U/K_* \to K : (k_1, k_2)K_* \mapsto k_1 k_2^{-1}$ and the left K_*-action on U/K_* corresponds to the action of K on itself by conjugation. The zonal spherical functions are, up to a scalar, the characters on K [34] which are parametrized by pairs (ℓ_1, ℓ_2) with $\ell_1 = \ell_2$ and we write $\varphi_\ell = \Phi^0_{\ell,\ell}$. Note that $\varphi_\ell = (-1)^{-j+l}(2\ell+1)^{-1/2}U_{2\ell}(\cos t)$ by (2.5) and $C^{\ell,\ell,0}_{j,-j,0} = (-1)^{-j+l}(2\ell+1)^{-1/2}$, where U_n is the Chebyshev polynomial of the second kind of degree n. The zonal spherical function $\varphi_{\frac{1}{2}}$ plays an important role and we denote it by $\varphi = \varphi_{\frac{1}{2}}$. Any other zonal spherical function φ_n can be expressed as a polynomial in φ, see, for example, [34, 36]. For the spherical functions, we obtain a similar result. Namely, the product of φ and a spherical function of type ℓ can be written as a linear combination of at most four spherical functions of type ℓ.

Proposition 3.1. We have as functions on U

$$\varphi \cdot \Phi^{\ell}_{\ell_1,\ell_2} = \sum_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \sum_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} |a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}|^2 \Phi^{\ell}_{m_1,m_2}, \tag{3.1}$$

where the coefficients $a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}$ are given by

$$a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell} = \sum_{j_1,j_2,i_1,i_2,n_1,n_2} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,\ell} C^{\frac{1}{2},\frac{1}{2},0}_{i_1,i_2,0} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} C^{m_1,m_2,\ell}_{n_1,n_2,\ell}, \tag{3.2}$$

where the sum is taken over

$$|j_1| \le \ell_1, \quad |j_2| \le \ell_2, \quad |i_1| \le \tfrac{1}{2}, \quad |i_2| \le \tfrac{1}{2}, \quad |n_1| \le m_1 \quad \text{and} \quad |n_2| \le m_2. \tag{3.3}$$

Moreover, $a^{(\ell_1,\ell_2)}_{(\ell_1+1/2,\ell_2+1/2),\ell} \neq 0$. Note that the sum in (3.2) is a double sum because of Theorem 2.1. □

Proposition 3.1 should be compared with [29, Theorem 5.2], where a similar calculation is given for the case (SU(3), U(2)), see also [30].

Proof. On the one hand, the representation $T^{\ell_1,\ell_2} \otimes T^{\frac{1}{2},\frac{1}{2}}$ can be written as a sum of at most four irreducible U-representations that contain the representation T^{ℓ} upon restriction to K_*. On the other hand we can find a 'natural' copy $\mathscr{H}^{\ell}$ of H^{ℓ} in the space $H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}}$ that is invariant under the K_*-action. Projection onto this space transfers via α, defined below, to a linear combination of projections on the spaces H^{ℓ} in the irreducible summands. The coefficients can be calculated in terms of Clebsch–Gordan coefficients and these in turn give rise to the recurrence relation. The details are as follows.

Consider the U-representation $T^{\ell_1,\ell_2} \otimes T^{\frac{1}{2},\frac{1}{2}}$ in the space $H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}}$. By Theorem 2.1, we have

$$\alpha : H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}} \to \bigoplus_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \bigoplus_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} H^{m_1,m_2}$$

which is a U-intertwiner given by

$$\alpha : (\psi^{\ell_1}_{j_1} \otimes \psi^{\ell_2}_{j_2}) \otimes (\psi^{\frac{1}{2}}_{i_1} \otimes \psi^{\frac{1}{2}}_{i_2}) \mapsto \sum_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \sum_{n_1=-m_1}^{m_1} \sum_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} \sum_{n_2=-m_2}^{m_2} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} \psi^{m_1}_{n_1} \otimes \psi^{m_2}_{n_2}.$$

Let $\mathscr{H}^{\ell} \subset H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}}$ be the space that is spanned by the vectors

$$\{\phi^{\ell_1,\ell_2}_{\ell,j} \otimes \phi^{\frac{1}{2},\frac{1}{2}}_{0,0} : -\ell \leq j \leq \ell\}.$$

The element $\phi^{\ell_1,\ell_2}_{\ell,j} \otimes \phi^{\frac{1}{2},\frac{1}{2}}_{0,0}$ maps to

$$\sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} \sum_{i_1=-\frac{1}{2}}^{\frac{1}{2}} \sum_{i_2=-\frac{1}{2}}^{\frac{1}{2}} \sum_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \sum_{n_1=-m_1}^{m_1} \sum_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} \sum_{n_2=-m_2}^{m_2} \sum_{p=|m_1-m_2|}^{m_1+m_2} \sum_{u=-p}^{p}$$
$$\times C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{\frac{1}{2},\frac{1}{2},0}_{i_1,i_2,0} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} C^{m_1,m_2,p}_{n_1,n_2,u} \phi^{m_1,m_2}_{p,u}.$$

Note that $u = n_1 + n_2 = j_1 + i_1 + j_2 + i_2 = j$, so the last sum can be omitted. Also, since α is a K_*-intertwiner, we must have $p = \ell$. For every pair (m_1, m_2), we have a projection

$$P^{(m_1,m_2)}_{\ell} : \bigoplus_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \bigoplus_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} H^{m_1,m_2} \to \bigoplus_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \bigoplus_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} H^{m_1,m_2}$$

onto the ℓ-isotypical summand in the summand H^{m_1,m_2}. Hence

$$P^{m_1,m_2}_{\ell}(\alpha(\phi^{\ell_1,\ell_2}_{\ell,j} \otimes \phi^{\frac{1}{2},\frac{1}{2}}_{0,0}))$$
$$= \sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} \sum_{i_1=-\frac{1}{2}}^{\frac{1}{2}} \sum_{i_2=-\frac{1}{2}}^{\frac{1}{2}} \sum_{n_1=-m_1}^{m_1} \sum_{n_2=-m_2}^{m_2} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{\frac{1}{2},\frac{1}{2},0}_{i_1,i_2,0} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} C^{m_1,m_2,\ell}_{n_1,n_2,j} \phi^{m_1,m_2}_{\ell,j}.$$

The map $P^{m_1,m_2}_{\ell} \circ \alpha$ is a K_*-intertwiner so Schur's lemma implies that

$$\sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} \sum_{i_1=-\frac{1}{2}}^{\frac{1}{2}} \sum_{i_2=-\frac{1}{2}}^{\frac{1}{2}} \sum_{n_1=-m_1}^{m_1} \sum_{n_2=-m_2}^{m_2} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{\frac{1}{2},\frac{1}{2},0}_{i_1,i_2,0} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} C^{m_1,m_2,\ell}_{n_1,n_2,j}$$

is independent of j. Hence it is equal to $a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}$, taking $j = \ell$. We have

$$\alpha(\phi^{\ell_1,\ell_2}_{\ell,j} \otimes \phi^{\frac{1}{2},\frac{1}{2}}_{0,0}) = \sum_{m_1,m_2} a^{(\ell_1,\ell_2)}_{(m_1,m_2),j} \phi^{m_1,m_2}_{\ell,j}.$$

Moreover, the map

$$P=\sum_{m_1=|\ell_1-1/2|}^{\ell_1+1/2}\sum_{m_2=|\ell_2-1/2|}^{\ell_2+1/2}P_\ell^{m_1,m_2}\circ\alpha : H^{\ell_1,\ell_2}\otimes H^{\frac12,\frac12}\to\bigoplus_{m_1=|\ell_1-\frac12|}^{\ell_1+\frac12}\bigoplus_{m_2=|\ell_2-\frac12|}^{\ell_2+\frac12}H^{m_1,m_2}$$

is a K_*-intertwiner. To show that it is not the trivial map we note that $a^{(\ell_1,\ell_2)}_{(\ell_1+\frac12,\ell_2+\frac12),\ell}$ is nonzero. Indeed, the equalities

$$C^{\ell_1,\ell_2,\ell}_{j_1,j_2,\ell}=\frac{(-1)^{\ell_1-j_1}(\ell+\ell_2-\ell_1)!}{(\ell_1+\ell_2+\ell+1)!\Delta(\ell_1,\ell_2,\ell)}\left[\frac{(2\ell+1)(\ell_1+j_1)!(\ell_2+\ell-j_1)!}{(\ell_1-j_1)!(\ell_2-\ell+j_1)!}\right]^{1/2},$$

$$C^{\ell_1,\frac12,\ell_1+\frac12}_{j_1,\frac12,j_1+\frac12}=\left[\frac{\ell_1+j_1+1}{2\ell_1+1}\right]^{1/2}\quad\text{and}\quad C^{\ell_1,\frac12,\ell_1+\frac12}_{j_1,-\frac12,j_1-\frac12}=\left[\frac{\ell_1-j_1+1}{2\ell_1+1}\right]^{1/2},$$

where $\Delta(\ell_1,\ell_2,\ell)$ is a positive function, can be found in [34, Chapter 8] and plugging these into the formula for $a^{(\ell_1,\ell_2)}_{(\ell_1+\frac12,\ell_2+\frac12),\ell}$ shows that it is the sum of positive numbers, hence it is nonzero.

We conclude that P is nontrivial, so its restriction to $\mathscr{H}^\ell$ is an isomorphism and it intertwines the K_*-action. It maps K_*-isotypical summands to K_*-isotypical summands. Hence, $\alpha|\mathscr{H}^\ell=P|\mathscr{H}^\ell$. Define

$$\gamma_\ell^{\ell_1,\ell_2}:H^\ell\to H^{\ell_1,\ell_2}\otimes H^{\frac12,\frac12}:\psi_j^\ell\mapsto\phi^{\ell_1,\ell_2}_{\ell,j}\otimes\phi^{\frac12,\frac12}_{0,0}.$$

This is a K-intertwiner. It follows that

$$\alpha\circ\gamma_\ell^{\ell_1,\ell_2}=\sum_{m_1,m_2}a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}\beta_\ell^{m_1,m_2}. \tag{3.4}$$

Define the $\mathrm{End}(H^\ell)$-valued function

$$\Psi^\ell_{\ell_1,\ell_2}:U\to\mathrm{End}(H^\ell):x\mapsto(\gamma_\ell^{\ell_1,\ell_2})^*\circ(T^{\ell_1,\ell_1}\otimes T^{\frac12,\frac12})(x)\circ\gamma_\ell^{\ell_1,\ell_2}.$$

Note that $\Psi^\ell_{\ell_1,\ell_2}(x)=\varphi(x)\Phi^\ell_{\ell_1,\ell_2}(x)$. On the other hand, we have $T^{\ell_1,\ell_2}\otimes T^{\frac12,\frac12}=\alpha\circ(\bigoplus_{m_1,m_2}T^{m_1,m_2})\circ\alpha$. Together with (3.4) this yields

$$\Psi^\ell_{\ell_1,\ell_2}=\sum_{m_1,m_2}|a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}|^2(\beta_\ell^{\ell_1,\ell_2})^*\circ T^{m_1,m_2}\circ\beta_\ell^{m_1,m_2}.$$

Hence the result. ■

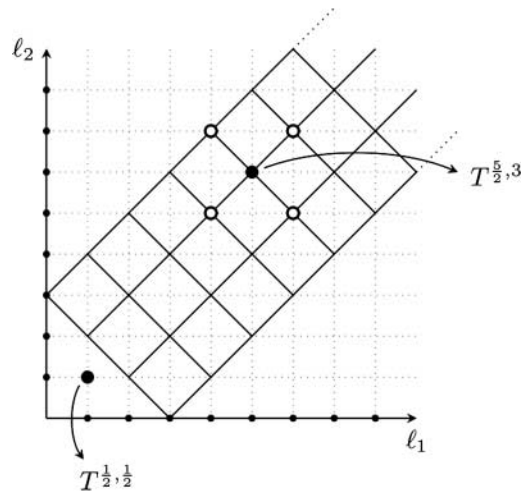

Fig. 2. Plot of how the tensor product $T^{\frac{5}{2},3} \otimes T^{\frac{1}{2},\frac{1}{2}}$ splits into irreducible summands.

In Figure 2, we have depicted the representations $(\frac{1}{2},\frac{1}{2})$ and $(\frac{5}{2},3)$ with black nodes. The tensor product decomposes into the four types $(\frac{5}{2} \pm \frac{1}{2}, 3 \pm \frac{1}{2})$ which are indicated with the white nodes.

Corollary 3.2. Given a spherical function $\Phi^{\ell}_{\ell_1,\ell_2}$ there exist $2\ell+1$ elements $q^{\ell,j}_{\ell_1,\ell_2}$, $j \in \{-\ell, -\ell+1, \dots, \ell\}$ in $\mathbb{C}[\varphi]$ such that

$$\Phi^{\ell}_{\ell_1,\ell_2} = \sum_{j=-\ell}^{\ell} q^{\ell,j}_{\ell_1,\ell_2} \Phi^{\ell}_{(\ell+j)/2,(\ell-j)/2}. \tag{3.5}$$

The degree of $q^{\ell,j}_{\ell_1,\ell_2}$ is $\ell_1+\ell_2-\ell$. □

Proof. We prove this by induction on $\ell_1+\ell_2$. If $\ell_1+\ell_2=\ell$, then the statement is true with the polynomials $q^{\ell,j}_{(\ell-k)/2,(\ell+k)/2} = \delta_{j,k}$. Suppose $\ell_1+\ell_2>\ell$ and that the statement holds for (ℓ_1',ℓ_2') with $\ell \le \ell_1'+\ell_2' < \ell_1+\ell_2$. We can write $|a^{(\ell_1-1/2,\ell_2-1/2)}_{(\ell_1,\ell_2),\ell}|^2 \Phi^{\ell}_{\ell_1,\ell_2}$ as

$$\varphi \cdot \Phi^{\ell}_{\ell_1-\frac{1}{2},\ell_2-\frac{1}{2}} - |a^{\ell_1-\frac{1}{2},\ell_2-\frac{1}{2}}_{(\ell_1-1,\ell_2),\ell}|^2 \Phi^{\ell}_{\ell_1-1,\ell_2} - |a^{\ell_1-\frac{1}{2},\ell_2-\frac{1}{2}}_{(\ell_1,\ell_2-1),\ell}|^2 \Phi^{\ell}_{\ell_1,\ell_2-1} - |a^{\ell_1-\frac{1}{2},\ell_2-\frac{1}{2}}_{(\ell_1-1,\ell_2-1),\ell}|^2 \Phi^{\ell}_{\ell_1-1,\ell_2-1}$$

by means of Proposition 3.1. The result follows from the induction hypothesis and $a^{(\ell_1-1/2,\ell_2-1/2)}_{(\ell_1,\ell_2),\ell} \neq 0$. ■

Remark 3.1. The fact that these functions $q^{\ell,j}_{\ell_1,\ell_2}$ are polynomials in $\cos(t)$ has also been shown by Koornwinder in [23, Theorem 3.4] using different methods. □

4 Restricted Spherical Functions

For the restricted spherical functions $\Phi^{\ell}_{\ell_1,\ell_2} : A_* \to \mathrm{End}(H^{\ell})$, we define a pairing:

$$\langle \Phi, \Psi \rangle_{A_*} = \frac{2}{\pi} \mathrm{tr} \left(\int_{A_*} \Phi(a)(\Psi(a))^* |D_*(a)| \, \mathrm{d}a \right), \tag{4.1}$$

where $D_*(a_t) = \sin^2(t)$, see [21]. In [23, Proposition 2.2], it is shown that on A_* the following orthogonality relations hold for the restricted spherical functions.

Proposition 4.1. The spherical functions on U of type ℓ, when restricted to A_*, are orthogonal with respect to (4.1). In fact, we have

$$\langle \Phi^{\ell}_{\ell_1,\ell_2}, \Phi^{\ell}_{\ell_1',\ell_2'} \rangle_{A_*} = \frac{(2\ell+1)^2}{(2\ell_1+1)(2\ell_2+1)} \delta_{\ell_1,\ell_1'} \delta_{\ell_2,\ell_2'}. \tag{4.2}$$

□

This is a direct consequence of the Schur orthogonality relations and the integral formula corresponding to the $U = K_* A_* K_*$ decomposition.

The parametrization of the U-types that contain a fixed K-type ℓ is given by (2.3). For later purposes, we reparamatrize (2.3) by the function $\zeta : \mathbb{N} \times \{-\ell, \ldots, \ell\} \to \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ given by

$$\zeta(d,k) = \left(\frac{d+\ell+k}{2}, \frac{d+\ell-k}{2} \right).$$

This new parametrization is pictured in Figure 3. For each degree d, we have $2\ell+1$ spherical functions. By Proposition 2.3, the restricted spherical functions take their values in the vector space $\mathrm{End}_M(H^{\ell})$ which is $2\ell+1$-dimensional. The appearance of the spherical functions in $2\ell+1$-tuples gives rise to the following definition.

Definition 4.2. Fix a K-type $\ell \in \frac{1}{2}\mathbb{N}$ and a degree $d \in \mathbb{N}$. The function $\Phi^{\ell}_d : A_* \to \mathrm{End}(H^{\ell})$ is defined by associating to each point $a \in A_*$ a matrix $\Phi^{\ell}_d(a)$ whose jth row is the vector $\Phi^{\ell}_{\zeta(d,j)}(a)$. More precisely, we have

$$(\Phi^{\ell}_d(a))_{p,q} = (\Phi^{\ell}_{\zeta(d,p)}(a))_{q,q} \quad \text{for all } a \in A_*. \tag{4.3}$$

The function Φ^{ℓ}_d is called the full spherical function of type ℓ and degree d. □

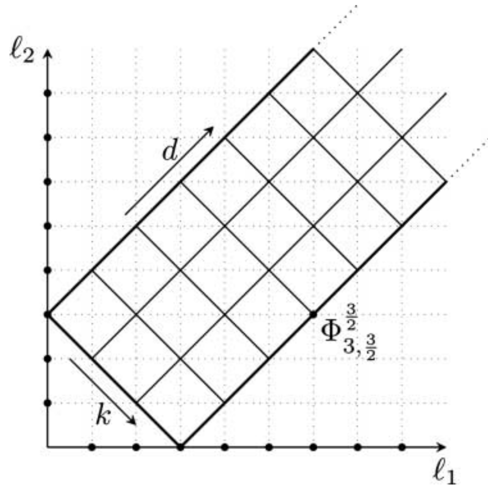

Fig. 3. Another parametrization of the pairs (ℓ_1, ℓ_2) containing ℓ; the steps of the ladder are paramatrized by d, the position on a given step by k.

The construction of a full spherical function from the spherical functions restricted to A_* has been previously employed by Grünbaum et al. [15] for the pair $(\mathrm{SU}(3), \mathrm{U}(2))$, see also Pacharoni and Tirao [29, 30] and Pacharoni and Román [28] for other cases.

Let A'_* be the open subset $\{a_t \in A_* : t \notin \pi\mathbb{Z}\}$. This is the regular part of A_*. The following proposition is shown in [23, Proposition 3.2]. In Proposition 5.7, we prove this result independently for general points in A_* in a different way.

Proposition 4.3. The full spherical function Φ^ℓ_0 of type ℓ and degree 0 has the property that its restriction to A'_* is invertible. □

Definition 4.4. Fix a K-type $\ell \in \frac{1}{2}\mathbb{N}$ and a degree $d \in \mathbb{N}$. Define the function

$$Q^\ell_d : A'_* \to \mathrm{End}(H^\ell) : a \mapsto \Phi^\ell_d(a)(\Phi^\ell_0(a))^{-1}. \tag{4.4}$$

The jth row is denoted by $Q^\ell_{\zeta(d,j)}(a)$. Q^ℓ_d is called the full spherical polynomial of type ℓ and degree d. □

The functions Q^ℓ_d and $Q^\ell_{\zeta(d,k)}$ are polynomials because $(Q^\ell_d)_{p,q} = q^{\ell,q}_{\zeta(d,p)}(\varphi)$. The degree of each row of Q^ℓ_d is d which justifies the name we have given these functions in Definition 4.4. Also Theorem 4.8 implies that Q^ℓ_d is indeed a matrix-valued polynomial of degree d in $\varphi(a)$ with a nonsingular leading coefficient, cf. Vretare [36]. In contrast, Φ^ℓ_d is not polynomial in $\phi(a)$ since these are Fourier polynomials which are not symmetric under the Weyl group S_2 for the reduced root system.

We shall show that the functions Φ^ℓ_d and Q^ℓ_d satisfy orthogonality relations that come from (4.1). We start with the Φ^ℓ_d. This function encodes $2\ell+1$ restricted spherical functions and to capture the orthogonality relations of (4.1), we need a matrix-valued inner product.

Definition 4.5. Let Φ and Ψ be $\mathrm{End}(H^\ell)$-valued functions on A_*. Define

$$\langle \Phi, \Psi\rangle := \frac{2}{\pi}\int_{A_*} \Phi(a)(\Psi(a))^*|D_*(a)|\,\mathrm{d}a. \tag{4.5}$$

□

Proposition 4.6. The pairing defined by (4.5) is a matrix-valued inner product. The functions Φ^ℓ_d with $d\in\mathbb{N}$ form an orthogonal family with respect to this inner product. □

Proof. The pairing satisfies all the linearity conditions of a matrix-valued inner product. Moreover, we have $\Phi(a)(\Phi(a))^*|D_*(a)|\geq 0$, for all $a\in A_*$. If $\langle\Phi,\Phi\rangle=0$, then $\Phi\Phi^*=0$ from which it follows that $\Phi=0$. Hence the pairing is an inner product. The orthogonality follows from the formula

$$(\langle \Phi_d, \Psi_{d'}\rangle)_{p,q} = \langle \Phi^\ell_{\zeta(d,p)}, \Phi^\ell_{\zeta(d',q)}\rangle_{A_*} = \delta_{d,d'}\delta_{p,q}\frac{(2\ell+1)^2}{(d+\ell+p+1)(d+\ell-p+1)}$$

and Proposition 4.1. ■

Define

$$V^\ell(a) = \Phi^\ell_0(a)(\Phi^\ell_0(a))^*|D_*(a)|, \tag{4.6}$$

with $D_*(a_t)=\sin^2 t$. This is a weight matrix and we have the following corollary.

Corollary 4.7. Let Q and R be $\mathrm{End}(H^\ell)$-valued functions on A_* and define the matrix-valued paring with respect to the weight V^ℓ by

$$\langle Q, R\rangle_{V^\ell} = \int_{A_*} Q(a)V^\ell(a)(R(a))^*\,\mathrm{d}a. \tag{4.7}$$

This pairing is a matrix-valued inner product and the functions Q^ℓ_d form an orthogonal family for this inner product. □

The functions Φ^ℓ_d and Q^ℓ_d being defined, we can now transfer the recurrence relations of Proposition 3.1 to these functions. Let $E_{i,j}$ be the elementary matrix with zeros everywhere except for the (i,j)th spot, where it has a one. If we write $E_{i,j}$ with $|i|>\ell$ or $|j|>\ell$, then we mean the zero matrix.

Theorem 4.8. Fix $\ell\in\frac{1}{2}\mathbb{N}$ and define the matrices A_d, B_d, and C_d by

$$\begin{aligned} A_d &= \sum_{k=-\ell}^{\ell} |a^{\zeta(d,k)}_{\zeta(d+1,k),\ell}|^2 E_{k,k}, \\ B_d &= \sum_{k=-\ell}^{\ell} \left(|a^{\zeta(d,k),}_{\zeta(d,k+1),\ell}|^2 E_{k,k+1} + |a^{\zeta(d,k)}_{\zeta(d,k-1),\ell}|^2 E_{k,k-1}\right), \\ C_d &= \sum_{k=-\ell}^{\ell} |a^{\zeta(d,k)}_{\zeta(d-1,k),\ell}|^2 E_{k,k}. \end{aligned} \tag{4.8}$$

For $a\in A_*$ we have

$$\varphi(a)\cdot\Phi^\ell_d(a) = A_d\Phi^\ell_{d+1}(a) + B_d\Phi^\ell_d(a) + C_d\Phi^\ell_{d-1}(a) \tag{4.9}$$

and similarly

$$\varphi(a)\cdot Q^\ell_d(a) = A_d Q^\ell_{d+1}(a) + B_d Q^\ell_d(a) + C_d Q^\ell_{d-1}(a). \tag{4.10}$$

Note $A_d\in GL_{2\ell+1}(\mathbb{R})$. □

Proof. It is clear that (4.10) follows from (4.9) by multiplying on the right with the inverse of Φ^ℓ_0. To prove (4.9), we look at the rows. Let $p\in\{-\ell,-\ell+1,\ldots,\ell\}$ and multiply (4.9) on the left by $E_{p,p}$ to pick out the pth row. The left-hand side gives $\varphi(a)E_{p,p}\Phi^\ell_d(a)$ while the right-hand side gives

$$\begin{aligned} &|a^{\zeta(d,p)}_{\zeta(d+1,p),\ell}|^2 E_{p,p}\Phi^\ell_{d+1}(a) + |a^{\zeta(d,p)}_{\zeta(d,p+1),\ell}|^2 E_{p,p+1}\Phi^\ell_d(a) + |a^{\zeta(d,p)}_{\zeta(d,p-1),\ell}|^2 E_{p,p-1}\Phi^\ell_d(a) \\ &\quad + |a^{\zeta(d,p)}_{\zeta(d-1,p),\ell}|^2 E_{p,p}\Phi^\ell_{d-1}(a). \end{aligned}$$

Now observe that these are equal by Proposition 3.1 and (4.3). This proves the result since p is arbitrary. ■

Finally, we discuss some symmetries of the full spherical functions. The Cartan involution corresponding to the pair (U, K_*) is the map $\theta(k_1, k_2) = (k_2, k_1)$. The representation T^{ℓ_1,ℓ_2} and $T^{\ell_2,\ell_1} \circ \theta$ are equivalent via the map $\psi^{\ell_1}_{j_1} \otimes \psi^{\ell_2}_{j_2} \mapsto \psi^{\ell_2}_{j_2} \otimes \psi^{\ell_1}_{j_1}$. It follows that $\theta^* \Phi^{\ell}_{\ell_1,\ell_2} = \Phi^{\ell}_{\ell_2,\ell_1}$. This has the following effect on the full spherical functions Φ^{ℓ}_d from Definition 4.2:

$$\theta^* \Phi^{\ell}_d = J \Phi^{\ell}_d, \tag{4.11}$$

where $J \in \mathrm{End}(H^{\ell})$ is given by $\psi^{\ell}_j \mapsto \psi^{\ell}_{-j}$. The Weyl group $\mathcal{W}(U, K_*) = \{1, s\}$ consists of the identity and the reflection s in $0 \in \mathfrak{a}_*$. The group $\mathcal{W}(U, K_*)$ acts on A_* and on the functions on A_* by pull-back.

Lemma 4.9. We have $s^* \Phi^{\ell}_{\ell_1,\ell_2}(a) = J \Phi^{\ell}_{\ell_1,\ell_2}(a)$ for all $a \in A_*$. The effect on the full spherical functions of type ℓ is

$$s^* \Phi^{\ell}_d = \Phi^{\ell}_d J. \tag{4.12}$$

□

Proof. This follows from (2.5) and the fact that $C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} = (-1)^{\ell_1+\ell_2-\ell} C^{\ell_1,\ell_2,\ell}_{-j_1,-j_2,-j}$. ■

Proposition 4.10. The functions Φ^{ℓ}_d commute with J. □

Proof. The action of θ and s_α on A_* is just taking the inverse. Formulas (4.11) and (4.12) now yield the result. ■

5 The Weight Matrix

We study the weight function $V^{\ell} : A_* \to \mathrm{End}(H^{\ell})$ defined in (4.6), in particular, its symmetries and explicit expressions for its matrix elements. First note that V^{ℓ} is real valued. Indeed, V^{ℓ} commutes with J,

$$J V^{\ell}(a) J = J \Phi^{\ell}_0(a) J (J \Phi^{\ell}_0(a) J)^* |D_*(a)| = \Phi^{\ell}_0(a) \Phi^{\ell}_0(a)^* |D_*(a)| = V^{\ell}(a), \tag{5.1}$$

since $J^* = J$ and $J^2 = 1$. This also shows that V is real valued,

$$\overline{V^{\ell}}(a_t) = V^{\ell}(a_{-t}) = J V^{\ell}(a_t) J = V^{\ell}(a_t). \tag{5.2}$$

Lemma 5.1. The weight has the symmetries $V^{\ell}_{p,q} = V^{\ell}_{q,p} = V^{\ell}_{-p,-q} = V^{\ell}_{-q,-p}$ for $p, q \in \{-\ell, \ldots, \ell\}$. □

Proof. The first equality follows since $V^{\ell}(a)$ is self-adjoint by (4.6) and real valued by (5.2). Since $V^{\ell}(a)$ commutes with J we see that $V^{\ell}_{p,q} = V^{\ell}_{-p,-q}$. ■

Set

$$v^{\ell}(a_t) = \Phi^{\ell}_0(a)\Phi^{\ell}_0(a)^* \tag{5.3}$$

so that $V^{\ell}(a_t) = v^{\ell}(a_t)|D_*(a_t)| = v^{\ell}(a_t)\sin^2 t$. Note that, for $-\ell \le p, q \le \ell$, the matrix coefficient

$$v^{\ell}(a_t)_{p,q} = \mathrm{tr}(\Phi^{\ell}_{\frac{\ell+p}{2},\frac{\ell-p}{2}}(a_t)(\Phi^{\ell}_{\frac{\ell+q}{2},\frac{\ell-q}{2}}(a_t))^*) \tag{5.4}$$

is a linear combination of zonal spherical functions by the following lemma.

Lemma 5.2. The function $U \to \mathbb{C} : x \mapsto \mathrm{tr}(\Phi^{\ell}_{\ell_1,\ell_2}(x)(\Phi^{\ell}_{m_1,m_2}(x))^*)$ is a bi-K-invariant function and

$$\mathrm{tr}(\Phi^{\ell}_{\ell_1,\ell_2}(a_t)(\Phi^{\ell}_{m_1,m_2}(a_t))^*) = \sum_{n=\max(|\ell_1-m_1|,|\ell_2-m_2|)}^{\min(\ell_1+m_1,\ell_2+m_2)} c_n U_{2n}(\cos t) \tag{5.5}$$

if $\ell_1 + m_1 - (\ell_2 + m_2) \in \mathbb{Z}$ and $\mathrm{tr}(\Phi^{\ell}_{\ell_1,\ell_2}(a_t)(\Phi^{\ell}_{m_1,m_2}(a_t))^*) = 0$, otherwise. □

Proof. It follows from Property (2) that the function is bi-K-invariant, so it is natural to expand the function in terms of the zonal spherical functions U_{2n} corresponding to the spherical representations $T^{n,n}$, $n \in \frac{1}{2}\mathbb{N}$. Since T^{ℓ_1,ℓ_2} is equivalent to its contragredient representation, we see that the only spherical functions occurring in the expansion of $\mathrm{tr}(\Phi^{\ell}_{\ell_1,\ell_2}(x)(\Phi^{\ell}_{m_1,m_2}(x))^*)$ are the ones for which $(n, n) \in A = \{(n_1, n_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N} : \ell_i + m_i - n_i \in \mathbb{Z}, |\ell_1 - m_1| \le n_1 \le \ell_1 + m_1, |\ell_2 - m_2| \le n_2 \le \ell_2 + m_2\}$ since the right-hand side corresponds to the tensor product decomposition $T^{\ell_1,\ell_2} \otimes T^{m_1,m_2} = \bigoplus_{(n_1,n_2)\in A} T^{n_1,n_2}$, see Proposition 3.1 and Figure 4. ■

Given $d, e \in \mathbb{N}$ and $-\ell \le p, q \le \ell$, we write $\zeta(d, p) = (\ell_1, \ell_2)$, $\zeta(e, q) = (m_1, m_2)$. Then we have

$$(\Phi^{\ell}_d(a_t)(\Phi^{\ell}_e)(a_t)^*)_{p,q} = \mathrm{tr}(\Phi^{\ell}_{\zeta(d,p)}(a_t)(\Phi^{\ell}_{\zeta(e,q)}(a_t))^*) = \sum_{j,j_1,j_2,i_1,i_2} (C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{m_1,m_2,\ell}_{i_1,i_2,j})^2\, \mathrm{e}^{\mathrm{i}(j_2-j_1+i_1-i_2)t},$$

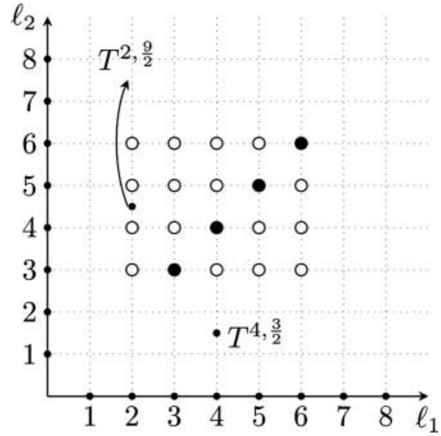

Fig. 4. Plot of the decomposition of the tensor product $T^{4,\frac{3}{4}} \otimes T^{2,\frac{9}{2}}$ into irreducible representations. The big nodes indicate the irreducible summands, the big black nodes the ones that contain the trivial K_*-type upon restricting.

where the sum is taken over

$$|j| \le \ell, \quad |j_1| \le \ell_1, \quad |j_2| \le \ell_2, \quad |i_1| \le m_1, \quad |i_2| \le m_2,$$

satisfying $j_1 + j_2 = i_1 + i_2 = j$. This equals

$$(\Phi_d^\ell(a_t)(\Phi_e^\ell(a_t))^*)_{p,q} = \sum_{|s| \le \min(\ell_1+m_1,\ell_2+m_2)} d^\ell_{\ell_1,\ell_2,m_1,m_2,s}\, \mathrm{e}^{\mathrm{i}st}, \tag{5.6}$$

where

$$d^\ell_{\ell_1,\ell_2,m_1,m_2,s} = \sum_{j,j_1,j_2,i_1,i_2} (C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{m_1,m_2,\ell}_{i_1,i_2,j})^2, \tag{5.7}$$

where the sum is taken over

$$|j| \le \ell, \quad |j_1| \le \ell_1, \quad |j_2| \le \ell_2, \quad |i_1| \le m_1, \quad |i_2| \le m_2,$$

satisfying $j_1 + j_2 = i_1 + i_2 = j$ and $j_2 - j_1 + i_1 - i_2 = s$. Since $U_n(\cos t) = \mathrm{e}^{-\mathrm{i}nt} + \mathrm{e}^{-\mathrm{i}(n-2)t} + \cdots + \mathrm{e}^{\mathrm{i}nt}$, it follows from (5.6) and Lemma 5.2 that we have the following summation result.

Corollary 5.3. Let $|s| \leq \max(|\ell_1 - m_1|, |\ell_2 - m_2|)$. Then $d^{\ell}_{\ell_1,\ell_2,m_1,m_2,s}$ is independent of s. $\square$

Note that the sum in (5.7) is a double sum of four Clebsch–Gordan coefficients, which in general are ${}_3F_2$-series [35].

We now turn to the case $\ell_1 = \frac{\ell+p}{2}$, $\ell_2 = \frac{\ell-p}{2}$, $m_1 = \frac{\ell+q}{2}$, $m_2 = \frac{\ell-q}{2}$. Because of Lemma 5.1 the next theorem gives an explicit expression for the weight matrix.

Theorem 5.4. Let $q - p \leq 0$ and $q + p \leq 0$. For $n = 0, \ldots, \ell + q$, there are coefficients $c^{\ell}_n(p,q) \in \mathbb{Q}_{>0}$ such that

$$(V^{\ell}(a_t))_{p,q} = \sin^2(t) \sum_{n=0}^{\ell+q} c^{\ell}_n(p,q) U_{2\ell+p+q-2n}(\cos(t)), \tag{5.8}$$

where the coefficients are given by

$$c^{\ell}_n(p,q) = \frac{2\ell+1}{\ell+p+1} \frac{(\ell-q)!(\ell+q)!}{(2\ell)!} \frac{(p-\ell)_{\ell+q-n}}{(\ell+p+2)_{\ell+q-n}} (-1)^{\ell+q-n} \frac{(2\ell+2-n)_n}{n!}. \tag{5.9}$$

$\square$

By $c_{n+p+q}(p,q) = c_n(-q,-p)$ the expansion (5.8) with (5.9) remains valid for $q \leq p$.

Proof. Since $\min(\ell_1 + m_1, \ell_2 + m_2) = \ell + \frac{p+q}{2}$ and $\max(|\ell_1 - m_1|, |\ell_2 - m_2|) = \frac{p-q}{2}$ in this case we find the expansion of the form as stated in (5.8). It remains to calculate the coefficients. Specializing (5.6) and writing

$$(C^{(\ell+m)/2,(\ell-m)/2,\ell}_{j_1,j_2,j})^2 = \delta_{j,j_1+j_2} \frac{\binom{\ell+m}{j_1+(\ell+m)/2}\binom{\ell-m}{j_2+(\ell-m)/2}}{\binom{2\ell}{\ell-j}}, \tag{5.10}$$

we find

$$\begin{aligned} v^{\ell}_{pq}(\cos t) &= \sum_{j=-\frac{\ell+p}{2}}^{\frac{\ell+p}{2}} \sum_{i=-\frac{\ell+q}{2}}^{\frac{\ell+q}{2}} F^{\ell}_{ij}(p,q) \exp(i(-2(i+j)t)) \\ &= \sum_{r=-(\ell+\frac{p+q}{2})}^{\ell+\frac{p+q}{2}} \left(\sum_{i=\max(-\frac{\ell+q}{2}, r-\frac{\ell+p}{2})}^{\min(\frac{\ell+q}{2}, r+\frac{\ell+p}{2})} F^{\ell}_{i,r-i}(p,q) \right) \mathrm{e}^{-2irt}, \end{aligned} \tag{5.11}$$

with

$$F^{\ell}_{ij}(p,q)=\binom{\ell+p}{j+(\ell+p)/2}\binom{\ell+q}{i+(\ell+q)/2}\sum_{k=\max(-j-\frac{\ell-p}{2},i-\frac{\ell-q}{2})}^{\min(-j+\frac{\ell-p}{2},i+\frac{\ell-q}{2})}\frac{\binom{\ell-p}{-k-j+(\ell-p)/2}\binom{\ell-q}{k-i+(\ell-q)/2}}{\binom{2\ell}{\ell-k}^2}. \tag{5.12}$$

From this, we can obtain the explicit expression of $v^{\ell}(a_t)_{p,q}$ in Chebyshev polynomials. The details are presented in Appendix 1. ■

Proposition 5.5. The commutant

$$\{V^{\ell}(a):a\in A_*\}':=\{Y\in \mathrm{End}(H^{\ell}):V^{\ell}(a)Y=YV^{\ell}(a)\quad \text{for all } a\in A_*\}=\{v^{\ell}(a):a\in A_*\}'$$

is spanned by the matrices I and J. □

Proof. By Proposition 4.10, we have $J\in\{V^{\ell}(a):a\in A_*\}'$. It suffices to show that the commutant contains no other elements than those spanned by I and J.

Let $v^{\ell}(a_t)=\sum_{n=0}^{2\ell}U_n(\cos t)A_n$, with $A_n\in \mathrm{Mat}_{2\ell+1}(\mathbb{C})$ by Theorem 5.4. Then for B in the commutant it is necessary and sufficient that $A_nB=BA_n$ for all n. First, put $C=A_{2\ell}$. Then by Theorem 5.4 $C_{p,q}=\binom{2\ell}{\ell+p}^{-1}\delta_{p,-q}$. The equation $BC=CB$ leads to $B_{p,q}=(C^{-1}BC)_{p,q}=\frac{C_{q,-q}}{C_{p,-p}}B_{-p,-q}=\frac{C_{q,-q}C_{-q,q}}{C_{p,-p}C_{-p,p}}B_{p,q}$ by iteration. Since $C_{q,-q}=C_{p,-p}$ if and only if $p=q$ or $p=-q$ we find $B_{p,q}=0$ for $p\neq q$ or $p\neq -q$. Moreover, $B_{p,p}=B_{-p,-p}$ and $B_{p,-p}=B_{-p,p}$.

Secondly, put $C'=A_{2\ell-1}$. Then, by Theorem 5.4, we have

$$C'_{p,q}=\delta_{|p+q|,1}(2\ell+1)\binom{2\ell}{\ell-p}^{-1}\binom{2\ell}{\ell+q}^{-1},$$

so the nonzero entries are different up to the symmetries $C'_{p,q}=C'_{-p,-q}=C'_{q,p}=C'_{-q,-p}$. Now $BC'=C'B$ implies by the previous result $B_{p,p}C'_{p,q}+B_{p,-p}C'_{-p,q}=C'_{p,q}B_{q,q}+C'_{p,-q}B_{q,q}$. Take $q=1-p$ to find $B_{p,p}=B_{1-p,1-p}=B_{p-1,p-1}$ unless $p=0$ or $p=1$, and take $q=p-1$ to find $B_{p,-p}=B_{1-p,p-1}=B_{p-1,1-p}$ unless $p=0$ or $p=1$. In particular, for $\ell\in\frac{1}{2}+\mathbb{N}$ this proves the result. In case $\ell\in\mathbb{N}$ we obtain one more equation: $B_{0,0}=B_{1,1}+B_{1,-1}$. This shows that B is in the span of I and J. ■

The matrix J has eigenvalues ± 1 and two eigenspaces H^{ℓ}_- and H^{ℓ}_+. The dimensions are $\lfloor \ell+\frac{1}{2}\rfloor$ and $\lceil \ell+\frac{1}{2}\rceil$. A choice of (ordered) bases of the eigenspaces is given by

$$\{\psi^{\ell}_j-\psi^{\ell}_{-j}:-\ell\le j<0,\ell-j\in\mathbb{Z}\}\quad\text{and}\quad\{\psi^{\ell}_j+\psi^{\ell}_{-j}:0\le j\le \ell,\ell-j\in\mathbb{Z}\}. \tag{5.13}$$

Let Y_ℓ be the matrix whose columns are the normalized basis vectors of (5.13). Conjugating V^ℓ with Y_ℓ yields a matrix with two blocks, one block of size $\lceil \ell+1/2\rceil \times \lceil \ell+1/2\rceil$ and one of size $\lfloor \ell+1/2\rfloor \times \lfloor \ell+1/2\rfloor$.

Corollary 5.6. The family $(Q^\ell_d)_{d\geq 0}$ and the weight V^ℓ are conjugate to a family and a weight in block form. More precisely

$$Y_\ell^{-1}Q^\ell_d(a_t)Y_\ell = \begin{pmatrix} Q^\ell_{d,-}(a_t) & 0 \\ 0 & Q^\ell_{d,+}(a_t)\end{pmatrix}, \quad Y_\ell^{-1}V^\ell(a_t)Y_\ell = \begin{pmatrix} V^\ell_-(a_t) & 0 \\ 0 & V^\ell_+(a_t)\end{pmatrix}.$$

The families $(Q^\ell_{d,\pm})_{d\geq 0}$ are orthogonal with respect to the weight $V^\ell_\pm$. Moreover, there is no further possible reduction. □

Proof. The functions Q^ℓ_d can be conjugated by Y_ℓ. Since the Q^ℓ_d commute with J, we see that the $Y_\ell^{-1}Q^\ell_dY_\ell$ has the same block structure as $Y_\ell^{-1}V^\ell Y_\ell$. The blocks of the $Y_\ell^{-1}Q^\ell_dY_\ell$ are orthogonal with respect to the corresponding block of $Y_\ell^{-1}V^\ell Y_\ell$. The polynomials $Q^\ell_{d,-}$ take their values in the (-1)-eigenspace H^ℓ_- of J, the polynomials $Q^\ell_{d,+}$ in the $(+1)$-eigenspace H^ℓ_+ of J. The dimensions are $\lfloor \ell+\frac{1}{2}\rfloor$ and $\lceil \ell+\frac{1}{2}\rceil$, respectively.

A further reduction would require an element in the commutant $\{V^\ell(a) : a \in A_*\}'$ not in the span of I and J. This is not possible by Proposition 5.5. ■

The entries of the weight v^ℓ with the Chebyshev polynomials of the highest degree 2ℓ occur only on the antidiagonal by Theorem 5.4. This shows that the determinant of $v^\ell(a_t)$ is a polynomial in $\cos t$ of degree $2\ell(2\ell+1)$ with leading coefficient $(-1)^{\ell(2\ell+1)}\prod_{p=-\ell}^{\ell} c_0(p,-p)2^{2\ell}\neq 0$. Hence v^ℓ is invertible on A_* away from the zeros of its determinant, of which there are only finitely many. We have proved the following proposition which should be compared with Proposition 4.3.

Proposition 5.7. The full spherical function Φ^ℓ_0 is invertible on A_* except for a finite set. □

In particular, Q^ℓ_d is well defined in Definition 4.2, except for a finite set. Since Q^ℓ_d is polynomial, it is well defined on A.

Calculations in Mathematica lead to the following conjecture.

Conjecture 5.8. $\det(v^\ell(a_t)) = (1-\cos^2 t)^{\ell(2\ell+1)}\prod_{p=-\ell}^{\ell}(2^{2\ell}c^\ell_0(p,-p))$. □

Conjecture 5.8 is supported by Koornwinder [23, Proposition 3.2], see Proposition 4.3. Conjecture 5.8 has been verified for $\ell \leq 16$.

6 The Matrix Orthogonal Polynomials Associated to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$

The main goal of Sections 3–5 was to study the properties of the matrix-valued spherical functions of any K-type associated to the pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$. These functions, introduced in Definition 2.2, are the building blocks of the full spherical functions described in Definition 4.2. We have exploited the fact that the spherical functions diagonalize when restricted to the subgroup A. This allows us to identify each spherical function with a row vector and arrange them in a square matrix.

The goal of this section is to translate the properties of the full spherical functions obtained in the previous sections at the group level to the corresponding family of matrix-valued orthogonal polynomials.

6.1 Matrix-valued orthogonal polynomials

Let W be a complex $N \times N$ matrix-valued integrable function on the interval (a, b) such that W is positive definite almost everywhere and with finite moments of all orders. Let $\mathrm{Mat}_N(\mathbb{C})$ be the algebra of all $N \times N$ complex matrices. The algebra over $\mathbb{C}$ of all polynomials in the indeterminate x with coefficients in $\mathrm{Mat}_N(\mathbb{C})$ denoted by $\mathrm{Mat}_N(\mathbb{C})[x]$. Let $\langle \cdot, \cdot \rangle$ be the following Hermitian sesquilinear form in the linear space $\mathrm{Mat}_N(\mathbb{C})[x]$:

$$\langle P, Q \rangle = \int_a^b P(x) W(x) Q(x)^* \, \mathrm{d}x. \tag{6.1}$$

The following properties are satisfied:

- $\langle aP + bQ, R \rangle = a\langle P, R \rangle + b\langle Q, R \rangle$, for all $P, Q, R \in \mathrm{Mat}_N(\mathbb{C})[x]$, $a, b \in \mathbb{C}$;
- $\langle TP, Q \rangle = T\langle P, Q \rangle$, for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$, $T \in \mathrm{Mat}_N(\mathbb{C})$;
- $\langle P, Q \rangle^* = \langle Q, P \rangle$, for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$;
- $\langle P, P \rangle \geq 0$ for all $P \in \mathrm{Mat}_N(\mathbb{C})[x]$. Moreover if $\langle P, P \rangle = 0$ then $P = 0$.

Given a weight matrix W one constructs a sequence of matrix-valued orthogonal polynomials, which is a sequence $\{R_n\}_{n \geq 0}$, where R_n is a polynomial of degree n with nonsingular leading coefficient and $\langle R_n, R_m \rangle = 0$, if $n \neq m$.

It is worth noting that there exists a unique sequence of monic orthogonal polynomials $\{P_n\}_{n \geq 0}$ in $\mathrm{Mat}_N(\mathbb{C})[x]$. Any other sequence of $\{R_n\}_{n \geq 0}$ of orthogonal polynomials in $\mathrm{Mat}_N(\mathbb{C})[x]$ is of the form $R_n(x) = A_n P_n(x)$ for some $A_n \in \mathrm{GL}_N(\mathbb{C})$.

By following a well-known argument, see, for instance [26, 27], one shows that the monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ satisfy a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x), \quad n \geq 0,$$

where $P_{-1} = 0$ and B_n, C_n are matrices depending on n and not on x.

There is a notion of similarity between two weight matrices that was pointed out in [8]. The weights W and $\tilde{W}$ are said to be similar if there exists a nonsingular matrix M, which does not depend on x, such that $\tilde{W}(x) = MW(x)M^*$ for all $x \in (a, b)$.

Proposition 6.1. Let $\{R_{n,1}\}_{n\geq 0}$ be a sequence of orthogonal polynomials with respect to W and $M \in GL_N(\mathbb{C})$. Then the sequence $\{R_{n,2}(x) = R_{n,1}(x)M^{-1}\}_{n\geq 0}$ is orthogonal with respect to $\tilde{W} = MWM^*$. Moreover, if $\{P_{n,1}\}$ is the sequence of monic orthogonal polynomials orthogonal with respect to W then $\{P_{n,2}(x) = MP_{n,1}(x)M^{-1}\}$ is the sequence of monic orthogonal polynomials with respect to $\tilde{W}$. □

Proof. It follows directly by observing that

$$\begin{aligned}\int R_{n,2}(x)\tilde{W}(x)R_{m,2}(x)^*\,\mathrm{d}x &= \int R_{n,1}(x)M^{-1}\tilde{W}(x)(M^{-1})^*R_{m,1}(x)^*\,\mathrm{d}x \\ &= \int R_{n,1}(x)W(x)R_{m,1}(x)^*\,\mathrm{d}x = 0, \quad \text{if } n \neq m.\end{aligned}$$

The second statement follows by looking at the leading coefficient of $P_{n,2}$ and the unicity of the sequence of monic orthogonal polynomials with respect to $\tilde{W}$. ■

A weight matrix W reduces to a smaller size if there exists a matrix M such that

$$W(x) = M\begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x)\end{pmatrix}M^* \quad \text{for all } x \in (a, b),$$

where W_1 and W_2 are matrix weights of smaller size. In this case, the monic polynomials $\{P_n\}_{n\geq 0}$ with respect to the weight W are given by

$$P_n(x) = M\begin{pmatrix} P_{n,1}(x) & 0 \\ 0 & P_{n,2}(x)\end{pmatrix}M^{-1}, \quad n \geq 0,$$

where $\{P_{n,1}\}_{n\geq 0}$ and $\{P_{n,2}\}_{n\geq 0}$ are the monic orthogonal polynomials with respect to W_1 and W_2, respectively.

6.2 Polynomials associated to SU(2) × SU(2)

In the rest of the paper, we will be concerned with the properties of the matrix orthogonal polynomials Q_d. For this purpose, we find it convenient to introduce a new labeling in the rows and columns of the weight V. More precisely, for any $\ell \in \frac{1}{2}\mathbb{Z}$, let W be the $(2\ell+1)\times(2\ell+1)$ matrix given by

$$\sqrt{1-x^2}\, W(x)_{n,m} = V(a_{\arccos x})_{-\ell+n,-\ell+m}, \qquad n, m \in \{0, 1, \ldots, 2\ell\}. \tag{6.2}$$

It then follows from Theorem 5.4 that

$$\begin{aligned} W(x)_{n,m} &= (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} \frac{(2\ell+1)}{n+1} \frac{(2\ell-m)!m!}{(2\ell)!} \\ &\quad \times \sum_{t=0}^{m} (-1)^{m-t} \frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}} \frac{(2\ell+2-t)_t}{t!} U_{n+m-2t}(x), \end{aligned} \tag{6.3}$$

if $n \leq m$ and $W(x)_{n,m} = W(x)_{m,n}$ otherwise.

We also consider the sequence of monic polynomials $\{P_d\}_{d\geq 0}$ given by

$$P_d(x)_{n,m} = \Upsilon_d^{-1} Q_d(a_{\arccos x})_{-\ell+n,-\ell+m}, \qquad n, m \in \{0, 1, \ldots, 2\ell\}, \tag{6.4}$$

where Υ_d is the leading coefficient of the polynomial $Q_d(a_{\arccos x})$, which is nonsingular by Theorem 4.8. Now we can rewrite the results on Section 5 in terms of the weight W and the polynomials P_d.

Corollary 6.2. The sequence of matrix polynomials $\{P_d(x)\}_{d>0}$ is orthogonal with respect to the matrix-valued inner product

$$\langle P, Q\rangle = \int_{-1}^{1} P(x) W(x) Q(x)^* \, dx. \qquad \square$$

Theorem 4.8 states that there is a three-term recurrence relation defining the matrix polynomials Q_d. These polynomials are functions on the group A. We can use (6.4) to derive a three-term recurrence relation for the polynomials P_d.

Corollary 6.3. For any $\ell \in \frac{1}{2}\mathbb{N}$, the matrix-valued orthogonal polynomials P_d, are defined by the following three-term recurrence relation:

$$xP_d(x) = P_{d+1}(x) + \Upsilon_d^{-1} B_d \Upsilon_d P_d(x) + \Upsilon_d^{-1} C_d \Upsilon_{d-1} P_{d-1}(x), \tag{6.5}$$

where the matrices A_d, B_d, and C_d are given in Theorem 4.8 and taking into account the relabeling as in the beginning of this subsection. □

6.3 Symmetries of the weight and the matrix polynomials

In this section, we shall use the symmetries satisfied by the full spherical functions to derive symmetry properties for the matrix weight W and the polynomials P_d.

For any $n \in \mathbb{N}$, let I_n be the $n \times n$ identity matrix and let J_n and F_n be the following $n \times n$ matrices:

$$J_n = \sum_{i=0}^{n-1} E_{i,n-1-i}, \qquad F_n = \sum_{i=0}^{n-1} (-1)^i E_{i,i}. \tag{6.6}$$

For any $n \times n$ matrix X the transpose X^{t} is defined by $(X^{\mathrm{t}})_{ij} = X_{ji}$ (reflection in the diagonal) and we define the reflection in the antidiagonal by $(X^d)_{ij} = X_{n-j,n-i}$. Note that taking transpose and taking antidiagonal transpose commute, and that

$$(X^{\mathrm{t}})^d = (X^d)^{\mathrm{t}} = X^{d\mathrm{t}} = J_n X J_n.$$

Moreover, $(XZ)^d = Z^d X^d$ for arbitrary matrices X and Z. We also need to consider the $(2\ell+1) \times (2\ell+1)$ matrix Y defined by

$$\begin{aligned} Y &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_{\ell+\frac{1}{2}} & J_{\ell+\frac{1}{2}} \\ -J_{\ell+\frac{1}{2}} & I_{\ell+\frac{1}{2}} \end{pmatrix} \quad \text{if } \ell = \frac{2n+1}{2}, \quad n \in \mathbb{N}, \\ Y &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_\ell & 0 & J_\ell \\ 0 & \sqrt{2} & 0 \\ -J_\ell & 0 & I_\ell \end{pmatrix} \quad \text{if } \ell \in \mathbb{N}. \end{aligned} \tag{6.7}$$

Proposition 6.4. The weight matrix $W(x)$ satisfies the following symmetries:

(i) $W(x)^{\mathrm{t}} = W(x)$ and $W(x)^d = W(x)$, for all $x \in [-1, 1]$. Thus,

$$J_{2\ell+1} W(x) J_{2\ell+1} = W(x),$$

for all $x \in [-1, 1]$.

(ii) $W(-x) = F_{2\ell+1} W(x) F_{2\ell+1}$ for all $x \in [-1, 1]$.

Here, $F_{2\ell+1}$ is the $(2\ell+1) \times (2\ell+1)$ matrix given in (6.6). □

Proof. The symmetry properties of W in (i) follow directly from Lemma 5.1.

The proof of (ii) follows from (6.3) by using the fact that $U_n(-x) = (-1)^n U_n(x)$ for any Chebyshev polynomial of the second kind $U_n(x)$, so that $W(-x)_{n,m} = (-1)^{n+m} W_{n,m}(x)$. ■

The weight matrix W can be conjugated into a 2×2 block diagonal matrix. In Corollary 5.6, we have pointed out this phenomenon for the weight V. The following theorem translates Corollary 5.6 to the weight matrix W.

Theorem 6.5. For any $\ell \in \frac{1}{2}\mathbb{N}$, the matrix W satisfies

$$\tilde{W}(x) = YW(x)Y^{\mathrm{t}} = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix},$$

where Y is a matrix given by (6.7). Moreover, if $\{P_{d,1}\}_{d\geq 0}$ (resp. $\{P_{d,2}\}_{d\geq 0}$) is a sequence of monic matrix orthogonal polynomials with respect to the weight $W_1(x)$ (resp. $W_2(x)$), then

$$\tilde{P}_d(x) = \begin{pmatrix} P_{d,1}(x) & 0 \\ 0 & P_{d,2} \end{pmatrix}, \quad d \geq 0 \tag{6.8}$$

is a sequence of matrix orthogonal polynomials with respect to $\tilde{W}$. There is no further reduction. □

The case $\ell = (2n+1)/2$, $n \in \mathbb{N}$, leads to weight matrices W of even size. In this case, W splits into two blocks of size $\ell + \frac{1}{2}$. In Corollary 6.6, we prove that these two blocks are equivalent, hence the corresponding matrix orthogonal polynomials are equivalent.

It follows from Proposition 6.4(1) that there exist $(n+1) \times (n+1)$ matrices $A(x)$ and $B(x)$ such that $A(x)^{\mathrm{t}} = A(x)$ and

$$W(x) = \begin{pmatrix} A(x) & B(x) \\ B(x)^{dt} & A(x)^{dt} \end{pmatrix}, \tag{6.9}$$

for all $x \in [-1, 1]$.

Corollary 6.6. Let $\ell = (2n+1)/2$, $n \in \mathbb{Z}$. Then

$$YW(x)Y^{t} = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix},$$

where

$$W_1(x) = A(x) + B(x)J_{n+1}, \quad W_2(x) = J_{n+1}F_{n+1}W_1(-x)F_{n+1}J_{n+1}.$$

Here, $A(x)$ and $B(x)$ are the matrices described in (6.3) and (6.9). Moreover, if $\{P_{d,1}\}_{d\geq 0}$ is the sequence of monic orthogonal polynomials with respect to $W_1(x)$ then

$$P_{d,2}(x) = (-1)^{d} J_{n+1}F_{n+1}P_{d,1}(-x)F_{n+1}J_{n+1} \tag{6.10}$$

is the sequence of monic orthogonal polynomials with respect to $W_2(x)$. □

Proof. In this proof, we will drop the subindex in the matrices J_{n+1} and F_{n+1} and we will use J and F instead. It is a straightforward calculation that, for $(n+1)\times(n+1)$-matrices A, B, C, and D, the following holds:

$$Y\begin{pmatrix} A & B \\ C & D \end{pmatrix}Y^{t} = \frac{1}{2}\begin{pmatrix} A + D^{dt} + J(C + B^{dt}) & B - C^{dt} + (D^{dt} - A)J \\ J(D^{dt} - A) + C - B^{dt} & D + A^{dt} - (C + B^{dt})J \end{pmatrix}.$$

In particular, for the weight function W, we get

$$YW(x)Y^{t} = Y\begin{pmatrix} A(x) & B(x) \\ B(x)^{dt} & A(x)^{dt} \end{pmatrix}Y^{t} = \begin{pmatrix} A(x) + B(x)J & 0 \\ 0 & J(A(x) - B(x)J)J \end{pmatrix}.$$

This proves that

$$W_1(x) = A(x) + B(x)J, \quad W_2(x) = J(A(x) - B(x)J)J.$$

It follows from Proposition 6.4 (2) that $A(-x) = FA(x)F$ and $B(-x) = FB(x)F$. Therefore, we have

$$JFW_1(-x)FJ = JFA(-x)FJ + FJB(-x)FJ = JA(x)J - JB(x) = W_2(x).$$

This proves the first assertion of the theorem.

The last statement follows from Proposition 6.1 ■

7 Matrix-Valued Differential Operators

In the study of matrix-valued orthogonal polynomials, an important ingredient is the study of differential operators which have these matrix-valued orthogonal polynomials as eigenfunctions. In this section, we discuss some of the differential operators that have the matrix-valued orthogonal polynomials of the previous section as eigenfunctions. The calculations rest on the explicit form of the weight function (6.3).

7.1 Symmetric differential operators

We consider right-hand side differential operators

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \qquad \partial = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{7.1}$$

in such a way that the action of D on the polynomial $P(x)$ is

$$PD = \sum_{i=0}^{s} \partial^i (P)(x) F_i(x).$$

In [18, Propositions 2.6 and 2.7], one can find a proof of the following proposition.

Proposition 7.1. Let $W = W(x)$ be a weight matrix of size N and let $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials in $\mathrm{Mat}_N(\mathbb{C})[x]$. If D is a right-hand side ordinary differential operator as in (7.1) of order s such that

$$P_n D = \Lambda_n P_n \quad \text{for all } n \geq 0,$$

with $\Lambda_n \in A$, then

$$F_i = F_i(x) = \sum_{j=0}^{i} x^j F_j^i, \qquad F_j^i \in \mathrm{Mat}_N(\mathbb{C})$$

is a polynomial of degree less than or equal to i. Moreover, D is determined by the sequence $\{\Lambda_n\}_{n\geq 0}$ and

$$\Lambda_n = \sum_{i=0}^{s} [n]_i F_i^i(D) \quad \text{for all } n \geq 0,$$

where $[n]_i = n(n-1)\cdots(n-i+1)$, $[n]_0 = 1$. □

We consider the following algebra of right-hand side differential operators with coefficients in $\mathrm{Mat}_N(\mathbb{C})[x]$.

$$\mathcal{D} = \left\{ D = \sum_i \partial^i F_i : F_i \in \mathrm{Mat}_N(\mathbb{C})[x],\ \deg F_i \le i \right\}.$$

Given any sequence of matrix-valued orthogonal polynomials $\{R_n\}_{n\ge 0}$ with respect to W, we define

$$\mathcal{D}(W) = \{D \in \mathcal{D} : R_n D = \Gamma_n(D) R_n,\ \Gamma_n(D) \in \mathrm{Mat}_N(\mathbb{C}),\ \text{for all } n \ge 0\}.$$

We observe that the definition of $\mathcal{D}(W)$ does not depend on the sequence of orthogonal polynomials $\{R_n\}_{n\ge 0}$.

Remark 7.1. The mapping $D \mapsto \Gamma_n(D)$ is a representation of $\mathcal{D}(W)$ in $\mathbb{C}^N$ for each $n \ge 0$. Moreover, the family of representations $\{\Gamma_n\}_{n\ge 0}$ separates the points of $\mathcal{D}(W)$. Note that $\mathcal{D}(W)$ is an algebra. □

Definition 7.2. A differential operator $D \in \mathcal{D}$ is said to be symmetric if $\langle PD, Q\rangle = \langle P, QD\rangle$ for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$. □

Proposition 7.3 ([18]). If $D \in \mathcal{D}$ is symmetric, then $D \in \mathcal{D}(W)$. □

The main theorem in [18] says that, for any $D \in \mathcal{D}$, there exists a unique differential operator $D^* \in \mathcal{D}(W)$, the adjoint of D, such that $\langle PD, Q\rangle = \langle P, QD^*\rangle$ for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$. The map $D \mapsto D^*$ is a $*$-operation in the algebra $\mathcal{D}(W)$. Moreover, we have $\mathcal{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W)$, where $\mathcal{S}(W)$ denotes the set of all symmetric operators. Therefore it suffices, in order to determine all the algebra $\mathcal{D}(W)$, to determine the space of symmetric operators $\mathcal{S}(W)$.

The condition of symmetry in Definition 7.2 can be translated into a set of differential equations involving the weight W and the coefficients of the differential operator D. For differential operators of order 2 this was proved in [7, Theorem 3.1].

Theorem 7.4. Let $W(x)$ be a weight matrix supported on (a, b). Let $D \in \mathcal{D}$ be the differential operator

$$D = \partial^2 F_2(x) + \partial F_1(x) + F_0^0,$$

Then D is symmetric with respect to W if and only if

$$F_2 W = W F_2, \tag{7.2}$$

$$2(F_2 W)' = W F_1^* + F_1 W, \tag{7.3}$$

$$(F_2 W)'' - (F_1 W)' + F_0 W = W F_0^*, \tag{7.4}$$

with the boundary conditions

$$\lim_{x \to a,b} F_2(x) W(x) = 0, \qquad \lim_{x \to a,b} (F_2(x) W(x))' - F_1(x) W(x) = 0. \tag{7.5}$$

□

7.2 Matrix-valued differential operators for the polynomials P_n

As in the previous section, we will denote by $\{P_n\}_{n \geq 0}$ the sequence of monic orthogonal polynomials with respect to the weight matrix W. We can write the weight as $W(x) = \rho(x) Z(x)$, where $\rho(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}$ and $Z(x)$ is the $(2\ell+1) \times (2\ell+1)$ matrix whose (n, m)-entry is given by

$$\begin{aligned} Z(x)_{n,m} &= \frac{(2\ell+1)}{n+1} \frac{(2\ell-m)!m!}{(2\ell)!} \sum_{t=0}^{m} (-1)^{m-t} \frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}} \frac{(2\ell+2-t)_t}{t!} U_{n+m-2t}. \\ &= \sum_{t=0}^{m} c(n, m, t) U_{n+m-2t}(x), \end{aligned} \tag{7.6}$$

if $m \leq n$ and $Z(x)_{n,m} = Z(x)_{m,n}$, otherwise.

Once we have an explicit expression for the weight matrix W, we can use the symmetry equations in Theorem 7.4 to find symmetric differential operators. If we start with a generic second-order differential operator

$$D = \sum_{i=0}^{2} \partial^i F_i(x), \qquad F_i(x) = \sum_{j=0}^{i} x^j F_j^i, \qquad F_j^i \in \mathrm{Mat}_N(\mathbb{C})$$

then Equations (7.2)–(7.4) lead to linear equations in the coefficients F_j^i. It is easy to solve these equations for small values of N using any software tool such as Maple. We have used the general expressions for small values of N to make an ansatz for the expressions of a first-order and a second-order differential operator. Then we prove that these operators are symmetric for all N by showing that they satisfy the conditions in Theorem 7.4.

In the following theorem, we show the matrix polynomials P_n satisfy a matrix-valued first-order differential equation. This phenomenon, which does not appear in the scalar case, has recently been studied in the literature, see, for instance [3, 4].

Theorem 7.5. Let E be the first-order matrix-valued differential operator

$$E = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right) A_1(x) + A_0,$$

where the matrices $A_1(x)$ and A_0 are given by

$$A_1(x) = \sum_{i=0}^{2\ell} \left(\frac{2\ell - i}{2\ell}\right) E_{i,i+1} - \sum_{i=0}^{2\ell} x \left(\frac{\ell - i}{\ell}\right) E_{i,i} - \sum_{i=0}^{2\ell} \left(\frac{i}{2\ell}\right) E_{i,i-1},$$

$$A_0 = \sum_{i=0}^{2\ell} \frac{(2\ell+2)(i-2\ell)}{2\ell} E_{i,i}.$$

Then E is symmetric with respect to the weight W; hence, $E \in D(W)$. Moreover, for every integer $n \geq 0$,

$$P_n(x)E = \Lambda_n(E)P_n(x),$$

where

$$\Lambda_n(E) = \sum_{i=0}^{2\ell} \left(-\frac{n(\ell - i)}{\ell} + \frac{(2\ell+2)(i-2\ell)}{2\ell}\right) E_{i,i}.$$

□

Proof. The proof of the theorem is performed by showing that the differential operator E is symmetric with respect to the weight W. It follows from Theorem 7.4, with $F_2 = 0$, that E is symmetric if and only if

$$W(x)A_1(x)^* + A_1(x)W(x) = 0, \tag{7.7}$$

$$-(A_1(x)W(x))' + A_0W(x) = W(x)A_0^*, \tag{7.8}$$

with the boundary condition

$$\lim_{x\to\pm 1} A_1(x)W(x) = 0. \tag{7.9}$$

The second statement will then follow from Propositions 7.1 and 7.3.

The verification of (7.7) and (7.8) involves elaborate computations, see Appendix 2. ■

Theorem 7.6. Let D be the second-order matrix-valued differential operator

$$D = (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\right) B_1(x) + B_0,$$

where the matrices $B_1(x)$ and B_0 are given by

$$B_1(x) = -\sum_{i=0}^{2\ell} \frac{(4\ell+3)(i-2\ell)}{2\ell} E_{i,i+1} + \sum_{i=0}^{2\ell} x \frac{(2\ell+3)(i-2\ell)}{\ell} E_{i,i} - \sum_{i=0}^{2\ell} \left(\frac{3i}{2\ell}\right) E_{i,i-1},$$

$$B_0 = -\sum_{i=0}^{2\ell} \frac{(i-2\ell)(i\ell-2\ell^2-5\ell-3)}{\ell} E_{i,i}.$$

Then D is symmetric with respect to the weight $W(x)$; hence, $D \in D(W)$. Moreover, for every integer $n \geq 0$,

$$P_n(x) D = \Lambda_n(D) P_n(x),$$

where

$$\Lambda_n(D) = \sum_{i=0}^{2\ell} \left(-n(n-1) + n\frac{(2\ell+3)(i-2\ell)}{\ell} - \frac{(i-2\ell)(i\ell-2\ell^2-5\ell-3)}{\ell} \right) E_{i,i}. \qquad \square$$

Proof. The proof of the theorem is similar to that of Theorem 7.5, see Appendix 2. ■

Corollary 7.7. The differential operators D and E commute. □

Proof. To see that D and E commute it is enough to verify that the corresponding eigenvalues commute. The eigenvalues commute because they are diagonal matrices. ■

As we pointed out in Theorem 6.5, for any $\ell \in \frac{1}{2}\mathbb{N}$ the matrix weight W and the polynomials P_n are $(2\ell+1)\times(2\ell+1)$ matrices that can be conjugated into 2×2 block matrices. More precisely,

$$YW(x)Y^{\mathrm{t}} = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix}, \quad YP_n(x)Y^{-1} = \begin{pmatrix} P_{n,1}(x) & 0 \\ 0 & P_{n,2}(x) \end{pmatrix},$$

where Y is the orthogonal matrix introduced in (6.7) and W_1, W_2 are the square matrices described in Corollary 6.6. Here $\{P_{n,1}\}_{n\geq 0}$ and $\{P_{n,2}\}_{n\geq 0}$ are the sequences of monic orthogonal polynomials with respect to the weights W_1 and W_2, respectively.

Proposition 7.8. Suppose $\ell = (2n+1)/2$, for some integer n, then E splits in $(n+1)\times(n+1)$ blocks in the following way:

$$Y\,E\,Y^t = \tilde{E} = \begin{pmatrix} -(\ell+1)I_{n+1} & E_1 \\ E_2 & -(\ell+1)I_{n+1} \end{pmatrix},$$

where

$$E_1 = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\tilde{A}_1(x) + \tilde{A}_0,$$

$$E_2 = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)F_{n+1}J_{n+1}\tilde{A}_1(-x)J_{n+1}F_{n+1} + F_{n+1}J_{n+1}\tilde{A}_0J_{n+1}F_{n+1}.$$

Here, F_{n+1}, J_{n+1} are the matrices introduced in (6.6). The matrices A_1 and A_0 are given by

$$\tilde{A}_1(x) = -\sum_{i=0}^{n-1}\frac{(2\ell-i)}{2\ell}E_{i,n-i-1} + x\sum_{i=0}^{n}\frac{(\ell-i)}{\ell}E_{i,n-i} + \sum_{i=1}^{n}\frac{i}{2\ell}E_{i,n-i+1} + \frac{(2\ell+1)}{4\ell}E_{n,n+1},$$

$$\tilde{A}_0 = \sum_{i=0}^{n}\frac{(\ell+1)(\ell-i)}{\ell}E_{i,n-i}. \qquad \square$$

Proof. The proposition follows by a straightforward computation. ∎

Proposition 7.9. Suppose $\ell\in\mathbb{N}$, then we have

$$Y\,E\,Y^t = \tilde{E} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\begin{pmatrix} \multicolumn{2}{c}{O_{(\ell+1)}} & \begin{matrix}\tilde{A}_1(x)\\ {v_1}^{\mathrm{t}}\end{matrix} \\ F_\ell\tilde{A}_1(x)F_\ell & v_2 & O_{\ell\times\ell} \end{pmatrix} + \begin{pmatrix} \multicolumn{2}{c}{-(\ell+1)I_{(\ell+1)}} & \begin{matrix}\tilde{A}_0(x)\\ {v_0}^{\mathrm{t}}\end{matrix} \\ F_\ell\tilde{A}_0(x)F_\ell & v_0 & -(\ell+1)I_\ell \end{pmatrix},$$

where $\tilde{A}_1$ and $\tilde{A}_0$ are $n \times n$ matrices given by

$$\tilde{A}_1(x) = -\sum_{i=0}^{\ell-2} \frac{(2\ell - i)}{2\ell} E_{i,\ell-i-1} + x \sum_{i=0}^{\ell-1} \frac{(\ell - i)}{\ell} E_{i,\ell-i} + \sum_{i=1}^{\ell-1} \frac{i}{2\ell} E_{i,\ell-i+1},$$

$$\tilde{A}_0 = \sum_{i=0}^{\ell-1} \frac{(\ell+1)(\ell-i)}{\ell} E_{i,\ell-i},$$

and the vectors $v_0, v_1, v_2 \in \mathbb{C}^\ell$ are

$$v_0 = (0, 0, \ldots, 0), \quad v_1 = \left(\frac{(2\ell+1)\sqrt{2}}{4\ell}, 0, \ldots, 0 \right), \quad v_2 = \left(-\frac{(2\ell+1)\sqrt{2}}{4(\ell+1)}, 0, \ldots, 0 \right). \quad \square$$

Proof. The proposition follows by a straightforward computation. ■

Let us assume that $\ell = (2n+1)/2$ for some $n \in \mathbb{N}$ so that the weight W and the polynomials P_n are matrices of even dimension. Proposition 7.8 says that

$$\tilde{P}_n(x)\tilde{E} = Y\Lambda_n Y^{\mathrm{t}} \tilde{P}_n(x), \quad n \geq 0. \tag{7.10}$$

A simple computation shows that

$$Y\Lambda_n Y^{\mathrm{t}} = -(\ell+1) I_{2n+2} + \begin{pmatrix} 0 & \Lambda_{n,1} \\ \Lambda_{n,2} & 0 \end{pmatrix},$$

where $\Lambda_{n,1}$ is a $(n+1) \times (n+1)$ matrix (depending on n) and

$$\Lambda_{n,2} = F_{n+1} J_{n+1} \Lambda_{n,1} J_{n+1} F_{n+1}.$$

It follows from (7.10) that the following matrix equation is satisfied

$$\begin{pmatrix} -(\ell+1) P_{n,1}(x) & P_{n,1}(x) E_1 \\ P_{n,2}(x) E_2 & -(\ell+1) P_{n,2}(x) \end{pmatrix} = \begin{pmatrix} -(\ell+1) P_{n,1}(x) & \Lambda_{n,1} P_{n,2}(x) \\ \Lambda_{n,2} P_{n,1}(x) & -(\ell+1) P_{n,2}(x) \end{pmatrix}.$$

Therefore, the polynomials $P_{n,1}$ and $P_{n,2}$ satisfy the following differential equations

$$P_{n,1} E_1 - \Lambda_{n,1} P_{n,2} = 0, \tag{7.11}$$

$$P_{n,2} E_2 - \Lambda_{n,2} P_{n,1} = 0. \tag{7.12}$$

Finally, it follows from (7.11), (7.12), and (6.10) that, for every $n \geq 0$, the polynomial $P_{n,1}$ is a solution of the following second-order matrix-valued differential equation

$$P_{n,1} E_1 E_2 - \Lambda_{n,1} \Lambda_{n,2} P_{n,1} = 0.$$

We can also obtain a second order differential equation for $P_{n,2}$.

8 Examples

The purpose of this section is to study the properties of the monic orthogonal polynomials $\{P_n\}_{n \geq 0}$ presented in Section 6 for small dimension. For $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, we show that these polynomials are solutions of certain matrix-valued differential equations. We will show that the polynomials can be defined by means of Rodrigues' formulas and we will give explicit expressions for the three-term recurrence relations.

8.1 The case $\ell = 0$; the scalar weight

In this case, the polynomials $\{P_n\}_{n \geq 0}$ are scalar-valued. The weight W reduces to the real function

$$W(x) = (1-x)^{\frac{1}{2}} (1+x)^{\frac{1}{2}}, \qquad x \in [-1, 1].$$

Therefore, the polynomials P_n are a multiple of the Chebyshev polynomials of the second kind: $P_n(x) = 2^{-n} U_n(x)$, $n \in \mathbb{N}$.

8.2 The case $\ell = \frac{1}{2}$; weight of dimension 2

In this case, the polynomials $\{P_n\}_{n \geq 0}$ are 2×2 matrices. The weight W is given by

$$W(x) = (1-x)^{\frac{1}{2}} (1+x)^{\frac{1}{2}} \begin{pmatrix} 2 & 2x \\ 2x & 2 \end{pmatrix}, \qquad x \in [-1, 1].$$

It is a straightforward computation that

$$Y W(x) Y^{\mathrm{t}} = 2 \begin{pmatrix} (1-x)^{\frac{1}{2}} (1+x)^{\frac{3}{2}} & 0 \\ 0 & (1-x)^{\frac{3}{2}} (1+x)^{\frac{1}{2}} \end{pmatrix}, \qquad Y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Observe that $W_1(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}$ and $W_2(x) = (1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}$ are Jacobi weights and therefore we have

$$P_{n,1} = \frac{2^n n!(n+2)!}{(2n+2)!} P_n^{(\frac{1}{2},\frac{3}{2})}(x), \quad P_{n,2}(x) = \frac{2^n n!(n+2)!}{(2n+2)!} P_n^{(\frac{3}{2},\frac{1}{2})}, \quad n \in \mathbb{N}_0,$$

where $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ are the classical Jacobi polynomials

8.2.1 *Differential equations*

By Theorem 7.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n(x) \begin{pmatrix} -x & 1 \\ -1 & x \end{pmatrix} + P_n(x) \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -n-3 & 0 \\ 0 & n \end{pmatrix} P_n(x).$$

We can conjugate the differential operator E by the matrix Y to obtain

$$\tilde{E} = Y\,E\,Y^{\mathrm{t}} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \begin{pmatrix} 0 & 1+x \\ x-1 & 0 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The monic polynomials

$$\tilde{P}_n(x) = Y P_n(x) Y^t = \begin{pmatrix} P_{n,1}(x) & 0 \\ 0 & P_{n,2}(x) \end{pmatrix}, \quad n \in \mathbb{N}_0$$

satisfy

$$\tilde{P}_n(x)\tilde{E} = \tilde{\Lambda}_n \tilde{P}_n(x) \quad \text{where } \tilde{\Lambda}_n(x) = \begin{pmatrix} -\frac{3}{2} & n+\frac{3}{2} \\ n+\frac{3}{2} & -\frac{3}{2} \end{pmatrix}.$$

Now the fact that $\tilde{P}_n(x)$ is an eigenfunction of $\tilde{E}$ is equivalent to the following relations between Jacobi polynomials:

$$(1+x)\frac{\mathrm{d}}{\mathrm{d}x} P^{(\frac{1}{2},\frac{3}{2})}(x) + \frac{3}{2} P_n^{(\frac{1}{2},\frac{3}{2})}(x) - \left(n+\frac{3}{2}\right) P_n^{(\frac{3}{2},\frac{1}{2})}(x) = 0,$$
$$(1-x)\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\frac{3}{2},\frac{1}{2})}(x) + \frac{3}{2} P_n^{(\frac{3}{2},\frac{1}{2})}(x) - \left(n+\frac{3}{2}\right) P_n^{(\frac{1}{2},\frac{3}{2})}(x) = 0.$$

8.3 Case $\ell = 1$; weight of dimension 3

Here, we consider the simplest example of nontrivial matrix orthogonal polynomials for the weight W. The weight matrix W of size 3×3 is obtained by setting $\ell = 1$. We have

$$W(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} \begin{pmatrix} 3 & 3x & 4x^2-1 \\ 3x & x^2+2 & 3x \\ 4x^2-1 & 3x & 3 \end{pmatrix}. \tag{8.1}$$

We know from Theorem 6.5 that the weight $W(x)$ splits into a block of size 2×2 and a block of size 1×1, namely

$$YW(x)Y^t = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix} = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} \begin{pmatrix} 4x^2+2 & 3\sqrt{2}x & 0 \\ 3\sqrt{2}x & x^2+2 & 0 \\ 0 & 0 & 4(1-x^2) \end{pmatrix}.$$

From Theorem 6.5 the monic orthogonal polynomials $\tilde{P}_n(x)$ with respect to $\tilde{W}(x)$ reduce to

$$\tilde{P}_n = \begin{pmatrix} P_{n,1}(x) & 0 \\ 0 & P_{n,2}(x) \end{pmatrix},$$

where $\{P_{n,2}\}_{n \geq 0}$ are the monic polynomials with respect to $W_1(x)$ and $\{P_{n,2}\}_{n \geq 0}$ are the monic polynomials with respect to the weight $W_2(x)$.

Remark 8.1. The weight W_2 is a multiple of the Jacobi weight $(1-x)^{\alpha}(1+x)^{\beta}$ corresponding to $\alpha = \frac{3}{2}$ and $\beta = \frac{3}{2}$. The monic polynomials $\{P_{n,2}\}_{n \geq 0}$ are then a multiple of the Gegenbauer polynomials

$$P_{n,2}(x) = \frac{2^n n!(n+3)!}{(2n+3)!} P_n^{(\frac{3}{2},\frac{3}{2})}(x).$$

□

8.3.1 *The first-order differential operator*

By Theorem 7.5, we have that the monic polynomials P_n are eigenfunctions of the differential operator E. More precisely, the following equation holds:

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n(x) \begin{pmatrix} -x & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & x \end{pmatrix} + P_n(x) \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -n-4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & n \end{pmatrix} P_n(x).$$

Now, we can conjugate the differential operator E by the matrix Y to obtain a differential operator $\tilde{E} = Y E Y^t$. The fact that the polynomials P_n are eigenfunctions of E says that the polynomials $\tilde{P}_n$ are eigenfunctions of $\tilde{E}$. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{P}_n(x)\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ x & -\sqrt{2} & 0 \end{pmatrix} + \tilde{P}_n(x)\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & n+2 \\ 0 & -2 & 0 \\ n+2 & 0 & -2 \end{pmatrix}\tilde{P}_n(x).$$

We can now rewrite the equation above in terms of the polynomials $P_{n,1}$ and $P_{n,2}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{n,1}(x)\begin{pmatrix} x \\ \frac{\sqrt{2}}{2} \end{pmatrix} + P_{n,1}(x)\begin{pmatrix} 2 \\ 0 \end{pmatrix} = P_{n,2}(x)\begin{pmatrix} n+2 \\ 0 \end{pmatrix},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{n,2}(x)\begin{pmatrix} x & -\sqrt{2} \end{pmatrix} + P_{n,2}(x)\begin{pmatrix} 2 & 0 \end{pmatrix} = \begin{pmatrix} n+2 & 0 \end{pmatrix} P_{n,1}(x).$$

Since $P_{n,2}$ is a Gegenbauer polynomial, we see that the elements of the first row of $P_{n,2}$ can be written explicitly in terms of Gegenbauer polynomials.

8.3.2 *Second-order differential operators*

In this section, we describe a set of linearly independent differential operators that have the polynomials $P_{n,1}$ as eigenfunctions.

Proposition 8.1. The matrix orthogonal polynomials $\{P_{n,1}\}_{n\geq 0}$ satisfy

$$P_{n,1}D_j = \Lambda_n(D_j)P_{n,1}, \quad j = 1, 2, 3, \ \ n \geq 0,$$

where the differential operators D_j are

$$D_1 = (x^2-1)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\begin{pmatrix} 5x & -2\sqrt{2} \\ -\sqrt{2} & 5x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\begin{pmatrix} x^2 & -\sqrt{2}x \\ \frac{\sqrt{2}x}{2} & -1 \end{pmatrix} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\begin{pmatrix} 5x & -3\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_3 = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\begin{pmatrix} -\sqrt{2}x & 4x^2-2 \\ x^2-2 & \sqrt{2}x \end{pmatrix} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\begin{pmatrix} -4\sqrt{2} & 16x \\ 6x & -2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 & 8 \\ 6 & 0 \end{pmatrix}.$$

and the eigenvalues Λ_j are given by

$$\Lambda_n(D_1) = \begin{pmatrix} n(n+4)+1 & 0 \\ 0 & n(n+4) \end{pmatrix}, \quad \Lambda_n(D_2) = \begin{pmatrix} (n+2)^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_n(D_3) = \begin{pmatrix} 0 & 4(n+1)(n+2) \\ (n+3)(n+2) & 0 \end{pmatrix}.$$

Moreover, the differential operators D_1, D_2, and D_3 satisfy

$$D_1D_2 = D_2D_1, \quad D_1D_3 \neq D_3D_1, \quad D_2D_3 \neq D_3D_2.$$ □

Proof. The proposition follows by proving that the differential operators D_j, where $j = 1, 2, 3$, are symmetric with respect to the weight matrix W_1. This is accomplished by a straightforward computation, showing that the differential equations (7.2)–(7.4) and the boundary conditions (7.5) are satisfied. As a consequence of Remark 7.1, the commutativity properties of the differential operators follow by observing the commutativity of the corresponding eigenvalues. ■

8.3.3 *Rodrigues' Formula*

Proposition 8.2. The matrix orthogonal polynomials $\{P_{n,1}(x)\}_{n\geq 0}$ satisfy the Rodrigues' formula

$$P_{n,1}(x) = c_n \left[(1-x^2)^{\frac{1}{2}+n} \left(\begin{pmatrix} 4x^2+2 & 3\sqrt{2}x \\ 3\sqrt{2}x & x^2+2 \end{pmatrix} + \begin{pmatrix} \dfrac{2n}{n+2} & \dfrac{\sqrt{2}nx}{n+2} \\ -\dfrac{\sqrt{2}nx}{n+1} & -\dfrac{n}{n+1} \end{pmatrix} \right) \right]^{(n)} W_1^{-1}(x), \quad (8.2)$$

where

$$c_n = \frac{(-1)^n 2^{-2n-2}(n+2)(n+3)\sqrt{\pi}}{(2n+3)\Gamma(n+\frac{3}{2})}.$$ □

Proof. The proposition can be proved in a similar way to [9, Theorem 3.1]. We include a sketch of the proof for the sake of completeness. First of all, we recall that the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]^{(n)}.$$

Let $R(x)$ and Y_n be the matrix polynomials of degree 2 and 1

$$R(x) = \begin{pmatrix} 4x^2+2 & 3\sqrt{2}x \\ 3\sqrt{2}x & x^2+2 \end{pmatrix}, \quad Y_n(x) = \begin{pmatrix} \dfrac{2n}{n+2} & \dfrac{n\sqrt{2}x}{n+2} \\ -\dfrac{\sqrt{2}nx}{n+1} & -\dfrac{n}{n+1} \end{pmatrix},$$

so that the (8.2) can be rewritten as

$$P_{n,1}(x) = c_n[(1-x^2)^{\frac{1}{2}+n}(R(x)+Y_n(x))]^{(n)} W_1^{-1}(x).$$

Then by applying the Leibniz rule on the right-hand side of (8.2), it is not difficult to prove that

$$\begin{aligned} P_{n,1}(x) &= c_n \tfrac{1}{2} n(n-1)[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{1}{2}+n}]^{(n-2)}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} R''(x)R(x)^{-1} \\ &\quad + c_n n[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{1}{2}+n}]^{(n-1)}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}(R'(x)+Y_n'(x))R(x)^{-1} \\ &\quad + c_n[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{1}{2}+n}]^{(n)}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}(I+Y_n(x)R(x)^{-1}). \end{aligned}$$

Now by applying the Rodrigues' formula for the Jacobi polynomials we obtain

$$\begin{aligned} P_{n,1}(x) = 2^n n! (-1)^n c_n [P_n^{(\frac{1}{2},\frac{1}{2})}(x)(I+Y_n(x)R(x)^{-1}) \\ - \tfrac{1}{2}(1-x)(1+x)P_{n-1}^{(\frac{3}{2},\frac{3}{2})}(x)(Y_n'(x)+R'(x))R(x)^{-1} \\ + \tfrac{1}{8}(1-x)^2(1+x)^2 P_{n-2}^{(\frac{5}{2},\frac{5}{2})}(x)R''(x)R(x)^{-1}]. \end{aligned} \tag{8.3}$$

Now with a careful computation, we can show that the expression above is a matrix polynomial of degree n with nonsingular leading term. Using integration by parts it is easy to show the orthogonality of $P_{n,1}$ and x^m, $m = 0, 1, \ldots, n-1$, with respect to the weight W_1. ∎

8.3.4 *Three-term recurrence relations*

In Corollary 6.3, we show that the matrix polynomials $P_n(x)$ of any size satisfy a three term recurrence relation. The recurrence relation for the polynomials $P_{n,1}$ can then be obtained by conjugating the recurrence relation for $P_n(x)$ by the matrix Y. The recurrence coefficients (4.8) are given in terms of Clebsch–Gordan coefficients and are difficult to manipulate. For $\ell = 1$, we can use the Rodrigues' formula (8.2) to derive explicit formulas for the three-term recurrence relation for the polynomials $P_{n,1}$.

First we need to compute the norm of $P_{n,1}(x)$. The Rodrigues' formula (8.2) and integration by parts lead to

$$\|P_{n,1}\|^2 = 2^{-2n-1}\pi \begin{pmatrix} \dfrac{(n+3)}{(n+1)} & 0 \\ 0 & \dfrac{(n+3)^2}{4(n+1)^2} \end{pmatrix}.$$

If $\{\mathcal{P}_{n,1}\}_{n\geq 0}$ is a sequence of orthonormal polynomials with respect to W_1 with leading coefficients Ω_n, then it follows directly from the orthogonality relations for the monic polynomials that $\|P_{n,1}\|^2 = \Omega_n^{-1}(\Omega_n^*)^{-1}$. The orthonormal polynomials $\mathcal{P}_{n,1}$ with leading coefficient

$$\Omega_n = \begin{pmatrix} \sqrt{\dfrac{2^{2n+1}(n+1)}{\pi(n+3)}} & 0 \\ 0 & \dfrac{2^{n+2}(n+1)}{\sqrt{\pi}(n+3)} \end{pmatrix}$$

satisfy the three-term recurrence relation

$$x\mathcal{P}_n(x) = A_{n+1}\mathcal{P}_{n+1}(x) + B_n\mathcal{P}_n(x) + A_n^*\mathcal{P}_{n-1}(x),$$

where $A_n = \Omega_{n-1}\Omega_n^{-1}$ and

$$B_n = \Omega_n[\text{coef. of } x^{n-1} \text{ in } P_{n,1} - \text{coef. of } x^n \text{ in } P_{n+1,1}]\Omega_n^{-1}.$$

The coefficient of x^{n-1} in $P_{n,1}$ can be obtained from (8.3). Now a careful computation shows that

$$A_n = \begin{pmatrix} \dfrac{1}{2}\sqrt{\dfrac{n(n+3)}{(n+1)(n+2)}} & 0 \\ 0 & \dfrac{n(n+3)}{2(n+1)(n+2)} \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 & \dfrac{\sqrt{2}}{\sqrt{(n+1)(n+3)}(n+2)} \\ \dfrac{\sqrt{2}}{\sqrt{(n+1)(n+3)}(n+2)} & 0 \end{pmatrix}.$$

Therefore, the monic polynomials $P_{n,1}$ satisfy the three-term recurrence relation

$$xP_{n,1}(x) = P_{n+1,1}(x) + \tilde{B}_n P_{n,1}(x) + \tilde{C}_n P_{n-1,1}(x),$$

where

$$\tilde{B}_n = \begin{pmatrix} 0 & \frac{2\sqrt{2}}{(n+2)(n+3)} \\ \frac{\sqrt{2}}{2(n+1)(n+2)} & 0 \end{pmatrix}, \quad \tilde{C}_n = \begin{pmatrix} \frac{n(n+3)}{4(n+1)(n+2)} & 0 \\ 0 & \frac{n^2(n+3)^2}{4(n+1)^2(n+2)^2} \end{pmatrix}.$$

8.4 Case $\ell = \frac{3}{2}$; weight of dimension 4

The weight matrix W of size 4×4 is obtained by setting $\ell = \frac{3}{2}$.

$$W(x) = (1-x^2)^{\frac{1}{2}} \begin{pmatrix} 4 & 4x & \frac{4}{3}(4x^2-1) & 4x(2x^2-1) \\ 4x & \frac{4}{9}(4x^2+5) & \frac{4}{9}x(2x^2+7) & \frac{4}{3}(4x^2-1) \\ \frac{4}{3}(4x^2-1) & \frac{4}{9}x(2x^2+7) & \frac{4}{9}(4x^2+5) & 4x \\ 4x(2x^2-1) & \frac{4}{3}(4x^2-1) & 4x & 4 \end{pmatrix}.$$

We know from Corollary 6.5 that the weight $W(x)$ splits in two blocks of size 2×2, namely

$$\tilde{W}(x) = YW(x)Y^t = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix},$$

where

$$W_1(x) = 4(1-x)^{1/2}(1+x)^{3/2} \begin{pmatrix} 2x^2-2x+1 & \frac{1}{3}(4x-1) \\ \frac{1}{3}(4x-1) & \frac{1}{9}(2x^2+2x+5) \end{pmatrix},$$

and

$$W_2(x) = J_2 F_2 W_1(-x) F_2 J_2, \quad \text{where } F_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows from Corollary 6.6 that the monic orthogonal polynomials $P_{n,2}$ with respect to the weight W_2 are completely determined by the the monic orthogonal polynomials $P_{n,1}$ with respect to $P_{n,2}(x) = J_2 F_2 P_{n,1}(-x) F_2 J_2$. Therefore, we only need to study the polynomials $P_{n,1}$.

8.4.1 *Differential operators*

In this section, we describe a set of linearly independent differential operators that have the polynomials $P_{n,1}$ as eigenfunctions.

Proposition 8.3. The matrix orthogonal polynomials $\{P_{n,1}\}_{n\geq 0}$ satisfy

$$P_{n,1}D_j = \Lambda_n(D_j)P_{n,1}, \quad j=1,2,3, \quad n\geq 0,$$

where the differential operators D_j are

$$D_1 = (x^2-1)\left(\frac{d^2}{dx^2}\right) + \left(\frac{d}{dx}\right)\begin{pmatrix} 6x & -3 \\ -1 & 6x-2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} x^2-\frac{1}{4} & -\frac{3}{2}x+\frac{3}{4} \\ \frac{x}{2}+\frac{1}{4} & -\frac{3}{4} \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} 6x & \frac{9}{2} \\ \frac{3}{2} & 0 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_3 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} -3x+3 & 9x^2-9x \\ x^2+x-2 & 3x-3 \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} -9 & 36x-18 \\ 8x+4 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 18 \\ 12 & 0 \end{pmatrix},$$

and the eigenvalues Λ_j are given by

$$\Lambda_n(D_1) = \begin{pmatrix} n(n+5)+2 & 0 \\ 0 & n(n+5) \end{pmatrix}, \quad \Lambda_n(D_2) = \begin{pmatrix} (n+3)(n+2) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_n(D_3) = \begin{pmatrix} 0 & 9(n+2)(n+1) \\ (n+4)(n+3) & 0 \end{pmatrix}.$$

Moreover, the differential operators D_1, D_2, and D_3 satisfy

$$D_1D_2 = D_2D_1, \quad D_1D_3 \neq D_3D_1, \quad D_2D_3 \neq D_3D_2. \qquad \square$$

8.4.2 *Rodrigues' formula*

The monic orthogonal polynomials $\{P_{n,1}(x)\}_{n\geq 0}$ satisfy the Rodrigues' formula

$$P_{n,1}(x) = c_n[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{3}{2}+n}(R(x)+Y_n(x))]^{(n)}W_1^{-1}(x),$$

where

$$c_n = \frac{2^{-2n-2}(-1)^n(n+3)(n+4)\sqrt{\pi}}{\Gamma(n+\frac{5}{2})},$$

and

$$R(x) = \begin{pmatrix} 2x^2-2x+1 & \frac{1}{3}(4x-1) \\ \frac{1}{3}(4x-1) & \frac{1}{9}(2x^2+2x+5) \end{pmatrix},$$

$$Y_n(x) = \begin{pmatrix} \frac{n}{n+3} & \frac{n}{3(n+3)}(2x+1) \\ \frac{n}{3(n+1)}(1-2x) & -\frac{n}{3(n+1)} \end{pmatrix}.$$

8.4.3 *Three-term recurrence relations*

The orthonormal polynomials $\mathcal{P}_{n,1}(x) = \|P_{n,1}\|^{-1}P_{n,1}$, with leading coefficient

$$\Omega_n = \begin{pmatrix} \sqrt{\frac{2^{2n+1}(n+1)}{\pi(n+4)}} & 0 \\ 0 & \frac{9(n+1)\sqrt{2^{2n+1}(n+2)}}{\sqrt{\pi(n+3)}(n+4)} \end{pmatrix}$$

satisfy the three-term recurrence relation

$$x\mathcal{P}_n(x) = A_{n+1}\mathcal{P}_{n+1}(x) + B_n\mathcal{P}_n(x) + A_n^*\mathcal{P}_{n-1}(x),$$

where

$$A_n = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{n(n+4)}{(n+1)(n+3)}} & 0 \\ 0 & \frac{n(n+4)}{2(n+2)\sqrt{(n+1)(n+3)}} \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 & \frac{3}{2\sqrt{(n+1)(n+2)(n+3)(n+4)}} \\ \frac{3}{2\sqrt{(n+1)(n+2)(n+3)(n+4)}} & \frac{2}{(n+2)(n+3)} \end{pmatrix}.$$

Therefore, the monic polynomials $P_{n,1}(x)$ satisfy the three-term recurrence relation

$$xP_{n,1} = P_{n+1,1} + \tilde{B}_nP_{n,1} + \tilde{C}_nP_{n-1,1},$$

where

$$\tilde{B}_n = \begin{pmatrix} 0 & \dfrac{9}{2(n+3)(n+4)} \\ \dfrac{1}{2(n+1)(n+2)} & \dfrac{2}{(n+2)(n+3)} \end{pmatrix},$$

$$\tilde{C}_n = \begin{pmatrix} \dfrac{n(n+4)}{4(n+1)(n+3)} & 0 \\ 0 & \dfrac{n^2(n+4)^2}{4(n+1)(n+2)^2(n+3)} \end{pmatrix}.$$

8.5 Case $\ell = 2$; weight of dimension 5

In this section, we consider the 2×2 irreducible block in the case $\ell = 2$, where the matrix weight W is of dimension 5. This case completes the list of all irreducible 2×2 blocks obtained by conjugating the weight W by the matrix Y.

The 2×2 block is given by

$$W_1(x) = (1-x)^{\frac{3}{2}}(1+x)^{\frac{3}{2}} \begin{pmatrix} x^2+4 & 10x \\ 10x & 16x^2+4 \end{pmatrix}.$$

As before, we denote by $\{P_{n,1}\}_n$ the sequence of monic orthogonal polynomials with respect to W_1.

8.5.1 *Differential operators*

In this section, we describe a set of linearly independent differential operators that have the polynomials $P_{n,1}$ as eigenfunctions.

Proposition 8.4. The matrix orthogonal polynomials $\{P_{n,1}\}_{n\geq 0}$ satisfy

$$P_{n,1}D_j = \Lambda_n(D_j)P_{n,1}, \quad j = 1, 2, 3, \quad n \geq 0,$$

where the differential operators D_j are

$$D_1 = (x^2-1)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\begin{pmatrix} 7x & -1 \\ -4 & 7x \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} x^2 & -\frac{1}{2}x \\ 2x & -1 \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} 7x & -3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_3 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} \frac{3}{8}x & \frac{x^2}{16} - \frac{1}{4} \\ x^2 - \frac{1}{4} & -\frac{3}{8}x \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} -\frac{1}{4} & \frac{5}{8}x \\ 4x & -1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{5}{4} \\ 2 & 0 \end{pmatrix},$$

and the eigenvalues Λ_j are given by

$$\Lambda_n(D_1) = \begin{pmatrix} n(n+6)-3 & 0 \\ 0 & n(n+6) \end{pmatrix}, \quad \Lambda_n(D_2) = \begin{pmatrix} (n+1)(n+5) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_n(D_3) = \begin{pmatrix} 0 & \frac{1}{16}(n+5)(n+4) \\ (n+2)(n+1) & 0 \end{pmatrix}.$$

Moreover, the differential operators D_1, D_2, and D_3 satisfy

$$D_1 D_2 = D_2 D_1, \quad D_1 D_3 \neq D_3 D_1, \quad D_2 D_3 \neq D_3 D_2.$$ □

8.5.2 *Rodrigues' Formula*

The monic orthogonal polynomials $\{P_{n,1}(x)\}_{n\geq 0}$ satisfy the Rodrigues' formula

$$P_{n,1}(x) = c_n[(1-x)^{\frac{3}{2}+n}(1+x)^{\frac{3}{2}+n}(R(x)+Y_n(x))]^{(n)} W_1^{-1}(x),$$

where

$$c_n = \frac{(-1)^n 2^{-2n-4}(n+3)(n+4)(n+5)\sqrt{\pi}}{(2n+5)\Gamma(n+\frac{5}{2})},$$

and

$$R(x) = \begin{pmatrix} x^2+4 & 10x \\ 10x & 16x^2+4 \end{pmatrix}, \quad Y_n(x) = \begin{pmatrix} -\frac{3n}{n+1} & -\frac{6nx}{n+1} \\ \frac{6nx}{n+4} & \frac{12n}{n+4} \end{pmatrix}.$$

8.5.3 *Three-term recurrence relations*

The orthonormal polynomials $\mathcal{P}_{n,1}(x) = \|P_{n,1}\|^{-1} P_{n,1}$ with leading coefficient

$$\Omega_n = \begin{pmatrix} 2^{2n+2}\dfrac{\sqrt{(n+2)}(n+1)}{\sqrt{\pi(n+4)}(n+5)} & 0 \\ 0 & 2^n\dfrac{\sqrt{2(n+1)}}{\sqrt{\pi(n+5)}} \end{pmatrix}$$

satisfy the three term recurrence relation

$$x\mathcal{P}_n(x) = A_{n+1}\mathcal{P}_{n+1}(x) + B_n\mathcal{P}_n(x) + A_n^*\mathcal{P}_{n-1}(x),$$

where

$$A_n = \begin{pmatrix} \dfrac{n(n+5)}{2\sqrt{(n+1)(n+2)(n+3)(n+4)}} & 0 \\ 0 & \dfrac{\sqrt{n(n+5)}}{\sqrt{(n+1)(n+4)}} \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 & \dfrac{2}{\sqrt{(n+1)(n+2)(n+4)(n+5)}} \\ \dfrac{2}{\sqrt{(n+1)(n+2)(n+4)(n+5)}} & 0 \end{pmatrix}.$$

Therefore, the monic polynomials $P_{n,1}(x)$ satisfy the three-term recurrence relation

$$xP_{n,1} = P_{n+1,1} + \tilde{B}_n P_{n,1} + \tilde{C}_n P_{n-1,1},$$

where

$$\tilde{B}_n = \begin{pmatrix} 0 & \dfrac{1}{2(n+1)(n+2)} \\ \dfrac{8}{(n+4)(n+5)} & 0 \end{pmatrix},$$

$$\tilde{C}_n = \begin{pmatrix} \dfrac{n^2(n+5)^2}{4(n+1)(n+2)(n+3)(n+4)} & 0 \\ 0 & \dfrac{n(n+5)}{4(n+1)(n+4)} \end{pmatrix}.$$

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Appendix 1. Transformation Formulas

The goal of this appendix it to prove Theorem 5.4. We use the standard notation for the Pochhammer symbols and the hypergeometric series from [1]. In the manipulations, we only need the Chu–Vandermonde summation formula [1, Corollary 2.2.3] which reads

$$ {}_2F_1\left(\begin{matrix}-n, a\\ c\end{matrix}; 1\right) = \frac{(c-a)_n}{(c)_n} \tag{A.1}$$

and Sheppard's transformation formula for ${}_3F_2$'s [1, Corollary 3.3.4] written as

$$\begin{aligned}&\sum_{k=0}^{n}(e+k)_{n-k}(d+k)_{n-k}\frac{(-n)_k(a)_n(b)_n}{k!}\\ &\quad=\sum_{k=0}^{n}(d-a)_{n-k}(e-a)_{n-k}\frac{(-n)_k(a)_k(a+b-n-d-e+1)_k}{k!}.\end{aligned} \tag{A.2}$$

Proposition A.1. Let $\ell \in \frac{1}{2}\mathbb{N}$ and $p, q \in \frac{1}{2}\mathbb{Z}$ such that $|p|, |q| \leq \ell$, $\ell - p, \ell - q \in \mathbb{Z}$, $q - p \leq 0$ and $q + p \leq 0$. Let $s \in \{0, \ldots, \ell + q\}$ and define

$$e_s(p,q) = \sum_{n=0}^{\ell+q-s}\binom{\ell+p}{n}\binom{\ell+q}{n+s}\sum_{m=0}^{\ell-p-s}\frac{\binom{\ell-p}{m+s}\binom{\ell-q}{m}}{\binom{2\ell}{m+n+s}^2}. \tag{A.3}$$

Then we have

$$e_s^{\ell}(p,q) = \frac{(2\ell+1)}{(\ell+p+1)}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!}\sum_{T=0}^{\ell+q-s}(-1)^{\ell+q-T}\frac{(p-\ell)_{\ell+q-T}(2+2\ell-T)_T}{(\ell+p+2)_{\ell+q-T}T!}. \tag{A.4}$$

□

Proof. First we reverse the inner summation using $M = \ell - p - s - m$ to get

$$e_s(p,q) = \sum_{n=0}^{\ell+q-s} \binom{\ell+p}{n}\binom{\ell+q}{n+s} \sum_{M=0}^{\ell-p-s} \frac{\binom{\ell-p}{M}\binom{\ell-q}{\ell-p-M-s}}{\binom{2\ell}{\ell-p-M+n}^2}. \tag{A.5}$$

We rewrite the inner summation:

$$\sum_{M=0}^{\ell-p-s} \frac{\binom{\ell-p}{M}\binom{\ell-q}{\ell-p-M-s}}{\binom{2\ell}{\ell-p-M+n}^2} = \frac{(\ell-q)!}{(2\ell)!}(-1)^s(-\ell+p)_s \binom{2\ell}{\ell-p+n}^{-1} \times \sum_{M=0}^{\ell-p-s} \frac{(-\ell+p+s)_M(\ell+p-n+1)_M}{M!(-\ell+p-n)_M} B(M), \tag{A.6}$$

where $B(M) = (\ell-p-M+1)_n(p-q+M+s+1)_{\ell+q-s-n}$ is a polynomial in M of degree $\ell+q-s$ that depends on $\ell, p, q, n,$ and s. The polynomial $B(M)$ has an expansion in $(-1)^t(-M)_t$,

$$B(M) = \sum_{t=0}^{\ell+q} A_t \cdot (-1)^t(-M)_t. \tag{A.7}$$

The coefficients $A_t = A_t(\ell, p, q, n, s)$ can be found by repeated application of the difference operator $\Delta_M f = f(M+1) - f(M)$. Let Δ_M^i be its ith power. We have

$$\left.\frac{\Delta_M^t}{t!}\right|_{M=0} B = A_t. \tag{A.8}$$

In other words,

$$B(M) = \sum_{t=0}^{\ell+q} A_t \cdot (-1)^t(-M)_t \quad \text{with } A_t = \left.\frac{\Delta_M^t}{t!}\right|_{M=0} B. \tag{A.9}$$

We calculate (A.6) by substituting (A.7) in it. Interchanging summations, the inner sum can be evaluated using (A.1) (after shifting the summation parameter). We get

$$\sum_{M=0}^{\ell-p-s} \frac{\binom{\ell-p}{M}\binom{\ell-q}{\ell-p-M-s}}{\binom{2\ell}{\ell-p-M+n}^2} = \frac{(\ell-q)!}{(2\ell)!}(-1)^s(-\ell+p)_s \binom{2\ell}{\ell-p+n}^{-1} \times \sum_{t=0}^{\ell+q-s} A_t \frac{(-2\ell-1)_{\ell-p-s-t}(-\ell+p+s)_t(\ell+p-n+1)_t}{(-\ell+p-n)_{\ell-p-s}}. \tag{A.10}$$

Substituting (A.10) and (A.8) in (A.5) and simplifying gives

$$e_s(p,q) = \frac{(\ell-q)!(\ell+p)!}{(2\ell)!(2\ell)!}(-1)^{\ell-p-s}(-\ell+p)_s(-\ell-q)_s \\ \times \sum_{t=0}^{\ell+q-s}(\ell+p+1)_t(-2\ell-1)_{\ell-p-s-t}(-\ell+p+s)_t \left.\frac{\Delta_M^t}{t!}\right|_{M=0}(p-q+s+M+1)_{\ell+q-s} \\ \times \sum_{n=0}^{\ell+q-s}\frac{(-\ell-q-s)_n(-\ell-p)_n(\ell-p-M+1)_n}{n!(-\ell-p-t)_n(-\ell-p-M)_n}. \tag{A.11}$$

The inner sum over n is a ${}_3F_2$-series, which can be transformed using (A.2). Note that the t-order difference operator can now be evaluated yielding only one nonzero term in the sum over n. This gives

$$e_s(p,q) = \frac{2\ell+1}{\ell+p+1}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!}\sum_{t=0}^{\ell+q-s}(-1)^{-s-t}\frac{(-\ell+p)_{s+t}(2+\ell-q+s+t)_{\ell+q-s-t}}{(\ell+p+2)_{s+t}(\ell+q-s-t)!}. \tag{A.12}$$

Reversing the order of summation using $T=\ell+q-s-t$ yields

$$e_s^\ell(p,q) = \frac{(2\ell+1)}{(\ell+p+1)}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!}\sum_{T=0}^{\ell+q-s}(-1)^{\ell+q-T}\frac{(p-\ell)_{\ell+q-T}(2+2\ell-T)_T}{(\ell+p+2)_{\ell+q-T}T!} \tag{A.13}$$

as was to be shown. ∎

Proof of Theorem 5.4. We already argued that there is an expansion in Chebyshev polynomials (5.8). From (5.11) it follows that there are coefficients $d_r^\ell(p,q)$ such that

$$v_{p,q}^\ell(\cos t) = \sum_{r=-(\ell+\frac{p+q}{2})}^{\ell+\frac{p+q}{2}} d_r^\ell(p,q)\,\mathrm{e}^{-2irt}. \tag{A.14}$$

The coefficients $d_r^\ell(p,q)$ and $c_n^\ell(p,q)$ are related by

$$d_r^\ell(p,q) = \sum_{n=0}^{\ell+q-(r+\frac{q-p}{2})} c_n^\ell(p,q). \tag{A.15}$$

Let $q-p\leq 0$, $q+p\leq 0$ and $r\geq \frac{p-q}{2}$ and substitute $r(s)=s+\frac{p-q}{2}$ in (5.11). Comparing this to (A.14) shows

$$d^{\ell}_{r(s)}(p,q)=\sum_{n=0}^{\ell+q-s}\binom{\ell+p}{n}\binom{\ell+q}{n+s}\sum_{m=0}^{\ell-p-s}\frac{\binom{\ell-p}{m+s}\binom{\ell-q}{m}}{\binom{2\ell}{m+n+s}^{2}} \tag{A.16}$$

for $s=0,\ldots,\ell+q$. Now, we use Proposition A.1 to show that $d^{\ell}_{r}(p,q)$ equals

$$\frac{(2\ell+1)}{(\ell+p+1)}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!}\sum_{n=0}^{\ell+q-(r+\frac{q-p}{2})}(-1)^{\ell+q-n}\frac{(p-\ell)_{\ell+q-n}(2+2\ell-n)_{n}}{(\ell+p+2)_{\ell+q-n}n!} \tag{A.17}$$

for $r=\frac{p-q}{2},\ldots,\ell+\frac{p+q}{2}$. It follows that

$$c^{\ell}_{n}(p,q)=\frac{2\ell+1}{\ell+p+1}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!}\frac{(p-\ell)_{\ell+q-n}}{(\ell+p+2)_{\ell+q-n}}(-1)^{\ell+q-n}\frac{(2\ell+2-n)_{n}}{n!}. \tag{A.18}$$

This proves the theorem. ∎

We can reformulate Proposition A.1 in terms of hypergeometric series.

Corollary A.2. For $N\in\mathbb{N}$, $a,b,c\in\mathbb{N}$ so that $0\leq a\leq N$, $0\leq b\leq N$, and additionally $a\leq b$, $N\leq a+b$ and $0\leq c\leq N-b$, we have

$$\sum_{m=0}^{c}\frac{(-c)_m\,(b+1)_m\,(b+1)_m}{(N-a-c+1)_m\,m!\,(b-N)_m}\,{}_4F_3\left(\begin{matrix}-b,\,N-a-b-c,\,N-b-m+1,\,N-b-m+1\\ N-b-c+1,\,-b-m,\,-b-m\end{matrix};1\right)$$
$$=\frac{\binom{N+1}{a}}{\binom{b}{N-a-c}\binom{N-b}{N-b-c}}\sum_{n=0}^{c}\frac{(-a)_{N-b-n}\,(-1)^{N-b-n}\,(N+2-n)_n}{(N-a+2)_{N-b-n}\,n!}. \qquad\square$$

The ${}_4F_3$-series in the summand is not balanced. Note that the case $s=0$ leads to single sums, and the ${}_4F_3$ boils down to a terminating ${}_2F_1$ which can be summed by the Chu–Vandermonde sum, so Corollary A.2 can be viewed as an extension of Chu–Vandermonde sum (A.1).

The coefficients $d^{\ell}_{r}(p,q)$ of (5.11) with $|r|\leq\frac{p-q}{2}$ are independent of r. Corollary 5.3 in case $\ell=\ell_1+\ell_2=m_1+m_2$ can be stated as follows.

Corollary A.3. For $N \in \mathbb{N}$, $a, b, c \in \mathbb{N}$ so that $0 \le a \le N$, $0 \le b \le N$, $b \le a$, $a + b \le N$, and $0 \le c \le N - a - b$, we have

$$\frac{\binom{N-b}{a+c}\binom{N-a}{c}}{\binom{N}{a+c}^2} \sum_{n=0}^{b} \frac{(-b)_n\,(c+a-N)_n\,(a+c+1)_n\,(a+c+1)_n}{n!\,(c+1)_n\,(a+c-N)_n\,(a+c-N)_n}$$

$$\times\ {}_4F_3\left(\begin{matrix}-a, -a-c, N-a-c-n+1, N-a-c-n+1\\ N-a-b-c+1, -a-c-n, -a-c-n\end{matrix};1\right)$$

$$= \frac{N+1}{N-a+1}\binom{N}{b}^{-1} \sum_{m=0}^{b} \frac{(-a)_m\,(N-b+m+2)_{b-m}\,(-1)^m}{(N-a+2)_m\,(b-m)!}.$$

In particular, the left-hand side is independent of c in the range stated. □

Different proofs of Corollaries A.2, A.3 using transformation and summation formulas for hypergeometric series have been communicated to us by Mizan Rahman.

Appendix 2. Proof of the Symmetry for Differential Operators

Proof of Theorem 7.5. In terms of $\rho(x)$ and $Z(x)$, the Equations (7.7) and (7.8) are given by

$$0 = Z(x)A_1(x)^* + A_1(x)Z(x), \tag{A.19}$$

$$0 = -A_1'(x)Z(x) - \frac{\rho(x')}{\rho(x)}A_1(x)Z(x) - A_1(x)Z'(x) + A_0Z(x) - Z(x)A_0. \tag{A.20}$$

As a consequence of the properties of symmetry of the weight W, it suffices to verify the conditions above for all the (n, m)-entries with $m \le n$. Here, we assume that $m < n$. The case $m = n$ can be done similarly. The first Equation (A.19) holds true if and only if

$$Z_{n,m-1}A_1(x)_{m,m-1} + Z_{n,m}A_1(x)_{m,m} + Z_{n,m+1}A_1(x)_{m,m+1} + Z_{n-1,m}A_1(x)_{n,n-1}$$

$$+ Z_{n,m}A_1(x)_{n,n} + Z_{n+1,m}A_1(x)_{n,n+1} = 0$$

for all $m \le n$. In order to prove the expression above we replace the coefficients of A_1 and Z in the left-hand side and we obtain

$$-\frac{m}{2\ell}\sum_{t=0}^{m-1} c(n, m-1, t)U_{n+m-2t-1}(x) - \frac{\ell - m}{\ell}\sum_{t=0}^{m} c(n, m, t)\,x\,U_{n+m-2t}(x)$$

$$+\frac{2\ell-m}{2\ell}\sum_{t=0}^{m+1}c(n,m+1,t)U_{n+m-2t+1}(x)-\frac{n}{2\ell}\sum_{t=0}^{m}c(n-1,m,t)U_{n+m-2t-1}(x)$$
$$-\frac{\ell-n}{\ell}\sum_{t=0}^{m}c(n,m,t)\,x\,U_{n+m-2t}(x)+\frac{2\ell-n}{2\ell}\sum_{t=0}^{m}c(n+1,m,t)U_{n+m-2t+1}(x).$$

By using the recurrence relation $xU_r(x)=\frac{1}{2}U_{r-1}(x)+\frac{1}{2}U_{r+1}(x)$, we obtain

$$\sum_{t=0}^{m-1}\left[-\frac{m}{2\ell}\,c(n,m-1,t)-\frac{n}{2\ell}\,c(n-1,m,t)-\frac{2\ell-n-m}{2\ell}\,c(n,m,t)\right]U_{n+m-2t-1}(x)$$
$$\times\sum_{t=0}^{m}\left[\frac{2\ell-m}{2\ell}\,c(n,m+1,t)+\frac{2\ell-n}{2\ell}\,c(n+1,m,t)-\frac{2\ell-n-m}{2\ell}\,c(n,m,t)\right]U_{n+m-2t+1}(x)$$
$$+\left[-\frac{n}{2\ell}c(n,m+1,m)-\frac{2\ell-m}{2\ell}c(n,m+1,m+1)-\frac{2\ell-m-n}{2\ell}c(n,m,m)\right]U_{n-m-1}(x).$$

A simple computation shows that the coefficient of U_{n-m-1} in the expression above is zero. Now, by changing the index of summation t, we obtain

$$\left[\frac{2\ell-m}{2\ell}\,c(n,m+1,0)+\frac{2\ell-n}{2\ell}\,c(n+1,m,0)-\frac{2\ell-n-m}{2\ell}\,c(n,m,0)\right]U_{n+m+1}(x)$$
$$+\sum_{t=0}^{m-1}\left[-\frac{m}{2\ell}\,c(n,m-1,t)-\frac{n}{2\ell}\,c(n-1,m,t)-\frac{2\ell-n-m}{2\ell}\,c(n,m,t)\;\frac{2\ell-m}{2\ell}\right.$$
$$\left.\times\,c(n,m+1,t+1)+\frac{2\ell-n}{2\ell}\,c(n+1,m,t+1)-\frac{2\ell-n-m}{2\ell}\,c(n,m,t+1)\right]U_{n+m-2t-1}(x). \tag{A.21}$$

Using the explicit expression of $c(n,m,t)$ in (7.6), we obtain that (A.21) is given by

$$\sum_{t=0}^{m}c(n,m,t)\left[-\frac{(m+1-t+n)(2\ell-m+1)}{2\ell(-m+1+t-n+2\ell)}-\frac{(-n+1+2\ell)(m+1-t+n)}{2\ell(-m+1+t-n+2\ell)}-\frac{2\ell-n-m}{2\ell}\right.$$
$$\left.+\frac{(2\ell+1-t)(m+1)}{2\ell(t+1)}+\frac{(2\ell+1-t)(n+1)}{2\ell(t+1)}+\frac{(2\ell-n-m)(m+1-t+n)(2\ell+1-t)}{\ell(-m+1+t-n+2\ell)(t+1)}\right]$$
$$\times\,U_{n+m-2t-1}(x)=0,$$

since the sum of the terms in the square brackets is zero. This completes the proof of (A.19).

Now we prove (A.20). The (n, m)-entry of the right-hand side of (A.20) is given by

$$x(A_1(x)Z(x))_{n,m} - (1-x^2)(A_1(x)Z'(x))_{n,m} + (1-x^2)[(A_0)_{n,n} - A_1'(x)_{n,n} - (A_0)_{m,m}]Z_{n,m}.$$

Using (7.6) we obtain

$$\begin{aligned}(1-x^2)&[(A_0)_{n,n} - A_1'(x)_{n,n} - (A_0)_{m,m}]Z_{n,m}\\ &= \sum_{t=0}^{m} \frac{\ell(n-m+1)-m}{\ell} c(n,m,t)(1-x^2)U_{n+m-2t}(x),\end{aligned} \tag{A.22}$$

$$\begin{aligned}x(A_1(x)Z(x))_{n,m} = \sum_{t=0}^{m} \Bigg[&-\frac{n}{2\ell}c(n-1,m,t)xU_{n+m-2t-1}(x) - \frac{\ell-n}{\ell}c(n,m,t)x^2U_{n+m-2t}(x)\\ &+\frac{2\ell-n}{2\ell}c(n+1,m,t)xU_{n+m-2t+1}(x)\Bigg],\end{aligned} \tag{A.23}$$

$$\begin{aligned}(1-x^2)&(A_1(x)Z'(x))_{n,m}\\ &= \sum_{t=0}^{m}\Bigg[-\frac{n}{2\ell}c(n-1,m,t)(1-x^2)U'_{n+m-2t-1}(x) - \frac{\ell-n}{\ell}c(n,m,t)x(1-x^2)U'_{n+m-2t}(x)\\ &\quad+\frac{2\ell-n}{2\ell}c(n+1,m,t)(1-x^2)U'_{n+m-2t+1}(x)\Bigg].\end{aligned} \tag{A.24}$$

Now we proceed as in the proof of condition (A.19). In (A.22) and (A.23), we use the three-term recurrence relation for the Chebyshev polynomials to get rid of the factors x and x^2. Equation (A.24) involves the derivative of the polynomials U. For this, we use the following identity

$$U_n'(x) = \frac{(n+2)U_{n-1}(x) - nU_{n+1}(x)}{2(1-x^2)}, \quad n \geq 0, \quad (U_{-1} \equiv 0).$$

Finally, we change the index of summation t and we use the explicit expression of the coefficients $c(n, m, t)$ to complete the proof.

The boundary condition (7.9) can easily be checked. ∎

Proof of Theorem 7.6. We will show that the conditions of symmetry in Theorem 7.4 hold true. The first Equation (7.2) is satisfied because $A_2(x)$ is a scalar matrix.

Equation (7.3) can be written in terms of $\rho(x)$ and $Z(x)$ in the following way:

$$(6x - B_1(x))Z(x) + 2(x^2 - 1)Z'(x) - Z(x)B_1(x)^* = 0.$$

This can be checked by a similar computation to that of the proof of Theorem 7.5.

Now we give the proof of the third condition for symmetry. If we take the derivative of (7.3), we multiply it by 2 and we add it to (7.4), we obtain the following equivalent condition

$$(W(x)B_1(x)^* - B_1(x)W(x))' - 2(W(x)B_0 - B_0W(x)) = 0. \tag{A.25}$$

We shall prove instead that

$$W(x)B_1(x)^* - B_1(x)W(x) - 2\left(\int W(x)\,\mathrm{d}x\right)B_0 - 2B_0\left(\int W(x)\,\mathrm{d}x\right) = 0,$$

which is obtained by integrating (A.25) with respect to x. Then (A.25) will follow by taking the derivative with respect to x.

We assume $m < n$. The other cases can be proved similarly. We proceed as in the proof of (A.19) in Theorem 7.5 to show that

$$\begin{aligned}&(W(x)B_1(x)^* - B_1(x)W(x))_{n,m}\\ &\quad= -\rho(x)\sum_{t=0}^{m} c(n,m,t)\frac{(m-n)(\ell+1)(4\ell^2 - \ell m - \ell n + 5\ell + 3)}{\ell(-m+1+t-n+2\ell)(t+1)}U_{n+m-2t-1}(x).\end{aligned} \tag{A.26}$$

On the other hand, we have

$$\begin{aligned}&-\left(2\left(\int W(x)\,\mathrm{d}x\right)B_0 + 2B_0\left(\int W(x)\,\mathrm{d}x\right)\right)_{n,m}\\ &\quad= \sum_{t=0}^{m}\left[2c(n,m,t)((B_0)_{m,m} - (B_0)_{n,n})\int \rho(x)U_{n+m-2t}(x)\,\mathrm{d}x\right].\end{aligned} \tag{A.27}$$

It is easy to show that the following formula for the Chebyshev polynomials holds

$$\int \rho(x)U_i(x) = \rho(x)\left(\frac{U_{i+1}(x)}{2(i+2)} - \frac{U_{i-1}(x)}{2i}\right), \quad (U_{-1} \equiv 0).$$

Therefore, we have that (A.27) is given by

$$\rho(x)\sum_{t=0}^{m} c(n,m,t)\frac{(B_0)_{m,m}-(B_0)_{n,n}}{(n+m-2t)}\left(\frac{c(n,m,t+1)}{c(n,m,t)}-1\right)U_{n+m-2t-1}(x)$$

$$=\rho(x)\sum_{t=0}^{m} c(n,m,t)\frac{(m-n)(\ell+1)(4\ell^2-\ell m-\ell n+5\ell+3)}{\ell(-m+1+t-n+2\ell)(t+1)}U_{n+m-2t-1}(x). \qquad \text{(A.28)}$$

Now (A.28) is exactly the negative of (A.26). This completes the proof of the theorem. ■

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