

BELGIAN MATHEMATICAL  
SOCIETY

Reprint from the Bulletin of the Belgian Mathematical Society – Simon Stevin

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*Bull. Belg. Math. Soc. Simon Stevin 19 (2012), 415–443*

The Bulletin of the Belgian Mathematical Society - Simon Stevin is published by The Belgian Mathematical Society, with financial support from the Universitaire Stichting van Belgie – Fondation Universitaire de Belgique and the Fonds National de la Recherche Scientifique (FNRS). It appears quarterly and is indexed and/or abstracted in Current Contents, Current Mathematical Publications, Mathematical Reviews, Science Citation Index Expanded and Zentralblatt für Mathematik.

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# On a family of Hopf algebras of dimension 72\*

Nicolás Andruskiewitsch

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## Abstract

We investigate a family of Hopf algebras of dimension 72 whose coradical is isomorphic to the algebra of functions on  $S_3$ . We determine the lattice of submodules of the so-called Verma modules and as a consequence we classify all simple modules. We show that these Hopf algebras are unimodular (as well as their duals) but not quasitriangular; also, they are cocycle deformations of each other.

## Introduction

The study of finite dimensional Hopf algebras over an algebraically closed field  $\mathbb{k}$  of characteristic 0 is split into two different classes: the class of semisimple Hopf algebras and the rest. The Lifting Method from [AS] is designed to deal with non-semisimple Hopf algebras whose coradical is a Hopf subalgebra<sup>1</sup>. Pointed Hopf algebras, that is Hopf algebras whose coradical is a group algebra, were intensively studied by this Method. It is natural to consider next the class of Hopf algebras whose coradical is the algebra  $\mathbb{k}^G$  of functions on a non-abelian group  $G$ . This class seems to be interesting at least by the following reasons:

- The categories of Yetter-Drinfeld modules over the group algebra  $\mathbb{k}G$  and  $\mathbb{k}^G$ ,  $G$  a finite group, are equivalent. Thence, a lot sensible information needed for the Lifting Method (description of Yetter-Drinfeld modules, determination of finite dimensional Nichols algebras) can be translated from the pointed case to this case –or vice versa.

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\*This work was partially supported by ANPCyT-Foncyt, CONICET, Ministerio de Ciencia y Tecnología (Córdoba) and Secyt (UNC)

<sup>1</sup>An adaptation to general non-semisimple Hopf algebras was recently proposed in [AC].

Received by the editors November 2011.

Communicated by M. Van den Bergh.

2010 *Mathematics Subject Classification* : 16W30.

*Key words and phrases* : Representations of Hopf algebras.

- The representation theory of Hopf algebras whose coradical is the algebra of functions on a non-abelian group looks easier than the representation theory of pointed Hopf algebras with non-abelian group, because the representation theory of  $\mathbb{k}^G$  is easier than that of  $G$ . Indeed,  $\mathbb{k}^G$  is a semisimple abelian algebra and we may try to imitate the rich methods in representation theory of Lie algebras, with  $\mathbb{k}^G$  playing the role of the Cartan subalgebra. We believe that the representation theory of Hopf algebras with coradical  $\mathbb{k}^G$  might be helpful to study Nichols algebras and deformations.

We have started the consideration of this class in [AV], where finite dimensional Hopf algebras whose coradical is  $\mathbb{k}^{S_3}$  were classified and, in particular, a new family of Hopf algebras of dimension 72 was defined. The purpose of the present paper is to study these Hopf algebras. We first discuss in Section 1 some general ideas about modules induced from simple  $\mathbb{k}^G$ -modules, that we call Verma modules. We introduce in Section 2 a new family of Hopf algebras, as a generalization of the construction in [AV], attached to the class of transpositions in  $S_n$  and depending on a parameter  $a$ .

Our main contributions are in Section 3: we determine the lattice of submodules of the various Verma modules and as a consequence we classify all simple modules over the Hopf algebras of dimension 72 introduced in [AV]. Some further information on these Hopf algebras is given in Section 4 and Section 5.

We assume that the reader has some familiarity with Yetter-Drinfeld modules and Nichols algebras  $\mathcal{B}(V)$ ; we refer to [AS] for these matters.

## Conventions

If  $V$  is a vector space,  $T(V)$  is the tensor algebra of  $V$ . If  $S$  is a subset of  $V$ , then we denote by  $\langle S \rangle$  the vector subspace generated by  $S$ . If  $A$  is an algebra and  $S$  is a subset of  $A$ , then we denote by  $(S)$  the two-sided ideal generated by  $S$  and by  $\mathbb{k}\langle S \rangle$  the subalgebra generated by  $S$ . If  $H$  is a Hopf algebra, then  $\Delta$ ,  $\epsilon$ ,  $\mathcal{S}$  denote respectively the comultiplication, the counit and the antipode. We denote by  $\widehat{R}$  the set of isomorphism classes of a simple  $R$ -modules,  $R$  an algebra; we identify a class in  $\widehat{R}$  with a representative without further notice. If  $S$ ,  $T$  and  $M$  are  $R$ -modules, we say that  $M$  is an extension of  $T$  by  $S$  when  $M$  fits into an exact sequence  $0 \rightarrow S \rightarrow M \rightarrow T \rightarrow 0$ .

## 1 Preliminaries

### 1.1 The induced representation

We collect well-known facts about the induced representation. Let  $B$  be a subalgebra of an algebra  $A$  and let  $V$  be a left  $B$ -module. The induced module is  $\text{Ind}_B^A V = A \otimes_B V$ . The induction has the following properties:

- Universal property: if  $W$  is an  $A$ -module and  $\varphi : V \rightarrow W$  is a morphism of  $B$ -modules, then it extends to a morphism of  $A$ -modules  $\overline{\varphi} : \text{Ind}_B^A V \rightarrow W$ .

$W$ . Hence, there is a natural isomorphism (called Frobenius reciprocity):  $\text{Hom}_B(V, \text{Res}_B^A W) \simeq \text{Hom}_A(\text{Ind}_B^A V, W)$ ; in categorical terms, *induction is left-adjoint to restriction*.

- Any finite dimensional simple  $A$ -module is a quotient of the induced module of a simple  $B$ -module.

Indeed, let  $S$  be a finite dimensional simple  $A$ -module and let  $T$  be a simple  $B$ -submodule of  $S$ . Then the induced morphism  $\text{Ind}_B^A T \rightarrow S$  is surjective.

- If  $B$  is semisimple, then any induced module is projective.

The induction functor, being left adjoint to the restriction one, preserves projectives, and any module over a semisimple algebra is projective.

- If  $A$  is a free right  $B$ -module, say  $A \simeq B^{(I)}$ , then  $\text{Ind}_B^A V = B^{(I)} \otimes_B V = V^{(I)}$  as  $B$ -modules, and a fortiori as vector spaces.

We summarize these basic properties in the setting of finite dimensional Hopf algebras, where freeness over Hopf subalgebras is known [NZ]. Also, finite dimensional Hopf algebras are Frobenius, so that injective modules are projective and vice versa.

**Proposition 1.** *Let  $A$  be a finite dimensional Hopf algebra and let  $B$  be a semisimple Hopf subalgebra.*

- If  $T \in \widehat{B}$ , then  $\dim \text{Ind}_B^A T = \frac{\dim T \dim A}{\dim B}$ .
- Any finite dimensional simple  $A$ -module is a quotient of the induced module of a simple  $B$ -module.
- The induced module of a finite dimensional  $B$ -module is injective and projective. ■

## 1.2 Representation theory of Hopf algebras with coradical a dual group algebra

An optimal situation to apply the Proposition 1 is when the coradical of the finite dimensional Hopf algebra  $A$  is a Hopf subalgebra; in this case  $B = \text{coradical of } A$  is the best choice. It is tempting to say that the induced module of a simple  $B$ -module is a *Verma module* of  $A$ .

Assume now the coradical  $B$  of the finite dimensional Hopf algebra  $A$  is the algebra of functions  $\mathbb{k}^G$  on a finite group  $G$ . In this case, we have:

- Any simple  $B$ -module has dimension 1 and  $\widehat{B} \simeq G$ ; for  $g \in G$ , the simple module  $\mathbb{k}_g$  has the action  $f \cdot 1 = f(g)1$ ,  $f \in \mathbb{k}^G$ . Thus any simple  $A$ -module is a quotient of a Verma module  $M_g := \text{Ind}_{\mathbb{k}^G} \mathbb{k}_g$ , for some  $g \in G$ .

- The ideal  $A\delta_g$  is isomorphic to  $M_g$  and  $A \simeq \bigoplus_{g \in G} M_g$ ; here  $\delta_g$  is the characteristic function of the subset  $\{g\}$ .

- Let  $g \in G$  such that  $\delta_g$  is a primitive idempotent of  $A$ . Since  $A$  is Frobenius,  $M_g \simeq A\delta_g$  has a unique simple submodule  $S$  and a unique maximal submodule  $N$ ;  $M_g$  is the injective hull of  $S$  and the projective cover of  $M_g/N$ . See [CR, (9.9)].

- In all known cases,  $\text{gr}A \simeq \mathcal{B}(V)\#\mathbb{k}^G$ , where  $V$  belongs to a concrete and short list. Hence,  $\dim M_g = \dim \mathcal{B}(V)$  for any  $g \in G$ . More than this, in all known cases we dispose of the following information:

- There exists a rack  $X$  and a 2-cocycle  $q \in Z^2(X, \mathbb{k}^\times)$  such that  $V \simeq (\mathbb{k}X, c^q)$  as braided vector spaces, see [AG] for details.
- There exists an epimorphism of Hopf algebras  $\phi : T(V)\#\mathbb{k}^G \rightarrow A$ , see [AV, Subsection 2.5] for details. Note that  $\phi(f \cdot x) = \text{ad } f(\phi(x))$  for all  $f \in \mathbb{k}^G$  and  $x \in T(V)$ .
- Let  $\mathbb{X}$  be the set of words in  $X$ , identified with a basis of the tensor algebra  $T(V)$ . There exists  $\mathbb{B} \subset \mathbb{X}$  such that the classes of the monomials in  $\mathbb{B}$  form a basis of  $\mathcal{B}(V)$ . The corresponding classes in  $A$  multiplied with the elements  $\delta_g \in \mathbb{k}^G, g \in G$ , form a basis of  $A$ .
- If  $x \in X$ , then there exists  $g_x \in G$  such that  $\delta_h \cdot x = \delta_{h, g_x} x$  for all  $h \in G$ . We extend this to have  $g_x \in G$  for any  $x \in \mathbb{X}$ .
- If  $x \in X$ , then  $x^2 = 0$  in  $\mathcal{B}(V)$  and there exists  $f_x \in \mathbb{k}^G$  such that  $x^2 = f_x$  in  $A$ .

Let  $g \in G$ . If  $x \in \mathbb{B}$ , then we denote by  $m_x$  the class of  $x$  in  $M_g$ . Hence  $(m_x)_{x \in \mathbb{B}}$  is a basis of  $M_g$ . We may describe the action of  $A$  on this basis of  $M_g$ , at least when we know explicitly the relations of  $A$  and the monomials in  $\mathbb{B}$ . To start with, let  $f \in \mathbb{k}^G$  and  $x \in \mathbb{B}$ . Then

$$\begin{aligned} f \cdot m_x &= \overline{fx \otimes 1} = \overline{f_{(1)} \cdot x f_{(2)} \otimes 1} = \overline{f_{(1)} \cdot x \otimes f_{(2)} \cdot 1} \\ &= f(g_x g) m_x. \end{aligned} \tag{1}$$

Let now  $x = x_1 \dots x_t$  be a monomial in  $\mathbb{B}$ , with  $x_1, \dots, x_t \in X$ . Set  $y = x_2 \dots x_t$ ; observe that  $y$  need not be in  $\mathbb{B}$ . Then

$$x_1 \cdot m_x = \overline{x_1^2 x_2 \dots x_t \otimes 1} = \overline{f_{x_1} y \otimes 1} = f_{x_1}(g_y g) \overline{y \otimes 1}. \tag{2}$$

Let now  $M$  be a finite dimensional  $A$ -module. It is convenient to consider the decomposition of  $M$  in isotypic components as  $\mathbb{k}^G$ -module:  $M = \bigoplus_{g \in G} M[g]$ , where  $M[g] = \delta_g \cdot M$ . Note that

$$x \cdot M[g] = M[g_x g] \quad \text{for all } x \in \mathbb{B}, g \in G. \tag{3}$$

For instance, (1) says that the isotypic components of the Verma module  $M_g$  are  $M_g[h] = \langle m_x : x \in \mathbb{B}, g_x g = h \rangle$ .

## 2 Hopf algebras related to the class of transpositions in the symmetric group

### 2.1 Quadratic Nichols algebras

Let  $n \geq 3$ ; denote by  $\mathcal{O}_2^n$  the conjugacy class of  $(12)$  in  $S_n$  and by  $\text{sgn} : C_{S_n}(12) \rightarrow \mathbb{k}$  the restriction of the sign representation of  $S_n$  to the centralizer of  $(12)$ . Let  $V_n = M((12), \text{sgn}) \in \frac{\mathbb{k}^{S_n}}{\mathbb{k}^{S_n}} \mathcal{YD}$ ;  $V_n$  has a basis  $(x_{(ij)})_{(ij) \in \mathcal{O}_2^n}$  such that the action  $\cdot$  and the coaction  $\delta$  are given by

$$\delta_h \cdot x_{(ij)} = \delta_{h,(ij)} x_{(ij)} \quad \forall h \in S_n \quad \text{and} \quad \delta(x_{(ij)}) = \sum_{h \in S_n} \text{sgn}(h) \delta_h \otimes x_{h^{-1}(ij)h}.$$

Let  $n = 3, 4, 5$ . By [MS, G], we know that  $\mathcal{B}(V_n)$  is quadratic and finite dimensional; actually, the ideal  $\mathcal{J}_n$  of relations of  $\mathcal{B}(V_n)$  is generated by

$$x_{(ij)}^2, \tag{4}$$

$$R_{(ij)(kl)} := x_{(ij)}x_{(kl)} + x_{(kl)}x_{(ij)}, \tag{5}$$

$$R_{(ij)(ik)} := x_{(ij)}x_{(ik)} + x_{(ik)}x_{(jk)} + x_{(jk)}x_{(ij)} \tag{6}$$

for  $(ij), (kl), (ik) \in \mathcal{O}_2^n$  with  $\#\{i, j, k, l\} = 4$ .

For  $n \geq 6$ , we define the *quadratic Nichols algebra*  $\mathcal{B}_n$  in the same way, that is as the quotient of the tensor algebra  $T(V_n)$  by the ideal generated by the quadratic relations (4), (5) and (6) for  $(ij), (kl), (ik) \in \mathcal{O}_2^n$  with  $\#\{i, j, k, l\} = 4$ . It is however open whether:

- $\mathcal{B}(V_n)$  is quadratic, i. e. isomorphic to  $\mathcal{B}_n$ ;
- the dimension of  $\mathcal{B}(V_n)$  is finite;
- the dimension of  $\mathcal{B}_n$  is finite.

But we do know that the only possible finite dimensional Nichols algebras<sup>2</sup> over  $S_n$  are related to the orbit of transpositions and a pair of characters [AFGV, Th. 1.1]. Also, the Nichols algebras related to these two characters are twist-equivalent [Ve].

### 2.2 The parameters

We consider the set of parameters

$$\mathfrak{A}_n := \left\{ \mathbf{a} = (a_{(ij)})_{(ij) \in \mathcal{O}_2^n} \in \mathbb{k}^{\mathcal{O}_2^n} : \sum_{(ij) \in \mathcal{O}_2^n} a_{(ij)} = 0 \right\}.$$

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<sup>2</sup>There is one exception when  $n = 4$  that is finite dimensional and two exceptions when  $n = 5$  and  $6$  that are not known.

The group  $\Gamma_n := \mathbb{k}^\times \times \text{Aut}(\mathbb{S}_n)$  acts on  $\mathfrak{A}_n$  by

$$(\mu, \theta) \triangleright \mathbf{a} = \mu(a_{\theta(ij)}), \quad \mu \in \mathbb{k}^\times, \quad \theta \in \text{Aut}(\mathbb{S}_n), \quad \mathbf{a} \in \mathfrak{A}_n. \quad (7)$$

Let  $[\mathbf{a}] \in \Gamma_n \backslash \mathfrak{A}_n$  be the class of  $\mathbf{a}$  under this action. Let  $\triangleright$  denote also the conjugation action of  $\mathbb{S}_n$  on itself, so that<sup>3</sup>  $\mathbb{S}_n < \{e\} \times \text{Aut}(\mathbb{S}_n) < \Gamma_n$ . Let  $\mathbb{S}_n^{\mathbf{a}} = \{g \in \mathbb{S}_n \mid g \triangleright \mathbf{a} = \mathbf{a}\}$  be the isotropy group of  $\mathbf{a}$  under the action of  $\mathbb{S}_n$ .

We fix  $\mathbf{a} \in \mathfrak{A}_n$  and introduce

$$f_{ij} = \sum_{g \in \mathbb{S}_n} (a_{(ij)} - a_{g^{-1}(ij)g}) \delta_g \in \mathbb{k}^{\mathbb{S}_n}, \quad (ij) \in \mathcal{O}_2^n. \quad (8)$$

Clearly,

$$f_{ij}(ts) = f_{ij}(s) \quad \forall t \in C_{\mathbb{S}_n}(ij), \quad s \in \mathbb{S}_n. \quad (9)$$

**Definition 2.** We say that  $g$  and  $h \in \mathbb{S}_n$  are  $\mathbf{a}$ -linked, denoted  $g \sim_{\mathbf{a}} h$ , if either  $g = h$  or else there exist  $(i_m j_m), \dots, (i_1 j_1) \in \mathcal{O}_2^n$  such that

- $g = (i_m j_m) \cdots (i_1 j_1) h$ ,
- $f_{i_s j_s}((i_s j_s)(i_{s-1} j_{s-1}) \cdots (i_1 j_1) h) \neq 0$  for all  $1 \leq s \leq m$ .

In particular,  $f_{i_1 j_1}(h) \neq 0$  by (9). We claim that  $\sim_{\mathbf{a}}$  is an equivalence relation. For, if  $g$  and  $h \in \mathbb{S}_n$  are  $\mathbf{a}$ -linked, then  $h = (i_1 j_1) \cdots (i_m j_m) g$  and

$$\begin{aligned} f_{i_s j_s}((i_s j_s)(i_{s+1} j_{s+1}) \cdots (i_m j_m) g) &= f_{i_s j_s}((i_s j_s)(i_{s-1} j_{s-1}) \cdots (i_1 j_1) h) \\ &\stackrel{(9)}{=} f_{i_s j_s}((i_s j_s)(i_{s-1} j_{s-1}) \cdots (i_1 j_1) h) \neq 0. \end{aligned}$$

In the same way, we see that if  $g \sim_{\mathbf{a}} h$  and also  $h \sim_{\mathbf{a}} z$ , then  $g \sim_{\mathbf{a}} z$ .

### 2.3 A family of Hopf algebras

We fix  $\mathbf{a} \in \mathfrak{A}_n$ ; recall the elements  $f_{ij}$  defined in (8). Let  $\mathcal{I}_{\mathbf{a}}$  be the ideal of  $T(V_n) \# \mathbb{k}^{\mathbb{S}_n}$  generated by (5), (6) and

$$x_{(ij)}^2 - f_{ij}, \quad (10)$$

for all  $(ij), (kl), (ik) \in \mathcal{O}_2^n$  such that  $\#\{i, j, k, l\} = 4$ . Then

$$\mathcal{A}_{[\mathbf{a}]} := T(V_n) \# \mathbb{k}^{\mathbb{S}_n} / \mathcal{I}_{\mathbf{a}}$$

is a Hopf algebra, see Remark 3. Also, if  $\text{gr} \mathcal{A}_{[\mathbf{a}]} \simeq \mathcal{B}(V_n) \# \mathbb{k}^{\mathbb{S}_n} \simeq \text{gr} \mathcal{A}_{[\mathbf{b}]}$ , then  $\mathcal{A}_{[\mathbf{a}]} \simeq \mathcal{A}_{[\mathbf{b}]}$  if and only if  $[\mathbf{a}] = [\mathbf{b}]$ , what justifies the notation. If  $n = 3$ , then  $\text{gr} \mathcal{A}_{[\mathbf{a}]} \simeq \mathcal{B}(V_3) \# \mathbb{k}^{\mathbb{S}_3}$  and  $\dim \mathcal{A}_{[\mathbf{a}]} = 72$  [AV]; for  $n = 4, 5$  the dimension is finite but we do not know if it is the "right" one; for  $n \geq 6$ , the dimension is unknown to be finite.

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<sup>3</sup>It is well-known that  $\mathbb{S}_n$  identifies with the group of inner automorphisms and that this equals  $\text{Aut} \mathbb{S}_n$ , except for  $n = 6$ .



**Remark 3.** A straightforward computation shows that

$$\begin{aligned} \Delta(x_{(ij)}^2) &= x_{(ij)}^2 \otimes 1 + \sum_{h \in S_n} \delta_h \otimes x_{h^{-1}(ij)h}^2 \quad \text{and} \\ \Delta(f_{ij}) &= f_{ij} \otimes 1 + \sum_{h \in S_n} \delta_h \otimes f_{h^{-1}(i)h^{-1}(j)}. \end{aligned}$$

Then  $J = \langle x_{(ij)}^2 - f_{ij} : (ij) \in \mathcal{O}_2^n \rangle$  is a coideal. Since  $f_{ij}(e) = 0$ , we have that  $J \subset \ker \epsilon$  and  $\mathcal{S}(J) \subseteq \mathbb{k}^{S_n} J$ . Thus  $\mathcal{I}_a = (J)$  is a Hopf ideal and  $\mathcal{A}_{[a]}$  is a Hopf algebra quotient of  $T(V_n) \# \mathbb{k}^{S_n}$ . We shall say that  $\mathbb{k}^{S_n}$  is a subalgebra of  $\mathcal{A}_{[a]}$  to express that the restriction of the projection  $T(V_n) \# \mathbb{k}^{S_n} \twoheadrightarrow \mathcal{A}_{[a]}$  to  $\mathbb{k}^{S_n}$  is injective.

Let us collect a few general facts on the representation theory of  $\mathcal{A}_{[a]}$ .

**Remark 4.** Assume that  $\mathbb{k}^{S_n}$  is a subalgebra of  $\mathcal{A}_{[a]}$  and let  $M$  be an  $\mathcal{A}_{[a]}$ -module. Hence

- (a) If  $(ij) \in \mathcal{O}_2^n$  satisfies  $f_{ij}(h) \neq 0$ , then  $\rho(x_{(ij)}) : M[h] \rightarrow M[(ij)h]$  is an isomorphism.
- (b) Let  $g \sim_a h \in S_n$ . Then  $\rho(x_{(i_m j_m)}) \circ \cdots \circ \rho(x_{(i_1 j_1)}) : M[h] \rightarrow M[g]$  is an isomorphism.

*Proof.*  $\rho(x_{(ij)}) : M[h] \rightarrow M[(ij)h]$  is injective and  $\rho(x_{(ij)}) : M[(ij)h] \rightarrow M[h]$  is surjective, by (10). Interchanging the roles of  $h$  and  $(ij)h$ , we get (a). Now (b) follows from (a). ■

This Remark is particularly useful to compare Verma modules.

**Proposition 5.** Assume that  $\dim \mathcal{A}_{[a]} < \infty$  and  $\mathbb{k}^{S_n}$  is a subalgebra of  $\mathcal{A}_{[a]}$ . If  $g$  and  $h$  are  $\mathbf{a}$ -linked, then the Verma modules  $M_g$  and  $M_h$  are isomorphic.

*Proof.* The Verma module  $M_h$  is generated by  $m_1 = 1 \otimes_{\mathbb{k}^{S_n}} 1 \in M_h[h]$ . By Remark 4 (b), there exists  $m \in M_h[g]$  such that  $M_h = \mathcal{A}_{[a]} \cdot m$ . Therefore, there is an epimorphism  $M_g \twoheadrightarrow M_h$ . Since  $\mathcal{A}_{[a]}$  is finite dimensional, all the Verma modules have the same dimension; hence  $M_g \simeq M_h$ . ■

**Definition 6.** We say that the parameter  $\mathbf{a}$  is *generic* when any of the following equivalent conditions holds.

- (a)  $a_{(ij)} \neq a_{(kl)}$  for all  $(ij) \neq (kl) \in \mathcal{O}_2^n$ .
- (b)  $a_{(ij)} \neq a_{h \triangleright (ij)}$  for all  $(ij) \in \mathcal{O}_2^n$  and all  $h \in S_n - C_{S_n}(ij)$ .
- (c)  $f_{ij}(h) \neq 0$  for all  $(ij) \in \mathcal{O}_2^n$  and all  $h \in S_n - C_{S_n}(ij)$ .

*Proof.* (a)  $\implies$  (b) is clear, since  $(ij) \neq h \triangleright (ij)$  by the assumption on  $h$ . (b)  $\implies$  (a) follows since any  $(kl) \neq (ij)$  is of the form  $(kl) = h \triangleright (ij)$ , for some  $h \notin S_n^{(ij)}$ . (b)  $\iff$  (c): given  $(ij)$ , we have

$$\{h \in S_n : a_{(ij)} = a_{h \triangleright (ij)}\} = \{h \in S_n : f_{ij}(h) = 0\};$$

hence, one of these sets equals  $C_{S_n}(ij)$  iff the other does. ■

**Lemma 7.** Assume that  $\mathbf{a}$  is generic, so that  $g \sim_{\mathbf{a}} h$  for all  $g, h \in \mathbb{S}_n - \{e\}$ . If  $\mathbb{k}^{\mathbb{S}_n}$  is a subalgebra of  $\mathcal{A}_{[\mathbf{a}]}$ , then

- (a) If  $\mathcal{A}_{[\mathbf{a}]}$  is finite dimensional, then the Verma modules  $M_g$  and  $M_h$  are isomorphic, for all  $g, h \in \mathbb{S}_n - \{e\}$ .
- (b) If  $M$  is an  $\mathcal{A}_{[\mathbf{a}]}$ -module, then  $\dim M[h] = \dim M[g]$  for all  $g, h \in \mathbb{S}_n - \{e\}$ . Thus  $\dim M = (n! - 1) \dim M[(ij)] + \dim M[e]$ .
- (c) If  $M$  is simple and  $n = 3$ , then  $\dim M[h] \leq 1$  for all  $h \in \mathbb{S}_3 - \{e\}$ .

*Proof.* Let  $(ij) \in \mathbb{S}_n$  and  $g \in \mathbb{S}_n - \{e\}$ .

- If  $g = (ik)$ , then  $g \sim_{\mathbf{a}} (ij)$ , as  $(ik) = (ij)(jk)(ij)$  and  $\mathbf{a}$  is generic.
- If  $g = (kl)$  with  $\#\{i, j, l, k\} = 4$ , then  $(ij) \sim_{\mathbf{a}} (ik)$  and  $(ik) \sim_{\mathbf{a}} (kl)$ , hence  $(ij) \sim_{\mathbf{a}} (kl)$ .
- If  $g = (i_1 i_2 \cdots i_r)$  is an  $r$ -cycle, then  $g = (i_1 i_r)(i_1 i_2 \cdots i_{r-1})$ . Hence  $g \sim_{\mathbf{a}} (ij)$  by induction on  $r$ .
- Let  $g = g_1 \cdots g_m$  be the product of the disjoint cycles  $g_1, \dots, g_m$ , with  $m \geq 2$ ; say  $g_1 = (i_1 \cdots i_r)$ ,  $g_2 = (i_{r+1} \cdots i_{r+s})$  and denote  $y = g_3 \cdots g_m$ . Then  $g = (i_1 i_{r+1})(i_1 \cdots i_{r+s})y$  and  $y \in C_{\mathbb{S}_n}(i_1 i_{r+1})$ . Hence  $g$  and  $(ij)$  are linked by induction on  $m$ .

Now (a) follows from Proposition 5 and (b) from Remark 4. If  $n = 3$  and  $M$  is simple, then  $\dim \mathcal{A}_{[\mathbf{a}]} = 72 > (\dim M)^2 \geq 25(\dim M[(12)])^2$  and the last assertion of the lemma follows. ■

The characterization of all one dimensional  $\mathcal{A}_{[\mathbf{a}]}$ -modules is not difficult. Let  $\approx$  be the equivalence relation in  $\mathcal{O}_2^n$  given by  $(ij) \approx (kl)$  iff  $a_{(ij)} = a_{(kl)}$ . Let  $\mathcal{O}_2^n = \coprod_{s \in Y} \mathcal{C}_s$  be the associated partition. If  $h \in \mathbb{S}_n$ , then

$$f_{ij}(h) = 0 \forall (ij) \in \mathcal{O}_2^n \iff h^{-1} \mathcal{C}_s h = \mathcal{C}_s \forall s \in Y \iff h \in \mathbb{S}_n^{\mathbf{a}}. \tag{11}$$

**Lemma 8.** Assume that  $\mathbb{k}^{\mathbb{S}_n}$  is a subalgebra of  $\mathcal{A}_{[\mathbf{a}]}$  and let  $h \in \mathbb{S}_n^{\mathbf{a}}$ . Then  $\mathbb{k}_h$  is a  $\mathcal{A}_{[\mathbf{a}]}$ -module with the action given by the algebra map  $\zeta_h : \mathcal{A}_{[\mathbf{a}]} \rightarrow \mathbb{k}$ ,

$$\zeta_h(x_{(ij)}) = 0, \quad (ij) \in \mathcal{O}_2^n \quad \text{and} \quad \zeta_h(f) = f(h), \quad f \in \mathbb{k}^{\mathbb{S}_n}. \tag{12}$$

The one-dimensional representations of  $\mathcal{A}_{[\mathbf{a}]}$  are all of this form.

*Proof.* Clearly,  $\zeta_h$  satisfies the relations of  $T(V_n) \# \mathbb{k}^{\mathbb{S}_n}$ , (5) and (6); (10) holds because  $h$  fulfills (11). Now, let  $M$  be a module of dimension 1. Then  $M = M[h]$  for some  $h$ ; thus  $f_{ij}(h) = 0$  for all  $(ij) \in \mathcal{O}_2^n$  by Remark 4. ■

### 3 Simple and Verma modules over Hopf algebras with coradical

$\mathbb{k}^{S_3}$

#### 3.1 Verma modules

In this Section, we focus on the case  $n = 3$ . Let  $\mathbf{a} \in \mathfrak{A}_3$ . Explicitly,  $\mathcal{A}_{[\mathbf{a}]}$  is the algebra  $(T(V_3)\#\mathbb{k}^{S_3})/\mathcal{I}_{\mathbf{a}}$  where  $\mathcal{I}_{\mathbf{a}}$  is the ideal generated by

$$R_{(13)(23)}, \quad R_{(23)(13)}, \quad x_{(ij)}^2 - f_{ij}, \quad (ij) \in \mathcal{O}_2^3, \quad (13)$$

where

$$\begin{aligned} f_{13} &= (a_{(13)} - a_{(23)})(\delta_{(12)} + \delta_{(123)}) + (a_{(13)} - a_{(12)})(\delta_{(23)} + \delta_{(132)}), \\ f_{23} &= (a_{(23)} - a_{(12)})(\delta_{(13)} + \delta_{(123)}) + (a_{(23)} - a_{(13)})(\delta_{(12)} + \delta_{(132)}), \\ f_{12} &= (a_{(12)} - a_{(13)})(\delta_{(23)} + \delta_{(123)}) + (a_{(12)} - a_{(23)})(\delta_{(13)} + \delta_{(132)}). \end{aligned} \quad (14)$$

We know from [AV] that  $\mathcal{A}_{[\mathbf{a}]}$  is a Hopf algebra of dimension 72 and coradical isomorphic to  $\mathbb{k}^{S_3}$ , for any  $\mathbf{a} \in \mathfrak{A}_3$ . Furthermore, any finite dimensional non-semisimple Hopf algebra with coradical  $\mathbb{k}^{S_3}$  is isomorphic to  $\mathcal{A}_{[\mathbf{a}]}$  for some  $\mathbf{a} \in \mathfrak{A}_3$ ;  $\mathcal{A}_{[\mathbf{b}]} \simeq \mathcal{A}_{[\mathbf{a}]}$  iff  $[\mathbf{a}] = [\mathbf{b}]$ . Let  $\Omega = f_{13}((12)\_ ) - f_{13}$ , that is

$$\begin{aligned} \Omega &= (a_{(23)} - a_{(13)})(\delta_{(12)} - \delta_e) \\ &\quad + (a_{(13)} - a_{(12)})(\delta_{(13)} - \delta_{(132)}) + (a_{(12)} - a_{(23)})(\delta_{(23)} - \delta_{(123)}). \end{aligned} \quad (15)$$

The following formulae follow from the defining relations:

$$x_{(12)}x_{(13)}x_{(12)} = x_{(13)}x_{(12)}x_{(13)} + x_{(23)}(a_{(13)} - a_{(12)}), \quad (16)$$

$$x_{(23)}x_{(12)}x_{(23)} = x_{(12)}x_{(23)}x_{(12)} - x_{(13)}(a_{(23)} - a_{(12)}) \text{ and} \quad (17)$$

$$x_{(23)}x_{(12)}x_{(13)} = x_{(13)}x_{(12)}x_{(23)} + x_{(12)}\Omega. \quad (18)$$

Let

$$\mathbb{B} = \left\{ \begin{array}{l} 1, \quad x_{(13)}, \quad x_{(13)}x_{(12)}, \quad x_{(13)}x_{(12)}x_{(13)}, \quad x_{(13)}x_{(12)}x_{(23)}x_{(12)}, \\ x_{(23)}, \quad x_{(12)}x_{(13)}, \quad x_{(12)}x_{(23)}x_{(12)}, \\ x_{(12)}, \quad x_{(23)}x_{(12)}, \quad x_{(13)}x_{(12)}x_{(23)}, \\ x_{(12)}x_{(23)} \end{array} \right\}.$$

Then  $\{x\delta_g | x \in \mathbb{B}, g \in S_3\}$  is a basis of  $\mathcal{A}_{[\mathbf{a}]}$  [AV]. Fix  $g \in G$ . The classes of the monomials in  $\mathbb{B}$  form a basis of the Verma module  $M_g$ . Denote by  $m_{(ij)\dots(rs)}$  the class of  $x_{(ij)} \dots x_{(rs)}$ ; we simply set  $m_{\text{top}} = m_{(13)(12)(23)(12)}$ . The action of  $\mathcal{A}_{[\mathbf{a}]}$  on  $M_g$  is described in this basis by the following formulae:

$$f \cdot m_1 = f(g)m_1, \quad f \in \mathbb{k}^{S_3}; \quad (19)$$

$$f \cdot m_{(ij)\dots(rs)} = f((ij)\dots(rs)g) m_{(ij)\dots(rs)}, \quad f \in \mathbb{k}^{S_3}; \quad (20)$$

$$x_{(ij)} \cdot m_1 = m_{(ij)}, \quad (ij) \in \mathcal{O}_2^3; \quad (21)$$

$$x_{(ij)} \cdot m_{(ij)} = f_{ij}(g)m_1, \quad (ij) \in \mathcal{O}_2^3; \quad (22)$$

$$x_{(13)} \cdot m_{(23)} = -m_{(23)(12)} - m_{(12)(13)}, \quad (23)$$

$$x_{(13)} \cdot m_{(12)} = m_{(13)(12)}, \quad (24)$$

$$x_{(23)} \cdot m_{(13)} = -m_{(12)(23)} - m_{(13)(12)}, \quad (25)$$

$$x_{(23)} \cdot m_{(12)} = m_{(23)(12)}, \quad (26)$$

$$x_{(12)} \cdot m_{(13)} = m_{(12)(13)}, \quad (27)$$

$$x_{(12)} \cdot m_{(23)} = m_{(12)(23)}; \quad (28)$$

$$x_{(13)} \cdot m_{(13)(12)} = f_{13}((12)g) m_{(12)}, \quad (29)$$

$$x_{(13)} \cdot m_{(12)(13)} = m_{(13)(12)(13)}, \quad (30)$$

$$x_{(13)} \cdot m_{(23)(12)} = -m_{(13)(12)(13)} - f_{13}((23)g) m_{(23)}, \quad (31)$$

$$x_{(13)} \cdot m_{(12)(23)} = m_{(13)(12)(23)}; \quad (32)$$

$$x_{(23)} \cdot m_{(13)(12)} = -m_{(12)(23)(12)} - f_{12}(g) m_{(13)}, \quad (33)$$

$$x_{(23)} \cdot m_{(12)(13)} = m_{(13)(12)(23)} + \Omega(g) m_{(12)}, \quad (34)$$

$$x_{(23)} \cdot m_{(23)(12)} = f_{23}((12)g) m_{(12)}, \quad (35)$$

$$x_{(23)} \cdot m_{(12)(23)} = m_{(12)(23)(12)} - m_{(13)} f_{23}((13)), \quad (36)$$

$$x_{(12)} \cdot m_{(13)(12)} = m_{(13)(12)(13)} + m_{(23)} f_{13}((23)), \quad (37)$$

$$x_{(12)} \cdot m_{(12)(13)} = f_{12}((13)g) m_{(13)}, \quad (38)$$

$$x_{(12)} \cdot m_{(23)(12)} = m_{(12)(23)(12)}, \quad (39)$$

$$x_{(12)} \cdot m_{(12)(23)} = f_{12}((23)g) m_{(23)}; \quad (40)$$

$$x_{(13)} \cdot m_{(13)(12)(13)} = f_{13}((12)(13)g) m_{(12)(13)}, \quad (41)$$

$$x_{(13)} \cdot m_{(12)(23)(12)} = m_{\text{top}}, \quad (42)$$

$$x_{(13)} \cdot m_{(13)(12)(23)} = f_{13}((12)(23)g) m_{(12)(23)}, \quad (43)$$

$$x_{(23)} \cdot m_{(13)(12)(13)} = m_{\text{top}} - (f_{12}\Omega + (a_{(13)} - a_{(12)})f_{23})(g) m_1, \quad (44)$$

$$x_{(23)} \cdot m_{(12)(23)(12)} = f_{12}(g) m_{(12)(23)} + (a_{(12)} - a_{(23)}) m_{(13)(12)}, \quad (45)$$

$$x_{(23)} \cdot m_{(13)(12)(23)} = f_{23}((23)(12)g) m_{(12)(13)} - \Omega(g) m_{(23)(12)}, \quad (46)$$

$$x_{(12)} \cdot m_{(13)(12)(13)} = (f_{13}(g) + f_{12}((23))) m_{(13)(12)} + f_{12}((23)) m_{(12)(23)}, \quad (47)$$

$$x_{(12)} \cdot m_{(12)(23)(12)} = f_{12}((23)(12)g) m_{(23)(12)}, \quad (48)$$

$$x_{(12)} \cdot m_{(13)(12)(23)} = -m_{\text{top}} + (f_{13}((23))f_{23} - f_{12}((13)\_\_)f_{13})(g) m_1; \quad (49)$$

$$x_{(13)} \cdot m_{\text{top}} = f_{13}(g) m_{(12)(23)(12)}, \quad (50)$$

$$x_{(23)} \cdot m_{\text{top}} = f_{23}(g) m_{(13)(12)(13)} + (f_{13}((23))f_{23} + \Omega f_{12})(g) m_{(23)}, \quad (51)$$

$$x_{(12)} \cdot m_{\text{top}} = -f_{12}(g) m_{(13)(12)(23)} + \quad (52)$$

$$+ (f_{13}((23))f_{23}((12)\_\_) - f_{12}((23)\_\_)f_{13}((12)\_\_))(g) m_{(12)};$$

To proceed with the description of the simple modules, we split the consideration of the algebras  $\mathcal{A}_{[a]}$  into several cases.

- $a_{(13)} = a_{(12)} = a_{(23)}$ . In this case, there is a projection  $\mathcal{A}_{[\mathbf{a}]} \rightarrow \mathbb{k}^{\mathbb{S}_3}$ . It is easy to see that any simple  $\mathcal{A}_{[\mathbf{a}]}$ -module is obtained from a simple  $\mathbb{k}^{\mathbb{S}_3}$ -module composing with this projection; thus,  $\widehat{\mathcal{A}}_{[\mathbf{a}]} \simeq \mathbb{S}_3$ .
- $a_{(13)} = a_{(12)}$  or  $a_{(23)} = a_{(12)}$  or  $a_{(13)} = a_{(23)}$ , but not in the previous case. Up to isomorphism, cf. (7), we may assume  $a_{(12)} \neq a_{(13)} = a_{(23)}$ . For shortness, we shall say that  $\mathbf{a}$  is *sub-generic*.
- $\mathbf{a}$  is generic.

In the next subsections, we investigate these two different cases. Let us consider the decomposition of the Verma module  $M_g$  in isotypic components as  $\mathbb{k}^{\mathbb{S}_3}$ -modules. The isotypic components of the Verma module  $M_e$  are

$$\begin{aligned} M_e[e] &= \langle m_1, m_{\text{top}} \rangle, & M_e[(12)] &= \langle m_{(12)}, m_{(13)(12)(23)} \rangle, \\ M_e[(13)] &= \langle m_{(13)}, m_{(12)(23)(12)} \rangle, & M_e[(23)] &= \langle m_{(23)}, m_{(13)(12)(13)} \rangle, \\ M_e[(123)] &= \langle m_{(13)(12)}, m_{(12)(23)} \rangle, & M_e[(132)] &= \langle m_{(12)(13)}, m_{(23)(12)} \rangle. \end{aligned} \tag{53}$$

Let  $g, h \in \mathbb{S}_3$ ,  $(ij) \in \mathcal{O}_2^3$ . By (20) and (3), we have

$$M_g[h] = M_e[hg^{-1}], \tag{54}$$

$$x_{(ij)} \cdot M_g[h] \subseteq M_g[(ij)h]. \tag{55}$$

It is convenient to introduce the following elements:

$$m_{\text{soc}} = f_{13}((23))f_{23}((13))m_1 - m_{\text{top}}, \tag{56}$$

$$m_o = m_{(13)(12)(13)} + f_{13}((23))m_{(23)}. \tag{57}$$

### 3.2 Case $\mathbf{a} \in \mathfrak{A}_3$ generic.

To determine the simple  $\mathcal{A}_{[\mathbf{a}]}$ -modules, we just need to determine the maximal submodules of the various Verma modules. By Lemma 7 (a), we are reduced to consider the Verma modules  $M_e$  and  $M_g$  for some fixed  $g \neq e$ . We choose  $g = (13)(23)$ ; for the sake of an easy exposition, we write the elements of  $\mathbb{S}_3$  as products of transpositions.

We start with the following observation. Let  $M$  be a cyclic  $\mathcal{A}_{[\mathbf{a}]}$ -module, generated by  $v \in M[(13)(23)]$ . By (55) and acting by the monomials in our basis of  $\mathcal{A}_{[\mathbf{a}]}$ , we see that

$$M[(23)(13)] = \langle x_{(13)}x_{(23)} \cdot v, x_{(23)}x_{(12)} \cdot v, x_{(12)}x_{(13)} \cdot v \rangle.$$

This weight space is  $\neq 0$  by Lemma 7 (b), and a further application of this Lemma gives the following result.

**Remark 9.** Let  $M$  be a cyclic  $\mathcal{A}_{[\mathbf{a}]}$ -module, generated by  $v \in M[(13)(23)]$ . If  $\dim M[(23)(13)] = 1$ , then

$$M[(23)] = \langle x_{(13)} \cdot v \rangle, \quad M[e] = \langle x_{(12)}x_{(23)} \cdot v, x_{(13)}x_{(12)} \cdot v \rangle,$$

$$\begin{aligned} M[(12)] &= \langle x_{(23)} \cdot v \rangle, & M[(13)] &= \langle x_{(12)} \cdot v \rangle, \\ M[(13)(23)] &= \langle v \rangle, & M[(23)(13)] &= \langle x_{(13)}x_{(23)} \cdot v \rangle. \end{aligned} \tag{58}$$

Thus, any cyclic module as in the Remark has either dimension 5, 6 or 7. Moreover, there is a simple module  $L$  like this;  $L$  has a basis  $\{v_g | e \neq g \in \mathbb{S}_3\}$  and the action is given by

$$v_g \in L[g], \quad x_{(ij)} \cdot v_g = \begin{cases} v_{(ij)g} & \text{if } \text{sgn } g = 1, \\ f_{ij}(g)v_{(ij)g} & \text{if } \text{sgn } g = -1. \end{cases} \tag{59}$$

Let  $\mathbb{k}_e$  be as in Lemma 8. We shall see that  $L$  and  $\mathbb{k}_e$  are the only simple modules of  $\mathcal{A}_{[a]}$ .

The Verma module  $M_e$  projects onto the simple submodule  $\mathbb{k}_e$ , hence the kernel of this projection is a maximal submodule; explicitly this is

$$N_e = \mathcal{A}_{[a]} \cdot M_e[(13)(23)] = \bigoplus_{g \sim_a (13)(23)} M_e[g] \oplus \langle m_{\text{top}} \rangle.$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of  $M_e$ .

**Lemma 10.** *The submodules of  $M_e$  are*

$$\langle m_{\text{top}} \rangle \subsetneq \mathcal{A}_{[a]} \cdot v \subsetneq N_e \subsetneq M_e$$

for any  $v \in M_e[(13)(23)] - 0$ . The submodules  $\mathcal{A}_{[a]} \cdot v$  and  $\mathcal{A}_{[a]} \cdot u$  coincide iff  $v \in \langle u \rangle$ . The quotients  $\mathcal{A}_{[a]} \cdot v / \langle m_{\text{top}} \rangle$  and  $N_e / \mathcal{A}_{[a]} \cdot v$  are isomorphic to  $L$ ; and  $M_e / N_e$  and  $\langle m_{\text{top}} \rangle$  are isomorphic to  $\mathbb{k}_e$ .

*Proof.* By (51), (50) and (52), we have  $x_{(ij)} \cdot m_{\text{top}} = 0$  for all  $(ij) \in \mathcal{O}_2^3$ . Let

$$\begin{aligned} v &= \lambda m_{(23)(12)} + \mu m_{(12)(13)} && \in M_e[(13)(23)] - 0, \\ w &= \mu m_{(12)(23)} + (\mu - \lambda) m_{(13)(12)} && \in M_e[(23)(13)]. \end{aligned}$$

Using the formulae (23) to (49), we see that  $x_{(13)}x_{(23)} \cdot v$ ,  $x_{(23)}x_{(12)} \cdot v$  and  $x_{(12)}x_{(13)} \cdot v$  are non-zero multiples of  $w$ . That is,  $\dim(\mathcal{A}_{[a]} \cdot v)[(23)(13)] = 1$ . Also,  $x_{(12)}x_{(23)} \cdot v = -\mu m_{\text{top}}$  and  $x_{(13)}x_{(12)} \cdot v = \lambda m_{\text{top}}$ . Hence

$$\left\{ v, x_{(23)} \cdot v, x_{(12)} \cdot v, x_{(13)} \cdot v, w, m_{\text{top}} \right\}$$

is a basis of  $\mathcal{A}_{[a]} \cdot v$  by Remark 9.

Let now  $N$  be a (proper, non-trivial) submodule of  $M_e$ . If  $N \neq \langle m_{\text{top}} \rangle$ , then there exists  $v \in N[(13)(23)] - 0$ . Hence  $\mathcal{A}_{[a]} \cdot v$  is a submodule of  $N$  and  $N[e] = \langle m_{\text{top}} \rangle$  because  $m_1 \in M_e[e]$  and  $\dim M_e[e] = 2$ . Therefore  $N = \mathcal{A}_{[a]} \cdot N[(13)(23)]$ . ■

It is convenient to introduce the following  $\mathcal{A}_{[a]}$ -modules which we will use in the Section 4.

**Definition 11.** Let  $\mathbf{t} \in \mathfrak{A}_3$ . We denote by  $W_{\mathbf{t}}(L, \mathbb{k}_e)$  the  $\mathcal{A}_{[\mathbf{a}]}$ -module with basis  $\{w_g : g \in \mathfrak{S}_3\}$  and action given by

$$w_g \in W_{\mathbf{t}}(L, \mathbb{k}_e)[g] \quad \text{for all } g \in \mathfrak{S}_3,$$

$$x_{(ij)} \cdot w_g = \begin{cases} 0 & \text{if } g = e, \\ w_{(ij)g} & \text{if } g \neq e \text{ and } \text{sgn } g = 1, \\ f_{ij}(g)w_{(ij)g} & \text{if } g \neq (ij) \text{ and } \text{sgn } g = -1, \\ t_{(ij)}w_e & \text{if } g = (ij). \end{cases}$$

The well-definition of  $W_{\mathbf{t}}$  follows from the next lemma.

**Lemma 12.** Let  $\mathbf{t}, \tilde{\mathbf{t}} \in \mathfrak{A}_3$ .

- (a) If  $\mathbf{t} = (0, 0, 0)$ , then  $W_{\mathbf{t}}(L, \mathbb{k}_e) \simeq \mathbb{k}_e \oplus L$ .
- (b) If  $\mathbf{t} \neq (0, 0, 0)$ , then there exists  $v \in M_e[(13)(23)] - 0$  such that  $W_{\mathbf{t}}(L, \mathbb{k}_e) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot v$ .
- (c) If  $v \in M_e[(13)(23)] - 0$ , then there exists  $\mathbf{t} \neq (0, 0, 0)$  such that  $W_{\mathbf{t}}(L, \mathbb{k}_e) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot v$ .
- (d)  $W_{\mathbf{t}}(L, \mathbb{k}_e)$  is an extension of  $L$  by  $\mathbb{k}_e$ .
- (e)  $W_{\mathbf{t}}(L, \mathbb{k}_e) \simeq W_{\tilde{\mathbf{t}}}(L, \mathbb{k}_e)$  if and only if  $\mathbf{t} = \mu \tilde{\mathbf{t}}$  with  $\mu \in \mathbb{k}^\times$ .

*Proof.* (a) is immediate. If we prove (b), then (d) follows from Lemma 10.

(b) We set  $w_{(13)(23)} = t_{(13)}m_{(23)(12)} - t_{(12)}m_{(12)(13)} \in M_e[(13)(23)] - 0$ ,

$$w_{(23)} = x_{(13)} \cdot w_{(13)(23)}, w_{(13)} = x_{(12)} \cdot w_{(13)(23)}, w_{(12)} = x_{(23)} \cdot w_{(13)(23)},$$

$$w_{(23)(13)} = \frac{1}{f_{23}((13))} x_{(23)} x_{(12)} \cdot w_{(13)(23)} \quad \text{and} \quad w_e = m_{\text{top}}.$$

By the proof of Lemma 10 and (17), we see that  $W_{\mathbf{t}}(L, \mathbb{k}_e) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot w_{(13)(23)}$ . (c) follows from the proof of Lemma 10. (e) Let  $\{\tilde{w}_g : g \in \mathfrak{S}_3\}$  be the basis of  $W_{\tilde{\mathbf{t}}}(L, \mathbb{k}_e)$  according to Definition 11. Let  $F : W_{\mathbf{t}}(L, \mathbb{k}_e) \rightarrow W_{\tilde{\mathbf{t}}}(L, \mathbb{k}_e)$  be an isomorphism of  $\mathcal{A}_{[\mathbf{a}]}$ -module. Since  $F$  is an isomorphism of  $\mathbb{k}^{\mathfrak{S}_3}$ -modules, there exists  $\mu_g \in \mathbb{k}^\times$  for all  $g \in \mathfrak{S}_3$  such that  $F(w_g) = \mu_g \tilde{w}_g$ . In particular,  $F$  induces an automorphism of  $L$ . Since  $L$  is simple (cf. Theorem 1),  $\mu_g = \mu_L$  for all  $g \neq e$ . Since  $F(x_{(ij)} \cdot w_{(ij)}) = x_{(ij)} \cdot F(w_{(ij)})$ , we see that  $\mathbf{t} = \frac{\mu_L}{\mu_e} \tilde{\mathbf{t}}$ . Conversely,  $F$  is well defined for all  $\mu_e$  and  $\mu_L$  such that  $\mu = \frac{\mu_L}{\mu_e}$ . ■

The Verma module  $M_{(13)(23)}$  projects onto the simple module  $L$ , hence the kernel of this projection is a maximal submodule; explicitly this is

$$N_{(13)(23)} = \mathcal{A}_{[\mathbf{a}]} \cdot M_{(13)(23)}[e] = M_{(13)(23)}[e] \oplus \mathcal{A}_{[\mathbf{a}]} \cdot m_{\text{soc}}.$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of  $M_{(13)(23)}$ . Recall  $m_{\text{soc}}$  from (56).

**Lemma 13.** *The submodules of  $M_{(13)(23)}$  are*

$$\mathcal{A}_{[a]} \cdot m_{\text{soc}} \subsetneq \mathcal{A}_{[a]} \cdot v \subsetneq N_{(13)(23)} \subsetneq M_{(13)(23)}$$

for all  $v \in M_{(13)(23)}[e] - 0$ . The submodules  $\mathcal{A}_{[a]} \cdot v$  and  $\mathcal{A}_{[a]} \cdot u$  coincide iff  $v \in \langle u \rangle$ . The quotients  $\mathcal{A}_{[a]} \cdot v / \mathcal{A}_{[a]} \cdot m_{\text{soc}}$  and  $N_{(13)(23)} / \mathcal{A}_{[a]} \cdot v$  are isomorphic to  $\mathbb{k}_e$ ; and  $M_{(13)(23)} / N_{(13)(23)}$  and  $\mathcal{A}_{[a]} \cdot m_{\text{soc}}$  are isomorphic to  $L$ .

*Proof.* Let  $v = \lambda m_1 + \mu m_{\text{top}} \in M_{(13)(23)}[(13)(23)] - 0$  and  $N = \mathcal{A}_{[a]} \cdot v$ . Using the formulae (23) to (49), we see that

$$\begin{aligned} x_{(12)}x_{(13)} \cdot v &= \lambda m_{(12)(13)} - \mu f_{13}((23))^2 m_{(23)(12)} \quad \text{and} \\ x_{(23)}x_{(12)} \cdot v &= \mu f_{23}((13))^2 m_{(12)(13)} + (\lambda + 2\mu f_{13}((23))f_{23}((13))) m_{(23)(12)}. \end{aligned}$$

Thus,  $\dim N[(23)(13)] = 1$  iff  $\lambda + \mu f_{13}((23))f_{23}((13)) = 0$ , that is iff  $v \in \langle m_{\text{soc}} \rangle - 0$ . In this case,

$$\left\{ v, x_{(23)} \cdot v, x_{(12)} \cdot v, x_{(13)} \cdot v, x_{(12)}x_{(13)} \cdot v \right\}$$

is a basis of  $\mathcal{A}_{[a]} \cdot m_{\text{soc}}$  by Remark 9.

Let now  $N$  be an arbitrary submodule of  $M_{(13)(23)}$ . Then  $N = M_{(13)(23)}$  if  $\dim N[(13)(23)] = 2$ . If  $\dim N[(13)(23)] = 0$ , then  $N \subset M_{(13)(23)}[e]$  by Lemma 7. But this is not possible since  $\ker x_{(13)} \cap \ker x_{(23)} \cap \ker x_{(12)} = 0$ , what is checked using the formulae (23) to (52). It remains the case  $\dim N[(13)(23)] = 1$ . By the argument at the beginning of the proof, the lemma follows. ■

It is convenient to introduce the following  $\mathcal{A}_{[a]}$ -modules which we will use in the Section 4.

**Definition 14.** Let  $\mathbf{t} \in \mathfrak{A}_3$ . We denote by  $W_{\mathbf{t}}(\mathbb{k}_e, L)$  the  $\mathcal{A}_{[a]}$ -module with basis  $\{w_g : g \in S_3\}$  and action given by

$$w_g \in W_{\mathbf{t}}(\mathbb{k}_e, L)[g], \quad x_{(ij)} \cdot w_g = \begin{cases} t_{(ij)} w_{(ij)} & \text{if } g = e, \\ f_{ij}(g) w_{(ij)g} & \text{if } g \neq e \text{ and } \text{sgn } g = 1, \\ w_{(ij)g} & \text{if } \text{sgn } g = -1. \end{cases}$$

The well-definition of  $W_{\mathbf{t}}(\mathbb{k}_e, L)$  follows from the next lemma.

**Lemma 15.** *Let  $\mathbf{t}, \tilde{\mathbf{t}} \in \mathfrak{A}_3$ .*

- (a) *If  $\mathbf{t} = (0, 0, 0)$ , then  $W_{\mathbf{t}}(\mathbb{k}_e, L) \simeq L \oplus \mathbb{k}_e$ .*
- (b) *If  $\mathbf{t} \neq (0, 0, 0)$ , then there exists  $v \in M_{(13)(23)}[e] - 0$  such that  $W_{\mathbf{t}}(\mathbb{k}_e, L) \simeq \mathcal{A}_{[a]} \cdot v$ .*
- (c) *If  $v \in M_{(13)(23)}[e] - 0$ , then there exists  $\mathbf{t} \neq (0, 0, 0)$  such that  $W_{\mathbf{t}}(\mathbb{k}_e, L) \simeq \mathcal{A}_{[a]} \cdot v$ .*
- (d)  *$W_{\mathbf{t}}(\mathbb{k}_e, L)$  is an extension of  $\mathbb{k}_e$  by  $L$ .*



(e)  $W_{\mathbf{t}}(\mathbb{k}_e, L) \simeq W_{\tilde{\mathbf{t}}}(\mathbb{k}_e, L)$  if and only if  $\mathbf{t} = \mu \tilde{\mathbf{t}}$  with  $\mu \in \mathbb{k}^\times$ .

*Proof.* (a) is immediate. If we prove (b), then (d) follows from Lemma 13.

(b) We set  $w_{(13)(23)} = m_{\text{soc}} \in M_{(13)(23)}[(13)(23)]$ ,

$$w_{(23)} = \frac{x_{(13)} \cdot w_{(13)(23)}}{f_{13}((13)(23))}, w_{(13)} = \frac{x_{(12)} \cdot w_{(13)(23)}}{f_{12}((13)(23))}, w_{(12)} = \frac{x_{(23)} \cdot w_{(13)(23)}}{f_{23}((13)(23))},$$

$w_{(23)(13)} = x_{(23)}x_{(12)} \cdot w_{(13)(23)}$  and  $w_e = -t_{(12)}m_{(13)(12)} + t_{(13)}m_{(12)(23)} \neq 0$  Using the formulae (23) to (49), it is not difficult to see that  $W_{\mathbf{t}}(\mathbb{k}_e, L) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot w_e$ . (c) follows using the formulae (23) to (49). The proof of (e) is similar to the proof of Lemma 12 (e). ■

**Theorem 1.** *Let  $\mathbf{a} \in \mathfrak{A}_3$  be generic. There are exactly 2 simple  $\mathcal{A}_{[\mathbf{a}]}$ -modules up to isomorphism, namely  $\mathbb{k}_e$  and  $L$ . Moreover,  $M_e$  is the projective cover, and the injective hull, of  $\mathbb{k}_e$ ; also,  $M_{(13)(23)}$  is the projective cover, and the injective hull, of  $L$ .*

*Proof.* We know that  $\mathbb{k}_e$  and  $L$  are the only two simple  $\mathcal{A}_{[\mathbf{a}]}$ -modules up to isomorphism by Proposition 1 and Lemmata 7 (a), 10 and 13. Hence, a set of primitive orthogonal idempotents has at most 6 elements [CR, (6.8)]. Since the  $\delta_g, g \in \mathfrak{S}_3$  are orthogonal idempotents, they must be primitive. Therefore  $M_e$  and  $M_{(13)(23)}$  are the projective covers (and the injective hulls) of  $\mathbb{k}_e$  and  $L$ , respectively by [CR, (9.9)], see page 418. ■

### 3.3 Case $\mathbf{a} \in \mathfrak{A}_3$ sub-generic.

Through this subsection, we suppose that  $a_{(12)} \neq a_{(13)} = a_{(23)}$ . Then the equivalence classes of  $\mathfrak{S}_3$  by  $\sim_{\mathbf{a}}$  are

$$\{e\}, \quad \{(12)\} \quad \text{and} \quad \{(13), (23), (13)(23), (23)(13)\}.$$

In fact,

- $e$  and  $(12)$  belong to the isotropy group  $\mathfrak{S}_3^{\mathbf{a}}$ .
- $(13) = (23)(12)(23)$  with  $f_{12}((23)) = a_{(12)} - a_{(13)} \neq 0$  and  $f_{23}((12)(23)) = a_{(23)} - a_{(12)} \neq 0$ .
- $(123) = (13)(23)$  with  $f_{13}((23)) = a_{(13)} - a_{(12)} \neq 0$ .
- $(132) = (23)(13)$  with  $f_{23}((13)) = a_{(23)} - a_{(12)} \neq 0$ .

To determine the simple  $\mathcal{A}_{[\mathbf{a}]}$ -modules, we proceed as in the subsection above; that is, we just need to determine the maximal submodules of the Verma modules  $M_e, M_{(12)}$  and  $M_{(13)(23)}$ , see Proposition 5.

Let  $M$  be a cyclic  $\mathcal{A}_{[a]}$ -module generated by  $v \in M[(13)(23)]$ . Here again, we can describe the weight spaces of  $M$ . By (55) and acting by the monomials in our basis, we see that

$$M[(23)(13)] = \langle x_{(13)}x_{(23)} \cdot v, x_{(23)}x_{(12)} \cdot v, x_{(12)}x_{(13)} \cdot v \rangle.$$

This weight space is  $\neq 0$  by Remark 4 applied to  $(13)(23) \sim_a (23)(13)$ , and a further application of this Remark gives the following result.

**Remark 16.** Let  $M$  be a cyclic  $\mathcal{A}_{[a]}$ -module generated by  $v \in M[(13)(23)]$ . If  $\dim M[(23)(13)] = 1$ , then

$$\begin{aligned} M[(13)] &= \langle x_{(12)} \cdot v \rangle, & M[(23)(13)] &= \langle x_{(12)}x_{(13)} \cdot v \rangle, \\ M[(23)] &= \langle x_{(13)} \cdot v \rangle, & M[(12)] &= \langle x_{(23)} \cdot v, (x_{(13)}x_{(12)}x_{(13)}) \cdot v \rangle, \\ M[(13)(23)] &= \langle v \rangle, & M[e] &= \langle x_{(23)}x_{(13)} \cdot v, (x_{(12)}x_{(23)}) \cdot v, x_{(13)}x_{(12)} \cdot v \rangle. \end{aligned} \tag{60}$$

There is a simple module  $L$  like this;  $\{v_{(13)}, v_{(23)}, v_{(13)(23)}, v_{(23)(13)}\}$  is a basis of  $L$  and the action is given by

$$v_g \in L[g], \quad x_{(ij)} \cdot v_g = \begin{cases} 0 & \text{if } g = (ij) \\ m_{(ij)g} & \text{if } g \neq (ij), \operatorname{sgn} g = -1, \\ f_{ij}(g)m_{(ij)g} & \text{if } \operatorname{sgn} g = 1. \end{cases} \tag{61}$$

Let  $\mathbb{k}_{(12)}$  and  $\mathbb{k}_e$  be as in Lemma 8. We shall see that  $L, \mathbb{k}_{(12)}$  and  $\mathbb{k}_e$  are the only simple modules of  $\mathcal{A}_{[a]}$ .

The Verma module  $M_e$  projects onto the simple module  $\mathbb{k}_e$ , hence the kernel of this projection is a maximal submodule; explicitly this is

$$\begin{aligned} N_e &= \mathcal{A}_{[a]} \cdot (M_e[(13)(23)] \oplus M_e[(12)]) \\ &= \oplus_{g \sim_a (13)(23)} M_e[g] \oplus M_e[(12)] \oplus \langle m_{\text{top}} \rangle. \end{aligned}$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of  $M_e$ .

**Lemma 17.** *The lattice of (proper, non-trivial) submodules of  $M_e$  is displayed in (62), where  $v$  and  $w$  satisfy*

$$M_e[(13)(23)] = \langle v, m_{(23)(12)} \rangle, \quad M_e[(12)] = \langle w, m_{(13)(12)(23)} \rangle.$$

*The submodules  $\mathcal{A}_{[a]} \cdot v$  (resp.  $\mathcal{A}_{[a]} \cdot w$ ) and  $\mathcal{A}_{[a]} \cdot v_1$  (resp.  $\mathcal{A}_{[a]} \cdot w_1$ ) coincide iff  $v \in \langle v_1 \rangle$  (resp.  $w \in \langle w_1 \rangle$ ). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.*

$$\begin{array}{ccccc}
 & & N_e & & \\
 & \swarrow^{\mathbb{k}_{(12)}} & & \searrow^L & \\
 \mathcal{A}_{[a]} \cdot M_e[(13)(23)] & & & & \mathcal{A}_{[a]} \cdot M_e[(12)] \\
 \downarrow^L & \searrow^L & & \swarrow^{\mathbb{k}_{(12)}} & \downarrow^{\mathbb{k}_{(12)}} \\
 \mathcal{A}_{[a]} \cdot v & & \mathcal{A}_{[a]} \cdot \langle m_{(13)(12)(23)}, m_{(23)(12)} \rangle & & \mathcal{A}_{[a]} \cdot w \\
 \downarrow^L & \swarrow^L & & \searrow^{\mathbb{k}_{(12)}} & \downarrow^{\mathbb{k}_{(12)}} \\
 \mathcal{A}_{[a]} \cdot m_{(13)(12)(23)} & & & & \mathcal{A}_{[a]} \cdot m_{(23)(12)} \\
 & \swarrow^{\mathbb{k}_{(12)}} & & \searrow^L & \\
 & & \langle m_{\text{top}} \rangle & & 
 \end{array} \tag{62}$$

Proof. Let

$$\begin{aligned}
 v &= \lambda m_{(23)(12)} + \mu m_{(12)(13)} && \in M_e[(13)(23)] - 0, \\
 \tilde{v} &= \mu m_{(12)(23)} + (\mu - \lambda) m_{(13)(12)} && \in M_e[(23)(13)].
 \end{aligned}$$

Using the formulae (23) to (49), we see that  $x_{(23)}x_{(12)} \cdot v$  and  $x_{(12)}x_{(13)} \cdot v$  are non-zero multiples of  $\tilde{v}$ . That is,  $\dim(\mathcal{A}_{[a]} \cdot v)[(23)(13)] = 1$ . Moreover,  $x_{(12)}x_{(23)} \cdot v = -\mu m_{\text{top}}$  and  $x_{(13)}x_{(12)} \cdot v = \lambda m_{\text{top}}$ ; and  $x_{(23)} \cdot v$  and  $(x_{(13)}x_{(12)}x_{(13)}) \cdot v$  are non-zero multiples of  $\mu m_{(13)(12)(23)}$ . By Remark 16, we obtain a basis for  $\mathcal{A}_{[a]} \cdot v$ :

$$\left\{ v, x_{(12)} \cdot v, x_{(13)} \cdot v, \tilde{v}, m_{\text{top}}, \mu m_{(13)(12)(23)} \right\}; \tag{63}$$

if  $\mu = 0$ , we obviate the last vector.

By (51), (50) and (52),  $x_{(ij)} \cdot m_{\text{top}} = 0$  for all  $(ij) \in \mathcal{O}_2^3$ . Then

$$\mathcal{A}_{[a]} \cdot m_{\text{top}} = \langle m_{\text{top}} \rangle$$

and  $\mathcal{A}_{[a]} \cdot u = \mathcal{A}_{[a]} \cdot m_1 = M_e$  if  $u \in M_e[e]$  is linearly independent to  $m_{\text{top}}$ .

By (43), (46) and (49),  $x_{(ij)} \cdot m_{(13)(12)(23)} = -\delta_{(12)}((ij))m_{\text{top}}$  for all  $(ij) \in \mathcal{O}_2^3$ . Then

$$\mathcal{A}_{[a]} \cdot m_{(13)(12)(23)} = \langle m_{\text{top}}, m_{(13)(12)(23)} \rangle.$$

By (22), (24) and (26),  $x_{(ij)} \cdot m_{(12)} = \delta_{(13)}((ij))m_{(13)(12)} + \delta_{(23)}((ij))m_{(23)(12)}$  for all  $(ij) \in \mathcal{O}_2^3$ . Then

$$\mathcal{A}_{[a]} \cdot w = \mathcal{A}_{[a]} \cdot m_{(23)(12)} \oplus \langle w \rangle$$

by (63) and Remark 4, if  $w \in M_e[(12)]$  is linearly independent to  $m_{(13)(12)(23)}$ .

Let now  $N$  be a (proper, non-trivial) submodule of  $M_e$  which is not  $\langle m_{\text{top}} \rangle$ . We set  $\tilde{N} = \mathcal{A}_{[a]} \cdot N[(12)] + \mathcal{A}_{[a]} \cdot N[(13)(23)]$ . Then  $\tilde{N}[g] = N[g]$  for all  $g \neq e$  by Remark 4. By the argument at the beginning of the proof,  $\langle m_{\text{top}} \rangle \subset \tilde{N}$ . Then

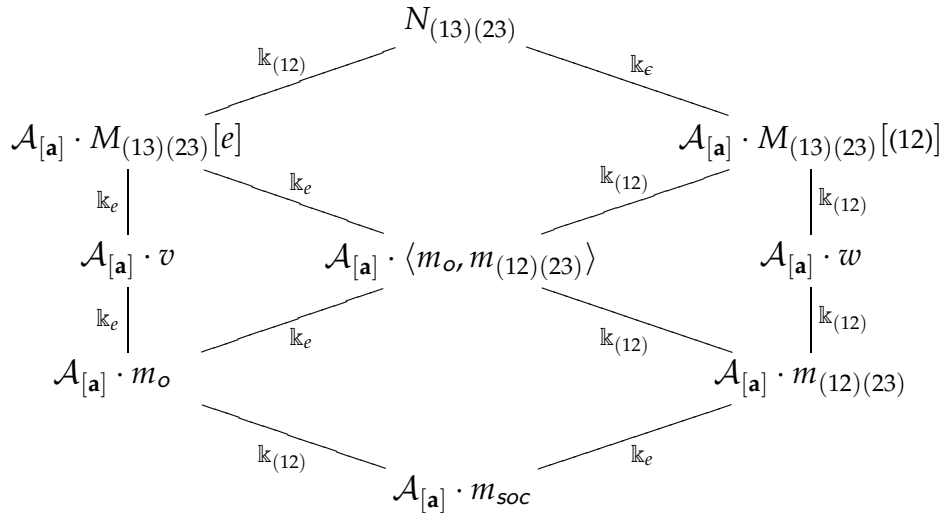
$\tilde{N}[e] = \langle m_{\text{top}} \rangle = N[e]$  because otherwise  $N = M_e$ . Therefore  $N = \tilde{N}$ . To finish, we have to calculate the submodules of  $M_e$  generated by homogeneous subspaces of  $M_e[(12)] \oplus M_e[(13)(23)]$ ; this follows from the argument at the beginning of the proof. ■

The Verma module  $M_{(13)(23)}$  projects onto the simple module  $L$ , hence the kernel of this projection is a maximal submodule; explicitly this is

$$\begin{aligned} N_{(13)(23)} &= \mathcal{A}_{[a]} \cdot \left( M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)] \right) \\ &= M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)] \oplus \mathcal{A}_{[a]} \cdot m_{\text{soc}}. \end{aligned}$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of  $M_{(13)(23)}$ .

**Lemma 18.** *The lattice of (proper, non-trivial) submodules of  $M_{(13)(23)}$  is*



Here  $v$  and  $w$  satisfy

$$M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle, \quad M_{(13)(23)}[(12)] = \langle w, m_o \rangle.$$

The submodules  $\mathcal{A}_{[a]} \cdot v$  (resp.  $\mathcal{A}_{[a]} \cdot w$ ) and  $\mathcal{A}_{[a]} \cdot v_1$  (resp.  $\mathcal{A}_{[a]} \cdot w_1$ ) coincide iff  $v \in \langle v_1 \rangle$  (resp.  $w \in \langle w_1 \rangle$ ). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.

*Proof.* Let  $u = \lambda m_1 + \mu m_{\text{top}} \in M_{(13)(23)}[(13)(23)] - 0$ . Using the formulae (23) to (49), we see that

$$\begin{aligned} x_{(12)}x_{(13)} \cdot u &= \lambda m_{(12)(13)} - \mu f_{13}((23))^2 m_{(23)(12)} \quad \text{and} \\ x_{(23)}x_{(12)} \cdot u &= \mu f_{23}((13))^2 m_{(12)(13)} + (\lambda + 2\mu f_{13}((23))f_{23}((13))) m_{(23)(12)}. \end{aligned}$$

Thus,  $\dim N[(23)(13)] = 1$  iff  $\lambda + \mu f_{13}((23))f_{23}((13)) = 0$ , that is iff  $u \in \langle m_{\text{soc}} \rangle - 0$ . By Remark 16,

$$\mathcal{A}_{[a]} \cdot m_{\text{soc}} = \langle m_{\text{soc}}, x_{(12)} \cdot m_{\text{soc}}, x_{(13)} \cdot m_{\text{soc}}, x_{(12)}x_{(13)} \cdot m_{\text{soc}} \rangle$$

and  $\mathcal{A}_{[a]} \cdot u = \mathcal{A}_{[a]} \cdot m_1 = M_{(13)(23)}$ , if  $u \in M_{(13)(23)}[(13)(23)]$  is linearly independent to  $m_{\text{soc}}$ .

By the formulae (23) to (52), if  $u \in (M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)]) - 0$ , then  $0 \neq \langle x_{(13)} \cdot u, x_{(23)} \cdot u \rangle \subset \mathcal{A}_{[a]} \cdot m_{\text{soc}}$ . Therefore

$$\mathcal{A}_{[a]} \cdot m_{\text{soc}} \subset \mathcal{A}_{[a]} \cdot u$$

by Remark 4. Also, if  $v$  and  $w$  satisfy  $M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle$  and  $M_{(13)(23)}[(12)] = \langle w, m_o \rangle$ , then

$$\langle x_{(12)} \cdot v \rangle = \langle m_o \rangle \quad \text{and} \quad \langle x_{(12)} \cdot w \rangle = \langle m_{(12)(23)} \rangle.$$

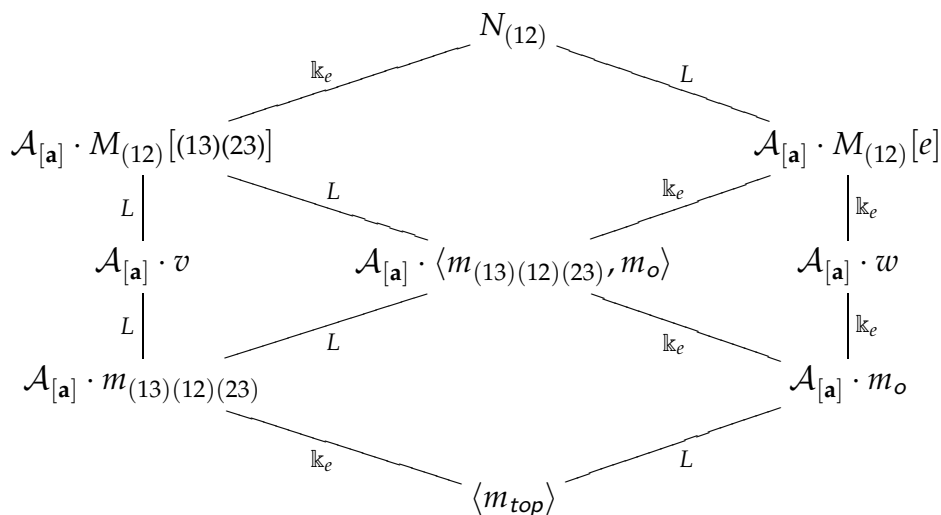
Let now  $N$  be a (proper, non-trivial) submodule of  $M_{(13)(23)}$  which is not  $\mathcal{A}_{[a]} \cdot m_{\text{soc}}$ . We set  $\tilde{N} = \mathcal{A}_{[a]} \cdot N[e] + \mathcal{A}_{[a]} \cdot N[(12)]$ . Then  $\tilde{N}[g] = N[g]$  for  $g = e, (12)$  by Remark 4. By the argument at the beginning of the proof,  $\mathcal{A}_{[a]} \cdot m_{\text{soc}} \subset \tilde{N}$ . Then  $\bigoplus_{g \sim_a (13)(23)} N[g] = \mathcal{A}_{[a]} \cdot m_{\text{soc}} = \bigoplus_{g \sim_a (13)(23)} \tilde{N}[g]$  because otherwise  $N = M_{(13)(23)}$ . Therefore  $N = \tilde{N}$ . To finish, we have to calculate the submodules of  $M_{(13)(23)}$  generated by homogeneous subspaces of  $M_{(13)(23)}[(12)] \oplus M_{(13)(23)}[e]$ ; this follows from the argument at the beginning of the proof. ■

The Verma module  $M_{(12)}$  projects onto the simple module  $\mathbb{k}_{(12)}$ , hence the kernel of this projection is a maximal submodule; explicitly this is

$$\begin{aligned} N_{(12)} &= \mathcal{A}_{[a]} \cdot \left( M_{(12)}[(13)(23)] \oplus M_{(12)}[e] \right) \\ &= \bigoplus_{g \sim_a (13)(23)} M_{(12)}[g] \oplus M_{(12)}[e] \oplus \langle m_{\text{top}} \rangle. \end{aligned}$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of  $M_{(12)}$ .

**Lemma 19.** *The lattice of (proper, non-trivial) submodules of  $M_{(12)}$  is*



Here  $v$  and  $w$  satisfy

$$M_{(12)}[(13)(23)] = \langle v, m_o \rangle, \quad M_{(12)}[e] = \langle w, m_{(13)(12)(23)} \rangle.$$

The submodules  $\mathcal{A}_{[a]} \cdot v$  (resp.  $\mathcal{A}_{[a]} \cdot w$ ) and  $\mathcal{A}_{[a]} \cdot v_1$  (resp.  $\mathcal{A}_{[a]} \cdot w_1$ ) coincide iff  $v \in \langle v_1 \rangle$  (resp.  $w \in \langle w_1 \rangle$ ). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.

*Proof.* Let  $v = \lambda m_{(23)} + \mu m_{(13)(12)(13)} \in M_{(12)}[(13)(23)]$  be a non-zero element. By Remark 16 and using the formulae (23) to (52), we see that

$$\begin{aligned}
 (\mathcal{A}_{[\mathbf{a}]} \cdot v)[(13)(23)] &= \langle v \rangle, \\
 (\mathcal{A}_{[\mathbf{a}]} \cdot v)[(13)] &= \langle (f_{13}((23))\mu - \lambda)m_{(12)(23)} - \mu f_{13}((23))m_{(13)(12)} \rangle, \\
 (\mathcal{A}_{[\mathbf{a}]} \cdot v)[(23)] &= \langle (f_{13}((23))\mu - \lambda)m_{(12)(13)} - \lambda m_{(23)(12)} \rangle, \\
 (\mathcal{A}_{[\mathbf{a}]} \cdot v)[(23)(13)] &= \langle (f_{13}((23))\mu - \lambda)f_{23}((13))m_{(13)} + \lambda m_{(12)(23)(12)} \rangle, \\
 (\mathcal{A}_{[\mathbf{a}]} \cdot v)[(12)] &= \langle m_{\text{top}} \rangle \text{ and} \\
 (\mathcal{A}_{[\mathbf{a}]} \cdot v)[e] &= \langle (f_{13}((23))\mu - \lambda)m_{(13)(12)(23)} \rangle.
 \end{aligned}
 \tag{64}$$

By (51), (50) and (52),  $x_{(ij)} \cdot m_{\text{top}} = 0$  for all  $(ij) \in \mathcal{O}_2^3$ . Then

$$\mathcal{A}_{[\mathbf{a}]} \cdot m_{\text{top}} = \langle m_{\text{top}} \rangle$$

and  $\mathcal{A}_{[\mathbf{a}]} \cdot u = \mathcal{A}_{[\mathbf{a}]} \cdot m_1 = M_e$ , if  $u \in M_{(12)}[(12)]$  is linearly independent to  $m_{\text{top}}$ . By (43), (46) and (49),  $x_{(ij)} \cdot m_{(13)(12)(23)} = -\delta_{(12)}((ij))m_{\text{top}}$  for all  $(ij) \in \mathcal{O}_2^3$ . Then

$$\mathcal{A}_{[\mathbf{a}]} \cdot m_{(13)(12)(23)} = \langle m_{\text{top}}, m_{(13)(12)(23)} \rangle.$$

By (22), (24) and (26),  $x_{(ij)} \cdot m_{(12)} = \delta_{(13)}((ij))m_{(13)(12)} + \delta_{(23)}((ij))m_{(23)(12)}$  for all  $(ij) \in \mathcal{O}_2^3$ . Then

$$\mathcal{A}_{[\mathbf{a}]} \cdot w = \mathcal{A}_{[\mathbf{a}]} \cdot m_o \oplus \langle w \rangle$$

by (64) and Remark 4, if  $w \in M_{(12)}[e]$  is linearly independent to  $m_{(13)(12)(23)}$ .

Let now  $N$  be a (proper, non-trivial) submodule of  $M_{(12)}$  which is not  $\langle m_{\text{top}} \rangle$ . We set  $\tilde{N} = \mathcal{A}_{[\mathbf{a}]} \cdot N[e] + \mathcal{A}_{[\mathbf{a}]} \cdot N[(13)(23)]$ . Then  $\tilde{N}[g] = N[g]$  for all  $g \neq (12)$  by Remark 4. By the argument at the beginning of the proof,  $\langle m_{\text{top}} \rangle \subset \tilde{N}$ . Then  $N[(12)] = \langle m_{\text{top}} \rangle = \tilde{N}[(12)]$  because otherwise  $N = M_{(12)}$ . Therefore  $N = \tilde{N}$ . To finish, we have to calculate the submodules of  $M_{(12)}$  generated by homogeneous subspaces of  $M_{(12)}[(13)(23)] \oplus M_{(12)}[e]$ ; this follows from the argument at the beginning of the proof. ■

As a consequence, we obtain the simples modules in the sub-generic case. The proof of the next theorem runs in the same way as that of Theorem 1.

**Theorem 2.** *Let  $\mathbf{a} \in \mathfrak{A}_3$  with  $a_{(12)} \neq a_{(13)} = a_{(23)}$ . There are exactly 3 simple  $\mathcal{A}_{[\mathbf{a}]}$  modules up to isomorphism, namely  $\mathbb{k}_e$ ,  $\mathbb{k}_{(12)}$  and  $L$ . Moreover,  $M_e$  is the projective cover, and the injective hull, of  $\mathbb{k}_e$ ;  $M_{(12)}$  is the projective cover, and the injective hull, of  $\mathbb{k}_{(12)}$ ; and  $M_{(13)(23)}$  is the projective cover, and the injective hull, of  $L$ .*

*Proof.* We know that  $\mathbb{k}_e$ ,  $\mathbb{k}_{(12)}$  and  $L$  are the only two simple  $\mathcal{A}_{[\mathbf{a}]}$ -modules up to isomorphism by Proposition 1 and Lemmata 17, 18 and 19. Hence, a set of primitive orthogonal idempotents has at most 6 elements [CR, (6.8)]. Since the  $\delta_g$ ,  $g \in \mathfrak{S}_3$  are orthogonal idempotents, they must be primitive. Therefore  $M_e$ ,  $M_{(12)}$  and  $M_{(13)(23)}$  are respectively the projective covers (and the injective hulls) of  $\mathbb{k}_e$ ,  $\mathbb{k}_{(12)}$  and  $L$  by [CR, (9.9)], see page 418. ■

### 4 Representation type of $\mathcal{A}_{[a]}$

In this section, we assume that  $n = 3$  as in the preceding one. We will determine the  $\mathcal{A}_{[a]}$ -modules which are extensions of simple  $\mathcal{A}_{[a]}$ -modules. As a consequence, we will show that  $\mathcal{A}_{[a]}$  is not of finite representation type for all  $\mathbf{a} \in \mathfrak{A}_3$ .

#### 4.1 Extensions of simple modules

By the following lemma, we are reduced to consider only submodules of the Verma modules for to determine the extensions of simple  $\mathcal{A}_{[a]}$ -modules. Then we shall split the consideration into three different cases like Section 3 and use the lemmata there.

**Lemma 20.** *Let  $\mathbf{a} \in \mathfrak{A}_3$  be non-zero. Let  $S$  and  $T$  be simple  $\mathcal{A}_{[a]}$ -modules and  $M$  be an extension of  $T$  by  $S$ . Hence either  $M \simeq S \oplus T$  as  $\mathcal{A}_{[a]}$ -modules or  $M$  is an indecomposable submodule of the Verma module which is the injective hull of  $S$ .*

*Proof.* If there exists a proper submodule  $N$  of  $M$  which is not  $S$ , then  $M \simeq S \oplus T$  as  $\mathcal{A}_{[a]}$ -modules. In fact,  $N \cap S$  is either 0 or  $S$  because  $S$  is simple. Let  $\pi$  be as in (65). Since  $T$  is simple,  $\pi|_N : N \rightarrow T$  results an epimorphism. Therefore  $M \simeq S \oplus T$  since  $\dim N = \dim(N \cap S) + \dim T$ .

Let  $M_S$  be the Verma module which is the injective hull of  $S$ . Then we have the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S & \xrightarrow{i} & M & \xrightarrow{\pi} & T \longrightarrow 0 \\
 & & \downarrow & \swarrow f & & & \\
 & & M_S & & & & 
 \end{array} \tag{65}$$

Therefore either  $M \simeq S \oplus T$  as  $\mathcal{A}_{[a]}$ -modules or  $f$  is injective. If  $f$  is injective, then  $M$  results indecomposable by Lemmata 10 and 13 in the generic case, and by Lemmata 17, 18 and 19 in the sub-generic case. ■

Recall the modules  $W_{\mathbf{t}}(L, \mathbb{k}_e)$  and  $W_{\mathbf{t}}(\mathbb{k}_e, L)$  from Definitions 11 and 14. The next results follow from Lemmata 10, 13, 17, 18 and 19 by Lemma 20.

**Lemma 21.** *Let  $\mathbf{a} \in \mathfrak{A}_3$  be generic. Let  $S$  and  $T$  be simple  $\mathcal{A}_{[a]}$ -modules and  $M$  be an extension of  $T$  by  $S$ .*

- (a) *If  $S \simeq T$ , then  $M \simeq S \oplus S$ .*
- (b) *If  $S \simeq \mathbb{k}_e$  and  $T \simeq L$ , then  $M \simeq W_{\mathbf{t}}(L, \mathbb{k}_e)$  for some  $\mathbf{t} \in \mathfrak{A}_3$ .*
- (c) *If  $S \simeq L$  and  $T \simeq \mathbb{k}_e$ , then  $M \simeq W_{\mathbf{t}}(\mathbb{k}_e, L)$  for some  $\mathbf{t} \in \mathfrak{A}_3$ .* ■

**Lemma 22.** *Let  $\mathbf{a} \in \mathfrak{A}_3$  with  $a_{(12)} \neq a_{(13)} = a_{(23)}$ . Let  $S$  and  $T$  be simple  $\mathcal{A}_{[a]}$ -modules and  $M$  be an extension of  $T$  by  $S$ .*

- (a) *If  $S \simeq T$ , then  $M \simeq S \oplus S$ .*
- (b) *If  $S \simeq \mathbb{k}_e$  and  $T \simeq \mathbb{k}_{(12)}$ , then  $M \simeq \mathcal{A}_{[a]} \cdot m_{(13)(12)(23)} \subset M_e$ .*

- (c) If  $S \simeq \mathbb{k}_{(12)}$  and  $T \simeq \mathbb{k}_e$ , then  $M \simeq \mathcal{A}_{[a]} \cdot m_{(13)(12)(23)} \subset M_{(12)}$ .
- (d) If  $S \simeq \mathbb{k}_e$  and  $T \simeq L$ , then  $M \simeq \mathcal{A}_{[a]} \cdot m_{(23)(12)} \subset M_e$ .
- (e) If  $S \simeq L$  and  $T \simeq \mathbb{k}_e$ , then  $M \simeq \mathcal{A}_{[a]} \cdot m_{(12)(23)} \subset M_{(13)(23)}$ .
- (f) If  $S \simeq \mathbb{k}_{(12)}$  and  $T \simeq L$ , then  $M \simeq \mathcal{A}_{[a]} \cdot m_o \subset M_{(12)}$ .
- (g) If  $S \simeq L$  and  $T \simeq \mathbb{k}_{(12)}$ , then  $M \simeq \mathcal{A}_{[a]} \cdot m_o \subset M_{(13)(23)}$ . ■

**Lemma 23.** Let  $\mathbb{k}_g$  and  $\mathbb{k}_h$  be one-dimensional simple  $\mathcal{A}_{[(0,0,0)]}$ -modules and  $M$  be an extension of  $\mathbb{k}_h$  by  $\mathbb{k}_g$ . Hence

- (a) If  $\text{sgn } g = \text{sgn } h$ , then  $M \simeq \mathbb{k}_g \oplus \mathbb{k}_h$ .
- (b) If  $\text{sgn } g \neq \text{sgn } h$  and  $M$  is not isomorphic to  $\mathbb{k}_g \oplus \mathbb{k}_h$ , then  $g = (st)h$  for a unique  $(st) \in \mathcal{O}_3^2$  and  $M$  has a basis  $\{w_g, w_h\}$  such that  $\langle w_g \rangle \simeq \mathbb{k}_g$  as  $\mathcal{A}_{[a]}$ -modules,  $w_h \in M[h]$  and  $x_{(ij)}w_h = \delta_{(ij),(st)}w_g$ .

*Proof.*  $M = M[g] \oplus M[h]$  as  $\mathbb{k}^{S_3}$ -modules and  $M[g] \simeq \mathbb{k}_g$  as  $\mathcal{A}_{[a]}$ -modules. Since  $x_{(ij)} \cdot M[h] \subset M[(ij)h]$ , the lemma follows. ■

## 4.2 Representation type

We summarize some facts about the representation type of an algebra.

Let  $R$  be an algebra and  $\{S_1, \dots, S_t\}$  be a complete list of non-isomorphic simple  $R$ -modules. The *separated quiver* of  $R$  is constructed as follows. The set of vertices is  $\{S_1, \dots, S_t, S'_1, \dots, S'_t\}$  and we write  $\dim \text{Ext}_R^1(S_i, S'_j)$  arrows from  $S_i$  to  $S'_j$ , cf. [ARS, p. 350]. Let us denote by  $\Gamma_R$  the underlying graph of the separated quiver of  $R$ .

A characterization of the hereditary algebras of finite and tame representation type is well-known, see for example [DR2]. As a consequence, the next well-known result is obtained. If  $R$  is of finite representation type, then it is Theorem D of [DR1] or Theorem X.2.6 of [ARS]. The proof given in [ARS] adapts immediately to the case when  $R$  is of tame representation type.

**Theorem 3.** Let  $R$  be a finite dimensional algebra with radical square zero. Then  $R$  is of finite (resp. tame) representation type if and only if  $\Gamma_R$  is a finite (resp. affine) disjoint union of Dynkin diagrams. ■

In order to use the above theorem, we know that

**Remark 24.** If  $\tau$  is the radical of  $R$ , then the separated quiver of  $R$  is equal to the separated quiver of  $R/\tau^2$ , see for example [GI, Lemma 4.5].

We obtain the following result by combining Corollary VI.1.5 and Proposition VI.1.6 of [ARS].

**Proposition 25.** Let  $R$  be an artin algebra,  $\chi$  an infinite cardinal and assume there are  $\chi$  non-isomorphic indecomposable modules of length  $n$ . Then  $R$  is not of finite representation type. ■



Here is the announced result.

**Proposition 26.**  $\mathcal{A}_{[(0,0,0)]}$  is of wild representation type. If  $\mathbf{a} \in \mathfrak{A}_3$  is non-zero, then  $\mathcal{A}_{[\mathbf{a}]}$  is not of finite representation type.

*Proof.* If  $\mathbf{a} \in \mathfrak{A}_3$  is generic, we can apply Proposition 25 by Lemma 12 and Lemma 15. Hence  $\mathcal{A}_{[\mathbf{a}]}$  is not of finite representation type for all  $\mathbf{a} \in \mathfrak{A}_3$  generic.

Let  $\mathbf{a} \in \mathfrak{A}_3$  be sub-generic or zero. Then  $\dim \text{Ext}_{\mathcal{A}_{[\mathbf{a}]}}^1(T, S) = 0$  if  $S \simeq T$  by Lemma 22 and 23, and  $\dim \text{Ext}_{\mathcal{A}_{[\mathbf{a}]}}^1(T, S) = 1$  in otherwise. In fact, suppose that  $a_{(12)} \neq a_{(13)} = a_{(23)}$ ,  $S \simeq \mathbb{k}_e$  and  $T \simeq L$ . By Lemma 18 and Theorem 2,  $L$  admits a projective resolution of the form

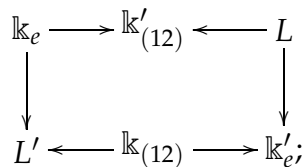
$$\dots \longrightarrow P^2 \longrightarrow M_e \oplus M_{(12)} \xrightarrow{F} M_{(13)(23)} \longrightarrow L \longrightarrow 0,$$

where  $F$  is defined by  $F|_{M_e}(m_1) = v$  and  $F|_{M_{(12)}}(m_1) = w$ ; here  $v$  and  $w$  satisfy  $M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle$ ,  $M_{(13)(23)}[(12)] = \langle w, m_o \rangle$ . Then

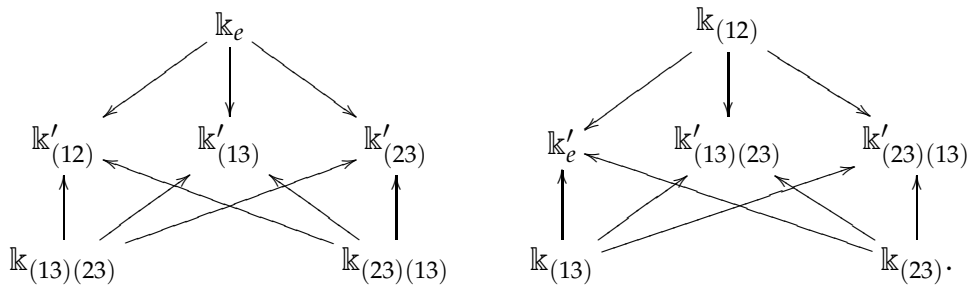
$$0 \longrightarrow \text{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_{(13)(23)}, \mathbb{k}_e) \xrightarrow{\partial_0} \text{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_e \oplus M_{(12)}, \mathbb{k}_e) \xrightarrow{\partial_1} \dots$$

and  $\text{Ext}_{\mathcal{A}_{[\mathbf{a}]}}^1(L, \mathbb{k}_e) = \ker \partial_1 / \text{Im } \partial_0$ . Since  $M_h$  is generated by  $m_1 \in M_h[h]$  for all  $h \in \mathfrak{S}_3$ ,  $\text{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_{(13)(23)}, \mathbb{k}_e) = 0$  and  $\dim \text{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_e \oplus M_{(12)}, \mathbb{k}_e) = 1$ . By Lemma 22, we know that there exists a non-trivial extension of  $L$  by  $\mathbb{k}_e$  and therefore  $\dim \text{Ext}_{\mathcal{A}_{[\mathbf{a}]}}^1(L, \mathbb{k}_e) = 1$  because it is non-zero. For other  $S$  and  $T$  and for the case  $\mathbf{a} = (0, 0, 0)$ , the proof is similar.

Hence if  $\mathbf{a} \in \mathcal{A}_{[\mathbf{a}]}$  is sub-generic and  $a_{(12)} \neq a_{(13)} = a_{(23)}$ , the separated quiver of  $\mathcal{A}_{[\mathbf{a}]}$  is



and the separated quiver of  $\mathcal{A}_{[(0,0,0)]}$  is



Therefore the lemma follows from Theorem 3 and Remark 24. ■

**Remark 27.** Let  $\mathbf{a} \in \mathfrak{A}_3$  be generic. It is not difficult to prove that the separated quiver of  $\mathcal{A}_{[\mathbf{a}]}$  is

$$\mathbb{k}_e \rightrightarrows L' \qquad L \rightrightarrows \mathbb{k}'_e.$$

## 5 On the structure of $\mathcal{A}_{[a]}$

In this section, we assume that  $n = 3$  as in the preceding one.

### 5.1 Cocycle deformations

We show in this subsection that the algebras  $\mathcal{A}_{[a]}$  are cocycle deformation of each other. For this, we first recall the following theorem due to Masuoka.

If  $K$  is a Hopf subalgebra of a Hopf algebra  $H$  and  $J$  is a Hopf ideal of  $K$ , then the two-sided ideal  $(J)$  of  $H$  is in fact a Hopf ideal of  $H$ .

**Theorem 4.** [M, Thm. 2], [BDR, Thm. 3.4]. *Suppose that  $K$  is Hopf subalgebra of a Hopf algebra  $H$ . Let  $I, J$  be Hopf ideal of  $K$ . If there is an algebra map  $\psi$  from  $K$  to  $\mathbb{k}$  such that*

- $J = \psi \rightharpoonup I \leftarrow \psi^{-1}$  and
- $H/(\psi \rightharpoonup I)$  is nonzero,

*then  $H/(\psi \rightharpoonup I)$  is a  $(H/(I), H/(J))$ -biGalois object and so the quotient Hopf algebras  $H/(I)$ ,  $H/(J)$  are monoidally Morita-Takeuchi equivalent. If  $H/(I)$  and  $H/(J)$  are finite dimensional, then  $H/(I)$  and  $H/(J)$  are cocycle deformations of each other. ■*

We will need the following lemma to apply the Masuoka's theorem.

**Lemma 28.** *If  $W$  is a vector space and  $U$  is a vector subspace of  $W^{\otimes n}$ , then the subalgebra of  $T(W)$  generated by  $U$  is isomorphic to  $T(U)$ .*

*Proof.* It is enough to prove the lemma for  $U = W^{\otimes n}$ . Fix  $n$  and let  $(x_i)_{i \in I}$  be a basis of  $W$ . Then  $\mathbf{B} = \{X_{\mathbf{i}} = x_{i_1} \cdots x_{i_n} : \mathbf{i} = (i_1, \dots, i_n) \in I^n\}$  forms a basis of  $W^{\otimes n}$ . Since the  $X_{\mathbf{i}}$ 's are all homogeneous elements of the same degree in  $T(W)$ , we only have to prove that  $\{X_{\mathbf{i}_1} \cdots X_{\mathbf{i}_m} : \mathbf{i}_1, \dots, \mathbf{i}_m \in I^n\}$  is linearly independent in  $T(W)$  for all  $m \geq 1$  and this is true because  $\mathbf{B}$  is a basis of monomials of the same degree. ■

Here is the announced result. Observe that this gives an alternative proof to the fact that  $\dim \mathcal{A}_{[a]} = 72$ , proved in [AV] using the Diamond Lemma.

**Proposition 29.** *For all  $\mathbf{a} \in \mathfrak{A}_3$ ,  $\mathcal{A}_{[a]}$  is a Hopf algebra monoidally Morita-Takeuchi equivalent to  $\mathcal{B}(V_3) \# \mathbb{k}^{S_3}$ .*

*Proof.* To start with, we consider the algebra  $\mathcal{K}_{\mathbf{a}} := T(V_3) \# \mathbb{k}^{S_3} / \mathcal{J}_{\mathbf{a}}$ ,  $\mathbf{a} \in \mathfrak{A}_3$ , where  $\mathcal{J}_{\mathbf{a}}$  is the ideal generated by

$$R_{(13)(23)}, \quad R_{(23)(13)} \quad \text{and} \quad x_{(ij)}^2 + \sum_{g \in S_3} a_{g^{-1}(ij)g} \delta_g, \quad (ij) \in \mathcal{O}_2^3. \quad (66)$$

Let  $M_3 = \mathbb{k}^{\mathcal{S}_3}$  with the regular representation. For all  $\mathbf{a} \in \mathfrak{A}_3$ ,  $M_3$  is an  $\mathcal{K}_{\mathbf{a}}$ -module with action given by

$$x_{(ij)} \cdot m_g = \begin{cases} m_{(ij)g} & \text{if } \text{sgn } g = -1, \\ -a_{g^{-1}(ij)g} m_{(ij)g} & \text{if } \text{sgn } g = 1. \end{cases}$$

We have to check that the relations defining  $\mathcal{K}_{\mathbf{a}}$  hold in the action. Then

$$\begin{aligned} \delta_h(x_{(ij)} \cdot m_g) &= \delta_h(\lambda_g m_{(ij)g}) = \lambda_g \delta_h((ij)g) m_{(ij)g} = \lambda_g \delta_{(ij)h}(g) m_{(ij)g} \\ &= x_{(ij)} \cdot (\delta_{(ij)h} \cdot m_g) \end{aligned}$$

with  $\lambda_g \in \mathbb{k}$  according to the definition of the action. Note that

$$x_{(ij)} \cdot (x_{(ik)} \cdot m_g) = \begin{cases} -a_{g^{-1}(ik)(ij)(ik)g} m_{(ij)(ik)g} & \text{if } \text{sgn } g = -1, \\ -a_{g^{-1}(ik)g} m_{(ij)(ik)g} & \text{if } \text{sgn } g = 1. \end{cases}$$

In any case, we have that  $x_{(ij)}^2 \cdot m_g = -a_{g^{-1}(ij)g} m_g$  and

$$R_{(ij)(ik)} \cdot m_g = -\left( \sum_{(st) \in \mathcal{O}_2^3} a_{g^{-1}(st)g} \right) m_{(ij)(ik)g} = 0.$$

Let  $W = \langle R_{(13)(23)}, R_{(23)(13)}, x_{(ij)}^2 : (ij) \in \mathcal{O}_2^3 \rangle$  and  $K$  be the subalgebra of  $T(V_3)$  generated by  $W$ ;  $K$  is a braided Hopf subalgebra because  $W$  is a Yetter-Drinfeld submodule contained in  $\mathcal{P}(T(V_3))$  the primitive elements of  $T(V_3)$ . Then  $K\#\mathbb{k}^{\mathcal{S}_3}$  is a Hopf subalgebra of  $T(V_3)\#\mathbb{k}^{\mathcal{S}_3}$ . For each  $\mathbf{a} \in \mathfrak{A}_3$ , by Lemma 28 we can define the algebra morphism  $\psi = \psi_K \otimes \epsilon : K\#\mathbb{k}^{\mathcal{S}_3} \rightarrow \mathbb{k}$  where

$$\psi_{K|W[g]} = 0 \text{ if } g \neq e \text{ and } \psi_K(x_{(ij)}^2) = -a_{(ij)} \forall (ij) \in \mathcal{O}_2^3.$$

If  $J$  denotes the ideal of  $K\#\mathbb{k}^{\mathcal{S}_3}$  generated by the generator of  $K$ , then  $\psi^{-1} \dashv J \leftarrow \psi$  is the ideal generated by the generators of  $\mathcal{I}_{\mathbf{a}}$ . In fact,  $\psi^{-1} = \psi \circ \mathcal{S}$  is the inverse element of  $\psi$  in the convolution group  $\text{Alg}(K\#\mathbb{k}^{\mathcal{S}_3}, \mathbb{k})$ ,  $\mathcal{S}(W)[g] \subset (K\#\mathbb{k}^{\mathcal{S}_3})[g^{-1}]$  and  $\mathcal{S}(x_{(ij)}^2) = -\sum_{h \in \mathcal{S}_3} \delta_{h^{-1}} x_{h^{-1}(ij)h}^2$ . Then our claim follows if we apply  $\psi \otimes \text{id} \otimes \psi^{-1}$  to  $(\Delta \otimes \text{id})\Delta(x_{(ij)}^2) =$

$$= x_{(ij)}^2 \otimes 1 \otimes 1 + \sum_{h \in \mathcal{S}_3} \delta_h \otimes x_{h^{-1}(ij)h}^2 \otimes 1 + \sum_{h,g \in \mathcal{S}_3} \delta_h \otimes \delta_g \otimes x_{g^{-1}h^{-1}(ij)hg}^2$$

and  $(\Delta \otimes \text{id})\Delta(x) = x \otimes 1 \otimes 1 + x_{-1} \otimes x_0 \otimes 1 + x_{-2} \otimes x_{-1} \otimes x_0$  for  $g \neq e$  and  $x \in W[g]$ ; note that also  $x_0 \in W[g]$ .

The ideal  $\psi^{-1} \dashv J$  is generated by

$$R_{(13)(23)}, R_{(23)(13)} \quad \text{and} \quad x_{(ij)}^2 + \sum_{g \in \mathcal{S}_3} a_{g^{-1}(ij)g} \delta_g \quad \forall (ij) \in \mathcal{O}_2^3.$$

Now  $\mathcal{K}_{\mathbf{a}} = T(V_3)\#\mathbb{k}^{\mathcal{S}_3} / \langle \psi^{-1} \dashv J \rangle \neq 0$  because it has a non-zero quotient in  $\text{End}(M_3)$ . Hence  $\mathcal{A}_{[\mathbf{a}]}$  is monoidally Morita-Takeuchi equivalent to  $\mathcal{B}(V_3)\#\mathbb{k}^{\mathcal{S}_3}$ , by Theorem 4. ■

### 5.2 Hopf subalgebras and integrals of $\mathcal{A}_{[a]}$

We collect some information about  $\mathcal{A}_{[a]}$ . Let

$$\chi = \sum_{g \in \mathbb{S}_3} \text{sgn}(g)\delta_g, \quad y = \sum_{(ij) \in \mathcal{O}_2^3} x_{(ij)}.$$

It is easy to see that  $\chi$  is a group-like element and that  $y \in \mathcal{P}_{1,\chi}(\mathcal{A}_{[a]})$ .

**Proposition 30.** *Let  $a \in \mathfrak{A}_3$ . Then*

- (a)  $G(\mathcal{A}_{[a]}) = \{1, \chi\}$ .
- (b)  $\mathcal{P}_{1,\chi}(\mathcal{A}_{[a]}) = \langle 1 - \chi, y \rangle$ .
- (c)  $\mathbb{k}\langle \chi, y \rangle$  is isomorphic to the 4-dimensional Sweedler Hopf algebra.
- (d) The Hopf subalgebras of  $\mathcal{A}_{[a]}$  are  $\mathbb{k}^{\mathbb{S}_3}$ ,  $\mathbb{k}\langle \chi \rangle$  and  $\mathbb{k}\langle \chi, y \rangle$ .
- (e)  $S^2(a) = \chi a \chi^{-1}$  for all  $a \in \mathcal{A}_{[a]}$ .
- (f) The space of left integrals is  $\langle m_{\text{top}}\delta_e \rangle$ ;  $\mathcal{A}_{[a]}$  is unimodular.
- (g)  $(\mathcal{A}_{[a]})^*$  is unimodular.
- (h)  $\mathcal{A}_{[a]}$  is not a quasitriangular Hopf algebra.

*Proof.* We know that the coradical  $(\mathcal{A}_{[a]})_0$  of  $\mathcal{A}_{[a]}$  is isomorphic to  $\mathbb{k}^{\mathbb{S}_3}$  by [AV]. Since  $G(\mathcal{A}_{[a]}) \subset (\mathcal{A}_{[a]})_0$ , (a) follows.

(b) Recall that  $V_3 = M((12), \text{sgn}) \in \frac{\mathbb{k}^{\mathbb{S}_3}}{\mathbb{k}^{\mathbb{S}_3}} \mathcal{YD}$ , see Subsection 2.1. Then  $\mathcal{P}_{1,\chi}(\mathcal{A}_{[a]})/\langle 1 - \chi \rangle$  is isomorphic to the isotypic component of the comodule  $V_3$  of type  $\chi$ . That is, if  $z = \sum_{(ij) \in \mathcal{O}_2^3} \lambda_{(ij)} x_{(ij)} \in (V_3)_\chi$ , then

$$\delta(z) = \sum_{h \in G, (ij) \in \mathcal{O}_2^3} \text{sgn}(h)\lambda_{(ij)}\delta_h \otimes x_{h^{-1}(ij)h} = \chi \otimes z.$$

Evaluating at  $g \otimes \text{id}$  for any  $g \in \mathbb{S}_3$ , we see that  $\lambda_{(ij)} = \lambda_{(12)}$  for all  $(ij) \in \mathcal{O}_2^n$ . Then  $z = \lambda_{(12)}y$ . The proof of (c) is now evident.

(d) Let  $A$  be a Hopf subalgebra of  $\mathcal{A}_{[a]}$ . Then  $A_0 = A \cap (\mathcal{A}_{[a]})_0 \subseteq \mathbb{k}^{\mathbb{S}_3}$  by [Mo, Lemma 5.2.12]. Hence  $A_0$  is either  $\mathbb{k}\langle \chi \rangle$  or else  $\mathbb{k}^{\mathbb{S}_3}$ . If  $A_0 = \mathbb{k}\langle \chi \rangle$ , then  $A$  is a pointed Hopf algebra with group  $\mathbb{Z}/2$ . Hence  $A$  is either  $\mathbb{k}\langle \chi \rangle$  or else  $\mathbb{k}\langle \chi, y \rangle$  by (b) and [N] or [CD]<sup>4</sup>. If  $A_0 = \mathbb{k}^{\mathbb{S}_3}$ , then  $A$  is either  $\mathbb{k}^{\mathbb{S}_3}$  or else  $A = \mathcal{A}_{[a]}$  by [AV].

To prove (e), just note that  $\chi x_{(ij)} \chi^{-1} = -x_{(ij)}$ .

(f) follows from Subsections 3.2 and 3.3. Let  $\Lambda$  be a non-zero left integral of  $\mathcal{A}_{[a]}$ . By Lemma 8, the distinguished group-like element of  $(\mathcal{A}_{[a]})^*$  is  $\zeta_h$  for some

<sup>4</sup>The classification of all finite dimensional pointed Hopf algebras with group  $\mathbb{Z}/2$  also follows easily performing the Lifting method [AS].

$h \in \mathbb{S}_3^a$ , hence  $\Lambda\delta_h = \zeta_h(\delta_h)\Lambda = \Lambda$ . Let us consider  $\mathcal{A}_{[a]}$  as a left  $\mathbb{k}^{\mathbb{S}_3}$ -module via the left adjoint action, see page 418. Let  $\Lambda_g \in (\mathcal{A}_{[a]})[g]$  such that  $\Lambda = \sum_{g \in \mathbb{S}_3} \Lambda_g$ . Then  $\Lambda = \delta_e \Lambda = \sum_{s,t \in \mathbb{S}_3} \text{ad } \delta_s(\Lambda_t)\delta_{s^{-1}}\delta_h = \Lambda_{h^{-1}}\delta_h$ . Since  $M_h \simeq \mathcal{A}_{[a]}\delta_h$ , we can use the lemmata of the Section 3 to compute  $\Lambda$ .

If  $\mathbf{a}$  is generic, then  $h = e$  by Theorem 1. Since  $x_{(ij)}\Lambda = 0$  for all  $(ij) \in \mathbb{S}_3$ ,  $\Lambda = m_{\text{top}}\delta_e$  by Lemma 10.

If  $\mathbf{a}$  is sub-generic, we assume that  $a_{(12)} \neq a_{(13)} = a_{(23)}$ , then either  $\Lambda = \Lambda_e\delta_e$  or  $\Lambda_{(12)}\delta_{(12)}$  by Theorem 2. Since  $x_{(ij)}\Lambda = 0$  for all  $(ij) \in \mathbb{S}_3$ ,  $\Lambda = m_{\text{top}}\delta_e$  by Lemma 17 and Lemma 19.

(g) By (e),  $\mathcal{S}^4 = \text{id}$ . By Radford’s formula for the antipode and (f), the distinguished group-like element of  $\mathcal{A}_{[a]}$  is central, hence trivial. Therefore,  $(\mathcal{A}_{[a]})^*$  is unimodular.

(h) If there exists  $R \in \mathcal{A}_{[a]} \otimes \mathcal{A}_{[a]}$  such that  $(\mathcal{A}_{[a]}, R)$  is a quasitriangular Hopf algebra, then  $(\mathcal{A}_{[a]}, R)$  has a unique minimal subquasitriangular Hopf algebra  $(A_R, R)$  by [R]. We shall show that such a Hopf subalgebra does not exist using (d) and therefore  $\mathcal{A}_{[a]}$  is not a quasitriangular Hopf algebra.

By [R, Prop. 2, Thm. 1] we know that there exist Hopf subalgebras  $H$  and  $B$  of  $\mathcal{A}_{[a]}$  such that  $A_R = HB$  and an isomorphism of Hopf algebras  $H^{\text{cop}} \rightarrow B$ . Then  $A_R \neq \mathcal{A}_{[a]}$ . In fact, let  $M(d, \mathbb{k})$  denote the matrix algebra over  $\mathbb{k}$  of dimension  $d^2$ . Then the coradical of  $(\mathcal{A}_{[a]})^*$  is isomorphic to

- $\mathbb{k}^6$  if  $\mathbf{a} = (0, 0, 0)$ .
- $\mathbb{k} \oplus M(5, \mathbb{k})^*$  if  $\mathbf{a}$  is generic by Theorem 1.
- $\mathbb{k}^2 \oplus M(4, \mathbb{k})^*$  if  $\mathbf{a}$  is sub-generic by Theorem 2.

Since  $(\mathcal{A}_{[a]})_0 \simeq \mathbb{k}^{\mathbb{S}_3}$ ,  $\mathcal{A}_{[a]}$  is not isomorphic to  $(\mathcal{A}_{[a]})^{\text{cop}}$  for all  $\mathbf{a} \in \mathfrak{A}_3$ . Clearly,  $A_R$  cannot be  $\mathbb{k}^{\mathbb{S}_3}$ . Since  $\mathcal{A}_{[a]}$  is not cocommutative,  $R$  cannot be  $1 \otimes 1$ . The quasitriangular structures on  $\mathbb{k}\langle \chi \rangle$  and  $\mathbb{k}\langle \chi, y \rangle$  are well known, see for example [R]. Then it remains the case  $A_R \subseteq \mathbb{k}\langle \chi, y \rangle$  with  $R = R_0 + R_\alpha$  where  $R_0 = \frac{1}{2}(1 \otimes 1 + 1 \otimes \chi + \chi \otimes 1 - \chi \otimes \chi)$  and  $R_\alpha = \frac{\alpha}{2}(y \otimes y + y \otimes \chi y + \chi y \otimes \chi y - \chi y \otimes y)$  for some  $\alpha \in \mathbb{k}$ . Since  $\Delta(\delta_g)^{\text{cop}}R = R\Delta(\delta_g)$  for all  $g \in \mathbb{S}_3$ , then

$$\Delta(\delta_g)^{\text{cop}}R_0 = R_0\Delta(\delta_g) = \Delta(\delta_g)R_0 \quad \text{in } \mathbb{k}^{\mathbb{S}_3};$$

but this is not possible because  $R_0^2 = 1 \otimes 1$  and  $\mathbb{k}^{\mathbb{S}_3}$  is not cocommutative. ■

### Acknowledgements

Part of the work of C. V. was done as a fellow of the Erasmus Mundus programme of the EU in the University of Antwerp. He thanks Prof. Fred Van Oystaeyen for his warm hospitality and help.

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