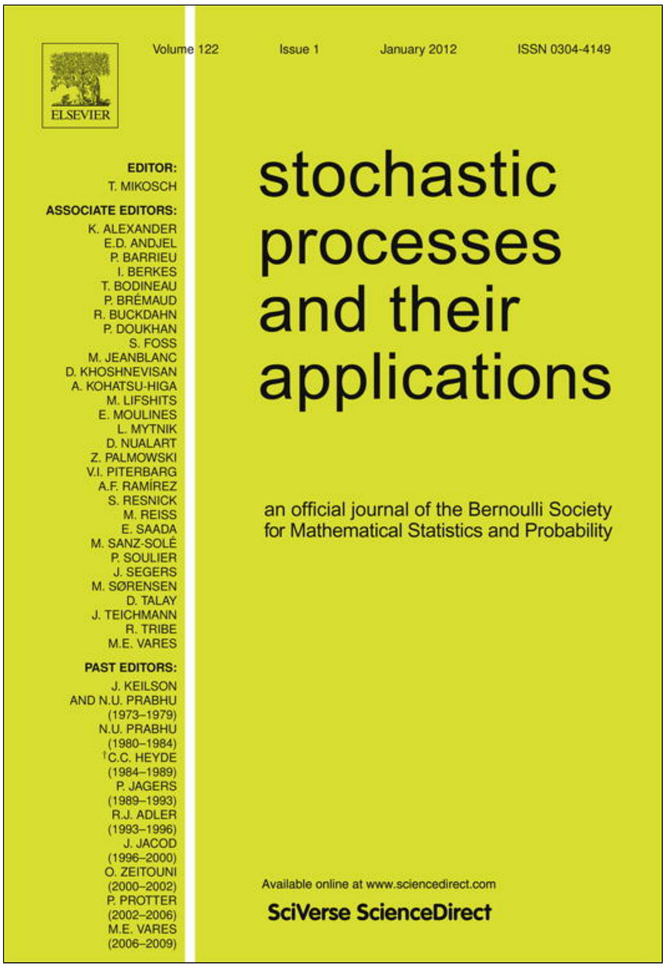
Volume 122 Issue 1 January 2012 ISSN 0304-4149
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Stochastic Processes and their Applications 122 (2012) 2185–2210
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Harmonic deformation of Delaunay triangulations

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Received 9 December 2010; received in revised form 2 September 2011; accepted 5 February 2012
Available online 11 February 2012

Abstract

We construct harmonic functions on random graphs given by Delaunay triangulations of ergodic point processes as the limit of the zero-temperature harness process.

MSC: 60F17; 60G55; 60K37

Keywords: Harness process; Point processes; Harmonic functions on graphs; Corrector

1. Introduction

Let S be an ergodic point process on $\mathbb{R}^d$ with intensity 1 and S° its Palm version. Call $\mathcal{P}$ and $\mathcal{E}$ the probability and expectation associated to S and S° (we think that S and S° are defined on a common probability space). The Voronoi cell of a point s in S° is the set of sites in $\mathbb{R}^d$ that are closer to s than to any other point in S°. Two points are *neighbors* if the intersection of the closure of the respective Voronoi cells has dimension $d-1$. The graph with vertices S° and edges given by pairs of neighbors is called the Delaunay triangulation of S°. The goal is to construct a function

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2012.02.003

$H: S^\circ \to \mathbb{R}^d$ such that the graph with vertices $H(S^\circ)$ and edges $\{(H(s), H(s')), s$ and s' are neighbors$\}$ has the following properties: (a) each vertex $H(s)$ is in the barycenter of its neighbors and (b) $|H(s) - s|/|s|$ vanishes as $|s|$ grows to infinity along any straight line. If such an H exists, the resulting graph is the *harmonic deformation* of the Delaunay triangulation of S°. The search of such H has been proposed by Biskup and Berger [5], who proved its existence in the graph induced by the supercritical percolation cluster in $\mathbb{Z}^d$; their approach was the motivation of this paper. The harmonic function H was tacitly present in Sidoravicius and Sznitman [22] and in Mathieu and Piatnitski [21]; the function $H(s) - s$ is called *corrector*. See also Caputo, Faggionato and Prescott [9] for a percolation-type graph in point processes on $\mathbb{R}^d$.

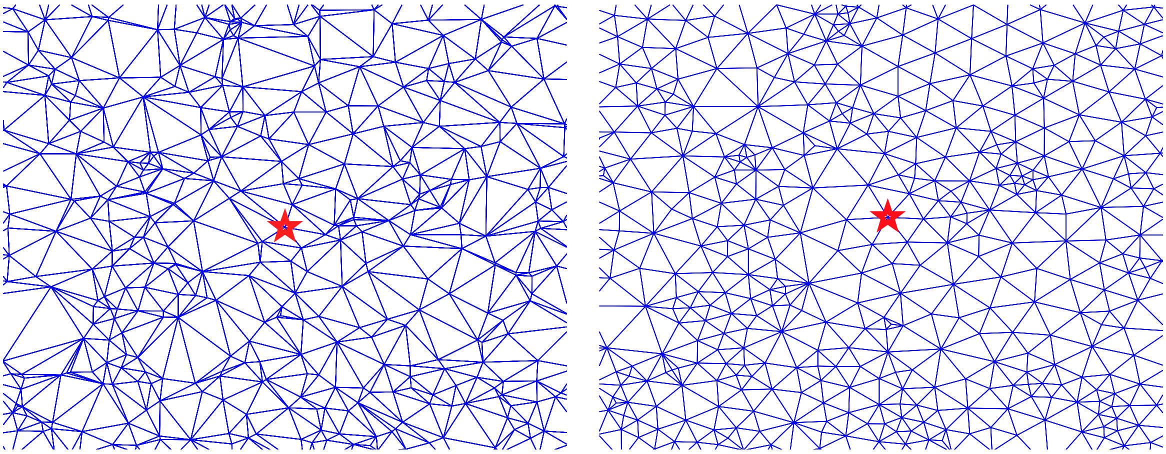

Fig. 1. Delaunay triangulation of a Poisson process and its harmonic deformation. The star indicates the origin (left) and the point $H(0)$ (right).

The functions from S° to $\mathbb{R}$ are called *surfaces*. The coordinates $h_1, \ldots, h_d$ of H are *harmonic* surfaces; that is $h_i(s)$ is the average of $\{h_i(s'), s'$ neighbor of $s\}$. The sublinearity of the corrector, requirement (b) above, amounts to ask that h_i have *tilt* e_i, the i-th canonical vector of $\mathbb{R}^d$. Roughly speaking, a surface f has *tilt* u (a unit vector) if $(f(K\tilde{u}) - K\tilde{u} \cdot u)/K$ converges to zero as K goes to $\pm$infinity for every $\tilde{u} \in \mathbb{R}^d$ (see [6,8,15]).

Fixing a direction u, we construct a harmonic surface h with tilt u as the limit (and a fixed point) of a stochastic process introduced by Hammersley called the harness process, [14,18]. The process is easily described by associating to each point s of S° a one-dimensional homogeneous Poisson process of rate 1. Fix an initial surface η_0 and for each point s at the epochs τ of the Poisson process associated to s update $\eta_\tau(s)$ to the average of the heights $\{\eta_{\tau-}(s'), s'$ is a neighbor of $s\}$. It is clear that if h is harmonic, then h is invariant for this dynamics. We start the harness process with $\eta_0 = \gamma$, the hyperplane defined by $\gamma(s) = s_i$, the i-th coordinate of s and show that $\eta_t(\cdot) - \eta_t(0)$ converges to h in $L_2(\mathcal{P} \times P)$, where $\mathcal{P}$ is the law of the point configuration S° and P is the law of the dynamics.

We prove that the tilt is invariant for the harness process for each t and in the limit when $t \to \infty$. In a finite graph the average of the square of the height differences of neighbors is decreasing with time for the harness process. Since essentially the same happens in infinite volume, the gradients of the surface converge under the harness dynamics. It remains to show that: (1) the limit of the gradients is a gradient field and (2) the limit is harmonic. Both statements follow from almost sure convergence along subsequences.

A key ingredient of the approach is the expression of the tilt of a surface as the scalar product of the gradient of the surface with a specific field (see Section 4). This implies that the limiting surface has the same tilt as the initial one.

2. Preliminaries and main result

Point processes and harmonic surfaces. Let S be an ergodic point process on $\mathbb{R}^d$ with intensity 1; call $\mathcal{P}$ its law and $\mathcal{E}$ the associated expectation. The process S takes values in $\mathcal{N}$, the space of locally finite point configurations of $\mathbb{R}^d$; we use the notation $\mathbf{s}$ for point configurations in $\mathcal{N}$ and S for random point processes in $\mathcal{N}$. The elements s of $\mathbf{s}$ are called *points* and the elements x of $\mathbb{R}^d$ are called *sites*. In the same way we use $\mathcal{N}^\circ$ for the space of configurations in $\mathcal{N}$ with a point at the origin and $\mathbf{s}^\circ$ for point configurations in that space. Let S° denote the Palm version of S. We can think of S° as S conditioned to have a point in the origin. If S is Poisson, then $S^\circ = S \cup \{0\}$. We abuse the notation and use $\mathcal{P}$ and $\mathcal{E}$ to denote the law of S° and its associated expectation. For $\mathbf{s} \in \mathcal{N}$ let the *Voronoi cell* of $s \in \mathbf{s}$ be defined by $\mathrm{Vor}(s) = \{x \in \mathbb{R}^d : |x - s| \le |x - s'|$, for all $s' \in \mathbf{s} \setminus \{s\}\}$. If the intersection of the Voronoi cells of s and s' is a $(d-1)$-dimensional surface, we say that s and s' are *Voronoi neighbors*. We consider the random graph with vertices $\mathbf{s}$ and edges $\{(s, s') : s$ and s' are Voronoi neighbors in $\mathbf{s}\}$. If S° is the Palm version of a Poisson process, the graph is a triangulation a.s. called the *Delaunay triangulation* of S°. To a site $x \in \mathbb{R}^d$ we associate the *center* of the Voronoi cell containing x: $\mathrm{Cen}(x) = \mathrm{Cen}(x, \mathbf{s}) = s \in \mathbf{s}$ if $x \in \mathrm{Vor}(s)$; if x belongs to the Voronoi cell of more than one point, use lexicographic order of the coordinates (or any other rule) to decide who is the center. Let

$$\Xi_1 := \{(s, \mathbf{s}) \in \mathbb{R}^d \times \mathcal{N} : s \in \mathbf{s}\}$$
$$\Xi_2 := \{(s, s', \mathbf{s}) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{N} : s, s' \in \mathbf{s}\}.$$

Functions $\eta : \Xi_1 \to \mathbb{R}$ are called *surfaces* and functions $\zeta : \Xi_2 \to \mathbb{R}$ are called *fields*. Denote by τ_x the translation operator: for x in $\mathbb{R}^d$, $\tau_x \mathbf{s} := \{s - x : s \in \mathbf{s}\}$. If $\eta(s, \mathbf{s}) = \eta(0, \tau_s \mathbf{s})$ for every $s \in \mathbf{s}$ we say that η is a *translation invariant* surface. A field ζ is *covariant* if $\zeta(s' - s, s'' - s, \tau_s \mathbf{s}) = \zeta(s', s'', \mathbf{s})$ for all $s, s', s'' \in \mathbf{s}$. A field ζ is a *flux* if $\zeta(s, s', \mathbf{s}) = -\zeta(s', s, \mathbf{s})$ for all $s, s' \in \mathbf{s}$. The *conductances* induced by $\mathbf{s}$ is the field a defined by

$$a(s, s', \mathbf{s}) := \mathbf{1}\{s \text{ and } s' \text{ are Voronoi neighbors in } \mathbf{s}\}. \tag{2.1}$$

The *Laplacian* operator is defined on surfaces η by

$$\Delta\eta(s, \mathbf{s}) = \sum_{s' \in \mathbf{s}} a(s, s', \mathbf{s})[\eta(s', \mathbf{s}) - \eta(s, \mathbf{s})]. \tag{2.2}$$

The *gradient* of a surface η is the field defined by

$$\nabla\eta(s, s', \mathbf{s}) = a(s, s', \mathbf{s})[\eta(s', \mathbf{s}) - \eta(s, \mathbf{s})].$$

For fields $\zeta : \Xi_2 \to \mathbb{R}$ the *divergence* $\mathrm{div}\zeta : \Xi_1 \to \mathbb{R}$ is given by

$$\mathrm{div}\zeta(s, \mathbf{s}) = \sum_{s' \in \mathbf{s}} a(s, s', \mathbf{s})\zeta(s, s', \mathbf{s}).$$

Hence $\Delta\eta = \mathrm{div}\nabla\eta$. To simplify notation we may drop the dependence on the point configuration when it is clear from the context. The Laplacian, gradient and divergence depend on the conductances, but we drop this dependence in the notation, as they are fixed by (2.1) along the paper.

A surface h is called *harmonic* for $\mathbf{s} \in \mathcal{N}$ if $\Delta h(s, \mathbf{s}) = 0$ for all $s \in \mathbf{s}$.

Pointwise tilt. We say that for $\mathbf{s} \in \mathcal{N}$ a surface η has *tilt* $I(\eta, \mathbf{s}) = (I_{e_1}(\eta, \mathbf{s}), \dots, I_{e_d}(\eta, \mathbf{s}))$ if for each $u \in \{e_1, \dots, e_d\}$ the following limits for $K \to \pm\infty$ exist, coincide and do not depend on $x \in \mathbb{R}^d$

$$I_u(\eta, \mathbf{s}) := \lim_{K \to \pm\infty} \frac{\eta(\mathrm{Cen}(x + Ku), \mathbf{s}) - \eta(\mathrm{Cen}(x), \mathbf{s})}{K}. \tag{2.3}$$

Harness process. Given a surface η, let $M_s \eta$ be the surface obtained by substituting the height $\eta(s)$ with the average of the heights at the neighbors of s:

$$(M_s \eta)(s') = \begin{cases} \dfrac{1}{a(s)} \displaystyle\sum_{s' \in \mathbf{s}} a(s, s') \eta(s') & \text{if } s' = s, \\ \eta(s') & \text{if } s' \neq s, \end{cases} \tag{2.4}$$

where $a(s) = \sum_{s' \in S} a(s, s')$. Take a point configuration $\mathbf{s}$ and define the generator

$$L_{\mathbf{s}} f(\eta) = \sum_{s \in \mathbf{s}} [f(M_s \eta) - f(\eta)]. \tag{2.5}$$

That is, at rate 1, the surface height at s is updated to the average of the heights at the neighbors of s. We construct this process as a function of a family of independent one-dimensional Poisson processes $T = (T_n, n = 1, 2 \dots)$ with law P. Take an arbitrary enumeration of the points, $\mathbf{s} = (s_n, n \geq 1)$ (for instance, s_n may be the n-th closest point to the origin) and update the surface at s_n at the epochs of T_n. When the point configuration is random, say S°, ask T to be independent of S° and define the process as above to obtain a process $(\eta_t, t \geq 0)$ as a function of (S°, T), with the product law $\mathcal{P} \times P$, and η_0. The resulting *noiseless harness process* is Markov on the space of surfaces with generator L_{S°. See Section 5 for a rigorous construction.

Assumptions. We assume that S is a stationary point process in $\mathbb{R}^d$ with Palm version S°, satisfying the following:

A1. The law of S is mixing.
A2. $\mathcal{P}(|S \cap \partial B| < d + 2$, for every ball $B \subset R^d) = 1$.
A3. $\mathcal{E} \exp(\beta a(0, S^\circ)) < \infty$ for some positive constant β. The number of neighbors of the origin has a finite positive exponential moment.
A4. $\mathcal{E}[(\ell_{d-1}(\partial \mathrm{Vor}(0, S^\circ)))^2] < \infty$. The $d-1$ Lebesgue measure of the boundary of the Voronoi cell of the origin has finite second moment.
A5. $\mathcal{E}[\sum_{s \in S^\circ} a(0, s)|s|^r] < \infty$ for some $r > 4$.
A6. $\mathcal{P}$ (S is periodic) $= 0$.

All these assumptions are satisfied if S is a homogeneous Poisson process. Assumption A1 guarantees "one dimensional" ergodicity as in (4.13) later. Assumption A2 is sufficient to define the Delaunay triangulation. Notice that A4 implies that the volume of the Voronoi cell of the origin has finite second moment: $\mathcal{E}[(\ell_d(\mathrm{Vor}(0, S^\circ)))^2] < \infty$.

Assumption A6 is used on the one hand in the Appendix to identify the motion of a random walk on the Delaunay triangulation with the motion of the environment as seen from the walker. On the other hand ergodicity and aperiodicity of the point process imply that there exist measurable functions $s_n : \mathcal{N} \to \mathbb{R}^d$ such that

B1. $s_{-n}(\tau_{s_n} S^\circ) = -s_n(S^\circ)$,

B2. $S^\circ = \{s_n(S^\circ); n \in \mathbb{Z}\}$, and
B3. $\tau_{s_n(S^\circ)} S^\circ$ has the same distribution as S° for every $n \in \mathbb{Z}$.

This is used to extend the properties of S° to $\tau_s S^\circ$, for all $s \in S^\circ$. The point is that $\tau_s S^\circ$ has the same law as S° only if s is correctly chosen as was shown in [13,20] for Poisson processes and by Timár [23] under the condition that S is ergodic and $\mathcal{P}$-a.s. aperiodic; see Heveling and Last [19].

Theorem 2.1. *Let S° be the Palm version of the stationary point process satisfying* A1–A6 *and let γ be a surface with covariant gradient, tilt $I(\gamma) \in \mathbb{R}^d$ and $\mathcal{C}(|\nabla\gamma|^r) < \infty$ for some $r > 4$. Then:* (a) *There exists a harmonic surface h with $h(0, S^\circ) = 0$ and $I(h) = I(\gamma)$ $\mathcal{P}$-a.s.* (b) *if η_t is the harness process with initial condition γ, then,*

$$\lim_{t\to\infty} \mathcal{E}E[\eta_t(s_n) - \eta_t(0) - h(s_n)]^2 = 0, \tag{2.6}$$

for any $n \in \mathbb{Z}$, with s_n as in B1–B3. (c) *In dimensions $d = 1$ and $d = 2$, h is the only harmonic surface with covariant gradient and tilt $I(\gamma)$.*

Let $c \in \mathbb{R}^d$; the *hyperplane* $\gamma(s, S^\circ) = c \cdot s$, $s \in S^\circ$ satisfies the hypotheses of the theorem with $I(\gamma) = c$. Items (a) and (b) of the theorem say that a surface with tilt c evolving along the harness process and seen from the height at the origin converges in $L_2(\mathcal{P} \times P)$ to a harmonic surface h with the same tilt and with $h(0) = 0$.

Let $H = (h_1, \ldots, h_d)$, where h_i is the harmonic surface obtained in Theorem 2.1 for the tilt e_i. The graph with vertices $H(S^\circ) = (H(s),\ s \in S^\circ)$ and conductances $\tilde{a}(H(s), H(s')) := a(s, s')$ is harmonic:

$$H(s) = \frac{1}{a(s)} \sum_{s' \in S^\circ} a(s, s')\, H(s') \tag{2.7}$$

that is, each point is in the barycenter of its neighbors in the neighborhood structure induced by the Delaunay triangulation of S°. This graph, called the *harmonic deformation* of the Delaunay triangulation, does not coincide with the Delaunay triangulation of $H(S^\circ)$.

Random walks in random graphs and martingales. Let $Y_t = Y_t^{S^\circ}$, be the random walk on S° which jumps from s to s' at rate $a(s, s')$. Since $H(S^\circ)$ is harmonic, the random walk $H(Y_t)$ on $H(S^\circ)$ is a martingale and it satisfies the conditions of the martingale central limit theorem [12, p. 417]. So, the invariance principle holds for $H(Y_t)$. The extension of the invariance principle from the walk $H(Y_t)$ to the walk Y_t requires the sublinearity in $|s|$ of the corrector $\chi(s) = H(s) - s$.

Corrector. Mathieu and Piatnitski [21] and Berger and Biskup [5] construct the corrector for the graph induced by the supercritical percolation cluster in $\mathbb{Z}^d$. Both papers prove sharp bounds on the asymptotic behavior of the corrector and, as a consequence, the quenched invariance principle for Y_t for every dimension $d \geq 2$. Key ingredients in those proofs are heat kernels estimates obtained by Barlow [1] (in [5] they are used just for $d \geq 3$). Sidoravicius and Sznitman [22] also used the corrector to obtain the quenched invariant principle for $d \geq 4$. Several papers obtain generalizations of similar results on subgraphs of $\mathbb{Z}^d$ [2,7,21]. Caputo, Faggionato and Prescott [9] use the corrector to prove a quenched invariance principle for random walks on random graphs with vertices in an ergodic point process on $\mathbb{R}^d$ and conductances governed by i.i.d. energy marks.

Uniqueness. The uniqueness of (the gradients of) a harmonic function with a given tilt has been proved by Biskup and Spohn [8] for the graph with conductances associated to the bonds of $\mathbb{Z}^d$ under "ellipticity conditions" (see (5.1) and Section 5.2 in that paper) and by Biskup and Prescott [7] in the bond percolation setting in $\mathbb{Z}^d$ using "heat kernel" estimates, see Section 7 later.

We obtain harmonic surfaces as limits of the zero temperature harness process. The tilt of a surface is obtained as a scalar product with a specific field and it is invariant for the process. This allows us to show that the harmonic limits have the same tilt as the initial surface.

The paper is organized as follows. In Section 3 we give basic definitions, define the space $\mathcal{H}$ of fields as a Hilbert space and show a useful integration by parts formula. In Section 4 we show that the coordinates of the tilt of a surface can be seen as the inner product of its gradient with a specific field in $\mathcal{H}$. In Section 5 we describe the Harris graphical construction of the Harness process. In Section 6 we prove the main theorem. Section 7 deals with the uniqueness of the harmonic surface in $d = 2$.

3. Point processes, fields and gradients

Let $\mathcal{N} = \mathcal{N}(\mathbb{R}^d)$ be the set of all locally finite subsets of $\mathbb{R}^d$, that is, for all $\mathbf{s} \in \mathcal{N}$, $|\mathbf{s} \cap B|$, the number of points in $\mathbf{s} \cap B$, is finite for every bounded set $B \subset \mathbb{R}^d$. We consider the σ-algebra $\mathcal{B}(\mathcal{N})$, the smallest σ-algebra containing the sets $\{\mathbf{s} \in \mathcal{N} : |\mathbf{s} \cap B| = k\}$, where B is a bounded Borel set of $\mathbb{R}^d$ and k is a positive integer.

Cesàro means and the space $\mathcal{H}$. Let $\mathcal{C}$ be the measure in Ξ_2 defined on $\zeta : \Xi_2 \to \mathbb{R}$ by

$$\mathcal{C}(\zeta) = \int_{\Xi_2} \zeta \, d\mathcal{C} = \frac{1}{2} \mathcal{E} \sum_{s \in S^\circ} a(0, s, S^\circ) \zeta(0, s, S^\circ). \tag{3.1}$$

This measure is absolutely continuous with respect to the second order Campbell measure associated to $\mathcal{P}$ with density $Z(u, v, \mathbf{s}) = a(u, v, \mathbf{s}) \delta_0(u)$. The space $\mathcal{H} := L_2(\Xi_2, \mathbb{R}, \mathcal{C})$ is Hilbert with inner product $\mathcal{C}(\zeta \cdot \zeta')$, where the field $(\zeta \cdot \zeta')$ is defined by

$$(\zeta \cdot \zeta')(s, s', S^\circ) = a(s, s', S^\circ) \zeta(s, s', S^\circ) \zeta'(s, s', S^\circ).$$

If two fields ζ and ζ' coincide in the pairs $(0, s)$ for all s neighbor of the origin, then their difference has zero $\mathcal{C}$-measure and hence a field in $\mathcal{H}$ is characterized by its values at $((0, s), s$ neighbor of the origin). Define the equivalence relation $\zeta \sim \zeta'$ if and only if $\zeta(0, s, \mathbf{s}^\circ) = \zeta'(0, s, \mathbf{s}^\circ)$, for all neighbor s of the origin. Each class of equivalence in $\mathcal{H}$ has a canonical covariant representant obtained by $\zeta(s, s', \mathbf{s}) := \zeta(0, s' - s, \tau_s \mathbf{s})$ for $s, s' \in \mathbf{s}$. So hereafter, when we refer to a field in $\mathcal{H}$, we assume that it is the covariant representant.

The space $\mathcal{H}$ was previously considered by Mathieu and Piatnitski [21] when (S°, a) are given by the infinite cluster for supercritical percolation in $\mathbb{Z}^d$. The Hilbert structure of this space is useful to obtain weak convergence for the dynamics.

Define the Cesàro limit of a field $\zeta : \Xi_2 \to \mathbb{R}$ by

$$C(\zeta) := \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{1}{2|\Lambda|} \sum_{\{s, s'\} \cap \Lambda \neq \emptyset} a(s, s', S) \zeta(s, s', S), \tag{3.2}$$

where $\Lambda = \Lambda(K) := [-K, K]^d \subset \mathbb{R}^d$. Since S is ergodic, the Pointwise Ergodic Theorem [10, p. 318] implies $C(\zeta) = \mathcal{C}(\zeta)$, $\mathcal{P}$-a.s. Analogously, for translation invariant surfaces η we define its Cesàro mean $C(\eta)$ (with a slight abuse of notation) and we have $C(\eta) = \mathcal{C}(\eta) = \mathcal{E}(\eta(0, S^\circ))$.

Lemma 3.1 (*Mass Transport Principle [3,4,17,20]*). *Let $\zeta : \Xi_2 \to \mathbb{R}$ be a covariant field such that either ζ is nonnegative or $\mathcal{E} \sum_{s \in S^\circ} |\zeta(0, s, S^\circ)| < \infty$. Then*

$$\mathcal{E} \sum_{s \in S^\circ} \zeta(0, s, S^\circ) = \mathcal{E} \sum_{s \in S^\circ} \zeta(s, 0, S^\circ). \tag{3.3}$$

Proof. Let s_n be the maps introduced in B1–B3. Use B2 and Fubini in the first identity and covariance of ζ in the second one to obtain

$$\begin{aligned}
\mathcal{E} \sum_{s \in S^\circ} \zeta(0, s, S^\circ) &= \sum_{n \in \mathbb{Z}} \mathcal{E}\zeta(0, s_n(S^\circ), S^\circ) = \sum_{n \in \mathbb{Z}} \mathcal{E}\zeta(-s_n(S^\circ), 0, \tau_{s_n(S^\circ)} S^\circ) \\
&= \sum_{n \in \mathbb{Z}} \mathcal{E}\zeta(s_{-n}(\tau_{s_n(S^\circ)} S^\circ), 0, \tau_{s_n(S^\circ)} S^\circ) = \sum_{n \in \mathbb{Z}} \mathcal{E}\zeta(s_{-n}(S^\circ), 0, S^\circ) \\
&= \mathcal{E} \sum_{s \in S^\circ} \zeta(s, 0, S^\circ),
\end{aligned}$$

where we used B1 in the third identity, B3 in the fourth one and Fubini and B2 again in the fifth one. □

Lemma 3.2 (*Integration By Parts Formula*). *Let $\zeta \in \mathcal{H}$ be a flux and ϕ be a translation invariant surface satisfying $\mathcal{E}[a(0)\phi^2(0)] < \infty$. Then*

$$\mathcal{C}(\nabla\phi \cdot \zeta) = -\mathcal{C}(\phi \cdot \mathrm{div}\zeta). \tag{3.4}$$

Proof. Note that

$$\begin{aligned}
\mathcal{C}(\nabla\phi \cdot \zeta) &= \frac{1}{2}\mathcal{E} \sum_{s \in S^\circ} a(0, s, S^\circ)\nabla\phi(0, s, S^\circ)\zeta(0, s, S^\circ) \\
&= \frac{1}{2}\mathcal{E} \sum_{s \in S^\circ} a(0, s, S^\circ)\phi(s, S^\circ)\zeta(0, s, S^\circ) \\
&\quad - \frac{1}{2}\mathcal{E} \sum_{s \in S^\circ} a(0, s, S^\circ)\phi(0, S^\circ)\zeta(0, s, S^\circ) \\
&= \frac{1}{2}\mathcal{E} \sum_{s \in S^\circ} a(0, s, S^\circ)\phi(s, S^\circ)\zeta(0, s, S^\circ) - \frac{1}{2}\mathcal{E}[\phi(0, S^\circ)\mathrm{div}\zeta(0, S^\circ)].
\end{aligned}$$

Since ζ and a are covariant and ϕ is translation invariant, $a(s, s', S^\circ)\phi(s', S^\circ)\zeta(s, s', S^\circ)$ is covariant and Lemma 3.1 implies

$$\begin{aligned}
\mathcal{E} \sum_{s \in S^\circ} a(0, s, S^\circ)\phi(s, S^\circ)\zeta(0, s, S^\circ) &= \mathcal{E} \sum_{s \in S^\circ} a(s, 0, S^\circ)\phi(0, S^\circ)\zeta(s, 0, S^\circ) \\
&= -\mathcal{E} \sum_{s \in S^\circ} a(0, s, S^\circ)\phi(0, S^\circ)\zeta(0, s, S^\circ) \\
&= -\mathcal{E}[\phi(0, S^\circ)\mathrm{div}\zeta(0, S^\circ)].
\end{aligned}$$

We used that ζ is a flux and a is symmetric. □

4. Tilt

We define here the "integrated tilt" $\mathcal{J}(\eta)$ for surfaces η with covariant gradient $\nabla\eta \in \mathcal{H}$. The coordinates of $\mathcal{J}(\eta)$ are defined as the inner product of the gradient field $\nabla\eta$ with a conveniently chosen field. We then prove that the pointwise tilt $I(\eta, S^\circ)$ coincides with $\mathcal{J}(\eta)$, $\mathcal{P}$-a.s.

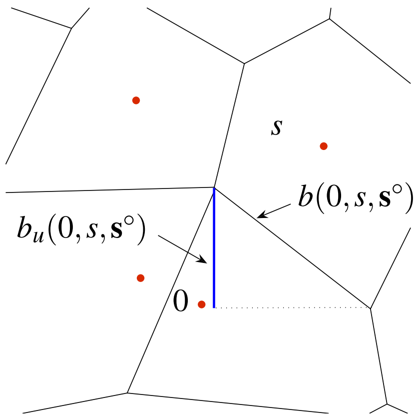

Fig. 2. Definition of the field ω_u for $u = e_1$.

Take a unit vector u and a point configuration $\mathbf{s}^\circ$. For neighbors s of the origin, let $b(0, s, \mathbf{s}^\circ)$ be the $(d-1)$-dimensional side in common of the Voronoi cells of 0 and s and let $b_u(0, s, \mathbf{s}^\circ)$ be the projection of $b(0, s, \mathbf{s}^\circ)$ over the hyperplane perpendicular to u, see Fig. 2. Define the field ω_u by

$$\omega_u(0, s, \mathbf{s}^\circ) := \mathrm{sg}(s \cdot u)\, a(0, s, \mathbf{s}^\circ)\, \ell_{d-1}(b_u(0, s, \mathbf{s}^\circ)), \tag{4.1}$$

where ℓ_{d-1} is the $(d-1)$-dimensional Lebesgue measure. By Assumption A4, $\omega_u \in \mathcal{H}$ and since $\nabla\eta$ is also in $\mathcal{H}$, we can define

$$\mathcal{J}_u(\eta) := \mathcal{C}(\nabla\eta \cdot \omega_u) \quad \text{and} \quad \mathcal{J}(\eta) := (\mathcal{J}_{e_1}(\eta), \ldots, \mathcal{J}_{e_d}(\eta)). \tag{4.2}$$

Proposition 4.1. *Let η be a surface with covariant $\nabla\eta \in \mathcal{H}$. Then*

$$I(\eta, S^\circ) = \mathcal{J}(\eta), \quad \mathcal{P}\text{-almost surely}. \tag{4.3}$$

Before proving the proposition we show a technical lemma. Let O_u be the $d-1$ dimensional hyperplane orthogonal to u: $O_u = \{y \in \mathbb{R}^d : y \cdot u = 0\}$.

For $y \in O_u$ let $l_u(y) = \{y + \alpha u; \alpha \in \mathbb{R}\}$, the line containing y with direction u. Fix $\mathbf{s} \in \mathcal{N}$, define $L_u(y, \mathbf{s}) := \{s \in \mathbf{s}: \mathrm{Vor}(s) \cap l_u(y) \neq \emptyset\}$, the set of centers of the Voronoi cells intersecting $l_u(y)$. Define $w: \mathbb{R}^d \times \Xi_2 \to \{0, 1\}$ by

$$w(y; s, s', \mathbf{s}) = \begin{cases} 1 & \text{if } b(s, s', \mathbf{s}) \cap l_u(y) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases}$$

the indicator that s and s' are neighbors and its boundary intersects the line $l_u(y)$. Define also $\theta: \mathbb{R}^d \times \Xi_1 \to \mathbb{R}$ by

$$\theta(y; s, \mathbf{s}) = \sum_{s' \in \mathbf{s}} s' a^+(s, s', \mathbf{s}) w(y; s, s', \mathbf{s}),$$

where $a^+(s, s', S) = a(s, s', S)\mathbf{1}\{(s' \cdot u) > (s \cdot u)\}$. In words, for $s \in L_u(y, \mathbf{s})$, $\theta(y; s, \mathbf{s})$ is the neighbor of s in the direction u such that their boundary intersects $l_u(y)$.

For $x \in \mathbb{R}^d$, let $x^* \in O_u$ be the projection of x over the hyperplane O_u. Observe that w satisfies

$$w(y; s, s', \mathbf{s}) = w(y - x^*; s - x, s' - x, \tau_x \mathbf{s}), \tag{4.4}$$

and

$$\theta(y; s, \mathbf{s}) - x = \theta(y - x^*; s - x, \tau_x \mathbf{s}), \tag{4.5}$$

for all $x \in \mathbb{R}^d$.

Lemma 4.2. *Let $\zeta \in \mathcal{H}$ be a flux, u a unit vector and $y \in \mathbb{R}^d$. Then*

$$\mathcal{E} \sum_{s \in S} \zeta(s, \theta(y; s, S), S) \mathbf{1}_{L_u(y,S)}(s) \mathbf{1}_A(s \cdot u) = \ell_1(A) \mathcal{C}(\zeta \cdot \omega_u) \tag{4.6}$$

for all $A \in \mathcal{B}(\mathbb{R})$ with 1-dimensional Lebesgue measure $\ell_1(A) < \infty$.

The random set $\{(s \cdot u) : s \in L_u(y, S)\}$ is the one-dimensional stationary point process obtained by projecting the points of $L_u(y, S)$ to $l_u(y)$. One can think that each point s has a weight $\zeta(s, \theta(y; s, S), S)$. The expression on the left of (4.6) is the average of these weights for the points projected over A. The expression on the right of (4.6) says that this average contributes to the expression as much as the Lebesgue measure of the projection over O_u of the boundary between s and its neighbor in L to its right.

Proof. By translation invariance we can take $y = 0$ and, for simplicity we take $u = e_1$, the other directions are treated analogously. In this case $O_u = \{x \in \mathbb{R}^d : x_1 = 0\}$, $s \cdot u = s_1$, the first coordinate of s and $x^* = (0, x_2, \ldots, x_d)$. Define

$$g(s, \mathbf{s}) := \zeta(s, \theta(0; s, \mathbf{s}), \mathbf{s}) \mathbf{1}_{L_u(0,\mathbf{s})}(s) \mathbf{1}_A(s_1).$$

From the Generalized Campbell formula, (4.5) and Fubini,

$$\begin{aligned}
\mathcal{E} \sum_{s \in S} |g(s, S)| &= \int_{\mathbb{R}^d} \mathcal{E} |g(x, \tau_{-x} S^\circ)| dx \\
&= \int_{\mathbb{R}^d} \mathcal{E} |\zeta(0, \theta(-x^*; 0, S^\circ), S^\circ)| \mathbf{1}_{L_u(-x^*, S^\circ)}(0) \mathbf{1}_A(x_1) dx \\
&= \ell_1(A) \int_{\mathbb{R}^{d-1}} \mathcal{E} \sum_{s \in S^\circ} |\zeta(0, s, S^\circ)| \mathbf{1}_{\{\theta(x^*; 0, S^\circ) = s\}} \mathbf{1}_{L_u(x^*, S^\circ)}(0) dx_2 \cdots dx_d \\
&= \ell_1(A) \mathcal{E} \sum_{s \in S^\circ} |\zeta(0, s, S^\circ)| \int_{\mathbb{R}^{d-1}} \mathbf{1}_{\{\theta(x^*; 0, S^\circ) = s\}} \mathbf{1}_{L_u(x^*, S^\circ)}(0) dx_2 \cdots dx_d.
\end{aligned} \tag{4.7}$$

For $s \in S^\circ$ such that $a^+(0, s, S^\circ) = 1$,

$$\{s = \theta(x^*; 0, S^\circ), 0 \in L_u(x^*, S^\circ)\} = \{l_u(x^*) \cap b(0, s, S^\circ) \neq \emptyset\} = \{x^* \in b_u(0, s, S^\circ)\}.$$

Hence, the integral in (4.7) gives $a^+(0, s, S^\circ) \ell_{d-1}(b_u(0, s, S^\circ))$ and

$$\mathcal{E} \left| \sum_{s \in S} g(s, S) \right| = \ell_1(A) \mathcal{E} \sum_{s \in S^\circ} a^+(0, s, S^\circ) |\zeta(0, s, S^\circ)| \ell_{d-1}(b_u(0, s, S^\circ)) < \infty,$$

because by Assumption A4 the field $\ell_{d-1}(b_u(0, s, S^\circ))$ is in $\mathcal{H}$. With the same computation,

$$\begin{aligned}
\mathcal{E} \sum_{s \in S} g(s, S) &= \ell_1(A) \mathcal{E} \sum_{s \in S^\circ} a^+(0, s, S^\circ) \zeta(0, s, S^\circ) \ell_{d-1}(b_u(0, s, S^\circ)) \\
&= \ell_1(A) \mathcal{E} \sum_{s \in S^\circ} a^+(0, s, S^\circ) \zeta(0, s, S^\circ) \omega_u(0, s, S^\circ).
\end{aligned} \tag{4.8}$$

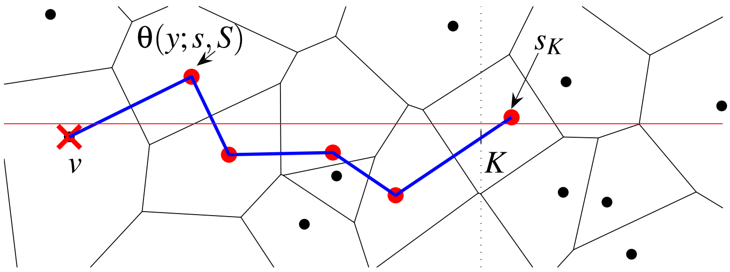

Fig. 3. Points of $L(0, \mathbf{s})$ (red and big). The horizontal line is l_y. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Since $a^+(s, 0, S^\circ)\zeta(s, 0, S^\circ)\omega_u(s, 0, S^\circ) = a^-(0, s, S^\circ)\zeta(0, s, S^\circ)\omega_u(0, s, S^\circ)$, Lemma 3.1 implies

$$\mathcal{E}\sum_{s\in S} g(s, S) = \ell_1(A)\frac{1}{2}\mathcal{E}\sum_{s\in S^\circ} a(0, s, S^\circ)\zeta(0, s, S^\circ)\omega_u(0, s, S^\circ) = \ell_1(A)\mathcal{C}(\zeta\cdot\omega_u). \quad \square$$

Proof of Proposition 4.1. Again without loosing generality we take $u = e_1$ and $u\cdot s = s_1$. Since $\nabla\eta$ is covariant and $\mathrm{Cen}(\cdot)$ is translation invariant (in the sense that $\mathrm{Cen}(x-z, \tau_z\mathbf{s}) = \mathrm{Cen}(x, \mathbf{s})$),

$$\begin{aligned}&\eta(\mathrm{Cen}(x + Ku, S), S) - \eta(\mathrm{Cen}(x, S), S)\\ &\quad = \eta(\mathrm{Cen}(Ku, \tau_x S), \tau_x S) - \eta(\mathrm{Cen}(0, \tau_x S), \tau_x S).\end{aligned} \tag{4.9}$$

Since S is stationary, if the limit in (2.3) exists, it is independent of $x\in\mathbb{R}^d$. Let $K > 0$, $A_K := [0, K]\times\mathbb{R}^{d-1}$ and define

$$\bar{s}_K = \mathrm{argmax}\{(s\cdot e_1): s\in L(0, S)\cap A_K\}, \qquad s_K = \theta(0; \bar{s}_K, S),$$

that is, s_K is the first point of $L(0, S)$ to the right of A_K. Let $Z_K = \eta(\mathrm{Cen}(Ku, S), S) - \eta(s_K, S)$ and observe that (see Fig. 3)

$$\eta(\mathrm{Cen}(Ku), S) - \eta(\mathrm{Cen}(0), S) = \sum_{s\in L(0,S)\cap A_K}(\eta(\theta(0; s, S), S) - \eta(s, S)) + Z_K. \tag{4.10}$$

Using (4.10), the limit as $K\to+\infty$ in (2.3) reads

$$\begin{aligned}I_u(\eta, S) &= \lim_{K\to\infty}\frac{1}{K}\sum_{s\in L(0,S)\cap A_K}\nabla\eta(s, \theta(0; s, S), S) + \lim_{K\to\infty}\frac{Z_K}{K}\\ &= \lim_{K\to\infty}\frac{1}{K}\sum_{s\in S}\nabla\eta(s, \theta(0; s, S), S)\mathbf{1}_{L(0,S)}(s)\mathbf{1}_{[0,K]}(s_1) + \lim_{K\to\infty}\frac{Z_K}{K}.\end{aligned} \tag{4.11}$$

Since $(Z_K)_{K\geq 0}$ is a stationary sequence, the same arguments as in the proof of Lemma 4.2 show that $\mathcal{E}|Z_0| \leq \mathcal{C}(\mathrm{div}|\nabla\eta|\ell(\mathrm{Vor}(0))) < \infty$, and so $Z_K/K\to 0$ almost surely as $K\to\infty$. Then it suffices to show that the first limit in (4.11) converges to $\mathcal{C}(\nabla\eta\cdot\omega_u)$. The sum in the first term in (4.11) can be telescoped as follows:

$$\sum_{k=0}^{K-1}\sum_{s\in S}\nabla\eta(s, \theta(0; s, S), S)\mathbf{1}_{L(0,S)}(s)\mathbf{1}_{[k,k+1]}(s_1) = \sum_{k=0}^{K-1}\phi(\tau_{ku}S) \tag{4.12}$$

where $\phi(S) := \sum_{s\in S} \nabla\eta(s, \theta(0; s, S), S)\mathbf{1}_{L(0,S)}(s)\mathbf{1}_{[0,1]}(s_1)$. Since the law of S is mixing, by Birkhoff's ergodic theorem, for integer K,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \phi(\tau_{ku} S) = \mathcal{E}[\phi(S)] = \mathcal{C}(\nabla\eta \cdot \omega_u) \quad \mathcal{P}\text{-a.s.}, \tag{4.13}$$

by Lemma 4.2. If $K \in \mathbb{R}$, the result follows from the above and

$$\lim_{K\to\infty} \frac{1}{K} |\eta(\text{Cen}(Ku, S)) - \eta(\text{Cen}([K]u), S)| = 0 \quad \mathcal{P}\text{-a.s.}$$

The same arguments work for $K < 0$. $\square$

5. The harness process

Given a configuration of points $\mathbf{s} \in \mathcal{N}$ we construct the process $\eta_t^\gamma(\cdot, \mathbf{s}) : \mathbf{s} \to \mathbb{R}$, with initial condition given by a surface $\gamma(\cdot, \mathbf{s}) : \mathbf{s} \to \mathbb{R}$ and generator given by (2.5).

Graphical construction. Let $T = (T_n, n = 1, 2, \ldots)$ be a family of independent Poisson processes of intensity 1, $T_n \subset \mathbb{R}$. For fixed $\mathbf{s} \in \mathcal{N}$ and an arbitrary enumeration of the points of $\mathbf{s} = (s_1, s_2, \ldots)$ we use the epochs of T_n to update the heights at s_n as follows. Fix $t > 0$ and define a family $(B^s_{[t,u]}; u \le t, s \in \mathbf{s})$ of backward simple random walks on $\mathbf{s}$ starting at $s \in \mathbf{s}$ at time t and jumping at the epochs in T as follows. Start with $B^s_{[t,t]} = s$; then, if $\tau \in T_n$ and at time $\tau+$ the walk is at s_n (that is, $B^s_{[t,\tau+]} = s_n$) then the walk chooses uniformly s', one of the neighbors of s_n with probability $\frac{1}{|a(s_n,\mathbf{s})|}$ and jumps over it, setting $B^s_{[t,\tau]} = s'$. Those jumps are performed with the aid of independent uniform in [0, 1] random variables $U = (U_n^k, k, n \ge 1)$; the variable U_n^k is used to perform the k-th jump from s_n. Now consider a random set of points S° with law $\mathcal{P}$ and assume U, T and S° independent. Call P and E the probability and expectation induced by (T, U), let $\mathbb{P} = \mathcal{P} \times P$ and call $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Denote

$$p_t(s, s', S^\circ, T) := \mathbb{P}(B^s_{[t,0]} = s' \mid S^\circ, T),$$

the probability that $B^s_{[t,0]} = s'$ conditioned on the sigma field generated by (S°, T). Define $\eta_t^\gamma(s, S^\circ, T)$ as the expectation of $\gamma(B^s_{[t,0]})$ conditioned on the sigma-field generated by (S°, T):

$$\eta_t^\gamma(s, S^\circ, T) := \sum_{s'\in S^\circ} p_t(s, s', S^\circ, T)\gamma(s'). \tag{5.1}$$

The η process has initial configuration $\eta_0^\gamma(s, S^\circ, T) = \gamma(s)$ and evolves as follows. If $s = s_n$ and $\tau \in T_n$ is an epoch of T_n, then

$$p_\tau(s, s', S^\circ, T) = \sum_{s''\in S^\circ} \frac{a(s, s'', S^\circ)}{a(s, S^\circ)} p_{\tau-}(s'', s', S^\circ, T),$$

and $\eta_\tau^\gamma(s, S^\circ, T)$ is updated by

$$\begin{aligned}\eta_\tau^\gamma(s, S^\circ, T) &= \sum_{s'\in S^\circ}\sum_{s''\in S^\circ} \frac{a(s, s'', S^\circ)}{a(s, S^\circ)} p_{\tau-}(s'', s', S^\circ, T)\gamma(s') \\ &= \sum_{s''\in S^\circ} \frac{a(s, s'', S^\circ)}{a(s, S^\circ)} \eta_{\tau-}(s'', S^\circ, T)\end{aligned} \tag{5.2}$$

while $\eta^\gamma_\tau(s', S^\circ, T)$ remains unchanged for $s' \neq s$. That is, $\eta^\gamma_\tau(\cdot, S^\circ, T) = M_s \eta^\gamma_{\tau-}(\cdot, S^\circ, T)$.

Lemma 5.1. *Given $\gamma: \Xi_1 \to \mathbb{R}$ with $\nabla\gamma \in \mathcal{H}$, the process $\eta^\gamma_t(\cdot, S^\circ, \cdot)$, is well defined $\mathcal{P}$-a.s. and has generator given by (2.5).*

Proof. To prove that the process is well defined we need to show that the sum on the right hand side of (5.1) is finite $\mathcal{P}$-a.s. Proposition A.2 in the appendix shows that

$$\mathbb{E}|\eta^\gamma_t(0, S^\circ, T)| \le \mathbb{E} \sum_{s' \in S^\circ} p_t(0, s', S^\circ, T)|\gamma(s', S^\circ)| \le t\mathcal{C}(|\nabla\gamma|).$$

This shows that the process is almost surely well defined at the origin. Using Assumption A6b the result is extended to all $s \in S^\circ$.

The fact that $\eta^\gamma_t(\cdot, S^\circ, \cdot)$ has generator given by (2.5), follows from (5.2) since S° is locally finite $\mathcal{P}$-a.s. □

We have constructed the process η^γ_t as a deterministic function of S° and T, the point configuration plus the time epochs associated to the points. That is, η^γ_t is a random surface. Let $(S^\circ, T) = ((s_n, T_n), n \ge 1)$ and $\tau_s(S^\circ, T) = ((s_n - s, T_n), n \ge 1)$, for $s \in S^\circ$. Since $p_t(s, s', (S^\circ, T)) = p_t(0, s' - s, \tau_s(S^\circ, T))$, $\gamma(0, S^\circ) = 0$ and $\nabla\gamma$ is covariant,

$$\begin{aligned}
\eta^\gamma_t(s, (S^\circ, T)) &= \sum_{s' \in S^\circ} p_t(s, s', (S^\circ, T))\, \gamma(s', S^\circ) \\
&= \sum_{s' \in S^\circ} p_t(0, s' - s, \tau_s(S^\circ, T))\, \gamma(s' - s, \tau_s S^\circ) + \gamma(s, S^\circ) \\
&= \sum_{s' \in \tau_s S^\circ} p_t(0, s', \tau_s(S^\circ, T))\, \gamma(s', \tau_s S^\circ) + \gamma(s, S^\circ) \\
&= \eta^\gamma_t(0, \tau_s(S^\circ, T)) + \gamma(s, S^\circ).
\end{aligned}$$

If we call

$$\psi_t(s, (S^\circ, T)) := \eta^\gamma_t(0, \tau_s(S^\circ, T)), \tag{5.3}$$

then the process at time t is the sum of the translation invariant surface ψ_t and the initial condition γ. That is,

$$\eta_t = \psi_t + \gamma. \tag{5.4}$$

In particular, it follows that $\nabla\eta^\gamma_t$ is a covariant (random) field $\mathbb{P}$-a.s.

The dependence of η^γ_t on (S°, T) will be dropped from the notation when clear from the context.

Extension of the Hilbert space $\mathcal{H}$ to include the randomness coming from the process. We consider the probabilistic space where S°, T, U are defined as independent processes and abuse notation by calling $\mathcal{C}$ the Campbell measure on Ξ_2 associated to S°, T, U:

$$\mathcal{C}(\nabla\eta^\gamma_t) := \mathcal{E}E(\nabla\eta^\gamma_t) = \mathbb{E}(\nabla\eta^\gamma_t).$$

The following bound – shown in the Appendix – implies that the process is well defined as an element in $\mathcal{H}$ for all time.

Lemma 5.2. *If $\mathcal{C}(|\nabla\gamma|^r) < \infty$ then*

$$\mathcal{C}(|\nabla\psi^\gamma_t|^r) \le 2^r \mathcal{C}(|\nabla\gamma|^r) m^r(t) < \infty,$$

where $m^r(t)$ denotes the r-th moment of a Poisson random variable with mean t.

As a consequence of (5.4) and Lemma 5.2 the tilt is invariant under the dynamics:

Proposition 5.3. *For all unitary $u \in \mathbb{R}^d$ and covariant surface γ with $\nabla\gamma \in \mathcal{H}$,*

$$\mathcal{J}_u(\eta_t^\gamma) = \mathcal{J}_u(\gamma),$$

for all $t \geq 0$.

Proof. First observe that with $\mathcal{P}$-probability one we have,

$$\begin{aligned}\operatorname{div}\omega_u(0, S^\circ) &= \sum_{s\in S^\circ} \omega_u(0, s, S^\circ)\\ &= \frac{1}{2}\sum_{\substack{s\in S^\circ\\(s\cdot u)>0}} \ell_{d-1}(b_u(0, s, S^\circ)) - \frac{1}{2}\sum_{\substack{s\in S^\circ\\(s\cdot u)<0}} \ell_{d-1}(b_u(0, s, S^\circ)) = 0,\end{aligned}$$

because each term in the subtraction is the $(d-1)$-dimensional Lebesgue measure of the projection of the Voronoi cell of the origin over the hyperplane orthogonal to u. Then,

$$\begin{aligned}\mathcal{J}_u(\eta_t^\gamma) &= \mathcal{C}(\nabla\eta_t^\gamma \cdot \omega_u) = \mathcal{C}(\nabla\gamma \cdot \omega_u) + \mathcal{C}(\nabla\psi_t \cdot \omega_u)\\ &= \mathcal{C}(\nabla\gamma \cdot \omega_u) - \mathcal{C}(\psi_t \cdot \operatorname{div}\omega_u) = \mathcal{C}(\nabla\gamma \cdot \omega_u) = \mathcal{J}_u(\gamma),\end{aligned}$$

where we used (5.4) in the second identity, the integration-by-parts Lemma 3.2 in the third identity as ψ_t^γ is a translation invariant surface and $\operatorname{div}\omega_u = 0$ in the fourth identity. □

6. The process converges to a harmonic surface

In this section we show that if γ is a surface with tilt $I(\gamma)$, whose gradient is in $\mathcal{H}$ and has more than 4 moments, then there exists a surface h with $\nabla h \in \mathcal{H}$ such that $\nabla\eta_t^\gamma$ converges strongly in $\mathcal{H}$ to ∇h. Furthermore h is harmonic and has the same tilt as γ. We split the proof into several lemmas.

Lemma 6.1. *If $\mathcal{C}(|\nabla\gamma|^r) < \infty$ for some $r > 4$, then for all $t > 0$*

$$\frac{d}{dt}\mathcal{C}(|\nabla\eta_t^\gamma|^2) = -2\mathbb{E}\left[a(0)^{-1}\left|\Delta\eta_t^\gamma(0, S^\circ, T)\right|^2\right]. \tag{6.1}$$

Proof. We drop the dependence on the initial condition γ, S° and T and write $\eta_t = \eta_t^\gamma(\cdot, S^\circ, T)$. Let $\mathcal{T}_2 = \bigcup_{s_n\in V_2} T_n$, the epochs corresponding to sites in V_2, the set of second neighbors of the origin. Define the events

$$\begin{aligned}F_1 &:= F_1(t,h) = \{|\mathcal{T}_2 \cap [t, t+h]| = 1\};\\ F_{1,s} &:= F_{1,s}(t, t+h) = F_1 \cap \{|\mathcal{T}(s) \cap [t, t+h]| = 1\} \cap \{s \in V_2\};\\ F_2 &:= F_2(t,h) = \{|\mathcal{T}_2 \cap [t, t+h]| \geq 2\}.\end{aligned}$$

Given S, $\mathcal{T}_2$ is a Poisson process with intensity $|V_2|$, hence

$$\mathbb{P}(F_1|S^\circ) = \mathbb{E}[\mathbf{1}_{F_1}|S^\circ] = |V_2|he^{-|V_2|h}, \tag{6.2}$$

$$\mathbb{P}(F_{1,s}|S^\circ) = he^{-|V_2|h}\mathbf{1}_{V_2}(s), \tag{6.3}$$

$$\mathbb{P}(F_2|S^\circ) \leq h^2|V_2|^2. \tag{6.4}$$

We have to compute

$$\mathbb{E}\sum_{s\in S^\circ} a(0,s)(|\nabla\eta_{t+h}(0,s)|^2-|\nabla\eta_t(0,s)|^2)(\mathbf{1}_{F_1}+\mathbf{1}_{F_2}) = I+II. \tag{6.5}$$

We use

$$\begin{aligned}|\nabla\eta_{t+h}(0,s)|^2-|\nabla\eta_t(0,s)|^2 &= [\nabla\eta_{t+h}(0,s)-\nabla\eta_t(0,s)]^2\\ &\quad+2\nabla\eta_t(0,s)[\nabla\eta_{t+h}(0,s)-\nabla\eta_t(0,s)],\end{aligned}$$

$$\Delta^\star\eta(s) := \frac{1}{|a(s)|}\sum_{s'\in S^\circ} a(s,s')(\eta(s')-\eta(s)) = M_s\eta(s)-\eta(s)$$

to compute each term in (6.5). Assume F_1 occurs.

- If the mark is neither at the origin nor at a neighbor of it, then $a(0,s)=0$, $\nabla\eta_{t+h}(0,s)=\nabla\eta_t(0,s)$, and the difference is zero.
- If the mark is at the origin and $a(0,s)=1$,

$$\begin{aligned}|\nabla\eta_{t+h}(0,s)|^2-|\nabla\eta_t(0,s)|^2 &= [-M_0\eta_t(0)+\eta_t(0)]^2+2\nabla\eta_t(0,s)[-M_0\eta_t(0)+\eta_t(0)]\\ &= -2\nabla\eta_t(0,s)\Delta^\star\eta_t(0)+|\Delta^\star\eta_t(0)|^2.\end{aligned} \tag{6.6}$$

- If the mark is at some s such that $a(0,s)=1$, we have $\nabla\eta_{t+h}(0,s')=\nabla\eta_t(0,s')$, for all $s'\neq s$. So

$$\begin{aligned}|\nabla\eta_{t+h}(0,s)|^2-|\nabla\eta_t(0,s)|^2 &= [M_s\eta_t(s)-\eta_t(s)]^2+2\nabla\eta_t(0,s)[M_s\eta_t(s)-\eta_t(s)]\\ &= 2\nabla\eta_t(0,s)\Delta^\star\eta_t(s)+|\Delta^\star\eta_t(s)|^2.\end{aligned} \tag{6.7}$$

Given S°, the process $\mathcal{T}_2\cap[t,t+h]$ is independent of η_t, so conditioning on S° by (6.2), (6.3), (6.6) and (6.7), we get that the first term in (6.5) equals

$$h\mathbb{E}\left(\mathrm{e}^{-|V_2|h}\sum_{s\in S^\circ} a(0,s)(2\nabla\eta_t(0,s)\nabla\Delta^\star\eta_t(0,s)+|\Delta^\star\eta_t(s)|^2+|\Delta^\star\eta_t(0)|^2)\right).$$

By monotone convergence,

$$\begin{aligned}&\lim_{h\to0}\frac{1}{2h}\mathbb{E}\left(\sum_{s\in S^\circ} a(0,s)(|\nabla\eta_{t+h}(0,s)|^2-|\nabla\eta_t(0,s)|^2)\mathbf{1}_{F_1}\right)\\ &= \mathbb{E}\left(\sum_{s\in S^\circ} a(0,s)\nabla\eta_t(0,s)\nabla\Delta^\star\eta_t(0,s)\right)\\ &\quad+\frac{1}{2}\mathbb{E}\left(\sum_{s\in S^\circ} a(0,s)(|\Delta^\star\eta_t(s)|^2+|\Delta^\star\eta_t(0)|^2)\right)\\ &= \mathbb{E}\left(\sum_{s\in S^\circ} a(0,s)\nabla\eta_t(0,s)\nabla\Delta^\star\eta_t(0,s)\right)+\mathbb{E}\Big(a(0)|\Delta^\star\eta_t|^2\Big),\end{aligned} \tag{6.8}$$

by the Mass Transport Principle (3.3). Let $1/p+1/q=1$ and $\zeta(0,s):=\mathbf{1}_{F_2}$, then for any time t', by means of (5.4) and Lemma 5.2, the second term in (6.5) reads

$$\frac{1}{h}\mathbb{E}\left(\sum_{s\in S^\circ}|\nabla\eta_{t'}(0,s)|^2\mathbf{1}_{F_2}\right) = \frac{2}{h}\mathcal{C}(|\nabla\eta_{t'}|^2\zeta)$$

$$\le \frac{2}{h}\mathcal{C}(|\nabla\eta_{t'}|^{2p})^{1/p}\mathcal{C}(\zeta^q)^{1/q}$$
$$= \frac{1}{h}\left[\mathbb{E}\left(\sum_{s\in S^\circ}|\nabla\eta_{t'}(0,s)|^{2p}\right)\right]^{1/p}\left[\mathbb{E}a(0)\mathbf{1}_{F_2}\right]^{1/q}$$
$$\le (Am^{2p}(t')+B)\left[\mathcal{E}|V_2|^3\right]^{1/q}h^{2/q-1},$$

for constants $A, B > 0$, where $m^r(t)$ is the r-th moment of a Poisson random variable with mean t. Choosing $q < 2$ and applying this bound for $t' = t$ and $t' = t + h$ we get

$$\lim_{h\to 0}\frac{1}{2h}\mathbb{E}\left(\sum_{s\in S^\circ}a(0,s)(|\nabla\eta_{t+h}(0,s)|^2 - |\nabla\eta_t(0,s)|^2)\mathbf{1}_{F_2}\right) = 0. \tag{6.9}$$

From (6.5), (6.8) and (6.9) and the integration by parts formula we obtain

$$\frac{d}{dt}\mathcal{C}(|\nabla\eta_t|^2) = 2\mathcal{C}(\nabla\eta_t\nabla\Delta^\star\eta_t) + \mathbb{E}[a(0)|\Delta^\star\eta_t|^2] = -\mathbb{E}[a(0)|\Delta^\star\eta_t|^2]. \quad \square$$

Corollary 6.2. *If γ satisfies the hypotheses in Lemma 6.1, then*

(a) *$\mathcal{C}(|\nabla\eta_t^\gamma|^2)$ is non-increasing in t;*
(b) *$\mathcal{C}(|\nabla\eta_t^\gamma|^2)$ is strictly decreasing at time t if and only if η_t^γ is not harmonic for (a, S°);*
(c) *$\lim_{t\to\infty} a(0)^{-1}\Delta\eta_t^\gamma(0) = 0$, $\mathbb{P}$-a.s. and in $L_2(P)$, $\mathcal{P}$-a.s.;*
(d) *$\lim_{t\to\infty}\Delta\eta_t^\gamma = 0$ $\mathbb{P}$-a.s.*

Proof. Let

$$Z_t := \frac{|\Delta\eta_t^\gamma(0)|^2}{a(0)} = a(0)|\Delta^\star\eta_t^\gamma(0)|^2.$$

Lemma 6.1 implies $\int_0^\infty \mathbb{E}[Z_t]dt < \infty$. Fix $t_0 = 0$ and denote $0 = t_0 < t_1 < t_2 < \cdots$ the ordered epochs of the superposition of the Poisson processes associated to the point at the origin and its neighbors. This is a Poisson process with intensity $a(0) + 1$. For each $n \ge 0$, given S°, Z_{t_n} is independent of $(t_{n+1} - t_n)$. Hence,

$$\int_0^\infty \mathbb{E}Z_t dt = \mathbb{E}\int_0^\infty Z_t dt = \sum_{k=0}^\infty \mathbb{E}[Z_{t_k}(t_{k+1}-t_k)] = \sum_{k=0}^\infty \mathbb{E}\left(\frac{Z_{t_k}}{a(0)+1}\right) < \infty.$$

Hence,

$$\sum_{k=0}^\infty \Delta\eta_{t_k}^\gamma(0) < \infty \quad \text{and} \quad \lim_{t\to\infty}\Delta\eta_t^\gamma(0) = 0 \quad \mathbb{P}\text{-a.s.}$$

The $L_2(P)$ convergence follows by dominate convergence using that $\Delta\eta_t^\gamma(0) \le \sum_{k=0}^\infty \Delta\eta_{t_k}^\gamma(0)$. $\square$

Proof of (a) and (b) of Theorem 2.1. In the notation we drop the dependence on the initial surface γ. We want to prove the existence of a harmonic surface h, with covariant ∇h and such that for all $n \in \mathbb{Z}$

$$\lim_{t\to\infty}\mathbb{E}a(0,s_n)[\nabla\eta_t(0,s_n) - \nabla h(0,s_n)]^2 = 0,$$

where $(s_n, n \in \mathbb{Z})$ is the enumeration of S° given in B1–B3.

Observe that

$$\mathbb{E}|a(0,s_n)(\nabla\eta_t(0,s_n) - \nabla h(0,s_n))|^2 \le \mathbb{E}\sum_{s\in S^\circ} a(0,s)|\nabla\eta_t(0,s) - \nabla h(0,s)|^2 = 2\mathcal{C}(|\nabla\eta_t - \nabla h|^2).$$

So, it is enough to show that $\nabla\eta_t \to \nabla h$ strongly in $\mathcal{H}$.

Existence of the limit. Since by Corollary 6.2, $\mathcal{C}(|\nabla\eta_t|^2)$ is bounded, $\nabla\eta_t$ is weakly compact, and hence for every sequence $\{t_k\}_{k\ge0}$, there exists a subsequence $\{t_{k_j}\}_{j\ge0}$ and a field $\zeta_\infty \in \mathcal{H}$ such that

$$\lim_{j\to\infty} \mathcal{C}(\nabla\eta_{t_{k_j}} \cdot \zeta) = \mathcal{C}(\zeta_\infty \cdot \zeta), \quad \text{for all } \zeta \in \mathcal{H}. \tag{6.10}$$

Uniqueness of the limit. Let $\{t_k\}_{k\ge0}$ be a subsequence such that $\nabla\eta_{t_k} \rightharpoonup \zeta_\infty$.

By (5.4),

$$\mathcal{C}(|\zeta_\infty|^2) = \lim_{k\to\infty} \mathcal{C}(\nabla\eta_{t_k} \cdot \zeta_\infty) = \mathcal{C}(\nabla\gamma \cdot \zeta_\infty) + \lim_{k\to\infty} \mathcal{C}(\nabla\psi_{t_k} \cdot \zeta_\infty), \tag{6.11}$$

where ψ_t is defined in (5.3). Integrating by parts and using Hölder,

$$\begin{aligned} |\mathcal{C}(\nabla\psi_{t_k} \cdot \zeta_\infty)| &= \lim_{j\to\infty} |\mathcal{C}(\nabla\psi_{t_k} \cdot \nabla\eta_{t_j})| = \lim_{j\to\infty} |\mathcal{C}(\psi_{t_k} \cdot \Delta\eta_{t_j})| \\ &\le \lim_{j\to\infty} \mathbb{E}(a(0)|\psi_{t_k}|^2)^{1/2}\mathbb{E}(a(0)^{-1}|\Delta\eta_{t_j}|^2)^{1/2} = 0, \end{aligned} \tag{6.12}$$

by Corollary 6.2. Therefore,

$$\mathcal{C}(|\zeta_\infty|^2) = \mathcal{C}(\nabla\gamma \cdot \zeta_\infty). \tag{6.13}$$

Let $\nabla\eta_{t_k} \rightharpoonup \zeta_\infty$ and $\nabla\eta_{t_j} \rightharpoonup \zeta'_\infty$ subsequences converging to two weak limits ζ_∞ and ζ'_∞. By (6.10) and (6.11),

$$\mathcal{C}(\zeta_\infty \cdot \zeta'_\infty) = \lim_{k\to\infty} \mathcal{C}(\nabla\eta_{t_k} \cdot \zeta'_\infty) = \mathcal{C}(\nabla\gamma \cdot \zeta'_\infty) + \lim_{k\to\infty} \mathcal{C}(\nabla\psi_{t_k} \cdot \zeta'_\infty) = \mathcal{C}(|\zeta'_\infty|^2), \tag{6.14}$$

by (6.12) and (6.13). The same holds for ζ_∞ and so $\mathcal{C}(|\zeta'_\infty|^2) = \mathcal{C}(|\zeta_\infty|^2) = \mathcal{C}(\zeta_\infty \cdot \zeta'_\infty)$. This implies $\mathcal{C}(|\zeta_\infty - \zeta'_\infty|^2) = 0$, i.e. there is a unique limit point.

Strong convergence. By (5.4) and integration by parts,

$$\mathcal{C}(|\nabla\eta_t|^2) = \mathcal{C}(\nabla\gamma\nabla\eta_t) + \mathcal{C}(\nabla\psi_t\nabla\eta_t) = \mathcal{C}(\nabla\gamma\nabla\eta_t) - \mathcal{C}(\psi_t\Delta\eta_t). \tag{6.15}$$

From Hölder's inequality,

$$(\mathcal{C}(\psi_t\Delta\eta_t))^2 \le \mathbb{E}(a(0)|\psi_t(0)|^2)\mathbb{E}\left(\frac{|\Delta\eta_t(0)|^2}{a(0)}\right). \tag{6.16}$$

Since by Lemma 6.1 $\mathbb{E}\frac{|\Delta\eta_t|^2}{a(0)}$ is integrable, there exists a subsequence $(t_k)_{k\ge0}$ such that

$$\lim_{k\to\infty} t_k\mathbb{E}\left(\frac{|\Delta\eta_{t_k}(0)|^2}{a(0)}\right) = 0.$$

From Lemma A.3 in the appendix,

$$\lim_{k\to\infty} \frac{\mathcal{E}E|\gamma(B^0_{[t_k,0]})|^2}{t_k} t_k\mathbb{E}\frac{|\Delta\eta_{t_k}(0)|^2}{a(0)} = 0. \tag{6.17}$$

Using (6.13), (6.15) and (6.17), $C(|\nabla\eta_t|^2) \to \mathcal{C}(|\zeta_\infty|^2)$, and hence $\nabla\eta_t$ converges strongly in $\mathcal{H}$ to ζ_∞.

Zero divergence. By Jensen's inequality and using $a(0) \geq 2$, we get

$$\lim_{t\to\infty} \mathbb{E}(a(0)^{-2}|\Delta\eta_t - \mathrm{div}\zeta_\infty|^2) \leq \lim_{t\to\infty} \mathbb{E}\left(a(0)^{-1}\sum_{s\in S^\circ} a(0,s)(\nabla\eta_t(0,s) - \zeta_\infty(0,s))^2\right)$$
$$\leq \lim_{t\to\infty} \mathcal{C}(|\nabla\eta_t - \zeta_\infty|^2) = 0.$$

It follows by Corollary 6.2 that

$$\mathrm{div}\zeta_\infty = 0 \quad \mathbb{P}\text{-a.s.} \tag{6.18}$$

Covariance. A field $\zeta \in \mathcal{H}$ is characterized by its values on the edges leaving the origin. Therefore, by taking the covariant canonical representant defined by $\zeta_\infty(s, s', S^\circ) := \zeta_\infty(0, s' - s, \tau_s S^\circ)$, we can consider ζ_∞ to be covariant.

Gradient field. To show that ζ_∞ is a gradient field we prove that it verifies the co-cycle property, that is there exists $\mathcal{N}^\star \subseteq \mathcal{N}$, with $\mathcal{P}(S^\circ \in \mathcal{N}^\star) = 1$ and such that for all $\mathbf{s} \in \mathcal{N}^\star$ and every closed path $s_{i_0}, s_{i_1}, \ldots, s_{i_k} = s_{i_0} \in \mathbf{s}$ with $a(s_{i_j}, s_{i_{j-1}}) = 1$, $j = 1, \ldots, k$ we have $\sum_{j=1}^k \zeta_\infty(s_{i_j}, s_{i_{j-1}}, \mathbf{s}) = 0$.

Let $n, m \in \mathbb{Z}$. Since $a(s_n, s_m)\nabla\eta_t(s_n, s_m) \overset{L_2(\mathbb{P})}{\to} a(s_n, s_m)\zeta_\infty(s_n, s_m)$, we have a subsequence that converges almost surely. Denote by $\mathcal{N}_{n,m} \subset \mathcal{N}$ the set where convergence holds. Using a standard diagonal argument we get a subsequence $(t_k)_{k\geq 0}$ such that

$$a(s_n, s_m)\nabla\eta_{t_k}(s_n, s_m) \xrightarrow{\text{a.s}} a(s_n, s_m)\zeta_\infty(s_n, s_m) \quad \text{for all } n, m \in \mathbb{Z}.$$

Define $\mathcal{N}^\star = \bigcap_{n,m\in\mathbb{Z}} \mathcal{N}_{n,m}$. Since the co-cycle property holds for every t the a.s. convergence implies the co-cycle property for ζ_∞.

Tilt. The tilt is a continuous functional in $\mathcal{H}$ and it is constant for the dynamics by Proposition 5.3. Hence the limit ζ_∞ has the same tilt as the initial surface. This completes the proof of (a) and (b) of the theorem. □

7. Uniqueness of harmonic surfaces in $d = 2$

In this section we prove uniqueness (up to an additive constant) of the harmonic surface with covariant gradient for $d = 2$. Observe that in dimension one the harmonic function with a given tilt can be explicitly computed and hence the uniqueness follows immediately. To prove uniqueness for $d = 2$ we use the following result.

Theorem 7.1 (*Theorem 5.1 of Berger and Biskup [5]*)**.** *For $c \in \mathbb{R}^2$, let $\gamma(s) = c \cdot s$, and h be a harmonic surface for $a(\cdot, \cdot, S^\circ)$ with covariant gradient in $\mathcal{H}$ and tilt $I(h) = I(\gamma)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \max_{s\in S^\circ\cap[-n,n]^2} \{|h(s) - c \cdot s|\} = 0, \quad \mathcal{P}\text{-a.s.} \tag{7.1}$$

We omit the proof; it follows [5], details can be found in [16]. Berger and Biskup [5] use this theorem to show uniqueness of the harmonic surface on the supercritical bond-percolation cluster in $\mathbb{Z}^2$; we adapt their proof to our case. Theorem 2.4 of Biskup and Prescott [7] proves (7.1) for bond percolation in $\mathbb{Z}^d$ for all $d \geq 2$ under "heat kernel estimates" assumptions, see (2.17) and (2.18) in that paper. These estimates are to be established in our setting.

Proof of (c) of Theorem 2.1. It is enough to show that if h is a harmonic surface with $I(h) = 0$, then $\nabla h = 0$ or, equivalently, $\mathcal{C}(|\nabla h|^2) = 0$. From the considerations after (3.2), if $\nabla h \in \mathcal{H}$ then, with probability 1,

$$\mathcal{C}(|\nabla h|^2) = \lim_{n\to\infty} \frac{1}{2(2n)^2} \sum_{s\in S\cap[-n,n]^2} \sum_{s'\in S} a(s,s')|\nabla h(s,s')|^2.$$

Let $S_n = S \cap [-n, n]^2$. Using that h is harmonic rewrite the sum at the right hand side as

$$\begin{aligned}\sum_{\substack{s\in S_n\\ s'\in S}} a(s,s')|\nabla h(s,s')|^2 &= \sum_{\substack{s\in S_n\\ s'\in S}} a(s,s')h(s')\nabla h(s,s') - \sum_{s\in S_n} h(s) \sum_{s'\in S} a(s,s')\nabla h(s,s')\\ &= \sum_{\substack{s\in S_n\\ s'\in S}} a(s,s')h(s')\nabla h(s,s').\end{aligned}$$

Using harmonicity again, we obtain

$$\begin{aligned}\sum_{\substack{s\in S_n\\ s'\in S}} a(s,s')|\nabla h(s,s')|^2 &= \sum_{\substack{s\in S_n\\ s'\in S_n}} a(s,s')h(s')\nabla h(s,s') + \sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')h(s')\nabla h(s,s')\\ &= \sum_{\substack{s\in S_n\\ s'\in S_n}} a(s',s)h(s)\nabla h(s',s) + \sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')h(s')\nabla h(s,s')\\ &= -\sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s',s)h(s)\nabla h(s',s) + \sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')h(s')\nabla h(s,s')\\ &= \sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')(h(s)+h(s'))\nabla h(s,s').\end{aligned}$$

Then, with $\mathcal{P}$-probability 1,

$$\mathcal{C}(|\nabla h|^2) = \lim_{n\to\infty} \frac{1}{8n^2} \sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')(h(s')+h(s))\nabla h(s,s').$$

Since this limit exists a.s., we are done if we can show that the r.h.s converges to zero in probability. Observe that

$$\begin{aligned}&\left|\sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')(h(s')+h(s))\nabla h(s,s')\right|\\ &\quad\le \max_{\substack{s\in S_n,\\ s'\in S\setminus S_n}} \{a(s,s')|h(s)+h(s')|\} \sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')|\nabla h(s,s')|.\end{aligned}$$

Let $A_n :=$ {There exists $s \in S_n$ and $s' \in S \setminus S_{2n}$ such that $a(s, s') = 1$}, and observe that

$$\begin{aligned}\mathcal{P}(A_n) &\le \mathcal{E} \sum_{s\in S_n} \sum_{s'\in S\setminus S_{2n}} a(s,s') \le \mathcal{E} \sum_{s\in S_n} \sum_{s'\in S\setminus S_{2n}} a(s,s') \frac{|s'-s|^4}{n^4}\\ &\le \frac{1}{n^2} \mathcal{E}\left[\sum_{s\in S^\circ} a(0,s)|s|^4\right].\end{aligned}$$

Therefore, by Borel–Cantelli, the fact that $I(h) = 0$ and Theorem 7.1, given ε we can take n big enough such that

$$(2n)^{-1} \max_{s\in S_n, s'\in S\setminus S_n} \{a(s,s')|h(s)+h(s')|\} \le \frac{1}{n} \max_{s\in S_{2n}} \{|h(s)|\} < \varepsilon.$$

It follows that

$$\lim_{n\to\infty} (2n)^{-1} \max_{s\in S_n, s'\in S\setminus S_n} \{a(s,s')|h(s)+h(s')|\} = 0, \quad \mathcal{P}\text{-a.s.}$$

and therefore it is enough to show that there exists a sequence $(Z_n)_{n\ge 1}$ such that

$$Z_n \ge \frac{1}{n}\phi_n(S) := \frac{1}{n}\sum_{s\in S_n}\sum_{s\in S\setminus S_n} a(s,s')|\nabla h(s,s')|,$$

almost surely and Z_n converges in probability.

Given $B, B' \in \mathcal{B}(\mathbb{R}^2)$, let $\phi_{B,B'}(S) := \sum_{s\in S}\sum_{s'\in S} a(s,s',S)|\nabla h(s,s',S)|\mathbf{1}_B(s)\mathbf{1}_{B'}(s')$, and observe that by the refined Campbell formula and the covariance of ∇h and a

$$\begin{aligned}
\mathcal{E}\phi_{B,B'} &= \int_{\mathbb{R}^2} \mathcal{E} \sum_{s'\in\tau_{-s}S} a(s,s',\tau_{-s}S)|\nabla h(s,s',\tau_{-s}S)|\mathbf{1}_B(s)\mathbf{1}_{B'}(s')ds \\
&= \int_{\mathbb{R}^2} \mathcal{E}\left[\sum_{s'\in S} a(s,s'+s,\tau_{-s}S)|\nabla h(s,s'+s,\tau_{-s}S)|\mathbf{1}_B(s)\mathbf{1}_{B'}(s'+s)\right]ds \\
&= \int_{\mathbb{R}^2} \mathcal{E}\left[\sum_{s'\in S} a(0,s',S)|\nabla h(0,s',S)|\mathbf{1}_B(s)\mathbf{1}_{B'}(s'+s)\right]ds \\
&= \mathcal{E}\sum_{s'\in S} a(0,s',S)|\nabla h(0,s',S)|\ell(B\cap\tau_{s'}B'). \qquad (7.2)
\end{aligned}$$

Let $B_n = [-n,n]^2$ and $\mathcal{X}_n$ be the family of half-planes defined by the borders of B_n, and disjoint from B_n. It is clear that

$$\sum_{\substack{s\in S_n\\ s'\in S\setminus S_n}} a(s,s')|\nabla h(s,s')| \le \sum_{B\in\mathcal{X}_n} \phi_{B_n,B}(S).$$

We show the convergence of $\frac{1}{n}\phi_{B_n,B}(S)$ for a fixed $B\in\mathcal{X}_n$. The convergence of the other terms follows from the same arguments.

Before proceeding, we have yet another approximation to take care of. Let $H_n = \mathbb{R}\times[n,+\infty)$, $G_n = [-n,n]\times(-\infty,n]$, and observe that

$$\phi_{B_n,H_n}(S) \le \phi_{G_n,H_n}(S), \quad \text{a.s.}$$

Let us see what happens with a fixed line first. To do that, let $G = [0,1]\times\mathbb{R}^-$ and $G_n^o = [-n,n]\times\mathbb{R}^-$. If we define $T = \tau_{e_1}$, by the covariance of ∇h and Birkhoff Ergodic Theorem, it follows that

$$\lim_{n\to\infty}\frac{1}{n}\phi_{G_n^o,H_0}(S) = \frac{1}{n}\lim_{n\to\infty}\sum_{k=-n}^{n-1}\phi_{G,H_0}(T^kS) = 2\mathcal{E}[\phi_{G,H_0}(S)] < \infty \quad \text{a.s.}$$

By the covariance of ∇h it follows that

$$\lim_{n\to\infty} \mathcal{P}(|\phi_{G_n,H_n}(S) - 2\mathcal{E}[\phi_{G,H_0}(S)]| > \varepsilon n)$$
$$= \lim_{n\to\infty} \mathcal{P}(|\phi_{G_n^o,H_0}(\tau_{ne_2}S) - 2\mathcal{E}[\phi_{G,H_0}(S)]| > \varepsilon n) = 0,$$

and the result follows. □

8. Final comments

8.1. Invariance principle

The key ingredient to obtain an invariance principle from the existence of a harmonic deformation of the original graph is a uniform sublinear bound of the corrector as in (7.1). Grisi [16] obtained this bound for the Poisson process following the arguments of Berger and Biskup [5] in $d = 2$. Hence the quenched invariance principle holds in the Delaunay triangulation of a Poisson process. Presumably this also holds for a non-periodic ergodic process satisfying Assumptions A1–A7. For $d \geq 3$ the proofs of a quenched invariance principle in the percolation setting and related models rely on heat kernel estimates like those obtained by Barlow [1], which do not follow from the sublinear behavior of the corrector along lines. An extension of these bounds to our case are to be obtained.

8.2. The process trajectory is orthogonal to the space of harmonic surfaces

Since the tilt in the direction $u \in \mathbb{R}^d$ is a continuous functional in $\mathcal{H}$, by Riesz Theorem, there exist a field $\omega_u \in \mathcal{H}$ such that the tilt is given by the scalar product with ω_u. In our case, we have found explicitly that field (the one given in (4.1)).

Given an initial condition γ, the process $\psi_t = \eta_t^\gamma - \gamma$ is a translation invariant surface and has zero tilt. The convergence of $\nabla\psi_t$ follows from the convergence of $\nabla\eta_t^\gamma$, and the limiting field is the gradient of the corrector $\nabla\chi_\gamma := \nabla h - \nabla\gamma$, for h given by Theorem 2.1. Integrating by parts and using translation invariance, for $\zeta \in \mathcal{H}$ with $\mathrm{div}\zeta \equiv 0$,

$$\mathcal{C}(\nabla(\gamma - \eta_t^\gamma)\zeta) = -\mathcal{C}((\gamma - \eta_t^\gamma)\mathrm{div}\zeta) = 0, \quad \text{for all } t \geq 0.$$

Hence $\gamma - \eta_t^\gamma$ is orthogonal to the subspace of fields in $\mathcal{H}$ with zero divergence (which contains the gradients of all harmonic surfaces). In fact, ∇h is the orthogonal projection of $\nabla\gamma$ over this subspace. In particular, we have

$$\nabla\gamma = \nabla h + (\nabla\gamma - \nabla h). \tag{8.1}$$

Mathieu and Piatnitski [21] also consider $L_2(\Xi_2, \mathcal{C})$. Eq. (8.1) corresponds to their decomposition of the space as $L_2(\Xi_2, \mathcal{C}) = L_2^{sol} \oplus L_2^{pot}$. Taking $\gamma_i(s) := (e_i \cdot s)$, $i \leq d$, the surface $\chi := (\chi_{\gamma_1}, \ldots, \chi_{\gamma_d})$ is what they call the *corrector*.

8.3. Regularization effect

The regularization effect observed in Fig. 1 can be explicitly formulated as follows. If one takes n arbitrary points $s_1, \ldots, s_n \in \mathbb{R}^2$, the barycenter minimizes the following sum of scalar products

$$\arg\min_{x\in\mathbb{R}^2} \sum_{k=1}^{n} [(s_k - x)\cdot(s_{k+1} - x)] = \frac{1}{n}\sum_{k=1}^{n} s_k, \tag{8.2}$$

where $s_{n+1} = s_1$. Take a point configuration $\mathbf{s}$ and $s, s' \in \mathbf{s}$ neighbors in the Delaunay triangulation of $\mathbf{s}$. The directed edge (s, s') is shared by the triangles $ss'\alpha_+$ and $ss'\alpha_-$, where $\alpha_+(s, s')$ is the first common neighbor of s and s' in the clockwise direction from $s' - s$ and $\alpha_-(s, s')$ is the other common neighbor. We show the following extension of (8.2) to harmonic surfaces.

Lemma 8.1. *Let S be a stationary point process. Then the harmonic deformation of the Delaunay triangulation of S minimizes*

$$\mathcal{E} \sum_{s \in S^\circ} a(0, s)\big[G(s) \cdot G(\alpha_+(0, s))\big] \tag{8.3}$$

among deformations $G: S^\circ \mapsto \mathbb{R}^d$ of S° such that $G(0) = 0$ and the corrector $G(s) - s$ has coordinates with gradient in $\mathcal{H}$.

We prove this Lemma below. Given a surface η define the fields $\zeta_+^\eta, \zeta_-^\eta: \Xi_2 \to \mathbb{R}$ by

$$\zeta_\pm^\eta(s, s') := a(s, s')\nabla\eta(s, \alpha_\pm(s, s')).$$

Any two surfaces $\eta, \phi: \Xi_1 \to \mathbb{R}$ satisfy

$$\mathcal{C}(\nabla\eta\zeta_+^\phi) = \mathcal{C}(\zeta_-^\eta\nabla\phi). \tag{8.4}$$

Also note that

$$\sum_{s' \in S^\circ} a(s, s')\zeta_\pm^\eta(s, s') = \Delta\eta(s) \quad \text{and} \quad \sum_{s' \in S^\circ} a(s, s')\zeta_\pm^\eta(s', s) = 0. \tag{8.5}$$

If ϕ is a translation invariant surface (that is $\phi(s, \mathbf{s}) = \phi(0, \tau_s \mathbf{s})$) then, by the mass transport principle,

$$\begin{aligned} 2\mathcal{C}(\nabla\phi\zeta_\pm^\eta) &= \mathcal{E} \sum_{s \in S^\circ} a(0, s)\nabla\phi(0, s)\zeta_\pm^\eta(0, s) \\ &= \mathcal{E}\,\phi(0) \sum_{s \in S^\circ} a(0, s)\zeta_\pm^\eta(s, 0) - \mathcal{E}\phi(0) \sum_{s \in S^\circ} a(0, s)\zeta_\pm^\eta(0, s) \\ &= -2\mathcal{C}(\phi\Delta\eta) = 2\mathcal{C}(\nabla\phi\nabla\eta), \end{aligned} \tag{8.6}$$

where the first identity in the bottom line follows from (8.5) and the second one by the integration by parts formula.

Lemma 8.2.

$$\frac{d}{dt}\mathcal{C}(\nabla\eta_t\zeta_+^{\eta_t}) = \frac{1}{2}\frac{d}{dt}\mathcal{C}(|\nabla\eta_t|^2) = -\mathbb{E}\left[a(0)^{-1}\left|\Delta\eta_t(0, S^\circ)\right|^2\right]. \tag{8.7}$$

Proof. Using (8.4) and $\nabla\eta_t = \nabla\gamma + \nabla\psi_t$,

$$\begin{aligned} \mathcal{C}(\nabla\eta_t\zeta_+^{\eta_t}) &= \mathcal{C}(\nabla\gamma\zeta_+^\gamma) + \mathcal{C}(\zeta_-^\gamma\nabla\psi_t) + \mathcal{C}(\nabla\psi_t\zeta_+^{\eta_t}) \\ &= \mathcal{C}(\nabla\gamma\zeta_+^\gamma) + \mathcal{C}(\nabla\gamma\nabla\psi_t) + \mathcal{C}(\nabla\psi_t\nabla\eta_t) \\ &= \mathcal{C}(\nabla\gamma\zeta_+^\gamma) + \mathcal{C}(\nabla\gamma\nabla\eta_t) + \mathcal{C}(\nabla\psi_t\nabla\eta_t) - \mathcal{C}(|\nabla\gamma|^2) \\ &= \mathcal{C}(\nabla\gamma\zeta_+^\gamma) + \mathcal{C}(|\nabla\eta_t|^2) - \mathcal{C}(|\nabla\gamma|^2), \end{aligned}$$

where the second identity follows from (8.6). This shows the first identity in (8.7); the second identity is (6.1). □

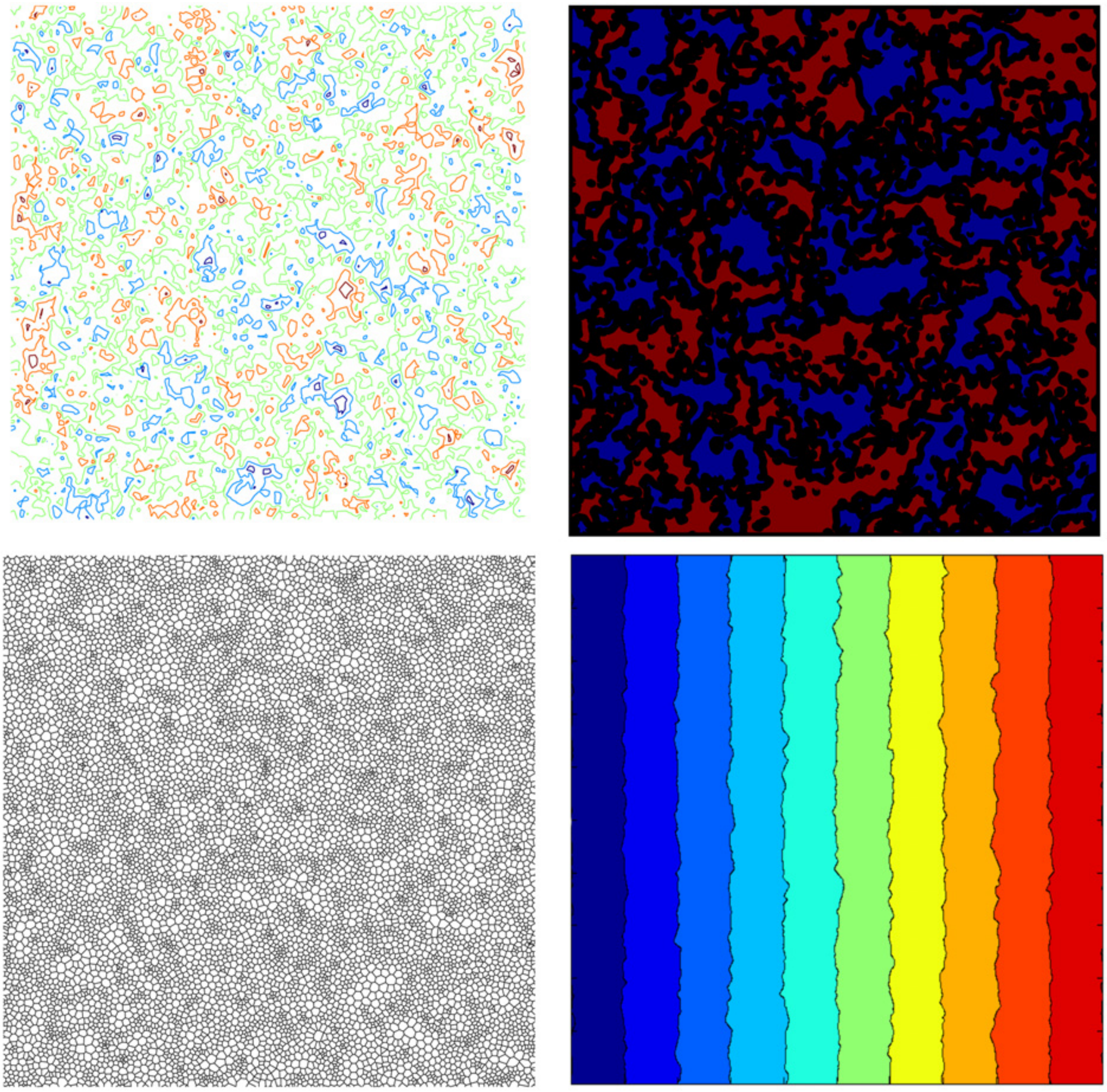

Fig. 4. Some harmonic pictures.

Proof of Lemma 8.1. Lemma 8.2 shows that $\mathcal{C}(\nabla \eta_t \zeta_+^{\eta_t})$ is non-increasing, and that it is strictly decreasing if and only if η_t is not harmonic and hence

$$\mathcal{C}(\nabla g \zeta_+^{g}) = 0 \quad \text{if and only if} \quad g \text{ is harmonic.}$$

Taking g_i as the coordinates of G and using that $G(0) = 0$, we get (8.3). □

8.4. Some simulations

The first two pictures in Fig. 4 show level curves of a linear interpolation of the surface $\gamma - h$. In the first one some level curves are drawn. From blue (minimum) to red (maximum). The level curve of zero is drawn in green. In the second one the sublevel set of zero is drawn in blue and the superlevel set is drawn in red. The black curve is the level set of zero.

The next picture is the Voronoi tessellation of the harmonic points. The Delaunay triangulation of this points does not necessarily coincide with the harmonic deformation of the original Delaunay triangulation. It is easy to construct examples where this in fact happens, and it can be seen in simulations. However, it can be appreciated in simulations that the density of triangles in

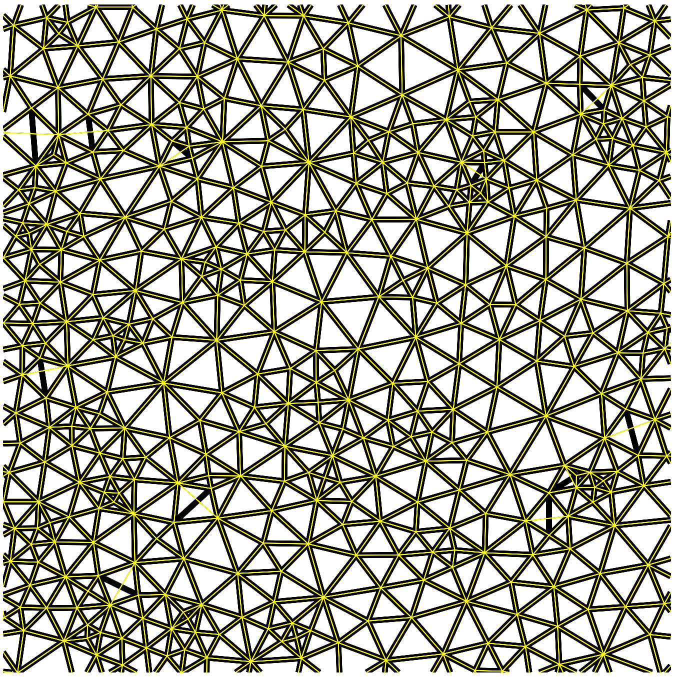

Fig. 5. Delaunay triangulation of harmonic points (black and thick) vs. harmonic graph (yellow). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the harmonic graph that are not Delaunay triangles is very low, as shown in Fig. 5. Finally, on the bottom-right of Fig. 4, the level curves of the harmonic surface with tilt (1, 0) is shown, that is the limit of the dynamics with initial condition given by the hyperplane $\gamma(x, y) = x$. Observe that the surface is pretty close to the original condition.

Acknowledgments

We thank Marek Biskup for his comments and to the referee for the careful reading of the manuscript and for many comments that helped us to improve the paper.

Appendix. The random walk and the environment process

This appendix collects some technical results used in Section 5. The *environment seen from the particle* was used by De Masi et al. [11] to show the annealed invariance principle for the random walk in the supercritical bond-percolation cluster. We adapt some of those results to our setting.

Let $\mathbf{s} \in \mathcal{N}$ and $s \in \mathbf{s}$. Let $\tilde{X}_n^s$ be a discrete time random walk on $\mathbf{s}$ with law $\tilde{P}_\mathbf{s}$ defined by $\tilde{X}_0^s = s$ and for $n \geq 1$,

$$\tilde{P}_\mathbf{s}(\tilde{X}_n^s = s'' | \tilde{X}_{n-1}^s = s') = \frac{a(s', s'', \mathbf{s})}{a(s', \mathbf{s})}.$$

That is, the walk starts at s and if it is at $s' \in \mathbf{s}$, then it chooses a neighbor uniformly at random and jumps over it. Let $\tilde{E}_\mathbf{s}$ be the expectation with respect to $\tilde{P}_\mathbf{s}$.

To build the continuous time walk, let $N = \{T_k; k \in \mathbb{N}\}$ be a rate 1 homogeneous Poisson Process in $\mathbb{R}_+$, independent of $(\tilde{X}_n)_{n\geq 0}$, and define

$$X_t := \tilde{X}_{N(t)}, \tag{A.1}$$

where $N(t) = |N \cap (0, t]|$ is the number of points of N in the interval $(0, t]$. Let $P_{\mathbf{s}} = \tilde{P}_{\mathbf{s}} \otimes Q$, where Q is the law of N in $(\mathcal{N}(\mathbb{R}^+), \mathcal{B}(\mathcal{N}(\mathbb{R}^+)))$. The law of X_t^0 coincides with the law of the walk $B^0_{[t,0]}$ defined in Section 5, so that the results below hold for $B^0_{[t,0]}$.

Given the process $\tilde{X}_n^s$ (with initial state $s \in \mathbf{s}$), define the process

$$\mathbf{s}_n^\circ = \tau_{\tilde{X}_n^s}\mathbf{s}.$$

This process can be thought as the *environment as seen from the particle* moving according to $\tilde{X}_n^s$. The process $\mathbf{s}_n^\circ$ is Markov with values in $\mathcal{N}^\circ$ (i.e. for all n, $0 \in \mathbf{s}_n^\circ$). We use $P_{\mathbf{s}}$ to denote the law of $\mathbf{s}_n^\circ$ in $\mathcal{N}^{\mathbb{Z}^+}$ with initial state $\mathbf{s}$.

Let $\mathcal{M}$ be the set of aperiodic $\mathbf{s}$:

$$\mathcal{M} = \{\mathbf{s} \in \mathcal{N} : \tau_x \mathbf{s} \neq \mathbf{s} \text{ for all } x \in \mathbb{R}^d, x \neq 0\}. \tag{A.2}$$

If $\mathbf{s}$ is aperiodic, then the trajectory of $\mathbf{s}_n$ determines univoquely the trajectory of the walk $\tilde{X}_n^0$. The Poisson Process is aperiodic almost surely.

Let S be an ergodic point process in $\mathbb{R}^d$, with Palm version S°. Denote by $\mathcal{Q}$ the probability measure on $(\mathcal{N}, \mathcal{B}(\mathcal{N}))$ given by

$$\int f(\mathbf{s})\mathcal{Q}(d\mathbf{s}) = \frac{1}{\mathcal{E}a(0)}\mathcal{E}[a(0)f(S^\circ)],$$

for bounded measurable $f : \mathcal{N} \to \mathbb{R}$.

Lemma A.1. *The process $(\mathbf{s}_n^\circ)_{n\geq 0}$ is reversible and ergodic under $\mathcal{Q}$.*

Proof. To check reversibility, let $f, g : \mathcal{N} \to \mathbb{R}$ be bounded measurable functions. Define $\phi(s, s', S^\circ) = a(s, s', S^\circ) f(\tau_s S^\circ) g(\tau_{s'} S^\circ)$ and observe that ϕ is covariant and integrable, and therefore, by the Mass Transport Principle (Lemma 3.1)

$$\begin{aligned}\int E_{\mathbf{s}^\circ} f(\mathbf{s}^\circ) g(\mathbf{s}_1^\circ)\mathcal{Q}(d\mathbf{s}^\circ) &= (1/\mathcal{E}a(0))\mathcal{E}\sum_{s\in S^\circ} a(0, s, S^\circ) f(S^\circ) g(\tau_s S^\circ)\\ &= (1/\mathcal{E}[a(0)])\mathcal{E}\sum_{s\in S^\circ} a(0, s, S^\circ) f(\tau_s S^\circ) g(S^\circ)\\ &= \int E_{\mathbf{s}^\circ} f(\mathbf{s}_1^\circ) g(\mathbf{s}^\circ)\mathcal{Q}(d\mathbf{s}^\circ).\end{aligned}$$

To show ergodicity, let $A \in \mathcal{B}(\mathcal{N}^\circ)$ be an invariant set for the dynamics, that is A is such that $\mathbf{s}_0^\circ \in A$ implies $\mathbf{s}_1^\circ \in A$. This implies that for any neighbor s of the origin, $\tau_s \mathbf{s}_0^\circ \in A$. Iterating the argument one shows that, if $\mathbf{s}^\circ \in A$ then $\tau_s \mathbf{s}^\circ \in A$ for every $s \in \mathbf{s}^\circ$. Therefore,

$$\mathcal{P}_o(A) := \mathcal{P}(S^\circ \in A) = \lim_{\Lambda\nearrow\mathbb{R}^d} \frac{1}{|\Lambda|}\sum_{s\in S^\circ} \mathbf{1}_{\tau_s S^\circ \in A} \in \{0, 1\}$$

and, as $\mathcal{Q} \ll \mathcal{P}_o$ and $\mathcal{P}_o \ll \mathcal{Q}$, it follows that $\mathcal{Q}(A) \in \{0, 1\}$. □

Proposition A.2. *Let $r \geq 1$ and γ be a surface with covariant gradient. If $c := 2\mathcal{C}(|\nabla\gamma|^r) < \infty$ then*

$$\begin{aligned}\mathbb{E}|\gamma(X_t) - \gamma(X_0)|^r &= \mathcal{E}E_{S^\circ}|\gamma(X_t) - \gamma(X_0)|^r \leq \mathcal{E}(a(0)E_{S^\circ}|\gamma(X_t) - \gamma(X_0)|^r)\\ &\leq cm^r(t),\end{aligned}$$

where $m^r(t)$ is the r-th moment of a Poisson random variable with mean t.

Proof. Suppose, without loss of generality, that $\gamma(0) \equiv 0$, and observe that

$$\mathcal{E}(E_{S^\circ}|\gamma(X_t)|^r) = \sum_{n=1}^{\infty} \mathcal{E}(E_{S^\circ}|\gamma(\tilde{X}_n)|^r \mathbf{1}_{N(t)=n}) = \sum_{n=1}^{\infty} \mathcal{E}(E_{S^\circ}|\gamma(\tilde{X}_n)|^r) e^{-t} \frac{t^n}{n!}.$$

On the other hand, by Hölder's inequality

$$|\gamma(\tilde{X}_n)|^r = \left| \sum_{k=1}^{n} (\gamma(\tilde{X}_k) - \gamma(\tilde{X}_{k-1})) \right|^r \le n^{r-1} \sum_{k=1}^{n} |\nabla\gamma(\tilde{X}_{k-1}, \tilde{X}_k)|^r.$$

Finally, as $\tilde{X}_k - \tilde{X}_{k-1}$ depends only on $\mathbf{s}_k$ and $\mathbf{s}_{k-1}$, by the stationarity of $\mathbf{s}_n$ under $\mathcal{Q}$ (Lemma A.1), it follows that

$$\begin{aligned}
\mathcal{E}(E_{S^\circ}|\gamma(X_t)|^r) &\le \sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathcal{E}(E_{S^\circ}|\nabla\gamma(\tilde{X}_{k-1}, \tilde{X}_k, S^\circ)|^r) e^{-t} \frac{t^n}{n!} \\
&\le \sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathcal{E}(a(0) E_{S^\circ}|\nabla\gamma(0, \tilde{X}_k - \tilde{X}_{k-1}, \mathbf{s}_{k-1})|^r) e^{-t} \frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} n^r \mathcal{E}(a(0) E_{S^\circ}|\nabla\gamma(0, \tilde{X}_1, S^\circ)|^r) e^{-t} \frac{t^n}{n!} \\
&= \mathcal{E}\left(\sum_{s \in S^\circ} a(0, s) |\nabla\gamma(0, s, S^\circ)|^r \right) m^r(t). \quad \square
\end{aligned}$$

To obtain estimates for $\mathcal{C}(|\nabla\eta_t^\gamma|^r)$ we study the process of the *environment as seen from the random walker* on S°, as in [11]. The law of S° is reversible and ergodic for this process, which allows us to make estimates on the original random walk. Let $B^0_{[t,0]}$ as in Section 5 be a random walk on the points of S° starting at $0 \in S^\circ$, and denote its law by $P^0_{S^\circ}$.

Proof of Lemma 5.2. From the covariance of $\nabla\gamma$ we can assume, without loss of generality, that $\gamma(0) \equiv 0$. By the Mass Transport Principle Lemma 3.1 and Proposition A.2,

$$\begin{aligned}
\mathcal{C}(|\nabla\psi_t|^r) &= \frac{1}{2} \mathbb{E} \sum_{s \in S^\circ} a(0, s) |\nabla\psi_t(0, s)|^r \le 2^{r-2} \mathbb{E} \sum_{s \in S^\circ} a(0, s) [|\psi_t(0)|^r + |\psi_t(0)|^r] \\
&\le 2^{r-1} \mathbb{E} a(0) |\psi_t(0)|^r \le 2^{r-1} \mathbb{E} a(0) |\gamma(B^0_{[t,0]})|^r \\
&\le 2^r \mathcal{C}(|\nabla\gamma|^r) m^r(t) \quad \mathcal{P}\text{-a.s.} \quad \square
\end{aligned}$$

The following Lemma is a part of the proof of Theorem 2.1 in [11]; we omit the proof.

Lemma A.3. *If a surface γ satisfies*

$$\mathcal{E} \sum_{s \in S} a(0, s) |\gamma(s)|^2 < \infty,$$

then

$$\lim_{n \to \infty} \frac{\mathcal{E}(a(0) E|\gamma(\tilde{X}_n)|^2)}{n} < \infty, \quad \text{and} \quad \lim_{t \to \infty} \frac{\mathcal{E}(a(0) E|\gamma(X_t)|^2)}{t} < \infty. \tag{A.3}$$

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