

HÖLDER FOLIATIONS, REVISITED

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ABSTRACT. We investigate transverse Hölder regularity of some canonical leaf conjugacies in normally hyperbolic dynamical systems and transverse Hölder regularity of some invariant foliations. Our results validate claims made elsewhere in the literature.

1. INTRODUCTION

A foliation \mathcal{F} that is normally hyperbolic and plaque expansive with respect to a diffeomorphism f is structurally stable in the following sense. For each C^1 small perturbation g of f there is a g -invariant foliation \mathcal{F}_g and a homeomorphism $h_g: M \rightarrow M$ sending the original foliation equivariantly to \mathcal{F}_g . Restricted to each \mathcal{F} -leaf, h_g is C^1 .

In [8] and [9] Damjanović and Katok assert that in the context of perturbations of Anosov actions (such as time one maps of Anosov flows) the homeomorphism h_g can be chosen to satisfy a Hölder condition. This does not follow from the standard fact proved by Anosov in [1] and the first two authors in [15] that the relevant g -invariant foliations are tangent to Hölder plane fields. It is a subtler issue, and is a consequence of Theorem A below.

In [16] Ilyashenko and Negut treat the case of skew products, such as perturbations of an Anosov diffeomorphism cross the identity. We generalize their result to uniformly compact laminations in Theorem B. See Section 3 for statements of Theorems A and B. The general question of when h_g is Hölder remains open.

2. BACKGROUND

A foliation \mathcal{F} of a manifold M is a division of M into disjoint submanifolds called leaves of the foliation with the following properties.

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- Each leaf is connected although it need not be a closed subset of the manifold.
- The leaves all have the same dimension, say c .
- For each point p in the manifold there exists a homeomorphism φ from $D^c \times D^{m-c}$ onto a neighborhood of p that carries each $D^c \times y$ to a subset of the leaf containing $\varphi(0, y)$.

D^c is the open c -dimensional disk, m is the dimension of the manifold M , and $m - c$ is the transverse dimension of the foliation. Such a φ is a *foliation box* and $\varphi(D^c \times y)$ is a *plaque* of the foliation.

DEFINITION. The *leaf topology* on M is generated by the plaques in an atlas of foliation boxes. It is denoted by (M, \mathcal{F}) . The leaves are the connected components of (M, \mathcal{F}) . The *leaf space* is the set of leaves. It is denoted by M/\mathcal{F} .

Each leaf is a c -dimensional manifold covered by plaque coordinate neighborhoods. The leaf topology (M, \mathcal{F}) projects to the discrete topology on M/\mathcal{F} .

The foliation \mathcal{F} is *f-invariant* by $f: M \rightarrow M$ if f permutes its leaves. That is,

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \pi \downarrow & & \downarrow \pi \\ M/\mathcal{F} & \xrightarrow{f} & M/\mathcal{F} \end{array}$$

commutes where π projects the point $p \in M$ to the leaf $\mathcal{F}(p)$ containing it.

DEFINITION. A *leaf conjugacy* from an f -invariant foliation \mathcal{F} to a g -invariant foliation \mathcal{G} is a homeomorphism $\mathfrak{h}: M \rightarrow M$ that sends \mathcal{F} -leaves to \mathcal{G} -leaves equivariantly in the sense that

$$\begin{array}{ccc} M/\mathcal{F} & \xrightarrow{f} & M/\mathcal{F} \\ \mathfrak{h} \downarrow & & \downarrow \mathfrak{h} \\ M/\mathcal{G} & \xrightarrow{g} & M/\mathcal{G} \end{array}$$

commutes. In other words, $\mathfrak{h}(f(\mathcal{F}(p))) = g(\mathcal{G}(\mathfrak{h}(p)))$.

A foliation is smooth if there exists a covering of the manifold by foliation boxes, each of which is a diffeomorphism. Smooth foliations are studied widely in differential topology, but in dynamics the naturally occurring foliations are only partially smooth. In [15] the term “lamination” is used for this kind of foliation. Here we suggest revised terminology.

DEFINITION. A foliation is *regular* if the manifold can be covered by foliation boxes $\varphi = \varphi(x, y)$ such that $\partial\varphi(x, y)/\partial x$ exists, is nonsingular, and depends continuously on $(x, y) \in D^c \times D^{m-c}$. A leaf conjugacy between regular invariant foliations is *regular* if its restriction to each leaf is C^1 , nonsingular, and these leaf derivatives are continuous on M .

The leaves of a regular foliation \mathcal{F} of M are C^1 and are assembled in a C^1 -continuous way. The vectors tangent to its leaves form a continuous subbundle $T\mathcal{F} \subset TM$. It is also natural to speak of a foliation being C^r regular for $r > 1$. Its leaves are C^r and are assembled C^r -continuously.

If \mathcal{F} is a regular foliation which is invariant under a diffeomorphism f then the tangent map $Tf: TM \rightarrow TM$ sends $T\mathcal{F}$ isomorphically to itself and the diagram

$$\begin{array}{ccc} T\mathcal{F} & \xrightarrow{Tf} & T\mathcal{F} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

commutes. (Here π is the projection $TM \rightarrow M$.)

DEFINITION. An f -invariant regular foliation \mathcal{F} of a compact manifold M is *normally hyperbolic* if the tangent bundle of M splits as a direct sum of continuous subbundles

$$TM = E^u \oplus E^c \oplus E^s$$

such that Tf carries E^u , $E^c = T\mathcal{F}$, and E^s to themselves isomorphically, and for some Riemann structure on TM we have

$$T_p^s f < 1 < T_p^u f \quad \text{and} \quad T_p^s f < T_p^c f < T_p^u f$$

for all $p \in M$. This is a shorthand expression where $T^u f$, $T^c f$, $T^s f$ are the restrictions of Tf to the subbundles E^u , E^c , E^s , and for linear transformations A, B we write $A < B$ and $A < c < B$ to indicate

$$\|A\| < \mathbf{m}(B) = \|B^{-1}\|^{-1} \quad \text{and} \quad \|A\| < c < \mathbf{m}(B).$$

$\mathbf{m}(B)$ is the *conorm* of B , the infimum of $|B(u)|$ as u varies over the unit vectors in the domain of B .

If we want to be more precise then we choose continuous functions $\mu, \nu, \hat{\nu}, \hat{\mu}: M \rightarrow (0, 1)$ and $\gamma, \hat{\gamma}: M \rightarrow (0, \infty)$ bracketting $T^s f, T^c f, T^u f$ in the sense that

$$\begin{aligned} \mu(p) &< T_p^s f < \nu(p) \\ \gamma(p) &< T_p^c f < (\hat{\gamma}(p))^{-1} \\ (\hat{\nu}(p))^{-1} &< T_p^u f < (\hat{\mu}(p))^{-1} \end{aligned}$$

and we choose them so that $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$.

A central result in [15] concerns perturbations of a normally hyperbolic foliation \mathcal{F} .

THEOREM 1. Foliation Stability *If \mathcal{F} is normally hyperbolic and plaque expansive (see below) then it is structurally stable in the following sense. For each diffeomorphism g that C^1 -approximates f there exists a unique g -invariant foliation \mathcal{F}_g near \mathcal{F} . The foliation \mathcal{F}_g is normally hyperbolic, plaque expansive, and (f, \mathcal{F}) is canonically leaf conjugate to (g, \mathcal{F}_g) by a homeomorphism $h: M \rightarrow M$.*

See Section 5 for more details about \mathfrak{h} and clarification of the word “canonically.”

DEFINITION. An f -invariant foliation is *plaque expansive* if there exist a plaquation (see below) \mathcal{P} of \mathcal{F} and a $\delta > 0$ such that any two δ -pseudo orbits of f that respect \mathcal{P} and δ -shadow each other necessarily belong to the same plaques of \mathcal{P} .

REMARK. A fundamental open question is whether normal hyperbolicity implies plaque expansivity. There are cases in which the implication is known, namely

- if the foliation \mathcal{F} is of class C^1 [15],
- or as shown by Carrasco, if the leaves of \mathcal{F} have uniformly bounded leaf volume [5],
- or as shown by Chillingworth and Hertz, Hertz and Ures, if $T^c f = Tf|_{T\mathcal{F}}$ is an isometry [7, 14],
- or, as shown by Hammerlindl, if the foliations \mathcal{W}^u and \mathcal{W}^s tangent to E^u and E^s are “undistorted,” for instance if M is the 3-torus [11].¹

Here is a more detailed description of plaque expansivity. In the first place it generalizes the concept that the map f is expansive, meaning there is a $\delta > 0$ such that for any distinct orbits $(f^n(x))$ and $(f^n(y))$ there exists a $k \in \mathbb{Z}$ with

$$d(f^k(x), f^k(y)) > \delta.$$

(d is a fixed metric on M .) In fact, if one considers the foliation of M by its own points then leaves are points, plaques are points, and the two concepts coincide.

The idea is to replace orbits of points by orbits of plaques. This is not quite possible because f need not send plaques to plaques. The f -image of a plaque may need to be adjusted (shrunk, stretched, or slid slightly along its leaf) in order to produce a new plaque of comparable size.

Formally, a *plaquation* of \mathcal{F} results from a choice of finitely many foliation boxes $\varphi: D^c \times D^{m-c} \rightarrow M$ such that the corresponding half size foliation boxes $\varphi(\frac{1}{2}D^c \times \frac{1}{2}D^{m-c})$ cover M . The plaquation \mathcal{P} consists of the unit size plaques $\varphi(D^c \times y)$. They cover the leaves of \mathcal{F} in a uniform fashion.

A δ -pseudo orbit of f is a bi-infinite sequence of points (x_n) such that for each $n \in \mathbb{Z}$, $d(f(x_n), x_{n+1}) < \delta$. It *respects the plaquation* \mathcal{P} if $f(x_n)$ and x_{n+1} always belong to a common plaque in \mathcal{P} . Plaque expansivity requires there to be a $\delta > 0$ such that if (x_n) and (y_n) are δ -pseudo orbits that respect \mathcal{P} and have $d(x_n, y_n) < \delta$ for all $n \in \mathbb{Z}$ then x_n and y_n always lie in a common plaque $\rho_n \in \mathcal{P}$.

Equivalently, plaque expansivity means there are a plaquation \mathcal{P} and a $\delta > 0$ such that for any sequences of plaques (ρ_n) and (σ_n) in \mathcal{P} with $f(\rho_n) \cap \rho_{n+1} \neq \emptyset$ and $f(\sigma_n) \cap \sigma_{n+1} \neq \emptyset$, either there exists an n such that the minimum distance

¹The definitions in the present paper are pointwise, not absolute as in [11], and do not imply undistortedness.

between ρ_n and σ_n exceeds δ or $\rho_n \cap \sigma_n \neq \emptyset$. In short, either plaque orbits spread apart to distance $> \delta$ or the plaques overlap.

It is not hard to see that plaque expansivity is independent of the metric d and the plaquation \mathcal{P} .

REMARK. More general than normal hyperbolicity of f is *partial hyperbolicity*. One assumes that TM has a Tf -invariant splitting $E^u \oplus E^c \oplus E^s$ as above, but E^c is not necessarily integrable. In this paper our focus is on normal hyperbolicity.

3. THEOREMS A AND B

The leaf conjugacy \mathfrak{h} in the Foliation Stability Theorem above is C^1 on leaves and the leaf derivative is transversely continuous, but what about general transverse regularity? Although \mathfrak{h} is not usually transversely differentiable [1], a natural guess would be that it can be chosen to satisfy a Hölder condition in the transverse direction. This is consistent with a remark of Jürgen Moser to the effect that *every conjugacy (and invariant structure) occurring naturally in smooth dynamics is Hölder*.

THEOREM A. *Suppose that $f: M \rightarrow M$ is normally hyperbolic at \mathcal{F} and the bundles E^{cu} , E^{cs} are of class C^1 . (This implies that $E^c = T\mathcal{F}$ is C^1 and therefore f is plaque expansive \mathcal{F} .) If g C^1 -approximates f then the canonical leaf conjugacy \mathfrak{h} in the Foliation Stability Theorem is biHölder-continuous. So are the holonomy maps along the leaves of the g -invariant foliations.*

BiHölder-continuity means what it says: The map and its inverse are Hölder-continuous. See Section 4 for estimates of the Hölder exponents.

Theorem B concerns laminations—foliations of compact sets. As defined in [15] a *lamination* \mathcal{L} of a compact set $\Lambda \subset M$ is a family of disjoint submanifolds (“leaves” of the lamination) whose union is Λ and which are assembled in a C^1 continuous fashion. That is, Λ is covered by “lamination boxes,” where a lamination box is a map $\varphi: D^c \times Y \rightarrow \Lambda$, Y is a fixed compact set, φ is a homeomorphism to a relatively open subset of Λ , and $\partial\varphi(x, y)/\partial x$ is nonsingular and continuous with respect to $(x, y) \in D^c \times Y$. The disks $D^c \times y$ are sent to plaques in the leaves. Normal hyperbolicity of a diffeomorphism at an invariant lamination is defined in the obvious way: $T_\Lambda M$ has a partially hyperbolic Tf -invariant splitting $E^u \oplus E^c \oplus E^s$ with $E^c = T\mathcal{L}$. An example is the orbits of an Axiom A flow on a basic set, such as a solenoid. As shown in [15] the invariant manifold theory and the Foliation Stability Theorem hold equally in the lamination case.

A simple type of lamination arises in the context of a skew product diffeomorphism $f: B \times Z \rightarrow B \times Z$ where B and Z are compact manifolds, and

$$f(b, z) = (f_0(b), f_1(b, z)).$$

Let Λ_0 be a hyperbolic set for f_0 with hyperbolic splitting $T_{\Lambda_0} B = E^u \oplus E^s$. Then $\Lambda = \Lambda_0 \times Z$ is a compact f -invariant set smoothly laminated by the compact manifolds $b \times Z$.

If the hyperbolicity of the base map f_0 dominates $\partial f_1(b, z)/\partial z$ then f is normally hyperbolic. Plaque expansiveness is automatic: Hyperbolicity of f_0 implies that f_0 -orbits separate, which in turn implies that f -orbits of leaves separate. Thus, f pseudo-orbits of plaques separate.

The following is the main result in [16], which was proved earlier in a somewhat more specific context by Nițică and Török [18].

THEOREM 2 ([16]). *For a normally hyperbolic skew product lamination as above, assume that E^u, E^s are trivial product bundles. Also assume that the hyperbolic set $\Lambda_0 \subset B$ has local product structure with respect to f_0 . If g C^1 -approximates f then the leaf conjugacy is Hölder and the holonomy maps along the leaves of the corresponding g -lamination are Hölder.*

A special case of Theorem 2 occurs when Z is a single point. The center unstable and center stable laminations are the unstable and stable laminations through the hyperbolic set Λ_0 of f_0 . The fact that their holonomy maps are Hölder was proved by Schmeling and Siegmund-Schultze in [21]. In particular, they showed that the stable and unstable foliations of an Anosov diffeomorphism have Hölder holonomy.

DEFINITION. A lamination whose leaves are compact and for which the leaf volume of the leaves is uniformly bounded is a *uniformly compact lamination*.

A skew product lamination is uniformly compact because its leaves are all the same, namely $b \times Z$.

Let \mathcal{L} be a normally hyperbolic lamination with splitting $E^u \oplus T\mathcal{L} \oplus E^s$. In [15] and elsewhere it is shown that there exist unique f -invariant local laminations \mathcal{W}^u and \mathcal{W}^s tangent at Λ to E^u and E^s . They are called the *strong* unstable and stable laminations, and are sometimes denoted as $\mathcal{W}^{uu}, \mathcal{W}^{ss}$. In this paper we denote them as $\mathcal{W}^u, \mathcal{W}^s$. Their leaves have plaques $W^u(p, r), W^s(p, r)$ for $p \in \Lambda$; f expands $W^u(p, r)$ across $W^u(f(p), r)$ and contracts $W^s(p, r)$ into $W^s(f(p), r)$. The *local center unstable manifold* and *local center stable manifold* of a leaf $L \in \mathcal{L}$ are

$$W^{cu}(L, r) = \bigcup_{p \in L} W^u(p, r) \quad \text{and} \quad W^{cs}(L, r) = \bigcup_{p \in L} W^s(p, r).$$

They are immersed but not necessarily embedded. Their plaques are

$$W^{cu}(p, r) = \bigcup_{q \in W^c(p, r)} W^u(q, r) \quad W^{cs}(p, r) = \bigcup_{q \in W^c(p, r)} W^s(q, r),$$

which depend continuously on p .

DEFINITION. A normally hyperbolic lamination \mathcal{L} is *dynamically coherent* if its center unstable and center stable plaques intersect in subplaques of the lamination. More precisely, if $z \in W^{cu}(p, r) \cap W^{cs}(q, r)$ then $z \in \Lambda$ and the intersection is an open subset of $\mathcal{L}(z)$.

REMARK. Dynamical coherence is automatic under the hypotheses of Theorem A and Theorem 2.

THEOREM B. *Suppose that $f: M \rightarrow M$ is normally hyperbolic at a dynamically coherent, uniformly compact lamination \mathcal{L} . Then \mathcal{L} is plaque expansive, the leaf conjugacy in the Lamination Stability Theorem is biHölder, and the leaf holonomies are Hölder.*

Of course Theorem B includes Theorem 2. See Section 4 for estimates of the Hölder exponents.

REMARK. The standard definition of dynamical coherence applies to foliations of the whole manifold, not to laminations of a subset. One assumes $f: M \rightarrow M$ is partially hyperbolic and its center unstable and center stable subbundles integrate to invariant foliations.² It follows that the leaves of these foliations intersect in a foliation at which f is normally hyperbolic. In contrast, the previous definition starts with a normally hyperbolic lamination and makes no assumption about integrability of the center unstable and center stable subbundles. After all, these bundles are only defined at the laminated set Λ , so it may not make sense to integrate them globally. But what about the case that the lamination is a foliation of M ?

PROPOSITION 3. *The two definitions of dynamical coherence are equivalent for foliations.*

Before we give the proof of this proposition, we remark that, in the case where the lamination \mathcal{L} is a foliation, the hypothesis of dynamical coherence can be dropped: it follows automatically from normal hyperbolicity and uniform compactness. This was recently proved by Bohnet in her PhD thesis (see Theorem 1.26 in [2]).

Proof. Suppose the foliation \mathcal{F} is dynamically coherent as defined above. The global center unstable manifolds $W^{cu}(L)$ for $L \in \mathcal{F}$ are tangent to E^{cu} , and we claim they foliate M . Suppose that $W^{cu}(L)$ intersects $W^{cu}(L')$ at $p \in L$. (Then the intersection contains $W^u(p)$.) Let ρ and ρ' be plaques of $W^{cu}(L)$ and $W^{cu}(L')$ at p . The new definition of dynamical coherence implies that $W^{cs}(p, r)$ meets these plaques in relatively open subsets of the \mathcal{F} -leaf through p , namely L . Therefore $L \cap W^{cu}(L')$ is relatively open in L . See Figure 1.

With respect to the leaf topology the intersection is closed in L . Since the leaves are connected, $L \subset W^{cu}(L')$. Symmetrically, $L' \subset W^{cu}(L)$, so $W^{cu}(L) = W^{cu}(L')$. Moreover, each $W^{cu}(L)$ is injectively immersed—it does not have any self-intersection—and thus the global center unstable manifolds form a foliation that integrates E^{cu} . Similarly the center stable manifolds form a foliation W^{cs} that integrates E^{cs} , so the new definition of dynamical coherence implies the standard one.

Conversely, suppose that f is partially hyperbolic and E^{cu} , E^{cs} integrate to foliations \mathcal{E}^{cu} , \mathcal{E}^{cs} . As shown in [4], the foliations are f -invariant, the intersection foliation \mathcal{F} subfoliates each of them, and f is normally hyperbolic at

²A current, frequently used definition of dynamical coherence does not include this invariance property. See [4].

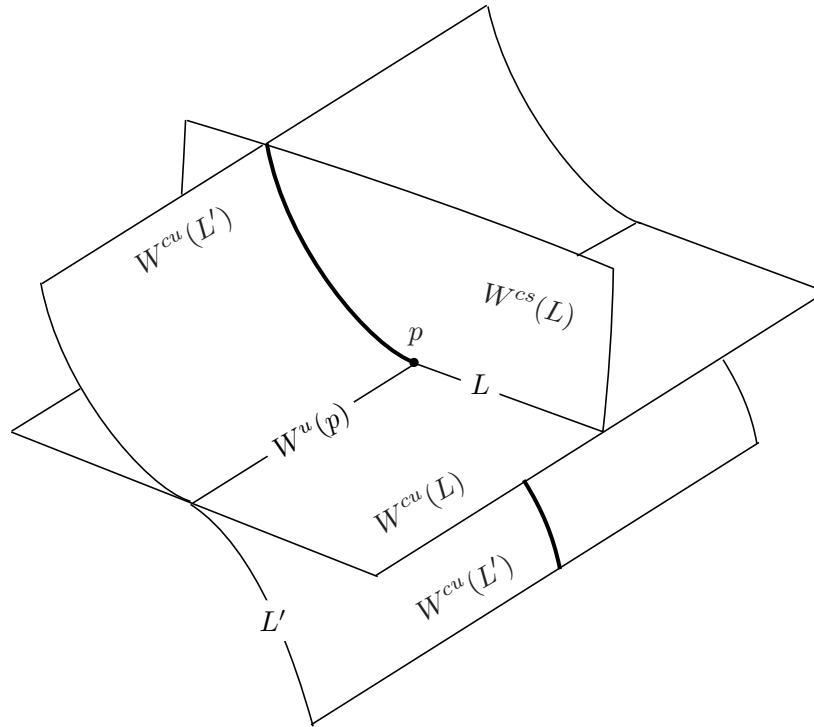


FIGURE 1. Locally, the intersection $W^{cu}(L') \cap W^{cs}(L)$ is a plaque of the \mathcal{F} -leaf L through p . It is the dark curve and the new dynamical coherence condition implies that actually it must equal L .

\mathcal{F} . Its normally hyperbolic splitting and its partially hyperbolic splitting coincide. By uniqueness of the strong unstable leaves (see [15]) W^u subfoliates \mathcal{E}^{cu} . Likewise W^s subfoliates \mathcal{W}^{cs} . Thus $\mathcal{E}^{cu} = \mathcal{W}^{cu}$ and $\mathcal{E}^{cs} = \mathcal{W}^{cs}$, which means the intersection foliation for \mathcal{W}^{cu} and \mathcal{W}^{cs} is \mathcal{F} , as required by the new definition of dynamical coherence. \square

Something of this survives for laminations.

PROPOSITION 4. *If \mathcal{L} is a normally hyperbolic, dynamically coherent lamination then its local center unstable manifolds meet in relatively open sets. So do its local center stable manifolds.*

Proof. The proof is the same as for foliations. For it is local. \square

4. THE HÖLDER EXPONENTS

A map $f: X \rightarrow Y$ from one metric space to another is θ -Hölder if there is a constant H such that for all $x, x' \in X$ we have

$$d_Y(f(x), f(x')) \leq H d_X(x, x')^\theta.$$

Bunching and separation conditions among the bracketing functions μ , ν , γ , $\hat{\gamma}$, $\hat{\nu}$, $\hat{\mu}$ described in Section 2 give estimates on the Hölder exponents of the normally hyperbolic summands, the leaf conjugacies, and the holonomies. Recall that

$$\begin{aligned}\mu(p) &< T_p^s f < \nu(p) \\ \gamma(p) &< T_p^c f < (\hat{\gamma}(p))^{-1} \\ (\hat{\nu}(p))^{-1} &< T_p^u f < (\hat{\mu}(p))^{-1}\end{aligned}$$

for all $p \in M$, and as functions

$$0 < \mu < \nu < 1 < \hat{\nu}^{-1} < \hat{\mu}^{-1} < \infty \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}.$$

First we describe the known Hölder results when f is C^2 . In [1, 15], and elsewhere (e.g., in the work of Hasselblatt [12]) it is shown that if f is a C^2 partially hyperbolic diffeomorphism then the summands in its splitting are Hölder. Specifically, Hölderness is implied by the bunching conditions

$$\begin{aligned}E^u &\text{ is } \theta\text{-Hölder when } \hat{\nu} < \hat{\gamma}\mu^\theta. \\ E^s &\text{ is } \theta\text{-Hölder when } \nu < \gamma\hat{\mu}^\theta. \\ E^{cu} &\text{ is } \theta\text{-Hölder when } \nu < \gamma\mu^\theta. \\ E^{cs} &\text{ is } \theta\text{-Hölder when } \hat{\nu} < \hat{\gamma}\hat{\mu}^\theta. \\ E^c &\text{ is } \theta\text{-Hölder when } \nu < \gamma\mu^\theta \text{ and } \hat{\nu} < \hat{\gamma}\hat{\mu}^\theta.\end{aligned}$$

The notation is chosen so the unstable conditions become the stable conditions by switching hats and non-hats. As shown in [12] and by Hasselblatt and Wilkinson in [13] the estimates are optimal in the C^2 case. The holonomy results for the strong foliations are similar: In [19] we show that if f is C^2 then

- \mathcal{W}^u has θ -Hölder holonomy when $\hat{\nu} < \hat{\gamma}\mu^\theta$.
- \mathcal{W}^s has θ -Hölder holonomy when $\nu < \gamma\hat{\mu}^\theta$.

If f is only C^1 then it makes little sense to hope the summands are Hölder. For they are Tf -invariant and Tf is only continuous.³ But it does make sense to ask whether holonomy is Hölder. For the invariant foliations are invariant under a C^1 diffeomorphism. In [24] Wilkinson shows that if f is C^1 then

- \mathcal{W}^u has θ -Hölder holonomy when $\hat{\nu} < \hat{\gamma}(\hat{\nu}\mu)^\theta$.
- \mathcal{W}^s has θ -Hölder holonomy when $\nu < \gamma(\nu\hat{\mu})^\theta$.

We believe these bunching conditions are optimal for the strong holonomies but we have no proof. We also have no proof that the other three types of holonomy (center unstable, center, and center stable) are Hölder in general. What we do prove in this paper are the following Hölder assertions when we perturb a normally hyperbolic diffeomorphism whose invariant foliations are C^1 or when the center foliation is uniformly compact.

If f is normally hyperbolic, dynamically coherent, its invariant foliations are of class C^1 , and g C^1 -approximates f then we will show

³In [13] it is shown that if the holonomy is Hölder then the bundles are Hölder, correcting an assertion in [19]. The converse is false, as shown by Wilkinson in [23].

- \mathcal{W}_g^{cu} has θ^2 -Hölder holonomy and the leaf conjugacy $\mathfrak{h}^{cu}: \mathcal{W}^{cu} \rightarrow \mathcal{W}_g^{cu}$ is θ -Hölder when $\nu < \mu^\theta$.
- \mathcal{W}_g^{cs} has θ^2 -Hölder holonomy and the leaf conjugacy $\mathfrak{h}^{cs}: \mathcal{W}^{cs} \rightarrow \mathcal{W}_g^{cs}$ is θ -Hölder when $\hat{\nu} < \hat{\mu}^\theta$.
- \mathcal{W}_g^c has θ^2 -Hölder holonomy and the leaf conjugacy $\mathfrak{h}^c: \mathcal{W}^c \rightarrow \mathcal{W}_g^c$ is θ -Hölder when $\nu < \mu^\theta$ and $\hat{\nu} < \hat{\mu}^\theta$.

(Recall that $\mathcal{W}^c = \mathcal{F}$ and $\mathcal{W}_g^c = \mathcal{F}_{g\cdot}$.) See Section 9 for the proofs.

If the center lamination of a C^1 normally hyperbolic, dynamically coherent diffeomorphism is uniformly compact then we will show that

- \mathcal{W}^c has θ -Hölder holonomy inside the center unstable leaves when $\hat{\nu} < \hat{\mu}^\theta$.
- \mathcal{W}^c has θ -Hölder holonomy inside the center stable leaves when $\nu < \mu^\theta$.
- \mathcal{W}^c has θ -Hölder holonomy when $\hat{\nu} < \hat{\mu}^\theta$ and $\nu < \mu^\theta$.

Furthermore, if g C^1 -approximates f then a canonical leaf conjugacy $\mathcal{L} \rightarrow \mathcal{L}_g$ is θ -Hölder when $\hat{\nu} < \hat{\mu}^\theta$ and $\nu < \mu^\theta$. See Section 10 for the proofs.

5. THE CANONICAL LEAF CONJUGACY

The leaf conjugacy in the Foliation Stability Theorem is constructed in [15] as follows. A smooth approximation \tilde{E} to $E^u \oplus E^s$ is chosen and exponentiated into M . This gives a smooth immersed tubular neighborhood $N(L, r)$ of each leaf $L \in \mathcal{F}$. It is the union of tubular fibers

$$N(p, r) = \exp \tilde{E}(p, r)$$

for $p \in L$. If r is uniformly small and distinct points p, q lie in a common plaque ρ of L then $N(p, r)$ and $N(q, r)$ are disjoint, but for points p, q in different plaques the fibers may meet badly.

Inside each $N(L, r)$ are local center unstable and center stable f -invariant manifolds that intersect in L . Applying graph transform ideas to the diffeomorphism g that approximates f we get local center unstable and center stable manifolds for g in $N(L, r)$. Their intersection is L_g . These L_g form a g -invariant foliation \mathcal{F}_g . It is the unique g -invariant foliation whose leaves approximate the leaves of \mathcal{F} .

The leaf conjugacy is a homeomorphism $\mathfrak{h}: M \rightarrow M$ sending L to L_g . Specifically it sends $p \in L$ to the unique point of the tubular fiber $N(p, r)$ whose g -orbit can be closely shadowed by an f pseudo-orbit that respects a fixed plaquation \mathcal{P} of \mathcal{F} . In terms of what \mathfrak{h} does to leaves, it is unique: $L_g = \mathfrak{h}(L)$ is uniquely determined by g and L . However, as a point map \mathfrak{h} depends on the choice of the tubular neighborhood structure $\mathcal{N} = \{N(p, r)\}$. A different choice of smooth approximation to $E^u \oplus E^s$ and a different choice of smooth exponential map give a different tubular neighborhood structure \mathcal{N}' , different tubular fibers, and consequently a different leaf conjugacy \mathfrak{h}' .

The relation between \mathfrak{h} and \mathfrak{h}' is simple. They are homotopic by a homotopy \mathfrak{h}_t that moves points a short distance in the plaques of \mathcal{F}_g . For $\mathfrak{h}(p)$ and $\mathfrak{h}'(p)$

belong to a common plaque ρ of $\mathfrak{h}(L)$, so we can draw the short geodesic $\gamma(t)$, $0 \leq t \leq 1$, from $\mathfrak{h}(p)$ to $\mathfrak{h}'(p)$ and project it to a path $\beta(t)$ in ρ using \mathcal{N} . The paths $\beta(t)$ give the homotopy.

So in this sense the leaf conjugacy is canonical: It is unique as a leaf map, it is unique as a plaque map, and as a point map it is unique up to a short plaque-preserving homotopy.

Finally, we weaken the smoothness of the tubular neighborhoods and speak also of laminations.

DEFINITION. A *tubular neighborhood structure* for a lamination \mathcal{L} is a choice of C^1 disks $N(p, r)$ which are uniformly approximately tangent to $E^u \oplus E^s$ at Λ such that for each plaque ρ in a plaquation of \mathcal{L} , the union of the tubular fibers $N(p, r)$ through points $p \in \rho$ forms a tubular neighborhood of ρ . If the tubular neighborhood structure results from exponentiating a smooth approximation to $E^u \oplus E^s$ it is said to be smooth.

The following summarizes to what extent leaf conjugacies for foliations and laminations are canonical.

PROPOSITION 5. *Suppose that f is normally hyperbolic and plaque expansive at the dynamically coherent lamination \mathcal{L} . Let \mathcal{N} and \mathcal{N}' be tubular neighborhood structures for \mathcal{L} . If g C^1 -approximates f then the leaf conjugacies $\mathcal{L} \rightarrow \mathcal{L}_g$ corresponding to \mathcal{N} and \mathcal{N}' are leaf canonical, plaque canonical, and homotopic by a short homotopy in the plaques of \mathcal{L}_g .*

6. HOLONOMY INTERSECTION

The following lemma is used to deduce properties of the center foliation from facts about the center unstable and center stable foliations.

LEMMA 6 (Holonomy Intersection). *If transverse regular foliations have θ -Hölder holonomy then so does the intersection foliation.*

Proof. Let \mathcal{F} and \mathcal{G} be the transverse foliations. The leaves of the intersection foliation \mathcal{H} are the connected components of the intersections of leaves of \mathcal{F} and \mathcal{G} . Take compact smooth local transversals τ and τ' to \mathcal{H} at x and x' such that x and x' belong to the same \mathcal{H} -leaf, $\mathcal{H}(x) = \mathcal{H}(x')$. The foliations \mathcal{F} and \mathcal{G} intersect τ in transverse foliations \mathcal{F}_τ and \mathcal{G}_τ of complementary dimensions. Their leaves meet in points. Likewise for τ' .

Let $h: \tau \rightarrow \tau'$ be an \mathcal{H} -holonomy map. It arises from choosing a path γ from x to x' in $\mathcal{H}(x)$, and then lifting γ to nearby \mathcal{H} -leaves. Since \mathcal{H} -leaves are contained in \mathcal{F} - and \mathcal{G} -leaves, h carries the leaves of \mathcal{F}_τ to leaves of $\mathcal{F}_{\tau'}$ and likewise for \mathcal{G} .

Transversality of \mathcal{F}_τ and \mathcal{G}_τ implies that there is a “foliation triangle inequality” for distance in τ , namely if $y = \mathcal{F}_\tau(p) \cap \mathcal{G}_\tau(q)$ then

$$\frac{1}{D} \max\{d_{\mathcal{F}_\tau}(p, y), d_{\mathcal{G}_\tau}(y, q)\} \leq d_\tau(p, q) \leq D(d_{\mathcal{F}_\tau}(p, y) + d_{\mathcal{G}_\tau}(y, q))$$

where D is a constant determined by the foliations and τ . A similar statement can be found in Proposition 19.1.1 of Katok and Hasselblatt’s book [17]. The distances are measured along τ or the intersection leaves. This does not use the fact that the intersection foliations are Hölder, but merely the fact that the angles between their leaves are bounded away from 0. A similar triangle inequality holds at τ' . See Figure 2.

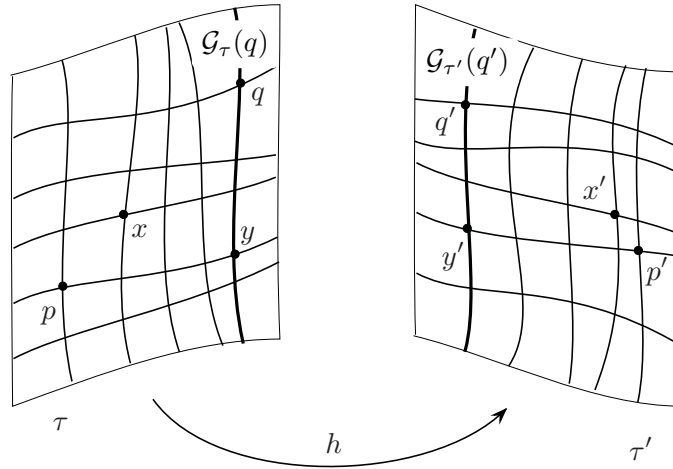


FIGURE 2. Transversality of the intersection foliations on τ and τ' implies modified triangle inequalities. The restriction of the \mathcal{H} -holonomy map h to $\mathcal{G}_\tau(q)$ is \mathcal{F} -holonomy.

τ is not a transversal to \mathcal{F} . Its dimension is wrong. Rather, $\mathcal{G}_\tau(q)$ is a transversal to \mathcal{F} at $q \in \tau$ and $\mathcal{G}_{\tau'}(q')$ is a transversal to \mathcal{F} at $q' = h(q) \in \tau'$. The map h restricted to $\mathcal{G}_\tau(q)$ is merely the \mathcal{F} -holonomy with respect to γ . Likewise for \mathcal{G} . These holonomy maps are Hölder. Thus, if $h(p) = p'$, $h(q) = q'$, $h(y) = y'$ as above, and $d_{\mathcal{F}_\tau}(p, y) \leq d_{\mathcal{G}_\tau}(y, q)$ then

$$\begin{aligned} d_{\tau'}(h(p), h(q)) &\leq D'(d_{\mathcal{F}_{\tau'}}(p', y') + d_{\mathcal{G}_{\tau'}}(y', q')) \\ &\leq D'(Cd_{\mathcal{F}_\tau}(p, y)^\theta + Cd_{\mathcal{G}_\tau}(y, q)^\theta) \\ &\leq 2CD'D^\theta d_\tau(p, q)^\theta \end{aligned}$$

shows that h is θ -Hölder. □

REMARK. In the proof of the Lemma 6 we did not need to know that all the holonomy maps of \mathcal{F} and \mathcal{G} are Hölder, only the ones that arise from \mathcal{H} -holonomy maps. It is possible that \mathcal{F} and \mathcal{G} have other holonomy maps which fail to be Hölder. This still permits the intersection foliation to be Hölder. See Remark 2 in Section 11.

A slight sharpening of the Lemma 6 replaces the assumption about \mathcal{F} and \mathcal{G} being Hölder by what was actually used in the proof, namely Hölderness of the slice maps. This also removes consideration of irrelevant holonomy maps. Likewise, the proof works just as well for laminations as for foliations.

THEOREM 7. *Suppose that \mathcal{F}, \mathcal{G} are transverse laminations which intersect in a lamination \mathcal{H} . If disks τ, τ' are transverse to \mathcal{H} and $h: \tau \rightarrow \tau'$ is a holonomy map along \mathcal{H} whose slice maps are uniformly θ -Hölder then h is θ -Hölder.*

Proof. The slice maps of h are its restrictions to the slices $\mathcal{F}_\tau(p)$ and $\mathcal{G}_\tau(q)$. It follows that h sends $\mathcal{F}_\tau(p)$ to $\mathcal{F}_{\tau'}(h(p))$ and similarly for \mathcal{G} . The inequality $d_{\tau'}(h(p), h(q)) \leq 2CD'D^\theta d_\tau(p, q)^\theta$ has exactly the same proof. \square

7. HOLONOMY VERSUS LEAF CONJUGACY

There is a natural relationship between leaf conjugacies arising in [15] and holonomies in certain skew products. It lets us deduce Hölderness of a leaf conjugacy in dimension n from Hölderness of holonomy maps of the suspended foliation in dimension $n + 1$. This suspension strategy cuts our work in half.

We consider compact manifolds T, M and a skew product diffeomorphism $G: T \times M \rightarrow T \times M$ covering the identity

$$G(t, x) = (t, g_t(x)).$$

$g_t: M \rightarrow M$ is the slice of G over t . In our application T is the circle or the segment. We assume that for some $0 \in T$,

- (a) $f = g_0$ is normally hyperbolic and plaque expansive at a foliation \mathcal{F} of M .
- (b) G C^1 -approximates the product diffeomorphism $F(t, x) = (t, f(x))$.

Then F is normally hyperbolic and plaque expansive at the product foliation $\mathcal{S} = T \times \mathcal{F}$ whose leaves are products $T \times L$ where L is a leaf of \mathcal{F} . Also, each g_t C^1 -approximates f . Fix a smooth bundle $E \subset TM$ complementary to $T\mathcal{F}$ and set $N(x, r) = \exp E(x, r)$. Then $\mathcal{N} = \{N(x, r) : x \in M\}$ is a convenient C^1 family of smooth disks transverse to the leaves of \mathcal{F} . Likewise $t \times N(x, r)$ is a small smooth disk through (t, x) in $T \times M$ transverse to the leaves of the product foliation \mathcal{S} .

We observe two things about these disks. First, by Topogonov's Triangle Theorem [6], there exists a uniform $r > 0$ such that $N(\rho, r) = \bigcup_{x \in \rho} N(x, r)$ is a tubular neighborhood of the plaque ρ . Its natural parameterization is the C^1 diffeomorphism $e_\rho: E(\rho, r) \rightarrow N(\rho, r)$ that sends (x, v) to $\exp_x(v)$. Second, e_ρ has uniformly bounded distortion. By this we mean that for some constant D and all plaques ρ , the map e_ρ neither expands distance by more than a factor D , nor contracts it by less than a factor $1/D$. This follows by further applications of the Toponogov Triangle Theorem. In particular, local holonomy maps from one plaque to another along the fibers of $N(\rho, r)$ have uniformly bounded distortion.

Then [15] implies two things, one about G and the other about g_t .

- (c) There is a unique G -invariant foliation \mathcal{S}_G near \mathcal{S} , and there is an equivariant leaf conjugacy $\mathfrak{h}: T \times M \rightarrow T \times M$ which approximates the identity map and sends \mathcal{S} -leaves to \mathcal{S}_G -leaves. In fact it sends $(t, x) \in T \times M$ to the unique point $(t, y) \in t \times N(x, r)$ whose G -orbit can be closely shadowed by an F pseudo-orbit that respects \mathcal{S} . Modulo the choice of \mathcal{N} , \mathfrak{h} is unique.

- (d) There is a unique g_t -invariant foliation \mathcal{F}_t near \mathcal{F} , and there is an equivariant leaf conjugacy $h_t: M \rightarrow M$ which approximates the identity map and sends \mathcal{F} -leaves to \mathcal{F}_t -leaves. In fact it sends $x \in M$ to the unique point $y \in N(x, r)$ whose g_t -orbit can be closely shadowed by an f pseudo-orbit that respects \mathcal{F} . Modulo the choice of \mathcal{N} , h_t is unique.

DEFINITION. The foliation \mathcal{S}_G is the *suspension foliation* for G .

Consider transversals $0 \times N(p, r)$ and $1 \times N(p, r)$ to an \mathcal{S} -leaf $T \times L$. For $x \in N(p, r)$ the straight line path $\sigma: t \mapsto (t, x)$ lies in the \mathcal{S} -leaf containing $(0, x)$. It lifts to a nearby path $t \mapsto (t, \alpha(t, x))$ in the \mathcal{S}_G -leaf through $(0, x)$ such that $\alpha(t, x) \in N(p, r)$ for $0 \leq t \leq 1$. The map $h_p: x \mapsto \alpha(1, x)$ is a *suspension holonomy map* for G . It sends $N(p, r')$ into $N(p, r)$ and is independent of the choice of α near σ . The radius r' is less than r so that $h_p(x)$ belongs to $N(p, r)$. (Actually, one should define the suspension holonomy as $H_p: (0, x) \rightarrow (1, h_p(x))$, but we abuse the concept for notational simplicity.) See Figure 3 in which σ actually “drops” to α .

If $x \in N(p, r') \cap N(q, r')$ then the straight line path σ from $(0, x)$ to $(1, x)$ lifts to two nearby paths

$$(t, \alpha(t)) \quad \text{and} \quad (t, \beta(t))$$

in the same \mathcal{S}_G -leaf. Thus $h_p(x)$ and $h_q(x)$ lie in the same \mathcal{F}_1 -plaque.

THEOREM 8. *The suspension leaf conjugacies are related by $h(t, p) = (t, h_t(p))$. The t -slice of \mathcal{S}_G equals \mathcal{F}_t , and h_t is a canonical leaf conjugacy $\mathcal{F} \rightarrow \mathcal{F}_t$. The suspension holonomy map satisfies*

$$h_p(x) = N(p, r) \cap h_1(\rho(x))$$

where $x \in N(p, r')$ and $\rho(x)$ is its \mathcal{F} -plaque. In particular, $h_p(p) = h_1(p)$. If the suspension holonomy maps h_p are uniformly θ -biHölder then so is the leaf conjugacy h_1 .

Proof. All this follows naturally from the dynamical characterization of leaf conjugacy. Here are the details. By (c), $h(t, p)$ is the unique $(t, y) \in t \times N(p, r)$ whose G -orbit $(t, g_t^n(y))$ is closely shadowed by an F pseudo-orbit (t_n, p_n) . Thus (p_n) is an f pseudo-orbit that closely shadows the g_t -orbit of $y \in N(p, r)$ and respects \mathcal{F} . By uniqueness in (d), $y = h_t(p)$, i.e., $h(t, p) = (t, h_t(p))$.

The leaves of \mathcal{S}_G approximate the product leaves and are transverse to the slice $t \times M$, so the t -slice of \mathcal{S}_G is a g_t -invariant foliation of M that approximates \mathcal{F} . By uniqueness it equals \mathcal{F}_t , and h_t is the canonical leaf conjugacy $\mathcal{F} \rightarrow \mathcal{F}_t$ with respect to the transversal family \mathcal{N} .

The leaf conjugacy h_1 sends \mathcal{F} -plaques to \mathcal{F}_1 -plaques. Since $h_p(x)$ and $h_x(x)$ lie in the same \mathcal{F}_1 -plaque, and since $h_x(x) = h_1(x)$ we see that $h_p(x)$ is the intersection of $N(p, r)$ with the \mathcal{F}_1 -plaque $h_1(\rho(x))$. See Figure 4.

By hypothesis, the holonomy maps h_p are uniformly θ -biHölder. We claim there is a constant H such that for all nearby $p, q \in M$ we have

$$\frac{d(p, q)^{1/\theta}}{H} \leq d(h_1(p), h_1(q)) \leq Hd(p, q)^\theta.$$

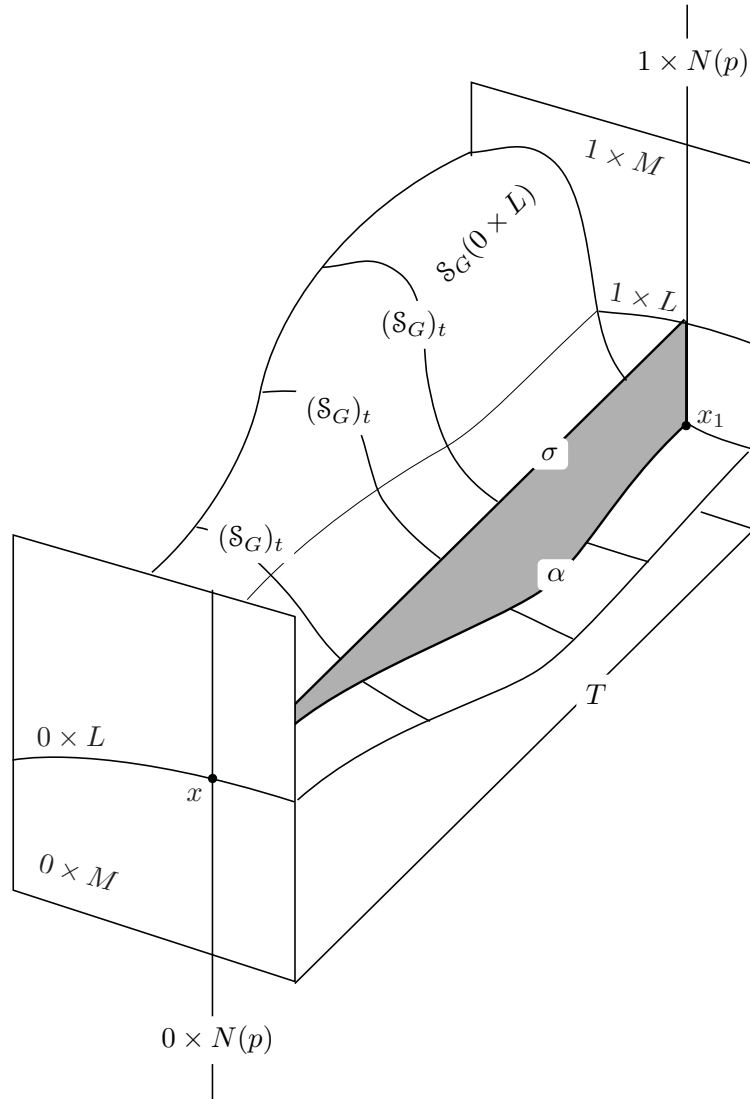


FIGURE 3. The suspension foliation \mathcal{S}_G and its suspension holonomy. The \mathcal{S}_G -leaf through $0 \times L$ is drawn. The slice $[0, 1] \times N(p)$ between σ and α is shaded. The holonomy sends x to x_1 .

Figure 5 indicates two geodesic triangles. They are effectively right triangles with hypotenuses shown as dotted lines. For their angles at x and $h_p(x)$ do not differ much from $\pi/2$. The ratio of the larger leg to the hypotenuse is bounded between $1/K$ and K where K is a constant.

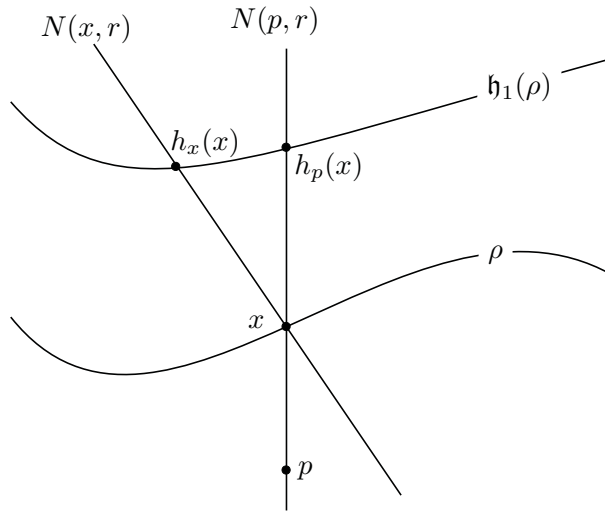


FIGURE 4. Both holonomy maps h_p and h_x send x into the same \mathcal{F}_1 -plaque, namely the plaque $h_1(\rho)$ through $h_1(x) = h_x(x)$.

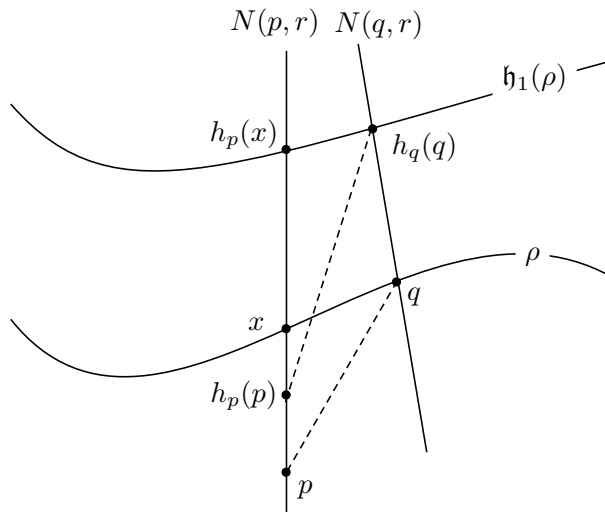


FIGURE 5. ρ is the plaque that contains q . It meets $N(p, r)$ at x . The triangles involved are essentially right triangles with the dotted lines as hypotenuses.

Thus

$$\begin{aligned}
 d(h_1(p), h_1(q)) &= d(h_p(p), h_q(q)) \\
 &\leq K \max\{d(h_p(p), h_p(x)), d(h_p(x), h_q(q))\} \\
 &\leq K \max\{H_0 d(p, x)^\theta, D d(x, q)\} \\
 &\leq K \max\{H_0, D\} \max\{d(p, x), d(x, q)\}^\theta \\
 &\leq K^{1+\theta} \max\{H_0, D\} d(p, q)^\theta
 \end{aligned}$$

where D bounds the distortion of the holonomy along $\mathcal{N}(\rho)$ from one plaque to another. Similarly

$$\begin{aligned} Kd(h_p(p), h_q(q)) &\geq \max\{d(h_p(p), h_p(x)), d(h_p(x), h_q(q))\} \\ &\geq \max\left\{\frac{d(p, x)^{1/\theta}}{H_0}, \frac{d(x, q)}{D}\right\} \\ &\geq \frac{\max\{d(p, x), d(x, q)\}^{1/\theta}}{\max\{H_0, D\}} \\ &\geq \frac{1}{K^{1/\theta} \max\{H_0, D\}} d(p, q)^{1/\theta}, \end{aligned}$$

which completes the proof that h_1 is θ -bihölder with biHölder constant $H = K^{1+1/\theta} \max\{H_0, D\}$. \square

REMARK. We have used the perturbation theory of [15] for a diffeomorphism of a manifold with boundary, namely $T \times M$ when $T = [0, 1]$. To avoid waiving our hands and saying that the whole theory in [15] works also on manifolds with boundary, it is simpler to replace $[0, 1]$ by the circle. The path of diffeomorphisms g_t with $0 \leq t \leq 1$ is replaced by a C^1 loop of diffeomorphisms, say g_t with $0 \leq t \leq 2$, all of which C^1 -approximate f . In this way F and G act on a compact manifold without boundary, namely $S^1 \times M$, and we get to treat the leaf conjugacies and holonomy as we did when $T = [0, 1]$.

ADDENDUM 9. *The preceding theorem holds also for laminations.*

Proof. The reasoning is exactly the same for laminations as for foliations. \square

8. A UNIFORM HÖLDER SECTION THEOREM

Consider a fiber contraction

$$\begin{array}{ccc} W & \xrightarrow{F} & W \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{h} & X \end{array}$$

It contracts the fibers uniformly and has a unique invariant section σ_F . Our goal here is a theorem asserting that σ_F is θ -Hölder when the fiber contraction dominates the base contraction at scale θ . Previous versions of such a result appear in [15], Shub’s book [22], and Wilkinson’s paper [23] under hypotheses sometimes involving a compact base space, C^2 differentiability, and global bundle triviality, all of which we need to relax. Compactness becomes uniformity, C^2 becomes C^1 , and global bundle triviality becomes local bundle triviality.

Existence, uniqueness, and continuity of the invariant section σ_F of a fiber contraction are straightforward. The standard assumptions are that

- (a) F is continuous, π is a continuous surjection, h is a homeomorphism, and each fiber $\pi^{-1}(x)$ is equipped with a metric d_x which makes the fiber complete, depends continuously on $x \in X$, and is uniformly bounded.

(b) There is a $k < 1$ such that

$$d_{h(x)}(F(w), F(w')) \leq kd_x(w, w')$$

for all $w, w' \in \pi^{-1}(x)$ and all $x \in X$.

(c) There exists a continuous section $\sigma_0: X \rightarrow W$.

Then the space Σ^c of continuous sections is metrized by

$$d(\sigma, \sigma') = \sup_x d_x(\sigma(x), \sigma'(x))$$

and is complete. It is contracted by the graph transform $F_\#: \Sigma^c \rightarrow \Sigma^c$,

$$F_\#: \sigma \mapsto F \circ \sigma \circ h^{-1}.$$

The unique fixed point of $F_\#$ is the invariant section σ_F . Under $F_\#$ -iteration every section $\sigma \in \Sigma^c$ converges uniformly to σ_F .

To show that σ_F is Hölder we want to justify the assertion that $F_\#$ leaves invariant a closed subspace of Hölder sections, and therefore σ_F lies in that subspace.

An initial Hölder assumption is that the fiber contraction θ -dominates the base contraction: If X is metrized this means that for all $x \in M$

$$k(x) < \mu(x)^\theta$$

where $k(x) \leq k < 1$ is the Lipschitz constant of F restricted to the fiber $\pi^{-1}(x)$ and $\mu(x)$ is the reciprocal of the Lipschitz constant of h^{-1} at $h(x)$. Without such a dominance condition Hölderity can fail.

A second Hölder assumption concerns the vertical shear of F . It measures how much F slides fibers up and down. For example, take

$$F: (x, y) \mapsto (x/9, y/3 + \sin(50x)),$$

and see Figure 6.

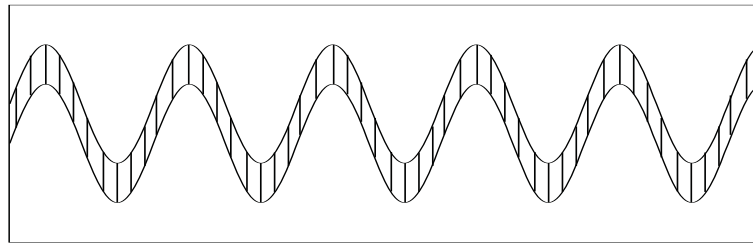


FIGURE 6. The fiber contraction is $1/3$, the base contraction is $1/9$, and for all $\theta < 1/2$, the fiber contraction θ -dominates the base contraction. The vertical shear is large but finite.

Bundle charts give the best way to quantify vertical shear. A bundle chart is a map φ that sends $U \times Y$ homeomorphically onto an open subset of W such

that each $\varphi(u \times Y)$ is a fiber $\pi^{-1}(\xi(u))$. This defines a base chart $\xi: U \rightarrow X$ so that

$$\begin{array}{ccc} U \times Y & \xrightarrow{\varphi} & W \\ \pi \downarrow & & \downarrow \pi \\ U & \xrightarrow{\xi} & X \end{array}$$

commutes. We assume U, Y are metric spaces, $\xi(U)$ is open in X , and $\xi: U \rightarrow \xi(U)$ is a homeomorphism.

A bundle atlas for W is a collection \mathcal{A} of bundle charts φ_i that cover W . The corresponding collection \mathcal{B} of base charts ξ_i is an atlas for X . We write $W_i = \varphi_i(U \times Y)$, $X_i = \xi_i(U)$, and $d_i(x, x') = d_U(\xi_i^{-1}(x), \xi_i^{-1}(x'))$. The base atlas covers X uniformly provided there is a $\delta > 0$ such that if $x, x' \in X_j$ and $d_j(x, x') < \delta$ then some X_i contains the pair $h^{-1}(x), h^{-1}(x')$.

If $F(W_i) \cap W_j \neq \emptyset$ then

$$F_{ij}(u, y) = \varphi_j^{-1} \circ F \circ \varphi_i(u, y) = (h_{ij}(u), v_{ij}(u, y)) \in U \times Y,$$

is a chart expression for F . The letters h and v indicate the horizontal and vertical components of F . The vertical shear of F_{ij} is

$$V_{ij}(u, u', y) = d_Y(v_{ij}(u, y), v_{ij}(u', y)).$$

If there is a constant L such that for all ij and $u, u' \in U, y \in Y$ we have

$$V_{ij}(u, u', y) \leq L d_U(u, u')^\theta$$

then F has θ -bounded vertical shear with respect to \mathcal{A} . It is natural to define the chartwise fiber contraction and base contraction as k_{ij} and μ_{ij} where k_{ij} is the supremum of the Lipschitz constants of $y \mapsto v_{ij}(u, y)$ with $u \in U$, and μ_{ij} is the reciprocal of the Lipschitz constant of h_{ij}^{-1} .

DEFINITION. The fiber contraction F is uniformly θ -Hölder with respect to \mathcal{A} if the base atlas covers X uniformly, F has uniformly θ -bounded vertical shear, and the fiber contraction uniformly θ -dominates the base contraction in the sense that

$$\sup_{ij} \frac{k_{ij}}{\mu_{ij}^\theta} < 1.$$

The principal part of a section $\sigma: X \rightarrow W$ in the bundle chart φ_i is the map $s_i: X_i \rightarrow Y$ such that

$$\varphi_i^{-1} \circ \sigma(x) = (\xi_i^{-1}(x), s_i(x)).$$

Its θ -Hölder constant is

$$H(s_i) = \sup \frac{d_Y(s_i(x), s_i(x'))}{d_i(x, x')^\theta}$$

where the supremum is taken over all $x, x' \in X_i$ with $x \neq x'$. If $\sup_i H(s_i) < \infty$ then the section is uniformly θ -Hölder with respect to \mathcal{A} .

THEOREM 10. *With respect to \mathcal{A} , if F is a uniformly θ -Hölder fiber contraction and there exist uniformly θ -Hölder sections then σ_F is uniformly θ -Hölder.*

REMARK. We refer to Theorem 10 as a pointwise (or relative) result in contrast to an absolute result because we are comparing fiber contraction to base contraction over small neighborhoods rather than comparing the weakest fiber contraction over all of W to the sharpest base contraction over all of X .

REMARK. The space X is built from homeomorphic copies of U but it may be nonseparable. In our application \mathcal{A} is locally finite but uncountable.

Proof. Let k_{ij} and μ_{ij} denote the fiber contraction and base contraction of $F_{ij} = (h_{ij}(u), v_{ij}(u, y))$,

$$k_{ij} = \sup_{y \neq y'} \frac{d_Y(v_{ij}(u, y), v_{ij}(u, y'))}{d_Y(y, y')}$$

$$\mu_{ij} = \inf_{u \neq u'} \frac{d_U(h_{ij}(u), h_{ij}(u'))}{d_U(u, u')}.$$

By assumption there exists a section σ_0 whose principal parts with respect to \mathcal{A} are uniformly θ -Hölder. Let H_0 be a bound for these Hölder constants. Choose

$$H = \max\{H_0, D/\delta^\theta, \sup_{ij} 1/(\mu_{ij}^\theta - k_{ij})\}$$

where D is the diameter of Y and δ is the base covering constant. (If $x, x' \in X_j$ and $d_j(x, x') \leq \delta$ then $h^{-1}(x), h^{-1}(x')$ lie in a common X_i . Since $\sup_{ij} k_{ij}/\mu_{ij}^\theta < 1$, H is finite.) Let $\Sigma^{\theta, H}$ be the set of sections whose principal parts have θ -Hölder constant $\leq H$. They satisfy

$$d_Y(s_j(x), s_j(x')) \leq H d_j(x, x')^\theta$$

for all $x, x' \in X_j$ and all j . The set $\Sigma^{\theta, H}$ is nonempty since it contains σ_0 . It is closed since if a sequence of sections converges with respect to the metric d on Σ then in any chart its principal parts converge uniformly, and Hölder conditions survive uniform convergence.

It remains to show that $\Sigma^{\theta, H}$ is $F_\#$ -invariant. Given $\sigma \in \Sigma^{\theta, H}$ and a chart φ_j we must show that the principal part of $\tilde{\sigma} = F_\#(\sigma)$ satisfies

$$d_Y(\tilde{s}_j(x), \tilde{s}_j(x')) \leq H d_j(x, x')^\theta.$$

Case 1. $d_j(x, x') > \delta$. Then

$$d_Y(\tilde{s}_j(x), \tilde{s}_j(x')) \leq D = (D/\delta^\theta) \delta^\theta \leq H d_j(x, x')^\theta$$

since $D/\delta^\theta \leq H$.

Case 2. $d_j(x, x') \leq \delta$. Then there is an X_i containing the pair $h^{-1}(x), h^{-1}(x')$. The principal parts of σ_i and $\tilde{\sigma}_j$ are related by the formula

$$\tilde{s}_j(x) = v_{ij}(u, s_i(h^{-1}(x)))$$

where $\xi_i(u) = h^{-1}(x)$. This is proved by formula manipulation. We have $F_{ij} = \varphi_j^{-1} \circ F \circ \varphi_i$, and thus $\varphi_j^{-1} \circ F = F_{ij} \circ \varphi_i^{-1}$, which gives

$$\begin{aligned} \varphi_j^{-1} \circ \tilde{\sigma}(x) &= \varphi_j^{-1} \circ F \circ \sigma \circ h^{-1}(x) \\ &= F_{ij} \circ \varphi_i^{-1} \circ \sigma \circ h^{-1} \\ &= F_{ij}(u, s_i(h^{-1}(x))) \\ &= (h_{ij}(u), v_{ij}(u, s_i(h^{-1}(x)))). \end{aligned}$$

Since $\varphi_j^{-1} \circ \tilde{\sigma}(x) = (\xi_j^{-1}(x), \tilde{s}_j(x))$, we equate the vertical components to get the relation between the principal parts of σ_i and $\tilde{\sigma}_j$ as stated.

Since $\sigma \in \Sigma^{\theta, H}$ we have

$$d_Y(s_i(h^{-1}(x)), s_i(h^{-1}(x'))) \leq Hd_i(h^{-1}(x), h^{-1}(x'))^\theta.$$

Thus

$$\begin{aligned} d_Y(\tilde{s}_j(x), \tilde{s}_j(x')) &= d_Y(v_{ij}(u, s_i(h^{-1}(x))), v_{ij}(u', s_i(h^{-1}(x')))) \\ &\leq d_Y(v_{ij}(u, s_i(h^{-1}(x))), v_{ij}(u, s_i(h^{-1}(x')))) \\ &\quad + d_Y(v_{ij}(u, s_i(h^{-1}(x'))), v_{ij}(u', s_i(h^{-1}(x')))) \\ &\leq k_{ij} d_Y(s_i(h^{-1}(x)), s_i(h^{-1}(x'))) + Ld_U(u, u')^\theta \\ &\leq k_{ij} Hd_i(h^{-1}(x), h^{-1}(x'))^\theta + Ld_i(h^{-1}(x), h^{-1}(x'))^\theta \\ &\leq ((k_{ij}H + L)/\mu_{ij}^\theta) d_j(x, x')^\theta \leq Hd_j(x, x')^\theta \end{aligned}$$

where $\xi_i(u) = h^{-1}(x)$ and $\xi_i(u') = h^{-1}(x')$. This completes the proof that $\Sigma^{\theta, H}$ is $F_\#$ -invariant and therefore that σ_F is θ -Hölder. \square

We will use the the Uniform Hölder Section Theorem as follows. The base manifold will be the disjoint union of the global strong stable manifolds of a partially hyperbolic diffeomorphism g . The fiber at $x \in W^s(p)$ is the local strong stable manifold at x , $W^s(x, r)$. (Although this makes the fiber a subset of the base we can still think of a bundle this way.) The bundle map F approximates the product $g \times g$. The base map is not g but is an amalgam a of g and a nearby diffeomorphism f . Both fiber and base are contracted, but for some θ , the fiber contraction dominates the θ^{th} power of the base contraction.

We cover the manifold M with finitely many charts inside which the bundle is trivial. This gives uncountably many charts φ_i that cover the bundle because the base manifold V has uncountably many components. However the chart expressions for F are uniform, and uniformity overcomes noncompactness.

9. THE PROOF OF THEOREM A

We are given a normally hyperbolic diffeomorphism f whose center unstable, center, and center stable foliations are C^1 . We intend to show that the corresponding foliations of a C^1 perturbation g of f are Hölder. Most of the work is contained in the following result, which will be applied to the suspension of \mathcal{W}^{cu} and the suspension of g .

Let \mathcal{E} be a C^1 foliation at which f is normally contracting, say

$$0 < \mu < T^s f < \nu < T_{T\mathcal{E}} f \quad \text{and} \quad \nu < \mu^\theta < 1,$$

as in the normally hyperbolic case. (The letter \mathcal{E} is meant to suggest a foliation along whose leaves f is at least somewhat expanding. An example is $\mathcal{E} = \mathcal{W}^{cu}$.) Let g C^1 -approximate f and let \mathcal{W}_g^s be its strong stable foliation. Being C^1 , f is plaque expansive with respect to \mathcal{E} , and so there is a unique nearby g -invariant foliation \mathcal{E}_g and there is a canonical leaf conjugacy $\mathfrak{h}: \mathcal{E} \rightarrow \mathcal{E}_g$ that respects \mathcal{W}_g^s .

PROPOSITION 11. *\mathfrak{h} is uniformly θ -biHölder when restricted to the \mathcal{W}_g^s -leaves and the \mathcal{E}_g -holonomy is θ^2 -Hölder.*

Proof. We define a map $a: M \rightarrow M$ which is an “amalgam” of f , \mathcal{E} , g , and \mathcal{W}_g^s as follows. For each $x \in M$ we set

$$a(x) = W_g^s(g(x), r) \cap \mathcal{E}(f(x), r).$$

Here $r > 0$ is small and fixed. r is a radius where the local invariant manifolds of f are fairly flat, meaning that distance measured along them approximates geodesic distance in M . If the sup-distance between f and g is $< r/2$ then $a(x)$ is uniquely defined by transversality. To get the necessary estimates on a we assume

$$d_{C^1}(f, g) \ll r.$$

The way to think of a in terms of \mathcal{W}_g^s is this. It takes the stable g -manifold at x and sends it to the stable g -manifold at $g(x)$, but it does not do so simply by applying g and it does not send the base point x to the base point $g(x)$. Rather, it first applies f and then slides along the \mathcal{E} -foliation to get to W_g^s . Since everything is local, the sliding distance is small and we see that

a -orbits are f pseudo-orbits.

Since \mathcal{E} is C^1 , the amalgam map $a: M \rightarrow M$ is C^1 along the \mathcal{W}_g^s -leaves. The C^1 hypothesis on \mathcal{E} is crucial here. Restricted to the \mathcal{W}_g^s -leaves the amalgam map C^1 -approximates g which C^1 -approximates f . Therefore, like g , the amalgam map contracts the stable g -manifolds by a factor $< \nu$. See Figure 7.

We next define two nonlinear fiber contractions and study their invariant sections. In both cases the total space is

$$W = \{(x, y) \in M \times M : y \in W_g^s(x, r)\}$$

and the projection $\pi: W \rightarrow M$ is $\pi(x, y) = x$. *A priori the bundle (W, π, M) is not C^1 .* For although M and π are smooth, the set W is locally the graph of \mathcal{W}_g^s , a foliation quite likely to be Hölder at best.

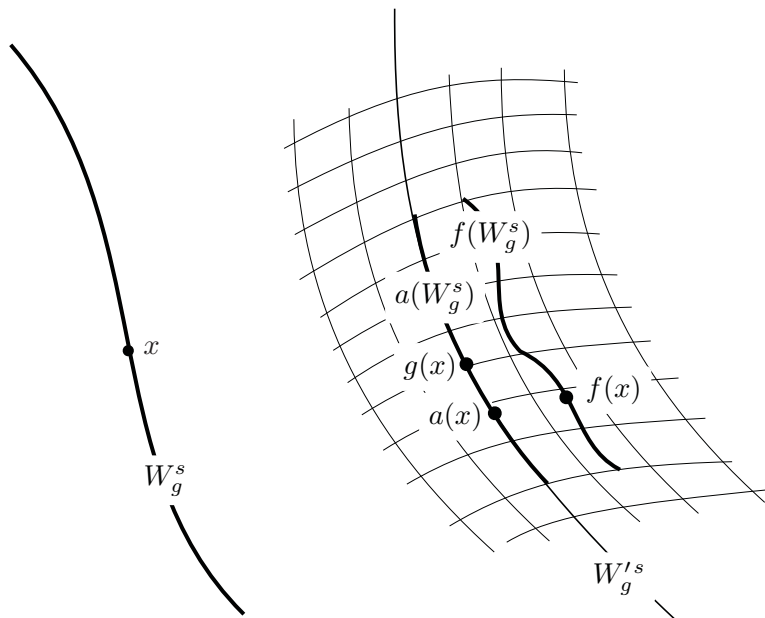


FIGURE 7. The amalgam map a and its effect on a local stable g -manifold W_g^s . The light horizontal curves are the \mathcal{E} -leaves and the light vertical curves are the \mathcal{W}_g^s -leaves. The curves labelled W_g^s and $W_g'^s$ represent \mathcal{W}_g^s -leaves of radius r based at x and $g(x)$ respectively. Note that f shrinks W_g^s by a factor $< \nu$ since W_g^s is approximately tangent to E^s .

The fiber maps are products, $F(x, y) = (a(x), g(y))$ and $G(x, y) = (g(x), a(y))$,

$$\begin{array}{ccc} W & \xrightarrow{F} & W \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{a} & M \end{array} \quad \begin{array}{ccc} W & \xrightarrow{G} & W \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{g} & M \end{array}$$

The π -fiber at x is $W_g^s(x, r)$, and it is contracted by g into $W_g^s(g(x), \nu r)$. The π -fiber over $a(x)$ is $W_g^s(a(x), r)$, and since $a(x) \in W_g^s(g(x), r)$ is much closer to $g(x)$ than νr , we have

$$g(W_g^s(x, r)) \subset W_g^s(g(x), \nu r) \subset W_g^s(a(x), r),$$

which means that F contracts fibers over a . For the same reasons we have

$$a(W_g^s(x, r)) \subset W_g^s(a(x), \nu r) \subset W_g^s(g(x), r),$$

which means that G contracts fibers over g . See Figure 7.

Fiber contractions have unique invariant sections. Let $\sigma: M \rightarrow W$ be the F -invariant section and $\tau: M \rightarrow W$ be the G -invariant section. Since $W \subset M \times M$ we can write

$$\sigma(x) = (x, s(x)) \quad \tau(x) = (x, t(x)).$$

The maps $s, t: M \rightarrow M$ are the *principal parts* of the sections σ, τ . We claim that s and t are inverse leaf conjugacies between \mathcal{E} and \mathcal{E}_g .

Since $\sigma(x) = (x, s(x))$ is F -invariant and $F = a \times g$, the a -orbit of x and the g -orbit of $s(x)$ shadow each other closely. Since a -orbits are f pseudo-orbits, the g -orbit of $s(x)$ is closely shadowed by an f pseudo-orbit through x . Therefore $s: \mathcal{E} \rightarrow \mathcal{E}_g$ is a canonical leaf conjugacy. It has an inverse map, s^{-1} . Now, F -invariance implies that $g(s(x)) = s(a(x))$, and therefore $s^{-1}(g(x)) = a(s^{-1}(x))$. Consequently

$$G(x, s^{-1}(x)) = (g(x), a(s^{-1}(x))) = (g(x), s^{-1}(g(x)))$$

which implies that $x \mapsto (x, s^{-1}(x))$ is a G -invariant section, i.e., it is the unique G -invariant section τ . Equal sections have equal principal parts, so $s^{-1} = t$.

The amalgam mapping $a: M \rightarrow M$ is a homeomorphism that carries $W_g^s(x, r)$ into $W_g^s(g(x), r)$ C^1 diffeomorphically. Therefore it sends the global stable manifold $W_g^s(x)$ diffeomorphically onto the global stable manifold $W_g^s(g(x))$. When we put the leaf topology on M with respect to the foliation \mathcal{W}_g^s we get a C^1 non-separable manifold V of dimension s whose connected components are the global stable manifolds $W_g^s(x)$, and $a: V \rightarrow V$ is a C^1 diffeomorphism.

We have a bundle \mathcal{W} over V . Its total space is W with the leaf topology on its base $V = \bigcup W_g^s$. Its fiber at x is $W_g^s(x, r)$. On \mathcal{W} we have two fiber contractions, F and G . We claim that \mathcal{W} is uniformly C^1 and the Uniform Hölder Section Theorem (Theorem 10) applies to it.

The fiber $W_g^s(x, r)$ varies in a fairly trivial C^1 fashion as x varies in the global stable manifold $W_g^s(x)$. Since each connected component $W_g^s \subset V$ is simply connected, any disk bundle over it (such as $\mathcal{W}|_{W_g^s}$) is trivial. A priori this triviality is not uniform. That is, there need be no relation between the trivializing vector fields at points of W_g^s which are nearby each other in M but distant along W_g^s . This is why we need an Invariant-Section Theorem in which the bundle is only locally trivial.

The proof that \mathcal{W} does have such a uniform C^1 bundle structure to which Theorem 10 applies is distractingly technical and relegated to the appendix following this section. Admitting this, we see that the leaf conjugacy $s: \mathcal{E} \rightarrow \mathcal{E}_g$ is uniformly θ -Hölder, and so is the inverse leaf conjugacy $t: \mathcal{E}_g \rightarrow \mathcal{E}$, when $\nu < \mu^\theta$.

To complete the proof of Proposition 11 we show that the \mathcal{E}_g -holonomy is θ^2 -Hölder.

Consider points p, p' with $p' \in \mathcal{E}(p)$ and draw a path $\alpha \subset \mathcal{E}(p)$ from p to p' . Corresponding to α we have the \mathcal{E}_g -holonomy map h_g . It sends $x \in W_g^s(p, r)$ to $x' \in W_g^s(p', r')$ such that x, x' lie on a common \mathcal{E}_g -leaf L_g and are joined by a path on L_g that approximates α . The radii $r \leq r'$ are small. Thus, $h_g = s' \circ h \circ t$ where t is the restriction of the inverse leaf conjugacy $\mathcal{E}_g \rightarrow \mathcal{E}$ to $W_g^s(p)$, h is the C^1 holonomy map for \mathcal{E} , and s' is the restriction of the forward leaf conjugacy $s: \mathcal{E} \rightarrow \mathcal{E}_g$ to $W_g^s(p', r')$. See Figure 8. Since Hölder exponents multiply under composition, the \mathcal{E}_g holonomy maps are θ^2 -Hölder. \square

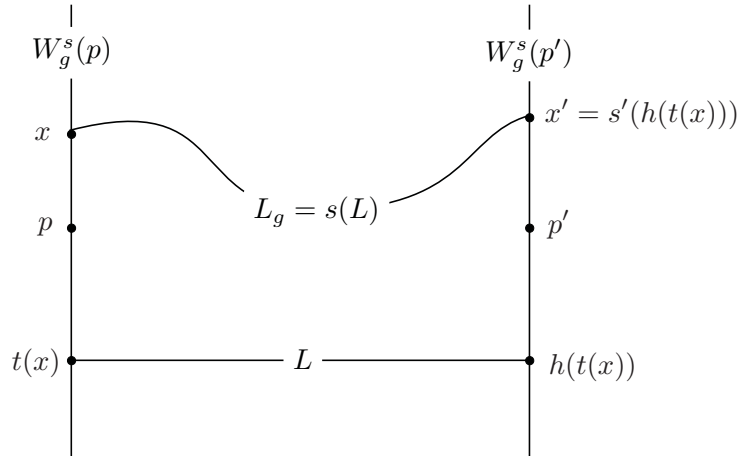


FIGURE 8. The leaf conjugacy $s: \mathcal{E} \rightarrow \mathcal{E}_g$ carries the \mathcal{E} -leaf L through $t(x)$ to the \mathcal{E}_g -leaf L_g through x . The map $x \mapsto x'$ is the \mathcal{E}_g -holonomy.

Proof of Theorem A. We are given a diffeomorphism f that is normally hyperbolic at a foliation \mathcal{F} . It is assumed that its splitting is C^1 , so in particular the summands E^{cu} , $E^c = T\mathcal{F}$, and E^{cs} integrate to unique C^1 f -invariant foliations \mathcal{W}^{cu} , $\mathcal{W}^c = \mathcal{F}$, and \mathcal{W}^{cs} . The diffeomorphism f is normally hyperbolic at \mathcal{W}^{cu} with respect to the splitting $T\mathcal{W}^{cu} \oplus E^s$. Likewise, f is normally hyperbolic at $\mathcal{W}^c = \mathcal{F}$ and at \mathcal{W}^{cs} . Since the foliations are C^1 they are plaque expansive and structurally stable: If g C^1 -approximates f then we have unique nearby g -invariant foliations \mathcal{W}_g^{cu} , $\mathcal{W}_g^c = \mathcal{F}_g$, \mathcal{W}_g^{cs} , and we have canonical leaf conjugacies from the f -invariant foliations to the corresponding g -invariant foliations.

We want to show that these leaf conjugacies are biHölder, and the holonomy maps along the g -invariant foliations are Hölder.

Let T be the circle of length 2. The f -invariant foliation \mathcal{W}^{cu} is C^1 and normally contracting, so the product foliation

$$\mathcal{S}^{cu} = T \times \mathcal{W}^{cu} \subset T \times M$$

is also C^1 and normally contracting with respect to the diffeomorphism

$$F(t, x) = (t, f(x)).$$

Its leaves are products $T \times W^{cu}(x)$. The contraction rates are the same for F and f . Since g C^1 -approximates f there is a C^1 loop of diffeomorphisms $g_t: M \rightarrow M$ that C^1 -approximate f such that $g_0 = f = g_2$ and $g_1 = g$. The diffeomorphism

$$G(t, x) = (t, g_t(x))$$

C^1 -approximates F . (These suspension diffeomorphisms F and G are different from the fiber maps F and G in Proposition 11.) Proposition 11 applies to the suspension diffeomorphisms so there exists a unique G -invariant foliation \mathcal{S}_G^{cu} near \mathcal{S}^{cu} all of whose holonomy maps are θ^2 -Hölder when $\nu < \mu^\theta$. Since \mathcal{W}_g^{cu}

is a slice of \mathcal{S}_G^{cu} , the \mathcal{W}_g^{cu} -holonomy maps are also θ^2 -Hölder. By Theorem 8 the suspension holonomy of \mathcal{S}_G^{cu} gives a canonical leaf conjugacy $\mathfrak{h}^{cu}: \mathcal{W}^{cu} \rightarrow \mathcal{W}_g^{cu}$, which is θ^2 -biHölder.

Similarly we have $\mathcal{S}^{cs} = T \times \mathcal{W}^{cs}$ and \mathcal{S}_G^{cs} whose holonomy maps are θ^2 -Hölder when $\widehat{\nu} < \widehat{\mu}^\theta$. Since \mathcal{W}_g^{cs} is a slice of \mathcal{S}_G^{cs} , the \mathcal{W}_g^{cs} -holonomy maps are also θ^2 -Hölder. By Theorem 8 the suspension holonomy of \mathcal{S}_G^{cs} gives a canonical leaf conjugacy $\mathfrak{h}^{cs}: \mathcal{W}^{cs} \rightarrow \mathcal{W}_g^{cs}$, which is θ^2 -biHölder.

To complete the proof we take intersections. Theorem 7 implies that the intersection foliation $\mathcal{S}_G^c = \mathcal{S}(\mathcal{F}_g)$ has θ^2 -Hölder holonomy. Since $\mathcal{W}_g^c = \mathcal{F}_g$ is a slice of \mathcal{S}_G^c , the \mathcal{W}_g^c -holonomy maps are also θ^2 -Hölder. By Theorem 8 the suspension holonomy of \mathcal{S}_G^c gives a canonical θ^2 -Hölder leaf conjugacy $\mathfrak{h}: \mathcal{F} \rightarrow \mathcal{F}_g$. □

REMARK. Why do we need suspension in the proof of Theorem A from Proposition 11? After all, Proposition 11 applies directly to \mathcal{W}^{cu} and \mathcal{W}^{cs} when they are C^1 and implies that the leaf conjugacies $\mathfrak{h}^{cu}: \mathcal{W}^{cu} \rightarrow \mathcal{W}_g^{cu}$ and $\mathfrak{h}^{cs}: \mathcal{W}^{cs} \rightarrow \mathcal{W}_g^{cs}$ are Hölder. Unfortunately we have no conjugacy intersection result to conclude from this that the center leaf conjugacy $\mathfrak{h}^c: \mathcal{W}^c \rightarrow \mathcal{W}_g^c$ is Hölder. But we do have a holonomy intersection result, namely Theorem 7. That is why we convert the leaf conjugacy question to a suspension holonomy question. See also Remark 1 in Section 11.

APPENDIX. UNIFORM BUNDLE CHARTS

In the proof of Proposition 11 the bundle \mathcal{W} has total space W , base space V , and dimension $2s$. We will construct bundle charts $\varphi: \mathbb{R}^s(r) \times \mathbb{R}^s(r) \rightarrow W$ which are uniformly C^1 and in which the fiber maps F, G are uniformly C^1 . ($F = a \times g$ and $G = g \times a$ are the fiber maps from Proposition 11. They are different from the diffeomorphisms F, G in the proof of Theorem A.) By this we mean two things:

- The chart transfer maps for overlapping charts are C^1 and their first derivatives are uniformly bounded and uniformly continuous.
- The chart expressions for F and G have uniformly bounded, uniformly continuous first derivatives.

The metric in which we measure everything will be a smooth metric on TM . In addition, all these derivatives and charts will be continuous from one component of V to another.

We start by smoothing the splitting. The original normally hyperbolic splitting $TM = E^u \oplus E^c \oplus E^s$ is orthogonal with respect to a continuous adapted Riemann structure. (The bundle E^c is $T\mathcal{F}$.) The smoothed splitting

$$TM = \widetilde{E}^u \oplus \widetilde{E}^c \oplus \widetilde{E}^s$$

uniformly approximates the original splitting and is orthogonal with respect to a smooth Riemann structure that approximates the original Riemann structure.

This is standard. Although the smoothed splitting is not invariant we have

$$Tf = \begin{bmatrix} \tilde{T}^u f & * & * \\ * & \tilde{T}^c f & * \\ * & * & \tilde{T}^s f \end{bmatrix} \text{ with respect to } \tilde{E}^u \oplus \tilde{E}^c \oplus \tilde{E}^s,$$

where the off-diagonal terms are small and $\tilde{T}^u f \oplus \tilde{T}^c f \oplus \tilde{T}^s f$ approximates $T^u f \oplus T^c f \oplus T^s f$.

Let ρ be a plaque of \mathcal{W}^s at p . If \exp_p is the smooth exponential at p then $\bar{\rho} = \exp_p^{-1}(\rho)$ is the graph of a C^1 function $\gamma: \tilde{E}^s(p, r) \rightarrow \tilde{E}^{cu}(p)$. The function γ has Lipschitz constant ≤ 1 , it has value 0 at the origin of $\tilde{E}^s(p)$, and its derivative with respect to $x \in \tilde{E}^s(p, r)$ is uniformly continuous with respect to x, p . This is all because \mathcal{W}^s is a regular foliation, the smoothed splitting approximates the original splitting, and r is small. Let $\text{gr}(\gamma): \tilde{E}^s(p, r) \rightarrow T_p M$ denote the function whose image is the graph of γ .

Now we define charts on \mathcal{W} in which to express this. We cover M with finitely many smooth charts $\mathbb{R}^m \rightarrow U \subset M$ over which the smoothed splitting is trivial. We choose smooth orthonormal trivializing vector fields Y_1, \dots, Y_s for \tilde{E}_U^s . Then for t_1, \dots, t_s we set

$$\eta(p, t_1, \dots, t_s) = t_1 Y_1(p) + \dots + t_s Y_s(p),$$

which makes η a linear isometry that sends $\mathbb{R}^s(r)$ to $\tilde{E}^s(p, r)$ and that depends smoothly on p . The composition

$$\exp \circ \text{gr}(\gamma) \circ \eta$$

is a uniformly C^1 parameterization of the strong stable plaque $W^s(p, r)$. We denote the parameterization by $\rho(p, t)$, and we define a corresponding \mathcal{W} -chart $\varphi_p: \mathbb{R}^s(r) \times \mathbb{R}^s(r) \rightarrow W$ by

$$\varphi_p(x, y) = (\rho(p, x), \rho(q, y)) \text{ where } q = \rho(p, x).$$

As U ranges through the finite set of chart neighborhoods U that cover M and p ranges through U this gives a bundle atlas \mathcal{A} for \mathcal{W} . Chart transfers are uniformly C^1 since they are just related by different choices of trivializing vector fields, and one set of trivializing vector fields is related to another by orthogonal maps $\tilde{E}^s(p) \rightarrow \tilde{E}^s(p)$ that depend smoothly on p .

The amalgam map a is defined from globally C^1 data, namely the uniformly C^1 transversals W_g^s , the diffeomorphism f , and the uniformly C^1 holonomy maps associated to the f -invariant C^1 foliation \mathcal{F} . Thus the fiber contractions $F = a \times g$ and $G = g \times a$ on \mathcal{W} are uniformly C^1 when represented in the charts $\varphi_p \in \mathcal{A}$. The key quantities in the hypotheses of Theorem 10 are the vertical shear and the contraction rates of fiber versus base. They are now easy to estimate.

The vertical shear is uniformly 1-bounded because F and G are uniformly C^1 . The fiber contraction of F is $\|T^s g\|$ and the base contraction is $\mathbf{m}(T^s a)$.

Since a C^1 -approximates g and g C^1 -approximates f the stable bunching condition $\nu < \mu^\theta$, which is

$$\sup_{p \in M} \frac{\|T_p^s f\|}{(\mathbf{m}(T_p^s f))^\theta} < 1,$$

gives

$$\sup_{p \in M} \frac{\|T_p^s g\|}{(\mathbf{m}(T_p^s a))^\theta} < 1 \quad \text{and} \quad \sup_{p \in M} \frac{\|T_p^s a\|}{(\mathbf{m}(T_p^s g))^\theta} < 1.$$

Thus the fiber contractions of F and G uniformly θ -dominate the base contractions, and Theorem 10 implies that the unique F - and G -invariant sections σ and τ are θ -Hölder.

10. THE PROOF OF THEOREM B

Theorem B concerns uniformly compact laminations, and the following result of David Epstein is used. See also [5].

THEOREM 12. [10] *Each leaf of a uniformly compact lamination has arbitrarily small laminated neighborhoods.*

A neighborhood of $L \in \mathcal{L}$ in Λ is laminated if it consists of whole leaves of \mathcal{L} . The idea of the proof is simple. Let L be a leaf of the uniformly compact lamination \mathcal{L} . If the assertion is false there exist leaves $L_n \in \mathcal{L}$ containing points p_n, q_n such that p_n converges to some $p \in L$ and q_n converges to some $q \notin L$. In order that L_n leaves the neighborhood of L it is necessary that L_n “spirals away” from L , which causes it to have a large volume, contrary to uniform compactness of \mathcal{L} . The details of the proof are not so simple.

A second result used in the proof of Theorem B concerns plaque expansivity.

PROPOSITION 13. [5] *Plaque expansivity is implied by uniform compactness, normal hyperbolicity, and dynamical coherence.*

REMARK. As noted above, plaque expansivity under the hypotheses of Theorem 2 is immediate. For in the skew product case we have leaf expansivity, which implies plaque expansivity. It is interesting to note that in the uniformly compact case leaf expansivity can fail while plaque expansivity remains true. See Remark 4 in Section 11 for further details.

Proof of Proposition 13. If ρ, σ are nearby plaques of leaves $P, Q \in \mathcal{L}$ then their local center unstable and center stable manifolds intersect in a plaque of a third leaf, say

$$\xi = W^{cu}(\rho, r) \cap W^{cs}(\sigma, r)$$

is a plaque of X . Here we use dynamical coherence as defined in Section 3 to assert that the intersection of the center unstable and center stable plaques is the plaque of a leaf. By Theorem 12 we can assume $X \subset W^{cu}(P, r)$. See Figure 9.

Consider plaque orbits (ρ_k) and (σ_k) starting at ρ and σ such that the distance between ρ_k and σ_k is always much less than r . Since the plaque orbits

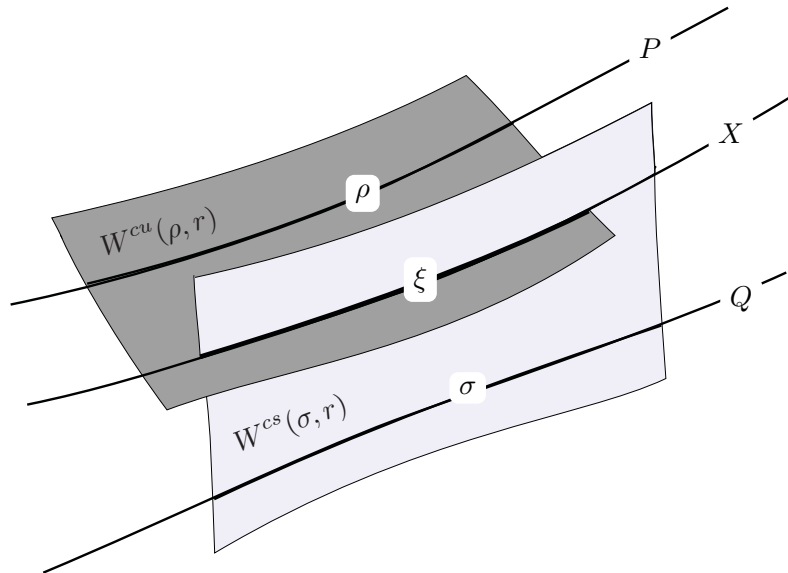


FIGURE 9. The radius r should be much less than the size of the plaques ρ , σ and much greater than the distance between them although in the figure the three quantities are not much different. The size of ξ will then be on the same order as that of ρ and σ .

respect the lamination and f preserves the laminations, the corresponding intersection plaque ξ_k is contained in $f^k(X)$. We must show that ρ and σ overlap.

Case 1. $X \neq P$. Since f overflows the family of local center unstable manifolds and $X \subset W^{cu}(P, r)$ is compact, forward iterates $f^k(X)$ are eventually pushed off $W^{cu}(f^k(P), r)$. The intersection plaque ξ_k is contained in $f^k(X)$ and is therefore eventually pushed off $W^{cu}(f^k(P), r)$. Correspondingly, ξ_k and σ_k become close together. Thus, for some large k the distance between ρ_k and σ_k is on the order of r , a contradiction to the assumption that the two plaque orbits stay much closer together than r .

Case 2. $X \neq Q$. Using inverse iterates, we arrive at the same contradiction.

Since the Cases 1 and 2 lead to contradictions, $P = X = Q$. Then ρ and σ are nearby plaques on a common leaf, and such plaques always overlap. \square

REMARK. We do not know whether dynamical coherence is necessary in Proposition 13. See Remark 7 in Section 11.

Proof of Theorem B. We are given a C^1 diffeomorphism f which is normally hyperbolic and dynamically coherent at a uniformly compact lamination \mathcal{L} . We assert that its center holonomy maps are Hölder, and if g C^1 -approximates f then a canonical leaf conjugacy $\mathcal{L} \rightarrow \mathcal{L}_g$ is biHölder.

We first examine the center holonomy maps restricted to the center unstable manifolds. By the proof Theorem 4.3 in [19] the choice of transversals affects the Hölder constant but not the Hölder exponent of a holonomy map. Thus we can use the local strong unstable manifolds of f (or of g if g perturbs f) as transversals. So let h be a center holonomy map $W^u(p, r) \rightarrow W^u(p', r')$. It is determined by a path $\xi: [0, 1] \rightarrow L$ from p to p' in the leaf $L \in \mathcal{L}$ containing p, p' . We claim that h is θ -Hölder when

$$\widehat{\nu} < \widehat{\mu}^\theta.$$

To cut down on the number of hats and reciprocal hats we set

$$\lambda = \widehat{\nu}^{-1} \quad \omega = \widehat{\mu}^{-1}.$$

Then $1 < \lambda < T^u f < \omega$ and we assert h is θ -Hölder when $\omega^\theta < \lambda$. We write

$$\begin{aligned} \lambda^k(p) &= \lambda(f^{k-1}(p)) \cdot \lambda(f^{k-2}(p)) \cdots \lambda(f(p)) \cdot \lambda(p) \\ \omega^k(p) &= \omega(f^{k-1}(p)) \cdot \omega(f^{k-2}(p)) \cdots \omega(f(p)) \cdot \omega(p) \end{aligned}$$

in order to make product formulas simple.

Fix a small $R > 0$ such that for all $p \in \Lambda$, all $x \in W^u(p, R)$, and all unit vectors $v \in T_x(W^u(p, R))$ we have

$$\lambda(p) < |T_x f(v)| < \omega(p).$$

Let $\Omega = \max_{x \in M} \|T_x f\|$. Given $a, b \in W^u(p, r) \cap \Lambda$ we choose paths α, β in the leaves A, B containing a, b that start at a, b , closely shadow ξ , and end at points $a', b' \in W^u(p', r')$. The center holonomy map h sends a, b to a', b' . We parameterize the paths so that $\beta(t) \in W^u(\alpha(t), R)$ for $0 \leq t \leq 1$. See Figure 10. Here

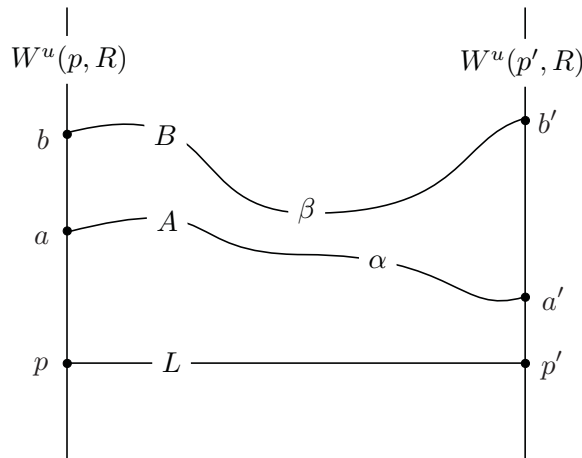


FIGURE 10. The center holonomy inside a local center unstable manifold $W^{cu}(L, R)$.

$0 < r < r' < R/2\Omega$ are sufficiently small: For each plaque ρ in a fixed plaquation of \mathcal{L} , the local strong unstable manifolds $W^u(x, R)$ with $x \in \rho$ give a tubular neighborhood of ρ in the local center unstable manifold.

Suppose that the center holonomy maps h inside the center unstable manifolds fail to be uniformly θ -Hölder. Then we can find such sequences $L_n, p_n, p'_n, \xi_n, a_n, a'_n, A_n, \alpha_n, b_n, b'_n, B_n, \beta_n, C_n$ such that

$$d'_n > C_n d_n^\theta \quad \text{and} \quad C_n \rightarrow \infty$$

where $d_n = d(a_n, b_n)$, $d'_n = d(a'_n, b'_n)$, and distance is measured along the local strong unstable manifolds. Since d'_n is bounded (by $2r'$) and $C_n \rightarrow \infty$, we get $d_n \rightarrow 0$. See Figure 11. Composing everything with f^k produces sequences

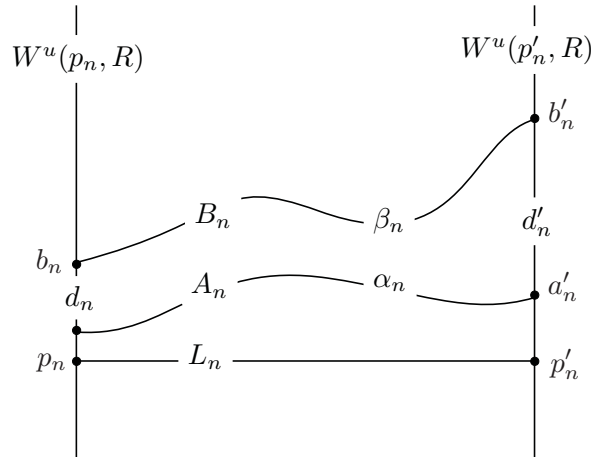


FIGURE 11. d_n is much less than d'_n .

$L_{n,k}, \dots, \beta_{n,k}$. Set

$$d_{n,k}(t) = d(\alpha_{n,k}(t), \beta_{n,k}(t)).$$

For each n let $k = k(n)$ be the smallest integer such that

- (a) $0 \leq t \leq 1$ and $0 \leq j \leq k$ imply

$$\beta_{n,j}(t) \in W^u(\alpha_{n,j}(t), R).$$

- (b) There exists $t \in [0, 1]$ such that $d_{n,k}(t) \geq r$.

The original paths α_n, β_n satisfy (a), f expands the unstable manifolds by a factor between λ and ω , and $\beta_n(t) \in W^u(\alpha_n(t), R)$, so k exists. (Here we use the assumption that $\Omega r' \leq R/2$.) Let $t = T_{n,k}$ be the smallest t such that

$$d(\alpha_{n,k}(t), \beta_{n,k}(t)) = r.$$

We claim that for large n , $T_{n,k}$ exists and $0 < T_{n,k} \leq 1$. See Figure 12. This is the heart of the proof.

Let $d_{n,j} = d(a_{n,j}, b_{n,j})$ and $d'_{n,j} = d(a'_{n,j}, b'_{n,j})$, where $a_{n,j} = f^j(a_n)$, $a'_{n,j} = f^j(a'_n)$, $b_{n,j} = f^j(b_n)$, $b'_{n,j} = f^j(b'_n)$, and $0 \leq j \leq k$. Then

$$\lambda < \frac{d_{n,j+1}}{d_{n,j}} < \omega \quad \lambda < \frac{d'_{n,j+1}}{d'_{n,j}} < \omega$$

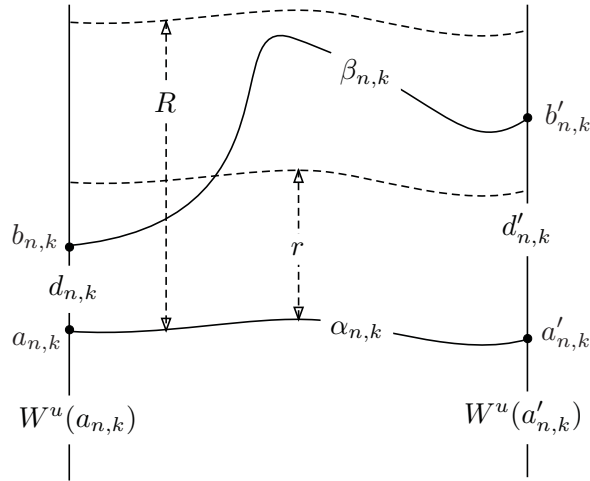


FIGURE 12. For $k = k(n)$ we have $d(\alpha_{n,k}(t), \beta_{n,k}(t)) = r$ for some $t \in (0, 1]$.

An over-estimate for $d_{n,j}$ imagines the expansion occurs at the fast rate ω . This gives $d_{n,j} \leq \omega^j d_n$ as long as $d_{n,j} \leq R$. Similarly, $d'_{n,j} \geq \lambda^j d'_n$. We claim that if n is large and $0 \leq j \leq k$ then we have $d_{n,j} < r$. This follows from formula manipulation. First of all, the hypothesis $\omega^\theta < \lambda$ implies that

$$\frac{\omega}{\lambda^{1/\theta}} < 1.$$

Then $\lambda^j d'_n \leq d'_{n,j} \leq R/2$ for $0 \leq j \leq k$ and $d'_n \geq C_n d_n^\theta$ imply that

$$\begin{aligned} d_{n,j} &\leq \omega^j d_n \leq \omega^j \left(\frac{d'_n}{C_n}\right)^{1/\theta} \leq \omega^j \left(\frac{d'_{n,j}}{\lambda^j C_n}\right)^{1/\theta} \\ &\leq \left(\frac{\omega}{\lambda^{1/\theta}}\right)^j \left(\frac{R}{2C_n}\right)^{1/\theta}. \end{aligned}$$

Since R is fixed and $C_n \rightarrow \infty$, this quantity tends to 0 as $n \rightarrow \infty$. Therefore, for all large n and all j , $0 \leq j \leq k = k(n)$,

$$d_{n,j} < r$$

and $d_{n,k} \rightarrow 0$ as $n \rightarrow \infty$.

The upshot is this: The first k iterates of f spread the pair (a'_n, b'_n) apart to distance $\geq r$ but don't spread the pair (a_n, b_n) apart much at all. By the Intermediate-Value Theorem there is a smallest $t \in (0, 1]$ with $d_{n,k}(t) = r$. This is $T_{n,k}$. Then we linearly reparameterize $\alpha_{n,k}$ and $\beta_{n,k}$ by $t \rightarrow t/T_{n,k}$ so that $T_{n,k}$ becomes 1 and, using the same notation for the reparameterized paths, we have $(a'_{n,k}, b'_{n,k}) = (\alpha_{n,k}(1), \beta_{n,k}(1))$ and

$$d(\alpha_{n,k}(1), \beta_{n,k}(1)) = r.$$

Since the lamination is uniformly compact there is a subsequence having a lot of convergence. For example, a subsequence of $f^{k(n)}(a_n)$ converges to some $p \in \Lambda$. Formally we should write $f^{k(n_\ell)}(a_{n_\ell}) \rightarrow p$ as $\ell \rightarrow \infty$ but we abbreviate it to $a_m \rightarrow p$ as $m = (n_\ell, k(n_\ell)) \rightarrow \infty$.

Since $d(a_m, b_m) \rightarrow 0$, b_m also converges to p . Set

$$d_m(t) = d(\alpha_m(t), \beta_m(t)).$$

Then $d_m(0) \rightarrow 0$ and $d_m(1) = r$ as $m \rightarrow \infty$. Let P be the leaf of \mathcal{L} through p , and let $\pi: U \rightarrow P$ be a small C^1 tubular neighborhood of P in M . We choose π so that its fibers at P are approximately parallel to $E_p^{us} = E_p^u \oplus E_p^s$ and have diameter $< r/2$. (In fact, by the Whitney Extension Theorem we can find π so that the $T_x(\pi^{-1}(x)) = E_x^{us}$ for all $x \in P$.)

Theorem 12 implies that P has a laminated neighborhood $N \subset \Lambda \cap U$, and N is much smaller than U . For each leaf $Q \subset N$, $\pi: Q \rightarrow P$ is a covering map. Since the leaves A_m, B_m contain points near P , they are wholly contained in N , and they cover P under π . The points $\alpha_m(t), \beta_m(t)$ may not lie on a common π -fiber, but we can project $\beta_m(t)$ along the plaque of B_m containing $\beta_m(t)$ to make this true. Let $\beta_m^*(t)$ be the projected path and set

$$d_m^*(t) = d(\alpha_m(t), \beta_m^*(t)).$$

Because the π -fibers are approximately tangent to E_p^{us} , $W^u(\alpha_m(t))$ is approximately parallel to $E_{\alpha_m(t)}^u$, and since

$$E_{\alpha_m(t)}^u \subset E_{\alpha_m(t)}^{us} \approx E_{\pi(\alpha_m(t))}^{us},$$

we have $d_m^*(t) \approx d_m(t)$ for $0 \leq t \leq 1$. Setting $b_m^* = \beta_m^*(1)$ gives

$$d(a'_m, b_m^*) = d_m^*(1) \approx d_m(1) = d(a'_m, b'_m) = r$$

which contradicts the fact that a'_m and b_m^* lie in a set of diameter $\leq r/2$. See Figure 13. Therefore the center holonomy maps along the center unstable manifolds are uniformly θ -Hölder.

Correspondingly, the center holonomy maps along the center stable manifolds are uniformly θ -Hölder when $\nu < \mu^\theta$. By the triangle inequality and dynamical coherence, the center holonomy maps are uniformly θ -Hölder when $\nu < \mu^\theta$ and $\hat{\nu} < \hat{\mu}^\theta$. See Figure 14.

Proposition 13 implies that \mathcal{L} is plaque expansive, so a canonical leaf conjugacy $h_g: \mathcal{L} \rightarrow \mathcal{L}_g$ exists when g C^1 -approximates f . We claim that h_g is θ -Hölder.

Reverting to the suspension considerations used in the proof of Theorem A, we have a C^1 -small homotopy loop at f , $t \mapsto g_t$, where $0 \leq t \leq 2$, $g_0 = g_2 = f$, and $g_1 = g$. The suspension diffeomorphism $G: S^1 \times M \rightarrow S^1 \times M$ is defined as $G(t, x) = (t, g_t(x))$ where S^1 is the circle of circumference 2. G C^1 -approximates the product diffeomorphism $F(t, x) = (t, f(x))$, which is normally hyperbolic at the product lamination $T \times \mathcal{L}$, while G is normally hyperbolic at the leaf conjugate suspension lamination \mathcal{S}_G . The leaves of the latter are uniformly compact, so, according to what was proved above, the \mathcal{S}_G -holonomy maps are θ -Hölder.

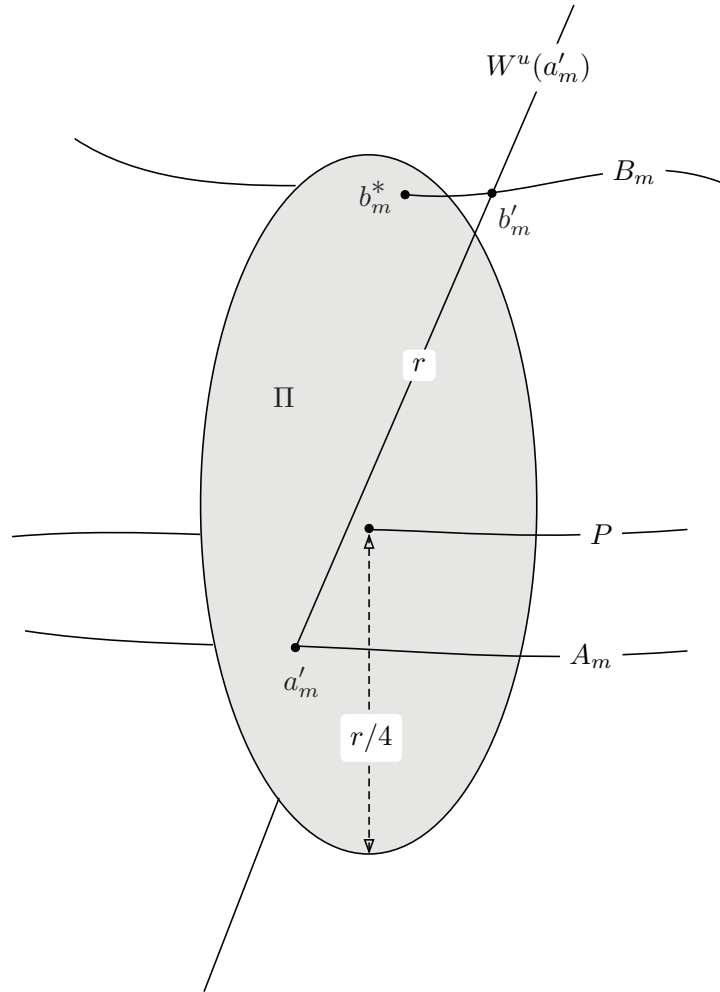


FIGURE 13. The point $b'_m = \beta_m(1)$ projects along B_m to b_m^* , so its distance to a'_m is approximately r , a contradiction to the fact that the pair a'_m, b_m^* lies in a set $\Pi = \pi^{-1}(\pi(a'_m))$ of diameter $\leq r/2$.

According to Theorem 8 and Addendum 9, one of these holonomy maps locally represents a leaf conjugacy from \mathcal{L} to \mathcal{L}_g , and this completes the proof. \square

REMARK. The geometric situation may be much more complex than a tubular neighborhood of P with nearby leaves projecting diffeomorphically to P . We need the Hölder estimate on transversals of uniformly positive radius. It is quite possible that L , A , and B double back on themselves and each other, repeatedly crossing a transversal. Their various branches may lie much closer to ξ than α and β do. These other branches may shadow ξ for a while and then

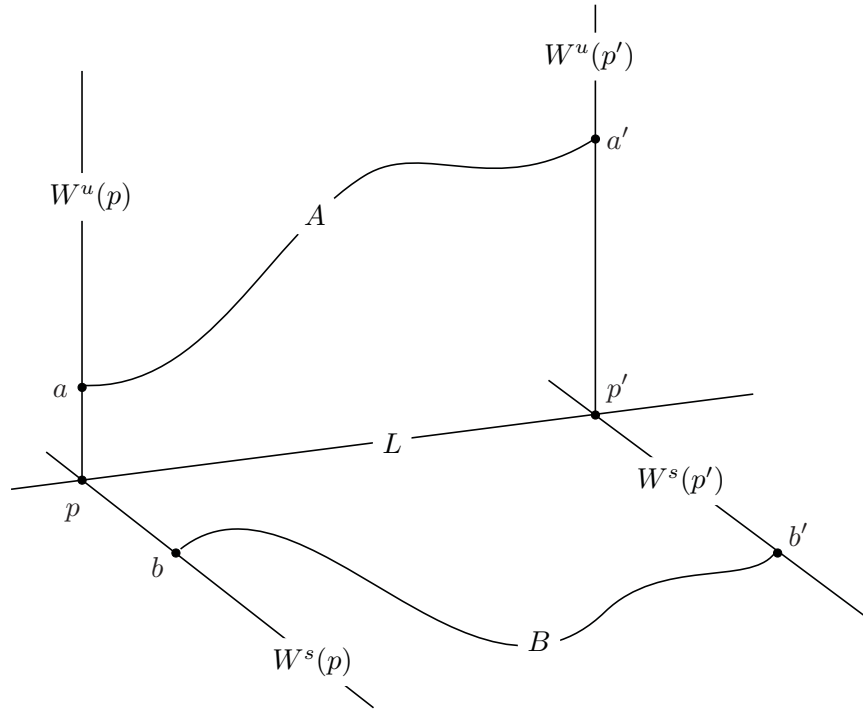


FIGURE 14. A, B are the center leaves through a, b . The distance between a' and b' is no more than $d(a', p') + d(p', b')$ which is Hölder controlled.

leave its neighborhood. The upshot is that we get R -sized tubular neighborhoods of the plaques but not of the leaves. See Remarks 4 and 5 of the next section.

11. CAUTIONARY REMARKS

REMARK 1. It is natural to ask whether there is an “Intersection Lemma” à la Theorem 7 for leaf conjugacies: For transverse foliations \mathcal{F} and \mathcal{G} intersecting in the foliation \mathcal{H} , can one deduce from the existence of a Hölder-continuous \mathcal{F} -conjugacy and a Hölder-continuous \mathcal{G} -conjugacy the existence of a Hölder-continuous \mathcal{H} -conjugacy? Such a general lemma would simplify considerably some of the arguments in this paper, but it appears that such a result cannot hold in complete generality. Here is a more detailed discussion.

In the course of proving Theorem A, we showed directly in Proposition 11 that there are Hölder-continuous leaf conjugacies for the center stable and center unstable foliations. It is tempting to try to combine these leaf conjugacies to obtain directly a leaf conjugacy for the intersection foliation $\mathcal{W}^c = \mathcal{W}^{cu} \cap \mathcal{W}^{cs}$. The issue is that tubular neighborhood structures for the two conjugacies—which consist of local unstable and stable manifolds—in general are not jointly integrable. They do not combine to give a tubular neighborhood structure for

the intersection foliation. One can choose a different tubular neighborhood structure for \mathcal{W}^c , one that is locally bifoliated by tubular neighborhoods for \mathcal{W}^{cs} and \mathcal{W}^{cu} , but then the question arises whether Hölder-continuity of the leaf conjugacy for one tubular neighborhood structure implies Hölder-continuity of the leaf conjugacy for *every* tubular neighborhood structure. The answer to this question, at least when posed in the setting of abstract foliations, is “no” as the following example shows.

Let \mathcal{F}_0 foliate the strip $\mathbb{R} \times [0, 1]$ in \mathbb{R}^2 by horizontal curves in such a way that the holonomy maps between vertical transversals are not Hölder-continuous. (As above, the proof of Theorem 4.3 in [19] shows that the choice of transversals has no effect on Hölderness of holonomy, so the holonomy maps with respect to all other transversals are also non-Hölder.) The leaf of \mathcal{F}_0 through $(0, y)$ is given by the graph of a smooth function $x \mapsto g_0(x, y)$, where for fixed $x \neq 0$, the map $y \mapsto g_0(x, y)$ is continuous but *not* Hölder-continuous. Arrange as well that the top and bottom leaves of \mathcal{F}_0 are horizontal, i.e., $g_0(x, 0) = 0$ and $g_0(x, 1) = 1$ for all x . Extend g_0 to \mathbb{R}^2 by setting

$$g(x, y + n) = g_0(x, y) + n$$

when $y \in [0, 1]$ and $n \in \mathbb{Z}$. Let \mathcal{F} be the foliation of \mathbb{R}^2 whose leaf through $(0, y)$ is $\{(x, g(x, y)) : x \in \mathbb{R}\}$. See Figure 15.

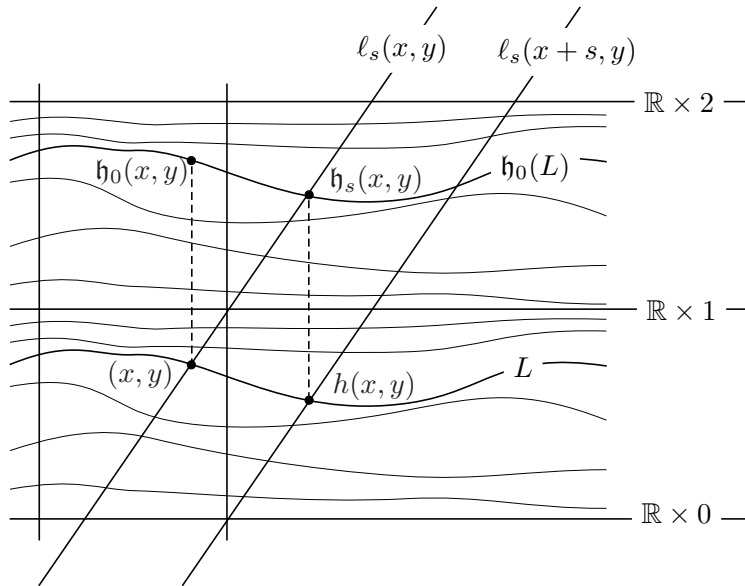


FIGURE 15. h_0 is vertical translation by 1. It commutes with the non-Hölder holonomy h from $\ell_s(x, y)$ to $\ell_s(x + s, y)$.

Vertical translation $(x, y) \mapsto (x, y + 1)$ is a smooth leaf conjugacy $h_0: \mathcal{F} \rightarrow \mathcal{F}$. It respects the vertical normal bundle. But if we use a different normal bundle things go bad. Let \mathcal{N}_s be the normal bundle whose fiber through (x, y) is the

line

$$\ell_s(t, x, y) = (x + st, y + t)$$

with vertical slope $s \neq 0$. Expressing the smooth leaf conjugacy h_0 with respect to \mathcal{N}_s gives a leaf conjugacy $h_s: \mathcal{F} \rightarrow \mathcal{F}$. It is a homeomorphism of \mathbb{R}^2 , smooth along the leaves of \mathcal{F} , but it is not transversely Hölder because

$$h_s(x, y) = h_0 \circ h(x, y)$$

where $h: \ell_s(x, y) \rightarrow \ell_s(x + s, y)$ is \mathcal{F} -holonomy. See Figure 15.

Hence, one needs to know something about holonomies to say anything about leaf conjugacies. And the smoothness of the conjugacy depends on the choice of tubular neighborhood structure if the foliation itself is not good.

REMARK 2. The hypothesis in the Intersection Lemma (Lemma 6) is unnecessarily strong. It requires all the holonomy maps of \mathcal{F} and \mathcal{G} to be Hölder in order that the intersection foliation $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ has Hölder holonomy. The following example shows we only need *some* of the holonomy maps to be Hölder.

Consider the unit cube I^3 with transverse foliations $\mathcal{F}_0, \mathcal{G}$ where the leaves of \mathcal{F}_0 are the horizontal squares $I^2 \times z$ and the leaves of \mathcal{G} are the vertical squares $x \times I^2$. The intersection foliation has segment leaves $x \times I \times z$. Approximate \mathcal{F}_0 by a foliation \mathcal{F} which meets every transversal $\tau = I \times y \times I$ in a family of curves shown in Figure 16. Choose \mathcal{F} so that its leaves are smooth but the Poincaré map $z \mapsto \varphi(z)$ of the flow shown on the transversals is non-Hölder.

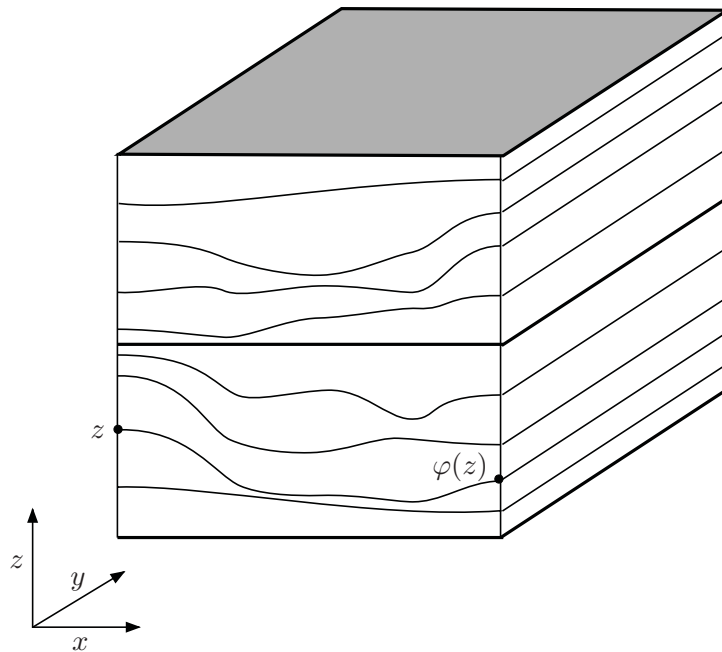


FIGURE 16. The intersection of \mathcal{F} with the faces of the cube.

Under the identifications that convert T^3 to the 3-torus, we get foliations \mathcal{F} and \mathcal{G} . The leaves of \mathcal{G} are “vertical” 2-tori. \mathcal{F} has two “horizontal” 2-torus leaves A, B . They correspond to the top/bottom face of the cube and the middle slice. The other leaves are cylinders that limit on A and B . The intersection foliation \mathcal{H} consists of circles $x \times S^1 \times z$. The \mathcal{H} -holonomy is the identity map, but the \mathcal{F} -holonomy includes the non-Hölder map φ . Thus \mathcal{F} and \mathcal{G} can have some bad holonomy although $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ has all good holonomy.

REMARK 3. In the proof of Theorem B we derived a contradiction from the assumption that the \mathcal{L} -holonomy is not Hölder. This involved the local center unstable and local center stable laminations. It might have seemed more natural to prove that \mathcal{W}^{cu} and \mathcal{W}^{cs} are Hölder and apply the Intersection Lemma to deduce that $\mathcal{L} = \mathcal{W}^{cu} \cap \mathcal{W}^{cs}$ is Hölder. However, \mathcal{W}^{cu} and \mathcal{W}^{cs} are only locally invariant and locally normally hyperbolic. The Intersection Lemma does not directly apply in this local situation.

REMARK 4. For quite a while we were confused about the relation between leaf expansivity and plaque expansivity for normally hyperbolic foliations in the uniformly compact case. If there is a $\delta > 0$ such that for each pair of distinct leaves, there is an iterate f^k of the normally hyperbolic diffeomorphism such that the distance between the f^k -iterates of the leaves exceeds δ then f is *leaf expansive*. A skew product (with compact fiber as in Theorem 2) over a hyperbolic set has this property. For the base map on the hyperbolic set is orbit expansive. It is obvious that leaf expansivity implies plaque expansivity. The converse, however, is false.

The example occurs on a 3-manifold. A similar example was used for other purposes by Bonatti and Wilkinson in [3]. Let M be $T^2 \times [0, 1]$ with $(x, y, 0)$ identified to $(-x, -y, 1)$. M is smooth and is double covered by the 3-torus. The vertical foliation $\{p \times [0, 1] : p \in T^2\}$ descends to a smooth, uniformly compact foliation \mathcal{F} of M by circles.

The standard Cat Map $f_A: T^2 \rightarrow T^2$ given by the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

lifts to a diffeomorphism $f: M \rightarrow M$,

$$f(z, t) = (f_A(z), t),$$

since $A(-v) = -A(v)$ for all $v \in \mathbb{R}^2$. It is normally hyperbolic and dynamically coherent at \mathcal{F} . We claim that f is plaque expansive but not leaf expansive.⁴

In [15] it is proved that every smooth normally hyperbolic foliation is plaque expansive, so \mathcal{F} is plaque expansive. A direct proof appears below.

⁴ \mathcal{F} is a Seifert fibration whose leaf space is the 2-sphere $T^2 / (x,y) \sim (-x,-y)$. The leaf map is a “two pronged pseudo Anosov” map on S^2 . The leaf L_0 through the origin is a circle of length 1. It is fixed by f . The leaves L_1, L_2, L_3 through $p_1 = (1/2, 0), p_2 = (0, 1/2), p_3 = (1/2, 1/2)$ are also circles of length 1. They are permuted cyclically by f as $L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow L_1$. All the other leaves are circles of length 2. The unstable and stable manifolds of the four special leaves are Möbius bands and the rest are cylinders.

To check that f is not leaf expansive, consider points $p, -p \in T^2$ near the origin. The \mathcal{F} -leaf through $(p, 0)$ is a circle of length 2 in M consisting of $p \times [0, 1]$ and $(-p) \times [0, 1]$. The local f_A -invariant manifolds of $p, -p$ meet at points $q, -q$ as shown in Figure 17. The leaves $P, Q \in \mathcal{F}$ corresponding to $\{p, -p\}$ and $\{q, -q\}$

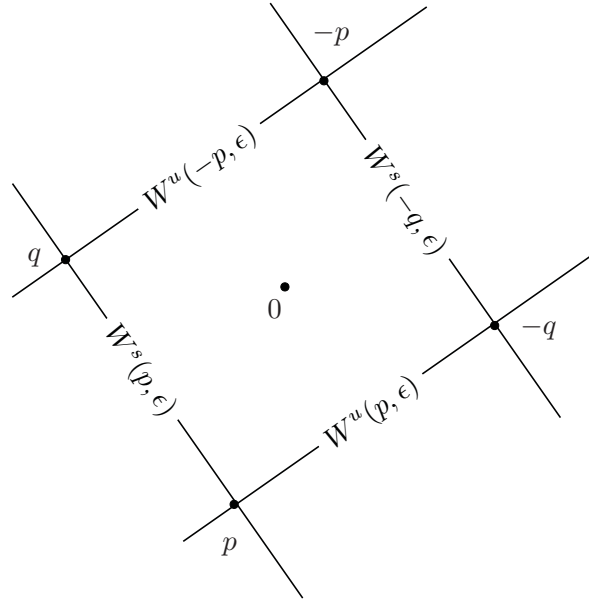


FIGURE 17. The Cat Map separates orbits of single points but does not separate orbits of pairs of points such as $\{p, -p\}$ and $\{q, -q\}$.

fail to separate under f -iteration. For under forward iterates, $f_A^k(p)$ and $f_A^k(q)$ are asymptotic, while under reverse iteration $f_A^k(p)$ and $f_A^k(-q)$ are asymptotic.

Here is a sketch of a direct proof that f is plaque expansive. Take nonoverlapping, nearby plaques ρ, σ in leaves P, Q . (The leaves can be equal without the plaques overlapping.) This gives plaques

$$\xi = W^{cu}(\rho, \epsilon) \cap W^{cs}(\sigma, \epsilon) \quad \eta = W^{cu}(\sigma, \epsilon) \cap W^{cs}(\rho, \epsilon).$$

Under forward f -iteration, ξ and ρ separate while η and σ are asymptotic. Under reverse f -iteration it is the opposite. See Figure 18. Let (ρ_k) and (σ_k) be plaque orbits starting at ρ and σ . If σ meets $W^{cu}(\rho, \epsilon)$ then $\sigma \approx \xi$ and $d(\rho_k, \sigma_k) > \delta$ for a suitable $k > 0$. (By approximate equality $\sigma \approx \xi$ we mean that $\sigma \cap \xi$ is a plaque of approximately the same size as σ and ξ .) If σ meets $W^{cs}(\rho, \epsilon)$ then $\sigma \approx \eta$ and $d(\rho_\ell, \sigma_\ell) > \delta$ for a suitable $\ell < 0$. Finally, if σ meets neither $W^{cu}(\rho, \epsilon)$ nor $W^{cs}(\rho, \epsilon)$ then $d(\rho_k, \sigma_k) > \delta$ for a suitable $k > 0$ and $d(\rho_\ell, \sigma_\ell) > \delta$ for a suitable $\ell < 0$.

REMARK 5. A phenomenon that can occur with normally hyperbolic, uniformly compact foliations is that the local center unstable manifold of a leaf can contain multiple branches of that leaf and other leaves. This occurs in the previous

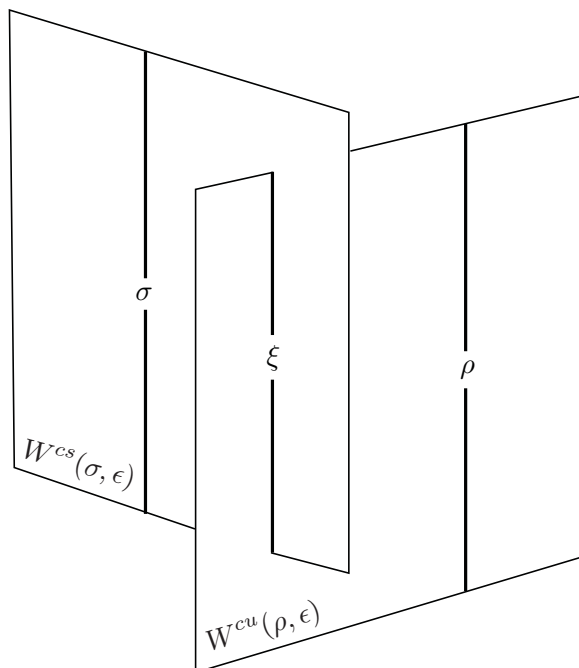


FIGURE 18. Plaque local product structure. The center unstable and center stable manifolds of nearby plaques intersect in plaques of approximately the same size.

example when the leaf lies in the local center unstable manifold of one of the special leaves—the circles of length 1. It is therefore difficult to assert in general that “under forward f -iteration, the center unstable manifold is overflowing.”

REMARK 6. As remarked above, each leaf of a uniformly compact foliation has a tubular neighborhood, but the radii of the tubular neighborhoods need not be bounded away from zero. It is tempting to expect that if these radii are indeed bounded away from zero then the foliation is nearly a skew product.

REMARK 7. Dynamical coherence was used in the proofs of plaque expansivity for normally hyperbolic, uniformly compact laminations (Proposition 13) and Hölderness of the leaf conjugacy (Theorem B). It appears to be a challenging task to see whether dynamical coherence is really necessary. The question is related to the concept in [5] of a foliation being complete. This means that nearby leaves do not splay apart infinitely, as do the orbits of an Anosov flow. Rather, they are somewhat parallel. Obviously, the leaves of a uniformly compact foliation have this completeness property, but we do not know about the intersections of their center unstable and center stable manifolds. The dynamical coherence assumption circumvents the problem.

UPSHOT. The structure of uniformly compact, normally hyperbolic foliations is yet to be well understood.

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