IP-DGFEM METHOD FOR THE $p(x)$- LAPLACIAN

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Abstract. In this paper we construct an "Interior Penalty" Discontinuous Galerkin method to approximate the minimizer of a variational problem related to the $p(x)-$Laplacian. The function $p : \Omega \to [p_1, p_2]$ is log Hölder continuous and $1 < p_1 \leq p_2 < \infty$. We prove that the minimizers of the discrete functional converge to the solution. We also make some numerical experiments in dimension one to compare this method with the Conforming Galerkin Method, in the case where p_1 is close to one. This example is motivated by its applications to image processing.

1. Introduction

In this paper we study a discontinuous Galerking method to approximate the minimizer of a non homogeneous functional. This functional is related to the so-called $p(x)-$Laplacian operator, i.e.

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u(x)|^{p(x)-2} \nabla u). \tag{1.1}$$

This operator extends the classical Laplacian ($p(x) \equiv 2$) and the so-called $p-$Laplacian ($p(x) \equiv p$ with $1 < p < \infty$) and it has been recently used in image processing and in the modeling of electrorheological fluids, see [3, 8, 23] .

In an image processing problem, the aim is to recover the real image I from an observed image ξ of the form $\xi = I + \eta$, where η is a noise.

Approaches to image denoising have been developed along three main lines: wavelet methods, stochastic methods and variational methods, see references in [3]. One variational approach that has attracted a great deal of attention is the total variation method of L. Rudin, S. Osher and E. Fetami [22]. The variational problem is

Minimize the functional $|Du|(\Omega)$ over all the functions in $BV(\Omega) \cap L^2(\Omega)$ such that

$$\int_\Omega u \, dx = \int_\Omega \xi \, dx \quad \text{and} \quad \int_\Omega |u - \xi|^2 \, dx = \sigma^2$$

for some $\sigma > 0$.

The conditions on the space come from the assumption that ξ is a function that represents a white noise with mean zero and variance σ. Moreover, the authors prove that this problem is equivalent to minimizing

$$|Du|(\Omega) + \frac{\lambda}{2} \int_\Omega |u - \xi|^2 \, dx$$

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for some nonnegative Lagrange multiplier $\lambda = \lambda(\sigma, \xi)$. This model works when the image is piecewise constant, but in some cases can cause a *staircasing* effect. See for instance [7].

An older approach consists in minimizing,

$$\int_\Omega |\nabla u|^2 + \frac{\lambda}{2} \int_\Omega |u - \xi|^2 \, dx.$$

This method solves the *staircasing* effect, but it has the problem that it does not preserve edges.

In [8], the authors introduce a model that involves the $p(x)-$Laplacian, for some function $p : \Omega \to [p_1, 2]$, with $p_1 > 1$. This function encodes the information on the regiones where the gradient is sufficiently large (at edges) and where the gradient is close to zero (in homogeneous regions). In this manner, the model avoids the *staircasing* effect still preserving the edges.

Recently, in [3] the authors propose a variant of the method of Chen, Levine and Rao [8]. More precisely, they consider the functional

$$\int_\Omega |\nabla u|^{p(x)} + \frac{\lambda}{2} \int_\Omega |u - \xi|^2 \, dx,$$

with $p : \Omega \to [1, 2]$ a function such that $p(x) = P_M(|\nabla G_\delta * \xi|(x))$, where $G_\delta(x)$ is an approximation of the identity, $M >> 1$ and P_M is a function that satisfies $P_M(0) = 2$ and $P_M(x) = 1$ for all $|x| > M$. Observe that, since $p(x) = 1$ for some values of x, the authors have to rewrite the functional in a form that allow for computation of weak derivatives.

Motivated by the above mentioned applications, we study a numerical method to approximate minimizers of a functional related to the $p(x)-$ Laplacian.

We work in the following setting:

Let Ω be a bounded Lipschitz domain. For functions p, s, t the following conditions will be assumed when necessary,

(H1) $p : \overline{\Omega} \to [p_1, p_2]$ $(1 < p_1 \leq p_2 < \infty)$ is log-Hölder continuous. That is, there exists a constant C_{log} such that

$$|p(x) - p(y)| \leq \frac{C_{log}}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \forall\, x, y \in \Omega;$$

(H2) $s \in L^\infty(\Omega)$, with $1 \leq s(x) < p^*(x) - \varepsilon$ for some $\varepsilon > 0$;

(H3) $t \in C^0(\partial\Omega)$ with $1 \leq t(x) < p_*(x)$.

Here, p^* and p_* are the Sobolev critical exponents for these spaces, i.e.

$$p^*(x) := \begin{cases} \frac{p(x)N}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \quad \text{and} \quad p_*(x) := \begin{cases} \frac{p(x)(N-1)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases} \tag{1.2}$$

Given p, q, r satisfying (H1),(H2) and (H3) respectively and $\xi \in L^{q(\cdot)}(\Omega)$, we want to minimize the functional

$$I(v) = \int_\Omega \left(|\nabla v(x)|^{p(x)} + |v(x) - \xi(x)|^{q(x)} \right) dx + \int_{\Gamma_N} |v|^{r(x)} \, dS$$

over all $v \in \mathcal{A}$, where

$$\mathcal{A} = \{ v \in W^{1,p(\cdot)}(\Omega) : v - u_D \in W^{1,p(\cdot)}_{\Gamma_D}(\Omega) \},$$

$u_D \in W^{1,p(\cdot)}(\Omega)$ and $\partial\Omega = \Gamma_D \cup \Gamma_N$. For the definitions of the variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$, see Section 2.

Let us observe that, considering the applications we have in mind, it is relevant to study the minimization problem in the case where p approaches the value 1 in some regions. We can see, by making some numerical experiments, that the minimizers have large derivative in these regions. For this reason, the Conforming Finite Element Method is not appropriate since, its use would imply the need of fine meshes in order to obtain good approximations, see Section 8.

We consider the so-called Discontinuous Galerkin Methods. These methods are relatively new from the theoretical point of view. In [1], we can find a unification of all methods of this type. In all the examples of that paper, the authors take as model a linear differential equation.

At this point we want to mention that, in [3] and [8] the authors find an approximation of the solutions by using an explicit finite difference scheme for the associated parabolic problem.

Our aim is to study, in the future, the minimization problem in the case where p approaches the value 1 in some regions (where there is no weak formulation). For this reason, we think that the best way to find approximations is by finding a good discretization of the minimization problem. We take a discretization similar to the one in [6] where the authors study a functional that includes the case $p =$ constant.

Our discrete functional is the following

$$I_h(v_h) = \int_\Omega \left(|\nabla v_h + R_h(v_h)|^{p(x)} + |v_h - \xi|^{q(x)} \right) dx + \int_{\Gamma_D} |v_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS + \int_{\Gamma_{int}} |[\![v_h]\!]|^{p(x)} \mathbf{h}^{1-p(x)}\, dS + \int_{\Gamma_N} |v_h|^{r(x)}\, dS,$$

where $\mathbf{h}$ is the local mesh size, h is the global mesh size, Γ_{int} is the union of the interior edges of the elements, $[\![v_h]\!]$ is the jump of the function between two edges and ∇v_h denotes the element-wise gradient of v_h, see Section 3 for a precise definition. Observe that the boundary condition is weakly imposed by the second term of the functional. Lastly, R_h is the lifting operator defined in Section 5.1, which represents the contributions of the jumps to the distributional gradient.

In the case $p =$ constant, the boundedness of this operator is proved by using an inf-sup condition. In our case, for technical reasons, we use a different approach, which consists of finding a local characterization of the operator.

With this setting the discrete problem is to find a minimizer u_h of I_h over the space $S^k(\mathcal{T}_h)$ of all the functions that are polynomials of degree at most k in each element, with $k \geq 1$, see Section 3.

In this work we show that the sequence u_h converges to the minimizer u of I over the space $\mathcal{A}$. We want to remark here that we are assuming that, for each $v_h \in S^k(\mathcal{T}_h)$, all the terms of $I_h(v_h)$ can be exactly computed.

In fact, we prove the following

Theorem 1.1. *Let Ω be a polyhedral domain. Let $p(x), q(x)$, and $r(x)$ be functions satisfying (H1), (H2) and (H3) respectively and let $u_D \in W^{2,p_2}(\Omega)$. For each $h \in (0,1]$, let $u_h \in S^k(\mathcal{T}_h)$*

be the minimizer of I_h. If u is the minimizer of I then

$$u_h \to u \text{ strongly in } L^{s(\cdot)}(\Omega) \quad \forall s \text{ satisfying (H2)}, \tag{1.3}$$

$$u_h \to u \text{ strongly in } L^{t(\cdot)}(\partial\Omega) \quad \forall t \text{ satisfying (H3)}, \tag{1.4}$$

$$I_h(u_h) \to I(u), \tag{1.5}$$

$$R(u_h) \to 0 \text{ strongly in } L^{p(\cdot)}(\Omega), \tag{1.6}$$

$$\int_{\Gamma_D} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS + \int_{\Gamma_{int}} |[\![u_h]\!]|^{p(x)} \mathbf{h}^{1-p(x)}\, dS \to 0, \tag{1.7}$$

$$\nabla u_h \to \nabla u \text{ strongly in } L^{p(\cdot)}(\Omega). \tag{1.8}$$

Lastly, we want to mention the places where we need the regularity hypotheses on the function p. First, in order to prove Theorem 1.1 we need to use the continuity of the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$, the continuity of the Trace operator $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$ and the Poincaré inequality. As we can see in Theorem 2.9, Theorem 2.10 and Theorem 2.8 for these results we need p to be log-Hölder and $r \in C^0(\partial\Omega)$.

We also use strongly that p is log-Hölder in Proposition 2.11. This result says that if κ is an element with diameter h_κ and p^κ_+ and p^κ_- are respectively the maximum and minimum of p over κ then $h_\kappa^{p^\kappa_- - p^\kappa_+}$ is bounded independent of h_κ. This property is crucial in the proof of several results along the paper.

On the other hand, in order to prove the convergence of the sequence u_h we need a technical hypothesis on the boundary condition u_D. In fact, Lemma 6.4 only covers the case where $u_D \in W^{2,p_2}(\Omega)$.

Outline of the paper. In Section 2 we state several properties of the Variable Exponent Sobolev Spaces.

In Section 3 we give some definitions and properties related to the mesh and to the Broken Sobolev Spaces.

In Section 4 we study the Reconstruction operator and we prove some error estimates that are crucial for the rest of the paper (Corollary 4.5).

In Section 5 we prove the boundedness of the Lifting operator (Theorem 5.3).

In Section 6 we prove the Broken Poincaré inequality (Theorem 6.1), the coercivity of the functional (Theorem 6.2) and finally we give the proof of Theorem 1.1.

In Section 7 we study the convergence of the Conforming Finite Element Method.

In Section 8 we give a 1d example and compare both conforming and non-conforming schemes.

2. Preliminaries: The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$

We now introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let $p\colon \Omega \to [p_1, p_2]$ be a measurable bounded function, called a variable exponent on Ω where $p_1 := essinf\, p(x)$ and $p_2 := esssup\, p(x)$ with $1 \le p_1 \le p_2 < \infty$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} := \inf\{k > 0 \colon \varrho_{p(\cdot)}(u/k) \leq 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

The following properties can be obtained directly from the definition of the norm,

Proposition 2.1. *If $u, u_n \in L^{p(\cdot)}(\Omega)$, $\|u\|_{p(\cdot)} = \lambda$, then*

(1) $\lambda < 1$ $(= 1, > 1)$ *if only if* $\displaystyle\int_\Omega |u(x)|^{p(x)}\,dx < 1$ $(= 1, > 1)$;

(2) *If* $\lambda \geq 1$, *then* $\lambda^{p_1} \leq \displaystyle\int_\Omega |u(x)|^{p(x)}\,dx \leq \lambda^{p_2}$;

(3) *If* $\lambda \leq 1$, *then* $\lambda^{p_2} \leq \displaystyle\int_\Omega |u(x)|^{p(x)}\,dx \leq \lambda^{p_1}$;

(4) $\displaystyle\int_\Omega |u_n(x)|^{p(x)}\,dx \to 0$ *if only if* $\|u_n\|_{p(\cdot)} \to 0$;

(5) $\|1\|_{p(\cdot)} \leq \max\left\{|\Omega|^{\frac{1}{p_1}}, |\Omega|^{\frac{1}{p_2}}\right\}$;

(6) *If* $\Omega = \bigcup_{i=1}^{m} \Omega_i$ *where* $\Omega_i \subset \Omega$ *are open sets then there exists a constant* $C > 0$ *depending on* m *such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \sum_{i=1}^{m} \|u\|_{L^{p(\cdot)}(\Omega_i)}.$$

Proof. See Theorem 1.3 and Theorem 1.4 in [18]. □

For the proofs of the following three theorems we refer the reader to [21].

Theorem 2.2. *Let $q(x) \leq p(x)$, then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.*

Theorem 2.3. *Let $p, q, r : \Omega \to [1, \infty)$ and $\varepsilon > 0$ be such that $p(x) \leq r(x) < q(x) - \varepsilon$ for all $x \in \Omega$. Then, there exists a positive constant C such that for every $u \in L^{p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega)$ the inequality*

$$\|u\|_{L^{r(\cdot)}(\Omega)} \leq C\|u\|^{\mu}_{L^{p(\cdot)}(\Omega)}\|u\|^{\nu}_{L^{q(\cdot)}(\Omega)}$$

holds, where $\mu > 0$ and $\nu \geq 0$ are define as

$$\mu = \begin{cases} esssup_\Omega \dfrac{p(x)}{r(x)}\dfrac{q(x)-r(x)}{q(x)-p(x)} & \text{if } \|u\|_{L^{p(\cdot)}(\Omega)} > 1, \\ essinf_\Omega \dfrac{p(x)}{r(x)}\dfrac{q(x)-r(x)}{q(x)-p(x)} & \text{if } \|u\|_{L^{p(\cdot)}(\Omega)} \leq 1, \end{cases}$$

$$\nu = \begin{cases} esssup_\Omega \dfrac{q(x)}{r(x)}\dfrac{r(x)-p(x)}{q(x)-p(x)} & \text{if } \|u\|_{L^{q(\cdot)}(\Omega)} > 1, \\ essinf_\Omega \dfrac{q(x)}{r(x)}\dfrac{r(x)-p(x)}{q(x)-p(x)} & \text{if } \|u\|_{L^{q(\cdot)}(\Omega)} \leq 1. \end{cases}$$

Theorem 2.4. *Let $p'(x)$ such that, $1/p(x) + 1/p'(x) = 1$. Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_1 > 1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.*

Now we give some well known inequalities,

Proposition 2.5. *For any x fixed we have the following inequalities*

$$|\eta-\xi|^{p(x)} \leq C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta-\xi) \qquad \text{if } p(x)\geq 2,$$

$$|\eta-\xi|^2\Big(|\eta|+|\xi|\Big)^{p(x)-2} \leq C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)(\eta-\xi) \qquad \text{if } p(x)< 2,$$

$$|\eta|^{p(x)} \leq 2^{p(x)-1}(|\eta-\xi|^{p(x)}+|\xi|^{p(x)}) \qquad \text{if } p(x)\geq 1.$$

These inequalities say that the function $A(x,q)=|q|^{p(x)-2}q$ is strictly monotone.

Proposition 2.6. *Let $F_n, F \in L^{p(\cdot)}(\Omega)$.*

(1) *If*

$$F_n \rightharpoonup F \text{ weakly in } L^{p(\cdot)}(\Omega)$$

then

$$\int_\Omega |F|^{p(x)}\,dx \leq \liminf_{n\to\infty}\int_\Omega |F_n|^{p(x)}\,dx.$$

(2) *If*

$$F_n \to F \text{ strongly in } L^{p(\cdot)}(\Omega)$$

then

$$\int_\Omega |F_n|^{p(x)}\,dx \to \int_\Omega |F|^{p(x)}\,dx.$$

(3) *If*

$$F_n \rightharpoonup F \text{ weakly in } L^{p(\cdot)}(\Omega) \quad \text{and} \quad \int_\Omega |F_n|^{p(x)}\,dx \to \int_\Omega |F|^{p(x)}\,dx \tag{2.9}$$

then

$$F_n \to F \text{ strongly in } L^{p(\cdot)}(\Omega).$$

Proof. For the proof of (1) and (3) see Theorem 3.9 and Lemma 2.4.17 in [13]. Finally (2) follows by Proposition 2.3 in [15]. □

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that, u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have the following version of Poincaré inequality, see Theorem 3.10 in [21].

Lemma 2.7. *If $p:\Omega\to[1,+\infty)$ is continuous in $\overline{\Omega}$, there exists a constant C such that for every $u\in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We also have the following version of the Poincaré inequality, see Lemma 2.1 in [20],

Theorem 2.8. *Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz domain and $p,q:\Omega\to[1,+\infty)$ with $p\leq q\leq p^*$. Then,*

$$\|u-(u)_\Omega\|_{L^{q(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u\in W^{1,p(\cdot)}(\Omega)$, where $(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$.

In order to have better properties of these spaces, we need more hypotheses on the regularity of p. For expample, it was proved in [11], Theorem 3.7, that if one assumes that $\partial\Omega$ is Lipschitz and p is log-Hölder continuous then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$, see also [10, 14, 16, 21, 24]. The local log-Hlöder condition was first used in the variable exponent context in [26].

We now state two Sobolev embedding Theorems (for the proofs see [12] and Corollary 2.4 in [17], respectively).

Theorem 2.9. *Let Ω be a Lipschitz domain. Let $p : \Omega \to [1, \infty)$ be log-Hölder continuous. Then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous.*

Theorem 2.10. *Let Ω be an open bounded domain with Lipschitz boundary. Suppose that $p \in C^0(\overline{\Omega})$ with $p_1 > 1$. If $r \in C^0(\partial\Omega)$ satisfies the condition $1 \leq r(x) < p_*(x)$ for all $x \in \partial\Omega$, then there is a compact boundary trace embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$.*

Let $\Gamma_D \subset \partial\Omega$, and p be log-Hölder. We define the space $W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$ as the closure of the space $\{\varphi \in C^\infty(\overline{\Omega}) : \varphi = 0 \text{ on } \Gamma_D\}$ in $W^{1,p(\cdot)}(\Omega)$.

The following proposition is crucial in order to prove the main result of this paper.

Proposition 2.11. *Let $p : \Omega \to [1, \infty)$ be* log*-Hölder continuous and bounded. Let $\alpha > 0$, $D \subset \Omega$ and $h = diam(D)$ then:*

(1) *There exists a constants C independent of h such that*

$$h^{\alpha(p(x)-p(y))} \leq C \quad \forall x, y \in \overline{D}; \tag{2.10}$$

(2) *If $A \geq h^\alpha$ then $A^{p(x)} \leq CA^{p(y)}$ for all $x, y \in \overline{D}$ such that $p(x) \leq p(y)$.*

Proof. Let $x, y \in \overline{D}$. If $p(x) \geq p(y)$ or $h \geq 1$ the result follows since Ω is bounded. If $p(x) \leq p(y)$ and $h < 1$, using that p is log-Hölder, we have

$$p(y) - p(x) \leq \frac{C}{\log\left(e + \dfrac{1}{|x-y|}\right)} \leq \frac{C}{\log\left(e + \dfrac{1}{h}\right)}.$$

Then, we get (2.10).

By (2.10) and as $A \geq h^\alpha$, we have that for all $x, y \in \overline{D}$ such that $p(x) \leq p(y)$,

$$A^{p(x)} = A^{p(y)} \left(\frac{A}{h^\alpha}\right)^{p(x)-p(y)} h^{\alpha(p(x)-p(y))} \leq CA^{p(y)}.$$

□

3. The mesh $\mathcal{T}_h$ and properties of $W^{1,p(\cdot)}(\mathcal{T}_h)$

In this section we give some definitions and properties related to the mesh and to the Broken Sobolev Space.

Hypothesis 3.1. *Let Ω be a polygonal Lipschitz domain and $(\mathcal{T}_h)_{h\in(0,1]}$ be a family of partitions of $\overline{\Omega}$ into polyhedral elements. We assume that there exist a finite number of reference polyhedral $\hat{\kappa}_1, ..., \hat{\kappa}_r$ such that for all $\kappa \in \mathcal{T}_h$ there exists an invertible affine map F_κ such that, $\kappa = F_\kappa(\hat{\kappa}_i)$. We assume that each $\kappa \in \mathcal{T}_h$ is closed and that $diam(\kappa) \leq h$ for all $\kappa \in \mathcal{T}_h$.*

Now we give some notation,

$$\mathcal{E}_h := \{\kappa \cap \kappa' : \dim_H(\kappa \cap \kappa') = N-1\} \cup \{\kappa \cap \partial\Omega : \dim_H(\kappa \cap \partial\Omega) = N-1\},$$
$$\Gamma_{int} := \bigcup \{e \in \mathcal{E}_h : \dim_H(e \cap \partial\Omega) < N-1\}.$$

$\mathcal{N}_h$ is the set of nodes of $\mathcal{T}_h$. For every $z \in \mathcal{N}_h$ and $e \in \mathcal{E}_h$ we define,

$$T_z := \bigcup\{\kappa \in \mathcal{T}_h \colon z \in \kappa\}, \quad T_\kappa := \bigcup\{T_z \colon z \in \kappa\}, \quad T_e := \bigcup\{T_\kappa \colon e \in \kappa\},$$
$$h_\kappa := \operatorname{diam}(\kappa), \qquad h_z := \operatorname{diam}(T_z) \qquad h_e := \operatorname{diam}(e),$$
$$p^\kappa_- := \operatorname*{ess\,inf}_{x\in\kappa} p(x) \qquad p^\kappa_+ := \operatorname*{ess\,sup}_{x\in\kappa} p(x), \quad p^e_- := \operatorname*{ess\,inf}_{x\in e} p(x) \text{ and } p^e_+ := \operatorname*{ess\,sup}_{x\in e} p(x).$$

We assume that the mesh satisfies the following hypotheses,

Hypothesis 3.2. *The family of partitions $(\mathcal{T}_h)_{h\in(0,1]}$ satisfies the Hypothesis 3.1 and*

(a) *There exist positive constants C_1 and C_2, independent of h, such that for each element $\kappa \in \mathcal{T}_h$*
$$C_1 h_\kappa^N \le |\kappa| \le C_2 h_\kappa^N.$$
(b) *There exists a constant $C_1 > 0$ such that for all $h \in (0,1]$ and for all face $e \in \mathcal{E}_h$ there exists a point $x_e \in e$ and a radius $\rho_e \ge C_1 diam(e)$ such that $B_{\rho_e}(x_e) \cap A_e \subset e$, where A_e is the affine hyperplane spanned by e. Moreover, there are positive constants such that*
$$ch_\kappa \le h_e \le Ch_\kappa, \quad ch_{\kappa'} \le h_e \le Ch_{\kappa'}$$
where $e = \kappa \cap \kappa'$.

We use the notation $\sim$ to compare quantities which differ only up to positive constants that do not depend on the local or global mesh size or on any function which appears in the estimate.

Remark 3.3. By the regularity assumption of the mesh we have the following,

$$\sharp\{z \in \mathcal{N}_h \colon z \in \kappa\} \sim 1, \qquad \sharp\{\kappa \in \mathcal{T}_h \colon \kappa \subset T_z\} \sim 1,$$
$$\sharp\{\kappa' \in \mathcal{T}_h \colon \kappa' \subset T_\kappa\} \sim 1, \quad \sharp\{e \in \mathcal{E}_h \colon e \subset T_z\} \sim 1 \quad \text{and} \quad \sharp\{e \in \mathcal{E}_h \colon e \subset T_\kappa\} \sim 1.$$

Remark 3.4. As a consequence, we have that $\operatorname{diam}(T_\kappa) \sim h_\kappa$ and for each $z \in \kappa$ and $e \subset \partial\kappa$, $h_z \sim h_\kappa$ and $h_e \sim h_\kappa$. See the discussion on Section 4.2 in [6].

Remark 3.5. By Proposition 2.11, we also have that for each edge $e \subset \partial\kappa$, $h_\kappa^{p(x)} \sim h_e^{p(y)}$ for any $x, y \in \kappa$. We will replace p^κ_-, p^e_- by p_- and p^κ_+, p^e_+ by p_+ when no confusion can arise.

Now, we introduce the finite element spaces associated with $\mathcal{T}_h$. We define the variable broken Sobolev space as

$$W^{1,p(\cdot)}(\mathcal{T}_h) := \{u \in L^1(\Omega) \colon u|_\kappa \in W^{1,p(\cdot)}(\kappa) \text{ for all } \kappa \in \mathcal{T}_h\},$$

and the subspaces

$$U^k(\mathcal{T}_h) := \{u \in C(\Omega) \colon u|_\kappa \in P^k \text{ for all } \kappa \in \mathcal{T}_h\},$$

$$S^k(\mathcal{T}_h) := \{u \in L^1(\Omega) \colon u|_\kappa \in P^k \text{ for all } \kappa \in \mathcal{T}_h\}$$

where P^k is the space of polynomials functions of degree at most $k \ge 1$.

We also define, for any $\kappa \in \mathcal{T}_h$, the space

$$W^{1,p(\cdot)}(T_\kappa) := \{u|_{T_\kappa} \colon u \in W^{1,p(\cdot)}(\mathcal{T}_h)\},$$

and in the same manner we define the spaces $W^{1,p(\cdot)}(T_z)$ and $W^{1,p(\cdot)}(T_e)$, for any $z \in \mathcal{N}_h$ and $e \in \mathcal{E}_h$.

For each face $e \in \mathcal{E}_h$, $e \subset \Gamma_{int}$ we denote by κ^+ and κ^- its neighboring elements. We write ν^+, ν^- to denote the outward normal unit vectors to the boundaries $\partial\kappa^\pm$, respectively. The jump of a function $u \in W^{1,p(\cdot)}(\mathcal{T}_h)$ and the average of a vector-valued function $\phi \in (W^{1,p(\cdot)}(\mathcal{T}_h))^N$, with traces $u^\pm$, $\phi^\pm$ from $k^\pm$ are, respectively, defined as the vectors

$$[\![u]\!] := u^+\nu^+ + u^-\nu^- \quad \text{and} \quad \{\phi\} := \frac{\phi^+ + \phi^-}{2}.$$

Let $\mathbf{h} : \partial\Omega \cup \Gamma_{int} \to \mathbb{R}$ a piecewise constant function define by

$$\mathbf{h}(x) = diam(e) \text{ if } x \in e,$$

where $e \in \mathcal{E}_h$.

We consider the following seminorms on $W^{1,p(\cdot)}(\mathcal{T}_h)$,

$$|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|[\![u]\!]\mathbf{h}^{\frac{-1}{p'(x)}}\|_{L^{p(\cdot)}(\Gamma_{int})},$$

$$|u|_{W^{1,p(\cdot)}_D(\mathcal{T}_h)} = |u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \|u\mathbf{h}^{\frac{-1}{p'(x)}}\|_{L^{p(\cdot)}(\Gamma_D)},$$

and the following local seminorm

$$|u|_{W^{1,p(\cdot)}(T_\kappa)} = \|\nabla u\|_{L^{p(\cdot)}(T_\kappa)} + \sum_{e \subset T_\kappa} \|[\![u]\!]\mathbf{h}^{\frac{-1}{p'(x)}}\|_{L^{p(\cdot)}(e)},$$

for any $\kappa \in \mathcal{T}_h$. Similarly, we define the seminorms $|u|_{W^{1,p(\cdot)}(T_z)}$ and $|u|_{W^{1,p(\cdot)}(T_e)}$ for any $z \in \mathcal{N}_h$ and $e \in \mathcal{E}_h$.

Lemma 3.6. *For all $p\colon [1,\infty) \to \mathbb{R}$, there exist a constant C, independent of h such that,*

$$|Du|(\Omega) \le C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h),\ \forall h \in (0,1].$$

Proof. For all $u \in W^{1,p(\cdot)}(\mathcal{T}_h)$, we have that

$$|Du|(\Omega) \le \int_\Omega |\nabla u|\, dx + \int_{\Gamma_{int}} |[\![u]\!]|\, ds.$$

Thus, by Hölder inequality, Proposition 2.1 (5) and the Hypothesis 3.2, there exists a constant C depending only of $|\Omega|$, p_1 and p_2 such that

$$|Du|(\Omega) \le C\left(\|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|\mathbf{h}^{\frac{-1}{p'(x)}}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})}\right).$$

The proof is now complete. □

Lemma 3.7. *Let $(\mathcal{T}_h)_{h\in(0,1]}$ be a family of partitions of Ω. Then, for each function $p, q\colon \Omega \to [1,\infty)$, there exists a constant $C > 0$ independent of h, such that for any $\kappa \in \mathcal{T}_h$*

$$\|u\|_{L^{p(\cdot)}(\kappa)} \le Ch_\kappa^{\frac{N}{p_+} - \frac{N}{q_-}} \|u\|_{L^{q(\cdot)}(\kappa)} \quad \forall u \in S^k(\mathcal{T}_h),\ \forall h \in (0,1].$$

Proof. Let $\kappa \in \mathcal{T}_h$, $\hat{\kappa}$ its corresponding reference element and $F_\kappa \colon \hat{\kappa} \to \kappa$ the associated affine mapping. We set $J = |\det(DF_\kappa)|$. Using Hypothesis 3.2, we have $C^{-1}h_\kappa^N \leq J \leq Ch_\kappa^N$, for some constant C which is independent of κ. Let $K > 0$, then we have

$$\int_\kappa \left(\frac{|u|}{K}\right)^{p(x)} dx = \int_{\hat{\kappa}} \left(\frac{|u \circ F_\kappa|}{K}\right)^{p\circ F_\kappa(x)} J\, dx \leq Ch_\kappa^N \int_{\hat{\kappa}} \left(\frac{|u \circ F_\kappa|}{K}\right)^{p\circ F_\kappa(x)} dx.$$

Thus,

$$\|(Ch_\kappa^N)^{-1/p(x)} u\|_{L^{p(\cdot)}(\kappa)} \leq \|u \circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{\kappa})}.$$

Using that $h_\kappa \ll 1$, we obtain

$$\|u\|_{L^{p(\cdot)}(\kappa)} \leq (Ch_\kappa^N)^{1/p_+} \|u \circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{\kappa})}. \tag{3.11}$$

Similarly, we have

$$\|u \circ F_\kappa\|_{L^{q\circ F_\kappa(\cdot)}(\hat{\kappa})} \leq (Ch_\kappa^{-N})^{1/q_-} \|u\|_{L^{q(\cdot)}(\kappa)}. \tag{3.12}$$

As on a finite dimensional space all the norms are equivalent, we have that there exists a constant $\bar{C}$ depending only on N and k such that,

$$\|u \circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{\kappa})} \leq C\|u \circ F_\kappa\|_{L^{p_2}(\hat{\kappa})} \leq C\|u \circ F_\kappa\|_{L^{q_1}(\hat{\kappa})} \leq \bar{C}\|u \circ F\|_{L^{q(\cdot)}(\hat{\kappa})}, \tag{3.13}$$

where in the first and last inequalities we are using Theorem 2.2.

Finally, by (3.11)–(3.13) we arrive at the desired result. □

Lemma 3.8. *If p is* log*-Hölder continuous then, for any $e \in \mathcal{E}_h \cap \partial\Omega$ and $z \in \mathcal{N}_h \cap e$ we have that,*

$$\|u\|_{L^{p(\cdot)}(e)} \leq Ch_z^{-\frac{1}{p_-}} \|u\|_{L^{p(\cdot)}(T_z)} \quad \forall u \in S^k(\mathcal{T}_h), \tag{3.14}$$

where $C = C(p_1, p_2, N, \Omega, C_{log})$.

Proof. Let $\kappa \in \mathcal{T}_h$ such that $e \subset \kappa$. Let F_κ and $\hat{\kappa}$ be as in the proof of Lemma 3.7 and let $\hat{e} = F_\kappa^{-1}(e)$.

Then,

$$\int_e \left(\frac{|u(x)|}{k}\right)^{p(x)} dS \leq Ch_\kappa^{N-1} \int_{\hat{e}} \left(\frac{|u \circ F_\kappa(x)|}{k}\right)^{p\circ F_\kappa(x)} dS.$$

Hence,

$$\left\| (C^{-1}h_\kappa)^{\frac{1}{p(x)}} \frac{u}{h_\kappa^{N/p(x)}} \right\|_{L^{p(\cdot)}(e)} \leq \|u \circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{e})}.$$

By using Theorem 2.2 and that all the norms are equivalent, we have

$$\|u \circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{e})} \leq C\|u \circ F_\kappa\|_{L^{p_2}(\hat{e})} \leq C\|u \circ F_\kappa\|_{L^1(\hat{e})}.$$

On the other hand, by the local inverse estimate in [6, page 837], we have

$$\|u \circ F_\kappa\|_{L^1(\hat{e})} \leq C\|u \circ F_\kappa\|_{L^1(\hat{\kappa})}.$$

By using again Theorem 2.2, we obtain

$$\|u \circ F_\kappa\|_{L^1(\hat{\kappa})} \leq C\|u \circ F_\kappa\|_{L^{p\circ F_\kappa(\cdot)}(\hat{\kappa})}.$$

By using all the inequalities and the definition of the Luxembourg norm, we arrive at

$$\left\| h_\kappa^{\frac{1}{p(x)}} \frac{u}{h_\kappa^{N/p(x)}} \right\|_{L^{p(\cdot)}(e)} \leq C \left\| \frac{u}{h_\kappa^{N/p(x)}} \right\|_{L^{p(\cdot)}(\kappa)}.$$

Finally, we obtain

$$\|h_\kappa^{\frac{1}{p(x)}} u\|_{L^{p(\cdot)}(e)} \leq C h_\kappa^{\frac{N(p_- - p_+)}{p_- p_+}} \|u\|_{L^{p(\cdot)}(\kappa)},$$

By Remark 2.11, we get

$$\|h_\kappa^{\frac{1}{p(x)}} u\|_{L^{p(\cdot)}(e)} \leq C e^{N \frac{C}{p_1^2}} \|u\|_{L^{p(\cdot)}(\kappa)}.$$

Now, inequality (3.14) follows immediately using Proposition 2.1(3) and the fact that $h_z \sim h_\kappa$. □

The next result establishes the existence of the local projector operator (for the proof see Subsection 3.1 of [6]).

Lemma 3.9. *For all $z \in \mathcal{N}_h$ there exists a linear map $\pi_z : BV(\Omega) \to \mathbb{R}$ such that*

$$\|u - \pi_z(u)\|_{L^1(T_z)} \leq C h_z |Du|(T_z) \quad \forall u \in BV(\Omega)$$

where C is a constant independent of h and z.

4. The reconstruction operator Q_h

In many Discontinuous Galerkin problems one uses a priori bounds in order to prove the Poincaré inequality for the discrete space. In order to prove these inequalities it is required to use a reconstruction operator. In this section we define, as in [6], a family of quasi-interpolant operators and prove some error estimates depending on the mesh size. These results are more general than the ones in [6], because we prove bounds in the variable $p(x)-$ norm. On the other hand these results are less general than the ones in [4] in the sense that they only cover the case of the finite dimensional space $\mathcal{S}^k(\mathcal{T}_h)$. This last restriction comes from the fact that in Lemma 3.7 we need to use the equivalence of the norms in the space of polynomials.

In order to prove these error estimates we strongly use Proposition 2.11. This is the reason why we need p to be log-Hölder continuous.

Now, we define and study the reconstruction operator. For each $h \in (0, 1]$, let

$$Q_h \colon S^k(\mathcal{T}_h) \to W^{1,\infty}(\Omega)$$

be the linear operator defined by

$$Q_h(u) = \sum_{z \in \mathcal{N}_h} \pi_z(u) \lambda_z,$$

where λ_z is the standard P^1 nodal basis function associated with the vertex z on the mesh $\mathcal{T}_h$.

In the next theorem, we give some local estimates of the $L^{q(\cdot)}(\kappa)$ and $L^{q(\cdot)}(e)$ norms in terms of the $W^{1,p(\cdot)}(T_\kappa)$ seminorm and h.

Theorem 4.1. *Let $p, q\colon \Omega \to [1,\infty)$ be* log*-Hölder continuous in $\overline{\Omega}$. Then, the operator Q_h satisfies*

$$\|u - Q_h(u)\|_{L^{q(\cdot)}(\kappa)} \leq C h_\kappa^{\frac{N}{q_-}-\frac{N}{p_-}+1} |u|_{W^{1,p(\cdot)}(T_\kappa)} \qquad \forall \kappa \in \mathcal{T}_h, \tag{4.15}$$

$$\|u - Q_h(u)\|_{L^{q(\cdot)}(e)} \leq C h_e^{\frac{N-1}{q_-}-\frac{N}{p_-}+1} |u|_{W^{1,p(\cdot)}(T_e)} \quad \forall e \in \mathcal{E}_h \cap \partial\Omega, \tag{4.16}$$

$$\|\nabla Q_h(u)\|_{L^{p(\cdot)}(\kappa)} \leq C |u|_{W^{1,p(\cdot)}(T_\kappa)} \qquad \forall \kappa \in \mathcal{T}_h, \tag{4.17}$$

for all $u \in S^k(\mathcal{T}_h)$ where C is a constant independent of h.

Proof. We proceed in three steps.

Step 1. We first show inequality (4.15).

Fix $\kappa \in \mathcal{T}_h$. For $z \in \mathcal{N}_h \cap \kappa$, by using Proposition 2.1 (6) and Lemma 3.7, we get

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C \sum_{\{\kappa' : \kappa' \subset T_z\}} h_{\kappa'}^{\frac{N}{q_+}-N} \|u - \pi_z(u)\|_{L^1(\kappa')}.$$

By Remark 3.4 and Proposition 2.11, we get

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C h_z^{\frac{N}{q_-}-N} \|u - \pi_z(u)\|_{L^1(T_z)}.$$

Thus, by Lemma 3.9, we have

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C h_z^{\frac{N}{q_-}-N+1} \left(\|\nabla u\|_{L^1(T_z)} + \sum_{e \subset T_z} \int_e |[\![u]\!]| \, ds \right).$$

Then, by using again Lemma 3.7 and Remark 3.3 we have

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C h_z^{\frac{N}{q_-}+1} \left(h_z^{-\frac{N}{p_-}} \|\nabla u\|_{L^{p(\cdot)}(T_z)} + h_z^{-N} \sum_{e \subset T_z} \int_e |[\![u]\!]| \, ds \right). \tag{4.18}$$

To estimate the second term, we use Hölder inequality, obtaining

$$\begin{aligned} \int_e |[\![u]\!]| \, ds &\leq 2 \| \, [\![u]\!] h_e^{-\frac{1}{p'(x)}} \|_{L^{p(\cdot)}(e)} \| h_e^{\frac{1}{p'(x)}} \|_{L^{p'(\cdot)}(e)} \\ &\leq C \| \, [\![u]\!] h_e^{-\frac{1}{p'(x)}} \|_{L^{p(\cdot)}(e)} h_e^{1-\frac{1}{p_-}} \|1\|_{L^{p'(\cdot)}(e)}. \end{aligned} \tag{4.19}$$

Now, by Proposition 2.1 (5), we have that

$$\|1\|_{L^{p'(\cdot)}(e)} \leq C h_e^{(N-1)(1-\frac{1}{p_-})}.$$

Then, we obtain

$$\int_e |[\![u]\!]| \, ds \leq C \| [\![u]\!] h_e^{-\frac{1}{p'(x)}} \|_{L^{p(\cdot)}(e)} h_z^{N(1-\frac{1}{p_-})}.$$

Therefore, summing over all $e \subset T_z$ and using (4.18), we arrive at

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)} \leq C h_z^{\frac{N}{q_-}-\frac{N}{p_-}+1} |u|_{W^{1,p(\cdot)}(T_z)}. \tag{4.20}$$

Now, as in the proof Theorem 3.1 in [6] and using Proposition 2.1 (6), we have the inequality (4.15).

Step 2. We now show the inequality (4.16).

Fix $e \in \mathcal{E}_h \cap \partial\Omega$ and let $z \in \mathcal{N}_h \cap e$. By the inequality (3.14),

$$\|u - \pi_z(u)\|_{L^{q(\cdot)}(e)} \leq C h_z^{-\frac{1}{q_-}} \|u - \pi_z(u)\|_{L^{q(\cdot)}(T_z)}.$$

Again, following the lines in [6] and using that p and q are log-Hölder continuous in $\overline{\Omega}$, we arrive at the inequality (4.16).

Step 3. Finally, we will show the inequality (4.17).

Fix $\kappa \in \mathcal{T}_h$. First, since $(\lambda_z)_{z\in\mathcal{N}_h\cap\kappa}$ is a partition of the unity in κ we have that for any $x \in \kappa$

$$\nabla Q_h u(x) - \nabla u(x) = \sum_{z\in\mathcal{N}_h\cap\kappa} (\pi_z(u) - u(x))\nabla\lambda_z(x) + \sum_{z\in\mathcal{N}_h\cap\kappa} \nabla u(x)\lambda_z(x)$$

$$\|\nabla Q_h u\|_{L^{p(\cdot)}(\kappa)} \leq \sum_{z\in\mathcal{N}_h\cap\kappa} \|(\pi_z(u) - u)\nabla\lambda_z\|_{L^{p(\cdot)}(\kappa)} + \sum_{z\in\mathcal{N}_h\cap\kappa} \|\nabla u\lambda_z\|_{L^{p(\cdot)}(\kappa)} + \|\nabla u\|_{L^{p(\cdot)}(\kappa)}.$$

Now, using Hypothesis (3.2), we have that there exists a constant C_1 such that $|\nabla\lambda_z| < C_1 h^{-1}$ in κ, and by (4.20) we get, using Remark 3.3,

$$\|\nabla Q_h u\|_{L^{p(\cdot)}(\kappa)} \leq C \sum_{z\in\mathcal{N}_h\cap\kappa} |u|_{W^{1,p(\cdot)}(T_z)} + |u|_{W^{1,p(\cdot)}(T_\kappa)} \leq (C+1)|u|_{W^{1,p(\cdot)}(T_\kappa)}.$$

The proof is now complete. □

Our next aim is to prove some global estimates. To this end we will need some definitions.

Definition 4.2. Let $p : \Omega \to [1,\infty)$. Given $q : \Omega \to [1,\infty)$ and $q \leq p^*$ in Ω, we define

$$\gamma = \sup\left\{\frac{q(x)}{p^*(x)} : x \in \Omega\right\}.$$

Observe that $0 \leq \gamma \leq 1$ and $\gamma = 0$ if $p(x) \geq N$ for all $x \in \Omega$ and $\gamma = 1$ if $p(x) < N$ and $q(x) = p^*(x)$ for all $x \in \Omega$.

Definition 4.3. Let $p : \Omega \to [1,\infty)$. Given $q : \Omega \to [1,\infty)$ and $q \leq p_*$ in Ω, we define

$$\beta = \sup\left\{\frac{q(x)}{p_*(x)} : x \in \Omega\right\}.$$

Observe that $0 \leq \beta \leq 1$ and $\beta = 0$ if $p(x) \geq N$ for all $x \in \Omega$ and $\beta = 1$ if $p(x) < N$ and $q(x) = p_*(x)$ for all $x \in \Omega$.

Lemma 4.4. *Let $p, q : \Omega \to [1,\infty)$ be* log*-Hölder continuous in $\overline{\Omega}$. Let $u \in S^k(\mathcal{T}_h)$ satisfy*

$$|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \leq 1. \tag{4.21}$$

Then, we have that

- *If $p \leq q \leq p^*$ in $\overline{\Omega}$, then*

$$\int_\Omega |u - Q_h(u)|^{q(x)}\, dx \leq C h^{N(1-\gamma)}, \tag{4.22}$$

$$\int_\Omega |\nabla Q_h(u)|^{p(x)}\, dx \leq C, \tag{4.23}$$

- *If $p \leq q \leq p_*$ in $\overline{\Omega}$, then*

$$\int_{\partial\Omega} |u - Q_h(u)|^{q(x)}\, dS \leq Ch^{(N-1)(1-\beta)}, \tag{4.24}$$

where $C = C(p_1, p_2, \Omega, C_{log}, N)$ and γ and β are given in Definitions 4.2 and 4.3, respectively.

Proof. First observe that, by (4.15), we have

$$\int_\kappa \frac{|u - Q_h(u)|^{q(x)}}{\left(Ch_k^{\frac{N}{q_-}-\frac{N}{p_-}+1}|u|_{W^{1,p(\cdot)}(T_\kappa)}\right)^{q(x)}} dx \leq 1 \quad \forall \kappa \in \mathcal{T}_h,$$

and by (4.21), we get

$$\frac{1}{Ch_k^{N-\frac{Nq_-}{p_-}+q_-}|u|^{q_-}_{W^{1,p(\cdot)}(T_\kappa)}} \int_\kappa |u - Q_h(u)|^{q(x)} dx \leq 1 \quad \forall \kappa \in \mathcal{T}_h.$$

Then, by Proposition 2.11,

$$\int_\kappa |u - Q_h(u)|^{q(x)}\, dx \leq Ch_\kappa^{N-\frac{Nq_-}{p_-}+q_-}|u|^{q_-}_{W^{1,p(\cdot)}(T_\kappa)} \leq Ch_\kappa^{N-\frac{Nq(x)}{p(x)}+q(x)}|u|^{q_-}_{W^{1,p(\cdot)}(T_\kappa)}$$

for any $\kappa \in \mathcal{T}_h$ and $x \in \kappa$. Therefore,

$$\int_\kappa |u - Q_h(u)|^{q(x)}\, dx \leq Ch^{N(1-\gamma)}|u|^{q_-}_{W^{1,p(\cdot)}(T_\kappa)} \quad \forall \kappa \in \mathcal{T}_h. \tag{4.25}$$

On the other hand, by Remark 3.3, the number of $\kappa \subset T_\kappa$ is uniformly bounded in h. Using this fact and Proposition 2.1 (6), we have that

$$|u|^{q_-}_{W^{1,p(\cdot)}(T_\kappa)} \leq C \sum_{\kappa \subset T_\kappa} \left(\|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)} + \|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \right). \tag{4.26}$$

On the other hand, if we suppose that $\|\nabla u\|_{L^{p(\cdot)}(\kappa)} \geq h_\kappa^{N/q_-}$, by Proposition 2.11 (2), we have that

$$\|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)} \leq C\|\nabla u\|^{q_+}_{L^{p(\cdot)}(\kappa)}. \tag{4.27}$$

Arguing as before, if $\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \geq h_\kappa^{N/q_-}$, we have that

$$\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \leq C\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_+}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})}. \tag{4.28}$$

Now, we take

$$A = \left\{ \kappa \in \mathcal{T}_h : \|\nabla u\|_{L^{p(\cdot)}(\kappa)} \geq h_\kappa^{N/q_-} \right\},$$

and

$$B = \left\{ \kappa \in \mathcal{T}_h : \|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \geq h_\kappa^{N/q_-} \right\}.$$

Observe that

$$\begin{aligned} \sum_{\kappa\in A^c} \|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)} &\leq \sum_{\kappa\in A^c} h_\kappa^N \leq C, \\ \sum_{\kappa\in B^c} \|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} &\leq C. \end{aligned} \tag{4.29}$$

On the other hand, by hypothesis (4.21), we have that $\|\nabla u\|_{L^{p(\cdot)}(\kappa)} \leq 1$ and then, for all $\kappa \in \mathcal{T}_h$

$$\begin{aligned} &\|\nabla u\|^{q_+}_{L^{p(\cdot)}(\kappa)} \leq \|\nabla u\|^{p_+}_{L^{p(\cdot)}(\kappa)} \leq \int_\kappa |\nabla u|^{p(x)}\, dx, \\ &\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_+}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \leq \int_{\kappa\cap\Gamma_{int}} |[\![u]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\, dx. \end{aligned} \tag{4.30}$$

Since each κ appears only in finitely many sets $T_{\kappa'}$ we have, by (4.26)–(4.30),

$$\begin{aligned} \sum_{\kappa\in\mathcal{T}_h} |u|^{q_-}_{W^{1,p(\cdot)}(T_\kappa)} \leq& C\left(\sum_{\kappa\in A}\|\nabla u\|^{q_+}_{L^{p(\cdot)}(\kappa)} + \sum_{\kappa\in A^c}\|\nabla u\|^{q_-}_{L^{p(\cdot)}(\kappa)}\right) \\ &+ C\left(\sum_{\kappa\in B}\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_+}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} + \sum_{\kappa\in B^c}\|[\![u]\!]\mathbf{h}^{\frac{1-p}{p}}\|^{q_-}_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})}\right) \\ \leq& C\left(\sum_{\kappa\in A}\int_\kappa |\nabla u|^{p(x)}\, dx + \sum_{\kappa\in B}\int_{\kappa\cap\Gamma_{int}} |[\![u]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\, ds + 1\right) \\ =& C\left(\int_\Omega |\nabla u|^{p(x)}\, dx + \int_{\Gamma_{int}} |[\![u]\!]|^{p(x)}\mathbf{h}^{1-p(x)}\, ds + 1\right). \end{aligned}$$

Thus, by (4.21) and (4.25) we get,

$$\int_\Omega |u - Q_h(u)|^{q(x)}\, dx = \sum_{\kappa\in\mathcal{T}_h}\int_\kappa |u - Q_h(u)|^{q(x)}\, dx \leq Ch^{N(1-\gamma)}.$$

Lastly, using the same argument, (4.16) and (4.17), we get

$$\int_{\partial\Omega} |u - Q_h(u)|^{q(x)}\, dS \leq Ch^{(N-1)(1-\beta)} \qquad \text{and} \qquad \int_\Omega |\nabla Q_h(u)|^{p(x)}\, dx \leq C,$$

where C is independent of h. □

The following corollary follows immediately

Corollary 4.5. *Let $p, q : \Omega \to [1,\infty)$ be log-Hölder continuous in $\overline{\Omega}$. Then, for all $u \in S^k(\mathcal{T}_h)$, we have,*

- *If $p \leq q \leq p^*$ in Ω, then*

$$\int_\Omega |u - Q_h(u)|^{q(x)}\, dx \leq Ch^{N(1-\gamma)} \max\left\{|u|^{q_1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}, |u|^{q_2}_{W^{1,p(\cdot)}(\mathcal{T}_h)}\right\}, \tag{4.31}$$

$$\int_\Omega |\nabla Q_h(u)|^{p(x)}\, dx \leq C \max\left\{|u|^{p_1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}, |u|^{p_2}_{W^{1,p(\cdot)}(\mathcal{T}_h)}\right\}. \tag{4.32}$$

- *If $p \leq q \leq p_*$ in Ω, then*

$$\int_{\partial\Omega} |u - Q_h(u)|^{q(x)}\, dS \leq Ch^{(N-1)(1-\beta)} \max\left\{|u|^{q_1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}, |u|^{q_2}_{W^{1,p(\cdot)}(\mathcal{T}_h)}\right\}. \tag{4.33}$$

Where $C = C(p_1, p_2, \Omega, C_{log}, N)$ and γ and β are given in Definitions 4.2 and 4.3, respectively.

Proof. It follows by Lemma 4.4, taking $v = u|u|^{-1}_{W^{1,p(\cdot)}(\mathcal{T}_h)}$. □

Remark 4.6. Under the same hypothesis of the last corollary, if $1 \leq q \leq p^*$ in Ω, we have that, for all $u \in S^k(\mathcal{T}_h)$,

$$\|u - Q_h(u)\|_{L^{q(\cdot)}(\Omega)} \leq Ch^{N(1-\gamma)}|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \quad \text{and} \quad \|\nabla Q_h(u)\|_{L^{p(\cdot)}(\Omega)} \leq C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)},$$

where $C = C(p_1, p_2, \Omega, C_{log}, N)$.

5. The lifting operator

We begin this section by defining, as in [6] (see also [1]), the lifting operator, i.e.

Definition 5.1. Let $l \geq 0$ and $R_h \colon W^{1,p(\cdot)}(\mathcal{T}_h) \to S^l(\mathcal{T}_h)^N$ defined as,

$$\int_\Omega \langle R_h(u), \phi\rangle \, dx = -\int_{\Gamma_{int}} \langle [\![u]\!], \{\phi\}\rangle \, dS \quad \forall \phi \in S^l(\mathcal{T}_h)^N.$$

This operator appears in the first term of the discretized functional I_h. As we can see from the definition, this operator represents the contribution of the jumps to the distributional gradient. This is the reason why it is crucial to add this term in order to have the consistency of the method.

We point out that this lifting operator was first used in [2] in order to describe the contributions of the jumps across the interelements of the computed solution on the (computed) gradient of the solution in a mixed formulation of Navier-Stokes equations. It was also used in [5] where a solid mathematical background for the method introduced in [2] was proposed.

Now, we give a bound of the $L^{p(\cdot)}(\Omega)$-norm of $R_h(u)$ in terms of the jumps of u in Γ_{int}.

When p is constant the proof follows from an inf-sup condition. Since in our case, we are dealing with the Luxemburg norm, we can't prove the boundedness directly from the definition. We can prove this inf-sup condition, but we can not use it to prove the result. Instead, we find a local characterization of R_h in order to prove a local bound and then, we prove the global bound.

We give first the local estimate.

Lemma 5.2. *There exists a constant C_1 such that, for any $\kappa \in \mathcal{T}_h$, we have*

$$\|R_h(u)\|_{L^{p(\cdot)}(\kappa)} \leq C\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof. We proceed in two steps.

Step 1. We first want to prove that,

$$|R_h(u)| \leq \frac{C}{h_\kappa^N} \sum_{e\subset\kappa} \int_e |[\![u]\!]| \, dS \quad \forall \kappa \in \mathcal{T}_h \tag{5.34}$$

where C is independent of κ and h.

We begin by observing that, by Hypothesis 3.1, there exists $m = m(k, N) \in \mathbb{N}$ such that for each $\kappa \in \mathcal{T}_h$,

$$R_h(u)|_\kappa \circ F_\kappa = \sum_{i=1}^m a_i \varphi_i(x),$$

where $\{\varphi_i\}$ is the standard nodal base of $\left(P^l\right)^N$ in the reference element $\hat{\kappa} := F_\kappa^{-1}(\kappa)$.

Using the definition of R_h we have that for each $1 \leq j \leq m$,

$$\int_\Omega R_h(u)\varphi_j \circ F_\kappa^{-1}(x)\,dx = \sum_{i=1}^m a_i \int_\kappa \varphi_i \circ F_\kappa^{-1}(x)\varphi_j \circ F_\kappa^{-1}(x)\,dx = -\sum_{e\subset\kappa}\int_e [\![u]\!]\{\varphi_j \circ F_\kappa^{-1}(x)\}\,dS.$$

On the other hand, if we change variables and we use Hypothesis 3.2 and the fact that $|\varphi_i(x)| \leq 1$, we get

$$\int_\kappa \varphi_i \circ F_\kappa^{-1}(x)\varphi_j \circ F_\kappa^{-1}(x)\,dx = h_\kappa^N \int_{\hat{\kappa}} \varphi_i(x)\varphi_j(x)\frac{|\det(DF_\kappa)|}{{h_\kappa}^N}\,dx = h_\kappa^N d_{ij}$$

with $d_{ij} \sim 1$.

Therefore,

$$R_h(u)|_\kappa \circ F_\kappa = \frac{1}{h_\kappa^N}\sum_{i=1}^m (D^{-1}b)_i\varphi_i(x)\,dx,$$

where $D = (d_{ij})$ and $b_j = -\sum_{e\subset\kappa}\int_e [\![u]\!]\{\varphi_j \circ F_\kappa^{-1}(x)\}\,dS$.

Thus, using that $|\varphi_i(x)| \leq 1$, we arrive at (5.34).

Step 2. Now, we show that there exists a constant C_1 such that, for any $\kappa \in \mathcal{T}_h$, we have

$$\|R_h(u)\|_{L^{p(\cdot)}(\kappa)} \leq C\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\kappa\cap\Gamma_{int})} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

By inequality (4.19), we have

$$\int_e |[\![u]\!]|\,ds \leq C{h_e}^{N(1-\frac{1}{p_-})}\|[\![u]\!]{h_e}^{-\frac{1}{p'(x)}}\|_{L^{p(\cdot)}(e)}.$$

Thus, by Hypothesis 3.2 and (5.34), we have that

$$|R_h(u)| \leq \frac{C}{h_\kappa^{N/p_-}}\sum_{e\subset\kappa}\|[\![u]\!]{h_e}^{-\frac{1}{p'(x)}}\|_{L^{p(\cdot)}(e)}.$$

Now, take $T = \sum_{e\in\kappa}\|[\![u]\!]{h_e}^{-\frac{1}{p'(x)}}\|_{L^{p(\cdot)}(e)}$. Then,

$$\int_\kappa \Big|\frac{R_h(u)}{T}\Big|^{p(x)}\,dx \leq C\int_\kappa h_\kappa^{-Np(x)/p_-}\,dx \leq Ch_\kappa^{N(1-p_+/p_-)} \leq C$$

where in the last inequality we are using Proposition 2.11.

The result follows now by Remark 3.3. □

Lemma 5.3. *Let $p : \Omega \to : [1,\infty)$ be log-Hölder continuous in Ω. Then, there exists a constant C such that,*

$$\|R_h(u)\|_{L^{p(\cdot)}(\Omega)} \leq C\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})} \quad \forall u \in W^{1,p(\cdot)}(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof. First, if we assume that $\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})} \leq 1$, we can prove using Lemma 5.2 and proceeding as in Lemma 4.4 that,

$$\int_\Omega |R_h(u)|^{p(x)}\,dx \leq C.$$

Finally, taking $v = u\left(\|\mathbf{h}^{-1/p'(x)}[\![u]\!]\|_{L^{p(\cdot)}(\Gamma_{int})}\right)^{-1}$, we obtain the desired result. □

6. Convergence of the method

In this section we first prove the broken Poincaré Sobolev inequality which is crucial to get compactness. We also prove the coercivity of the functional and we finally arrive at the proof of Theorem 1.1.

Theorem 6.1. *Let $p : \Omega \to [1, +\infty)$ be $\log-$Hölder continuous in $\overline{\Omega}$. There exists a constant C such that,*

$$\|u - (u)_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \quad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1],$$

where $(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$. In particular,

$$\|u\|_{L^{p^*(\cdot)}(\Omega)} \leq C\Big(\|u\|_{L^1(\Omega)} + |u|_{W^{1,p(\cdot)}(\mathcal{T}_h)}\Big) \quad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

Proof. We begin by observing that

$$\|u - (u)_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq \|u - Q_h(u)\|_{L^{p^*(\cdot)}(\Omega)} + \|Q_h(u) - (Q_h(u))_\Omega\|_{L^{p^*(\cdot)}(\Omega)} + C\|Q_h(u) - u\|_{L^1(\Omega)}.$$

Then, using the Remark 4.6 and Theorem 2.8, we have

$$\|u - (u)_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq C|u|_{W^{1,p(\cdot)}(\mathcal{T}_h)} \quad \forall u \in S^k(\mathcal{T}_h) \quad \forall h \in (0,1].$$

The proof is complete. □

Theorem 6.2. *For each $h \in (0,1]$, let $u_h \in W^{1,p(\cdot)}(\mathcal{T}_h)$. If there exists a constant C independent of h such that $I_h(u_h) \leq C$ for all $h \in (0,1]$, then*

$$\sup_{h\in(0,1]} \Big(\|u_h\|_{L^1(\Omega)} + |u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}\Big) < \infty.$$

Moreover,

$$\sup_{h\in(0,1]} \int_{\partial\Omega} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS < \infty.$$

Proof. Since $I_h(u_h) \leq C$ then, $\|\mathbf{h}^{-1/p'(x)}[\![u_h]\!]\|_{L^{p(\cdot)}(\Gamma_{int})} \leq C$. And, by Lemma 5.3 and Proposition 2.1, we have

$$\int_\Omega |R_h(u_h)|^{p(x)}\, dx \leq C.$$

Using the third inequality in Proposition 2.5 and the above inequality, we get

$$\int_\Omega |R_h(u_h) + \nabla u_h|^{p(x)}\, dx \geq 2^{1-p_2} \int_\Omega |\nabla u_h|^{p(x)}\, dx - C.$$

Therefore,

$$I_h(u_h) + C \geq 2^{1-p_2} \int_\Omega |\nabla u_h|^{p(x)}\, dx + \int_{\Gamma_D} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS + \int_{\Gamma_{int}} |[\![u_h]\!]|^{p(x)} \mathbf{h}^{1-p(x)}\, dS.$$

Thus, as $I_h(u_h) \leq C$, we obtain that $|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}$ and $\displaystyle\int_{\partial\Omega} |u_h - u_D|^{p(x)} \mathbf{h}^{1-p(x)}\, dS$ are uniformly bounded.

Finally, by Friedrichs inequality for BV, Lemma 3.6, Hölder inequality, Proposition 2.1 and the fact that $\mathbf{h} \leq 1$ we have,

$$\begin{aligned}\|u_h\|_{L^1(\Omega)} \leq & C\left(|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \int_{\Gamma_D} |u_h|\, dS\right)\\ \leq & C\left(|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \int_{\Gamma_D} |u_D|\, dS + \|(u_h - u_D)\mathbf{h}^{-1/p'(x)}\|_{L^{p(\cdot)}(\Gamma_D)}\|\mathbf{h}^{1/p'(x)}\|_{L^{p'(\cdot)}(\Gamma_D)}\right)\\ \leq & C\left(|u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)} + \int_{\Gamma_D} |u_D|\, dS + \|(u_h - u_D)\mathbf{h}^{-1/p'(x)}\|_{L^{p(\cdot)}(\Gamma_D)}\right).\end{aligned}$$

This completes the proof. □

Lemma 6.3. *Let p, s and t be functions satisfying (H1), (H2) and (H3) respectively. Let $u_h \in S^k(\mathcal{T}_h)$ be under the conditions of Theorem 6.2. Then, there exist a sequence $h_j \to 0$ and a function $u \in W^{1,p(\cdot)}(\Omega)$ such that*

$$u_{h_j} \overset{*}{\rightharpoonup} u \quad \text{weakly* in } BV(\Omega) \tag{6.35}$$

$$\nabla u_{h_j} + R_h(u_{h_j}) \rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega), \tag{6.36}$$

$$u_{h_j} \to u \quad \text{strongly in } L^{s(\cdot)}(\Omega), \tag{6.37}$$

$$u_{h_j} \to u \quad \text{strongly in } L^{t(\cdot)}(\partial\Omega). \tag{6.38}$$

Proof. We first observe that, by Theorem 6.2, we have that

$$\sup_{h\in(0,1]} (\|u_h\|_{L^1(\Omega)} + |u_h|_{W^{1,p(\cdot)}(\mathcal{T}_h)}) < \infty \quad \forall h \in (0,1].$$

Then, the proofs of (6.35) and (6.36) follows by applying Theorem 5.2 in [6] with value p_1, and using that $R_h(u_h)$ is bounded in $L^{p(\cdot)}(\Omega)$, see Lemma 5.3.

We now prove (6.37). By (6.35) and the compactness of the embedding $BV(\Omega) \subset L^1(\Omega)$, there exists a subsequence of u_{h_j}, still denoted by u_{h_j}, such that

$$u_{h_j} \to u \text{ in } L^1(\Omega).$$

Since $\|u_{h_j}\|_{L^1} + |u_{h_j}|_{W^{1,p(\cdot)}(\mathcal{T}_h)}$ is bounded, by Theorem 6.1, $\|u_{h_j}\|_{L^{p^*(\cdot)}(\Omega)}$ is bounded, and by Theorem 2.9, $u \in W^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$. Therefore, using Theorem 2.3, we obtain that

$$u_{h_j} \to u \quad \text{in } L^{s(\cdot)}(\Omega), \tag{6.39}$$

for all s satisfying (H2).

Finally, we prove (6.38). We begin by observing that, by Corollary 4.5,

$$\|u_h - Q_h(u_h)\|_{L^{p(\cdot)}(\Omega)} \to 0 \text{ as } h \to 0 \tag{6.40}$$

and $\{Q_h(u_h)\}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Then, there exists $v \in W^{1,p(\cdot)}(\Omega)$ and subsequence $\{Q_{h_j}(u_{h_j})\}$ such that $Q_{h_j}(u_{h_j}) \rightharpoonup v$ weakly in $W^{1,p(\cdot)}(\Omega)$. Therefore, by (6.39) and (6.40), $v = u$.

Using Theorem 2.10,

$$Q_{h_j}(u_{h_j}) \to u \text{ strongly in } L^{t(\cdot)}(\partial\Omega). \tag{6.41}$$

Now, taking $\bar{t} : \Omega \to [1,\infty)$ log-Hölder with $t \leq \bar{t} < p_*$ and by Corollary 4.5, we get $Q_{h_j}(u_{h_j}) - u_{h_j} \to 0$ strongly in $L^{t(\cdot)}(\partial\Omega)$. Therefore, $u_{h_j} \to u$ in $L^{t(\cdot)}(\partial\Omega)$. □

Before proving the convergence of the minimizers, we need an auxiliary lemma. It is in this step where we need more regularity of the boundary data.

Lemma 6.4. *Let $h \in (0,1]$, and $p : \Omega \to (1,\infty)$ satisfying (H1). Assume that $u_D \in W^{2,p_2}(\Omega)$ and let $v \in W^{2,p_2}(\Omega) \cap \mathcal{A}$ then, there exists $v_h \in U^1(\mathcal{T}_h)$, such that*

$$\|v_h - v\|_{W^{1,p(\cdot)}(\Omega)} \to 0 \quad \text{as } h \to 0,$$

and

$$I_h(v_h) \to I(v) \quad \text{as } h \to 0.$$

Proof. Since p is log-Hölder, we have that $C^\infty(\bar{\Omega})$ are dense in $W^{1,p(\cdot)}(\Omega)$. Then the first part follows by standard approximation theory, see in Theorem 3.1.5 [9].

Moreover, v_h satisfies

$$\|v - v_h\|_{L^{p_2}(\partial\kappa)} \le C\|v - v_h\|_{W^{1,p_2}(\kappa)} \le Ch_\kappa \|D^2 v\|_{L^{p_2}(\kappa)} \tag{6.42}$$

for each $\kappa \in \mathcal{T}_h$. Using Remark 3.4 and summing over all $e \in \partial\Omega$, we have

$$\int_{\partial\Omega} |v - v_h|^{p_2} \mathbf{h}^{1-p_2}\, ds \le Ch\|D^2 v\|^{p_2}_{L^{p_2}(\Omega)}. \tag{6.43}$$

In addition, by Hölder inequality and since $h \le 1$, we have

$$\int_{\partial\Omega} |v - v_h|^{p(x)} \mathbf{h}^{1-p(x)}\, ds \le C\||v - v_h|^{p(\cdot)} \mathbf{h}^{(1-p_2)p(\cdot)/p_2}\|_{L^{p_2/p(\cdot)}(\partial\Omega)}.$$

Since

$$\int_{\partial\Omega} (|v - v_h|^{p(x)} \mathbf{h}^{(1-p_2)p(x)/p_2})^{p_2/p(x)}\, ds = \int_{\partial\Omega} |v - v_h|^{p_2} \mathbf{h}^{(1-p_2)}\, ds,$$

then, by (6.43)

$$\int_{\partial\Omega} |v - v_h|^{p(x)} \mathbf{h}^{1-p(x)}\, ds \to 0 \quad \text{as } h \to 0.$$

Since $v_h \in W^{1,p(\cdot)}(\Omega)$ then $[\![v_h]\!] = 0$ and $R_h(v_h) = 0$. Finally, using (6.42) and Theorem 2.6, we obtain the desired result. □

Now, we are in a condition to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 6.4 there exist $w_h \in U^1(\mathcal{T}_h)$ such that $w_h \to u_D$ strongly in $W^{1,p(\cdot)}(\Omega)$ and $I_h(w_h) \to I(u_D)$. Therefore, since $I_h(u_h) \le I_h(w_h)$, we have that $I_h(u_h)$ is bounded.

Then, by Theorem 6.2 and Lemma 6.3 there exist a subsequence u_{h_j} and $u \in W^{1,p(\cdot)}(\Omega)$ such that

$$\begin{aligned}
u_{h_j} &\overset{*}{\rightharpoonup} u \quad \text{weakly* in } BV(\Omega),\\
\nabla u_{h_j} + R_h(u_{h_j}) &\rightharpoonup \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega),\\
u_{h_j} &\to u \quad \text{strongly in } L^{s(\cdot)}(\Omega), \quad \forall s \text{ satisfying } (H2)\\
u_{h_j} &\to u \quad \text{strongly in } L^{t(\cdot)}(\partial\Omega), \quad \forall t \text{ satisfying } (H3).
\end{aligned} \tag{6.44}$$

On the other hand, since the penalty term,

$$\int_{\Gamma_D} \mathbf{h}^{1-p} |u_h - u_D|^p\, dS$$

is bounded, we have that

$$\|u-u_D\|_{L^{p(\cdot)}(\Gamma_D)} \leq \|u-u_{h_j}\|_{L^{p(\cdot)}(\Gamma_D)} + \|u_{h_j}-u_D\|_{L^{p(\cdot)}(\Gamma_D)} \to 0.$$

Then $u \in \mathcal{A}$.

Taking $s=q$ and $t=r$ in (6.44), by Proposition 2.6 we have

$$\begin{aligned} I(u) &\leq \liminf_{j\to\infty} \left[\int_\Omega \left(|\nabla u_{h_j} + R_h(u_{h_j})|^{p(x)} + |u_{h_j}-\xi|^{q(x)} \right) dx + \int_{\Gamma_N} |u_{h_j}|^{r(x)}\, dS \right] \\ &\leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}). \end{aligned} \tag{6.45}$$

Now, we want to prove that u is the minimizer of I. Let $v \in \mathcal{A}\cap W^{2,p_2}(\Omega)$, and let $v_h \in U^1(\mathcal{T}_h)$ as in Lemma 6.4. Then $I_h(v_h) \to I(v)$. Therefore, by (6.45)

$$I(u) \leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}) \leq \lim_{j\to\infty} I_{h_j}(v_{h_j}) = I(v). \tag{6.46}$$

Now, let $w \in \mathcal{A}$, then for any $\varepsilon > 0$ there exists $v \in \mathcal{A}\cap W^{2,p_2}(\Omega)$ such that $\|v-w\|_{W^{1,p(\cdot)}(\Omega)} < \varepsilon$. By Theorem 2.6 we have that $I(v) < I(w) + \varepsilon$, therefore by (6.46)

$$I(u) \leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}) \leq I(w) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get

$$I(u) \leq \liminf_{j\to\infty} I_{h_j}(u_{h_j}) \leq \limsup_{j\to\infty} I_{h_j}(u_{h_j}) \leq I(w) \quad \forall\, w \in \mathcal{A}.$$

Therefore $I(u) \leq I(w)$.

Moreover, taking $w=u$, we have that all the inequalities in (6.45) are equalities, therefore we have that $I_{h_j}(u_{h_j}) \to I(u)$. Then

$$\int_{\Gamma_{int}} |[\![u_{h_j}]\!]|^{p(x)} \mathbf{h_j}^{1-p(x)}\, dS \to 0$$

and using Lemma 5.3 we have that $R_h(u_{h_j}) \to 0$. This fact and (6.44) imply that $\nabla u_{h_j} \rightharpoonup \nabla u$ weakly in $L^{p(\cdot)}(\Omega)$.

Since u is the unique minimizer of I, the whole sequence u_h converges to u.

Finally, since

$$\nabla u_h + R_h(u_h) \rightharpoonup \nabla u \text{ weakly in } L^{p(\cdot)}(\Omega) \text{ and } \int_\Omega (|\nabla u_h + R_h(u_h)|^{p(x)}\, dx \to \int_\Omega |\nabla u|^{p(x)}\, dx,$$

by Proposition 2.6, $\nabla u_h + R_h(u_h) \to \nabla u$ strongly in $L^{p(\cdot)}(\Omega)$. Therefore, since $R_h(u_h) \to 0$ strongly in $L^{p(\cdot)}(\Omega)$, we get that $\nabla u_h \to \nabla u$ strongly in $L^{p(\cdot)}(\Omega)$. □

7. The Continuous Galerkin Method

In order to make a complete study of this problem, we prove the convergence of the Continuous Galerkin finite element method for our problem. In the next section, we make a comparison of the two methods in an example.

For simplicity, we take the following functional:

$$I(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u-\xi|^{q(x)}}{q(x)} \right) dx$$

with $q(x)$ satisfying (H2). Then, since the functional I is strictly convex and coercive in $\mathcal{A}$ there exists a unique minimizer of the problem.

We take now a partition of Ω as in Hypothesis 3.1 and the usual conforming subspace $U^k(\mathcal{T}_h)$ of $W^{1,p(\cdot)}(\Omega)$. This subspace consists of all continuous functions such that they are polynomials of degree at most k in each $\kappa \in \mathcal{T}_h$. We assume that for some h', $u_D \in U^k(\mathcal{T}_{h'})$ (this assumption replaces the one on Lemma 6.4).

Let now $h \leq h'$ and

$$V_h^k = \{v_h \in U^k(\mathcal{T}_h) \colon v_h = u_D \text{ on } \partial\Omega\}.$$

For simplicity, we may assume that $h' = 1$.

Remark 7.1. Let $\Pi_h : C_0^\infty(\Omega) \to U_h^k$ be the interpolant mapping defined in Theorem 3.1.5 in [9]. Then,we have that

$$\|\Pi_h\phi - \phi\|_{W^{1,p(\cdot)}(\Omega)} \leq C\|\Pi_h\phi - \phi\|_{W^{1,p_2}(\Omega)} \to 0,$$

for any $\phi \in C_0^\infty(\Omega)$. We also have, by the continuity of I, that $I(\Pi_h\phi + u_D) \to I(\phi + u_D)$ as $h \to 0$.

By the strict convexity of I, for each $h \in (0, 1]$ there exists a function $u_h \in V_h^k$ such that u_h is a minimizer in V_h^k of I.

Now we prove the main result of this section.

Theorem 7.2. *The sequence $\{u_h\}$ converges to u strongly in $W^{1,p(\cdot)}(\Omega)$, where u is the unique minimizer of I.*

Proof. Since $\{I(u_h)\}$ is uniformly bounded, there exists a subsequence of $\{u_h\}$ (still denoted by $\{u_h\}$) such that

$$u_h \rightharpoonup u \text{ weakly in } W^{1,p(\cdot)}(\Omega), \tag{7.47}$$

$$u_h \to u \text{ strongly in } L^{p(\cdot)}(\Omega). \tag{7.48}$$

As in the proof of Theorem 1.1, using Remark 7.1 instead of Lemma 6.4, we can prove that u is the minimizer of I and that $I(u_h) \to I(u)$ as $h \to 0$. By the convexity of I and (7.47), we have that

$$\int_\Omega |\nabla u_h|^{p(x)}\, dx \to \int_\Omega |\nabla u|^{p(x)}\, dx \quad \text{as } h \to 0. \tag{7.49}$$

Then, by (7.48) and (7.49), using Proposition 2.6 (3), we have that $\{u_h\}$ converges to u strongly in $W^{1,p(\cdot)}(\Omega)$.

Observe that, as in Theorem 1.1, we can conclude that the whole sequence $\{u_h\}$ converges to u strongly in $W^{1,p(\cdot)}(\Omega)$. □

8. One dimensional example

In this section, we give an example in one dimension. Our idea is to compare the Continuous Galerkin Finite Element Method (CGFEM) versus the Discontinuous Galerkin Finite Element Method (DGFEM). We will see in the following example that, if the function p attains values close to one, our method converges faster to the solution.

Let $\Omega = (-1,1)$, $0 < \varepsilon, a < 1$ and $p : [-1,1] \to [1,2]$ given by

$$p(x) = \begin{cases} \frac{1-\varepsilon}{a}|x| + 1 + \varepsilon & \text{if} \quad |x| \le a, \\ 2 & \text{if} \quad a \le |x| \le 1. \end{cases}$$

For this function $p(x)$ and for a given $B > 0$, we study the following problem,

$$\begin{cases} (|u'(x)|^{p(x)-1}u'(x))' = 0 & \text{in } (-1,1), \\ u(1) = -u(-1) = B. \end{cases} \tag{8.50}$$

We begin by observing that, since the operator is strictly monotone, we have a unique solution of (8.50). Moreover, the solution satisfies $|u'(x)| = C^{\frac{1}{(p(x)-1)}}$ for some constant $C > 0$. Therefore, $|u'(x)| > \max\{C, C^{1/\varepsilon}\}$ and, using that $u \in C^{1,\alpha}([-1,1])$, we have that u' does not change sign. Then, since $u(1) > u(-1)$ we obtain that $u'(x) > 0$.

Thus,

$$u(x) = C(x+1) - B \quad \text{if } -1 \le x \le -a, \tag{8.51}$$

$$u(x) = C(x-1) + B \quad \text{if} \quad a \le x \le 1. \tag{8.52}$$

Since p is even, we have that u is odd, so $u(0) = 0$. Therefore, $u(x) = \int_0^x C^{\frac{1}{p(s)-1}}\, ds$ for all $x \in [-a, a]$.

On the other hand, since the derivative of u at zero has modulus $C^{1/\varepsilon}$, if $C > 1$ we have

$$\lim_{\varepsilon \to 0} |u'(0)| = +\infty.$$

This is reasonable since we expect to have big derivative when p approaches the value one.

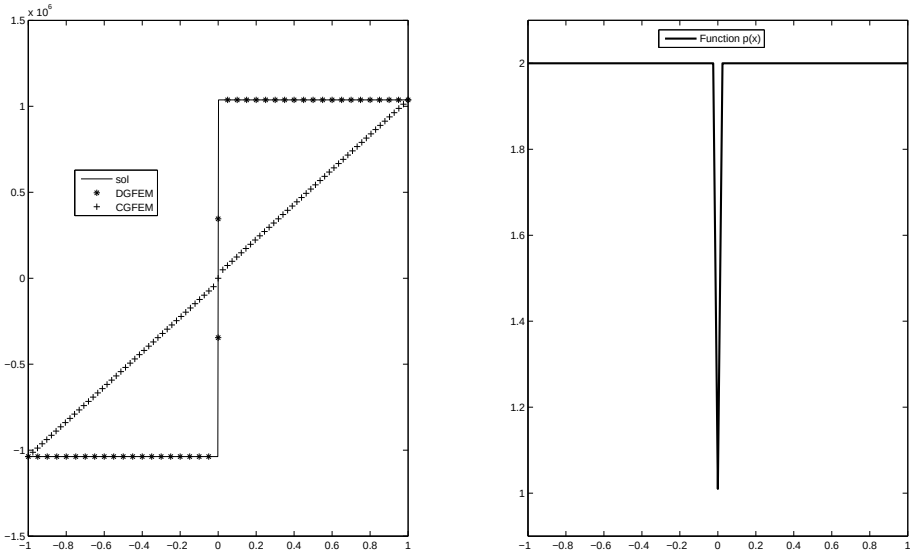
From now on, we take $\varepsilon = a = .01$ and since it is easier to get B from C, we impose $C = 1.3$. Then $B = \int_0^1 1.3^{\frac{100}{1+999s}}\, ds \simeq 1.03\ 10^6$. Observe that in this case, $|u'(0)| = 1.3^{100} \simeq 2.4 \times 10^{11}$.

Now, we find the corresponding solution for the CGFEM and the DGFEM. In both cases we take a uniform partition of $[-1,1]$ in n subintervals with size $2/n$ and $k = 1$. We use the trapezoidal numerical quadrature to compute the integrals appearing in the discrete functionals. The analysis of these integration errors falls beyond the scope of this paper.

Observe that for the continuous method, we impose the boundary conditions and then, the space where we find minimizers has dimension $n - 2$. For the Discontinuous method, since we do not impose conditions on the boundary, and the number of nodal basis are $2n - 2$, we are minimizing in a space of this dimension. Therefore, to make a comparison between both methods, we compare the discrete problem for the DGFEM in $n-$intervals with the CGFEM in $2n-$intervals.

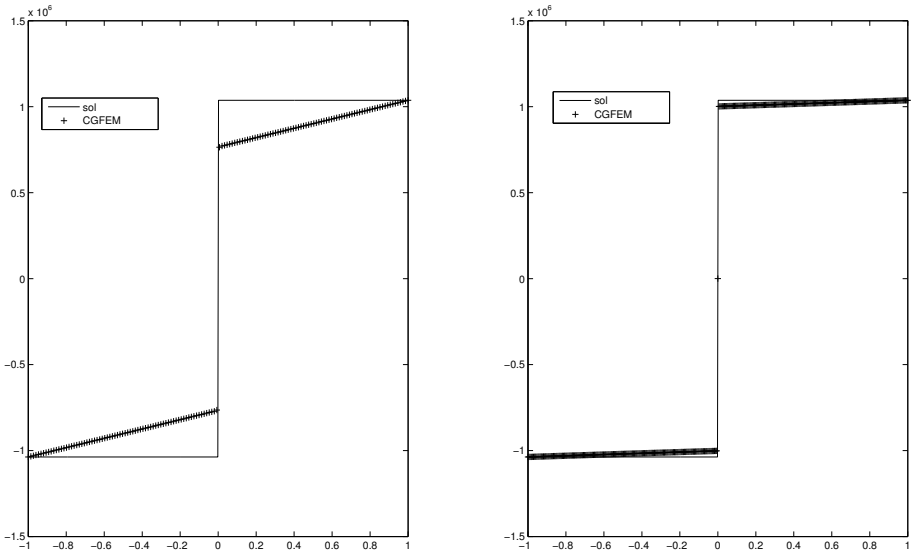
We want to mention that, in order to find minimizers of both discrete problems, we use a BFGS Quasi-Newton method (see [19, 25]).

In the next two figures, we first plot the solution versus the approximation using the DGFEM and CGFEM for the case $n = 41$. The second figure is the graphic of the function $p(x)$.



Note that, when we use the CGFEM the discrete solution is close to the function $y = x$ which is a solution of (8.50) with $p \equiv 2$, that means that this method needs a smaller step in order to see the points where p is close to one.

In the following figure we can see that, the minimizers of the continuous methods are far from the solution even for $n = 150$ (300-intervals). We need $n = 200$ (400-intervals) to arrive at a good approximation of u.



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