

Infinite divisibility, entanglement entropy, and minimal surfaces

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Abstract

The path integral representation for the Renyi entanglement entropies of integer index n implies these information measures define operator correlation functions in QFT. We analyze the question of whether the limit $n \rightarrow 1$, corresponding to the entanglement entropy, can also be represented in terms of a path integral with insertions on the region's boundary, at first order in $n - 1$. This conjecture has been used in the literature in several occasions, and specially in an attempt to prove the Ryu-Takayanagi holographic entanglement entropy formula. We show it leads to the infinite divisibility of the relevant correlation matrices in the $n \rightarrow 1$ limit, which is equivalent to an infinite series of polynomial inequalities for the entropies in QFT and the areas of minimal surfaces representing the entanglement entropy in the AdS-CFT context. We check these inequalities in several examples. No counterexample is found in the few known exact results for the entanglement entropy in QFT, but we find counterexamples in the minimal areas corresponding to more complicated geometries. We develop some analytic tools to test the inequalities, and as a byproduct, we show that positivity for the correlation functions is a local property when supplemented with analyticity. We also discuss general situations where infinite divisibility for correlation matrices appears in quantum field theory. This is the case of some exponential operators in the semiclassical and large N limits. This result has analogies to the central limit theorem in probability theory.

1 Introduction

The positivity of the Hilbert space scalar product gives place to an infinite series of inequalities for the correlators of operators in quantum mechanics. Given a family \mathcal{O}_i , $i = 1, \dots, m$, of operators, a state $|0\rangle$, and arbitrary complex numbers λ_i , we have

$$\langle 0 | \left(\sum_{i=1}^m \lambda_i \mathcal{O}_i \right)^\dagger \left(\sum_{j=1}^m \lambda_j \mathcal{O}_j \right) | 0 \rangle = \sum_{i,j=1}^m \lambda_i^* \lambda_j \langle 0 | \mathcal{O}_i^\dagger \mathcal{O}_j | 0 \rangle \geq 0. \quad (1)$$

Because the λ_i are arbitrary, the matrix of correlators have to be positive definite. This is equivalent to

$$\det \left(\{ \langle \Omega | \mathcal{O}_i^\dagger \mathcal{O}_j | \Omega \rangle \}_{i,j=1\dots m} \right) \geq 0, \quad (2)$$

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for any collection of m operators.

When the starting point in the description of the theory is given by some numerical functions, these inequalities become specially relevant. In fact, this property allows us to recover the Hilbert space from vacuum expectation values, and plays a central role in Wightman reconstruction theorem in quantum field theory (QFT) [1].

A remarkable case of a function unexpectedly satisfying the inequalities (2) is given by the traces of integer powers of the local density matrix $\text{tr}(\rho_V^n)$, $n = 2, 3, \dots$. Here ρ_V is the vacuum state density matrix $|0\rangle\langle 0|$ reduced to a region V of the space by tracing over the degrees of freedom outside V . These statistical measures are a priori unrelated to operators, and it is quite surprising they are given by operator correlators in QFT. It is convenient for later use to express these traces in terms of the ‘‘entanglement’’ Renyi entropies $S_n(V) = -(n-1)^{-1} \log(\text{tr}\rho_V^n)$, such that

$$\text{tr}(\rho_V^n) = e^{-(n-1)S_n(V)}. \quad (3)$$

Allowing for arbitrary real index n , the limit $\lim_{n \rightarrow 1} S_n(V) = S(V) = -\text{tr}\rho_V \log \rho_V$, gives the entanglement entropy corresponding to V . In the Euclidean version of QFT the positivity inequalities analogous to (2) are called reflection positivity inequalities [2]. For the integer n Renyi entropies these later write

$$\det \left(\{ \text{tr}\rho_{V_i \bar{V}_j}^n \}_{i,j=1\dots m} \right) = \det \left(\{ e^{-(n-1)S_n(V_i \bar{V}_j)} \}_{i,j=1\dots m} \right) \geq 0. \quad (4)$$

Here V_i , $i = 1\dots m$ is a collection of codimension one sets included in the half-space of positive Euclidean time, and \bar{V}_j is the Euclidean time-reflected region corresponding to V_j . The region $V_i \bar{V}_j$ is just the union of V_i and \bar{V}_j (see figure 1). A generalized version of these inequalities for the Renyi entropies (in a real time formulation) valid for any quantum mechanical system has been proved in [3].

Notice that the operator behind these inequalities is not $(\rho_{V_i \bar{V}_j})^{n-1}$, which nevertheless produces the correct expectation values $\langle (\rho_{V_i \bar{V}_j})^{n-1} \rangle = \text{tr}\rho_{V_i \bar{V}_j}^n$. The inequalities (4) further require that $\text{tr}\rho_{V_i \bar{V}_j}^n$ has a correlation function interpretation, $\text{tr}\rho_{V_i \bar{V}_j}^n = \langle \mathcal{O}_{V_i} \mathcal{O}_{\bar{V}_j} \rangle$, for operators \mathcal{O}_{V_i} and $\mathcal{O}_{\bar{V}_j}$ localized in V_i and \bar{V}_j , in some theory, not necessarily the original one in which ρ is calculated. In fact, the path integral formulation shows this n -dependent theory is a n -replicated model. The trace $\text{tr}\rho_V^n$ is given by the partition function in a n -replicated space identified along the n copies of V in such a way one goes from a copy to the other crossing V . The complete manifold is flat everywhere except at the boundaries of V where it displays a conical singularity of angle $2\pi n$. For a detailed description of this replica trick see [4, 5, 6].

These partition functions in non trivial manifolds have been interpreted in the two dimensional case in terms of operator expectation values [6, 7]. These are called twisting operators, and are defined to do the job of imposing the correct boundary conditions. From the point of view of this paper an operator interpretation follows from (4).

A short proof of the inequalities (4) follows directly in the usual way reflection positivity is proved in the path integral formulation. This involves splitting the path integral for positive and negative time. Writing ϕ^+ and ϕ^- for the restrictions of the field ϕ to positive and negative Euclidean time, and considering a family of regions V_i in the positive Euclidean time half-space,

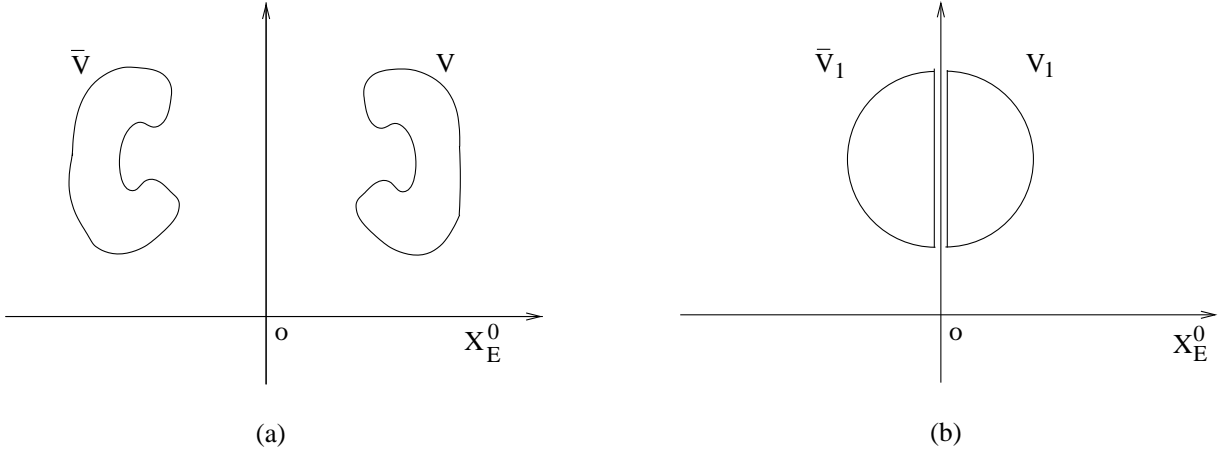


Figure 1: (a) Two regions V and \bar{V} in Euclidean space. One is the time-reflected image of the other. (b) If the regions V_1 and \bar{V}_1 are close to the $x_E^0 = 0$ plane, the common surface near this plane should give an overall numerical factor which drops out from the inequalities. This configuration gives information about the set formed by the union $V = V_1\bar{V}_1$ in the limit of no separation.

we have

$$\begin{aligned}
\sum_{i,j} \lambda_i \lambda_j^* \text{tr}(\rho_{V_i \bar{V}_j}^n) &= \mathcal{N}^{-n} \sum_{i,j} \lambda_i \lambda_j^* \int^{V_i \bar{V}_j} \mathcal{D}\phi e^{-S[\phi]} \\
&= \mathcal{N}^{-n} \int \mathcal{D}\phi_0(\vec{x}) \left(\sum_i \lambda_i \int_{\phi^+(0,\vec{x})=\phi_0(\vec{x})}^{V_i} \mathcal{D}\phi^+ e^{-S[\phi^+]} \right) \left(\sum_j \lambda_j^* \int_{\phi^-(0,\vec{x})=\phi_0(\vec{x})}^{\bar{V}_j} \mathcal{D}\phi^- e^{-S[\phi^-]} \right).
\end{aligned} \tag{5}$$

These are integrals in n replicated space, and the superscript V means the appropriate boundary conditions take place on V . The normalization factor $\mathcal{N} = \int \mathcal{D}\phi e^{-S[\phi]}$ is the one copy path integral without boundary conditions on V . The fields ϕ_0 , ϕ , ϕ^+ and ϕ^- have n independent components. $\phi_0(\vec{x})$ is the common value of $\phi^+(t, \vec{x})$ and $\phi^-(t, \vec{x})$ for $t = 0$. Provided the Euclidean action has the time reflection symmetry $S[\phi(t, \vec{x})] = S[\phi(-t, \vec{x})]^*$, a change of field variables in the path integral shows the two terms in the brackets are conjugate of each other, and the result is positive. This gives a proof of (4) in the path integral formulation (alternatively, a proof in the operator formalism is given in [3]).

The twisting operators \mathcal{O}_V are localized in V , but they do not really depend on the shape of V but only on its boundary ∂V . This is because different V with the same boundary give place to the same boundary conditions of the path integral. In the real time interpretation this is just causality: ρ_V and $\rho_{V'}$ are the same if the spatial sets V and V' have the same boundary because in that case these are equivalent Cauchy surfaces [8].

The hypersurfaces V have two possible sides. Crossing V from one side, the change of copies in the replicated manifold is in the sequence $1 \rightarrow 2 \rightarrow 3 \dots$ or in the opposite direction $1 \leftarrow 2 \leftarrow 3 \dots$, according to the chosen orientation for V . If the orientation is determined by a normal vector to the surface V , in the definition of \bar{V} it is assumed that the orientation vector has changed by a time reflection. We can take advantage of this to obtain inequalities for single component sets V . Arranging regions V_1 close to the surface $t = 0$, $V_1\bar{V}_1$ approaches a one component region V (see figure 1b). The sides along the $t = 0$ plane have opposite orientation

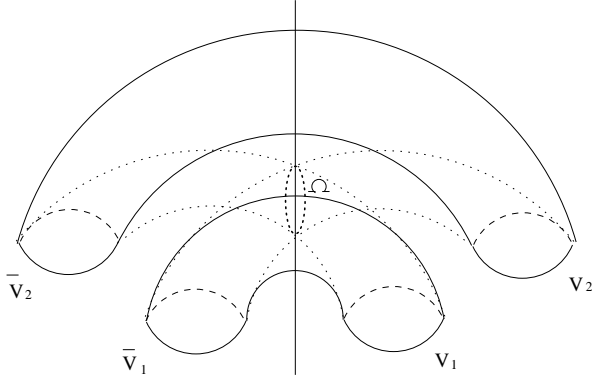


Figure 2: Two regions V_1 and V_2 , their time reflected conjugate ones \bar{V}_1 , \bar{V}_2 , and the minimal surfaces corresponding to the four combinations $V_i\bar{V}_j$. The surfaces corresponding to $V_1\bar{V}_2$ and $V_2\bar{V}_1$ cut at a surface Ω at $x_E^0 = 0$, because the points at $x_E^0 = 0$ are invariant under time reflection. The surface Ω divides these two surfaces in four pieces, and application of the triangle inequality for the minimal areas leads to (10). The same argument (though not involving Euclidean time-reflection symmetry) was given in [14] to prove strong subadditivity of the holographic entanglement entropy for regions lying in a single hyperplane.

and their contribution should decouple from the inequality (4) because the nearby operators should have an OPE proportional to the identity.

The entropy is obtained from the Renyi entropies in the limit $n = 1$. This requires an analytic extension in n from the knowledge of S_n for n integer. This extension is unique under reasonable assumptions (see [7, 9]). However, in general there is no path integral representation for S_n for general n . The naive approach would be imposing boundary conditions in a manifold, with conical angle $2\pi n$, for non integer n , in the boundary of V , and such that for integer n one recovers the above mentioned replicated manifolds. Unfortunately this cannot be generally achieved using manifolds which are flat outside ∂V . A notable exception is the case where there is a “rotational” symmetry with the angle $2\pi n$ as the parameter of a one dimensional symmetry group (see [10] for a nice account of these problems). This is the case of V being the half space in a general QFT, or a sphere in a conformal QFT. In curved space times this symmetry appears typically when the boundary of V is a Killing horizon.

However, as a working hypothesis, one may still think that for infinitesimal deficit angles $2\pi(n - 1)$, appropriate for the limit corresponding to the entropy, one could still use the path integral formula, and trust it to first order in $(n - 1)$. The conical curvature is then placed by insertion of an infinitesimal curvature tensor supported on the boundary of V . This construction has been used in the literature in several occasions, though up to the present there is no clear deductive way to this representation for the entropy in a general region.

A notable case where the representation of the entanglement entropy in terms of a path integral in a manifold with infinitesimal conical singularities is used is Fursaev’s proof of the holographic entanglement entropy [11], as given by the Ryu-Takayanagi ansatz [12]. This ansatz gives the entanglement entropy in V in the AdS boundary QFT as $(4G)^{-1}$ times the area of a minimal surface in the bulk AdS space, with boundary coinciding with ∂V ,

$$S(V) = (4G)^{-1} \min_{\partial\Sigma=\partial V} \mathcal{A}(\Sigma). \quad (6)$$

We often call the area of the minimal surface directly as $\mathcal{A}(\partial V)$. This holographic formula for

the entropy has passed several tests, including giving the correct coefficient of the logarithmically divergent term for the entropy of spheres in any dimension [10, 13], the strong subadditive inequality for sets in a single spatial hyperplane [14], and the thermal entropy of the boundary in presence of a bulk black hole [15]. The missing link between the replicated space for integer n and the small angle hypothesis sheds some doubts on one of the assumptions of Fursaev's proof for general geometries, as discussed in [10, 16].

Another example in which the small deficit angle hypothesis is applied is Solodukhin's formula for the logarithmically divergent term in the entropy for conformal theories and general regions in four dimensions [17]. This is given as a linear combination of two dimensionless, conformally invariant, and local terms, which are integrals of polynomials of the extrinsic curvature on ∂V . While this general geometric structure of the coefficient of the logarithmic term can be ascribed in general grounds to the nature of the ultraviolet divergences in the entropy, the hypothesis of path integrals for small deficit angle in Solodukhin's work is essential to relate the coefficients of the geometrical terms to the two trace anomaly coefficients of the QFT. The trace anomaly coefficients are defined for smooth metrics, thus the necessity of the limit of small conical defects to establish a connection with the entanglement entropy. For the case where V is a sphere, the coefficient turns out to be proportional to the coefficient of the Euler density in the trace anomaly. This has been later proved to be the case in any dimensions using purely QFT arguments, which rely on the existence of a rotational (conformal) symmetry for the sphere [10, 13] (see also [18]). For a cylinder, where this additional symmetry is absent, the other anomaly coefficient is selected in Solodukhin's formula. This result has been reproduced by holographic calculations of entanglement entropy for theories with higher derivative gravity actions [19], and also confirmed in numerical calculations of the logarithmic term for free massless scalars and fermions in four dimensions [20].

Thinking in terms of the positivity inequalities, the idea of the small deficit angle path integral representation, with insertions at the boundary of V is far from being innocent, and implies a large body of non trivial relations. We have seen above that $e^{-(n-1)S_n(V)}$ obeys the correlator inequalities due to its path integral representation. The assumption of the small deficit angle path integral representation can be then be rephrased (and generalized) as that $e^{-(n-1)S_n(V)}$ obeys the positivity inequalities at first order in $n - 1$ as $n \rightarrow 1$. This is equivalent to the matrices $\delta_{ij} - (n - 1)S(V_i\bar{V}_j)$ being positive definite for small enough $(n - 1)$. As explained in the next section this is the mathematical requirement for the matrices $-S(V_i\bar{V}_j)$ to be conditionally positive definite, or equivalently, the matrices of the exponentials $e^{-\lambda S(V_i\bar{V}_j)}$, with any $\lambda > 0$, to be infinitely divisible. Expanding the determinants in (4) for small $(n - 1)$ we get

$$\det(\{S(V_i\bar{V}_{j+1}) + S(V_{i+1}\bar{V}_j) - S(V_i\bar{V}_j) - S(V_{i+1}\bar{V}_{j+1})\}_{i,j=1,\dots,m-1}) \geq 0. \quad (7)$$

This is an infinite series of polynomial inequalities in the entropy, one for each polynomial order.

The entanglement entropies have ultraviolet divergences which might in principle obscure the physical content of these inequalities. Correspondingly, the minimal areas in AdS space diverge due to the infinite volume of the surfaces as they approach the asymptotic boundary. However, these divergences, being local and extensive along the boundaries ∂V , cancel out in the generic matrix element in (7). The inequalities can be equivalently written in terms of the mutual information $I(U, V) = S(U) + S(V) - S(UV)$, which is finite, as

$$\det(\{I(V_i, \bar{V}_j) + I(V_{i+1}, \bar{V}_{j+1}) - I(V_i, \bar{V}_{j+1}) - I(V_{i+1}, \bar{V}_j)\}_{i,j=1,\dots,m-1}) \geq 0. \quad (8)$$

The divergences can also be eliminated from (4) by computing the well defined connected

correlators

$$e^{(n-1)I_n(V_i, \bar{V}_j)} = e^{(n-1)S_n(V_i)} e^{(n-1)S_n(\bar{V}_j)} e^{-(n-1)S_n(V_i \bar{V}_j)}, \quad (9)$$

with $I_n(U, V) = S_n(U) + S_n(V) - S_n(UV)$. Multiplying the matrix elements in (4) by the factor $e^{(n-1)S_n(V_i)} e^{(n-1)S_n(\bar{V}_j)}$ does not change the positive definite character of the matrix.

The linear inequality in (7)(with $m = 2$) writes

$$2S(V_1 \bar{V}_2) \geq S(V_1 \bar{V}_1) + S(V_2 \bar{V}_2). \quad (10)$$

It also holds for the integer index Renyi entropies since it coincides with the $m = 2$ case in (4). When V_1 and V_2 are on the same hyperplane the inequality (10) coincides with some specially symmetric case of the strong subadditive inequality of the entropy. This last case has been discussed in the context of minimal areas [21]. The reason for the validity of (10) for minimal areas is very simple, and was first discussed in [14] in relation to the strong subadditive inequality for coplanar regions. It is due to the triangle inequality satisfied by the minimum of the area functional (see figure 2). However, in the present case, the time-symmetric position of the regions in (10) is important for the validity of the inequality, since it enforces the relevant minimal surfaces cut each other in order to apply the triangle inequality.

We hope that learning whether the $m > 2$ non-linear inequalities in (7) hold will teach us something about the fundamentation of Fursaev's proof and the Ryu-Takayanagi proposal for the holographic entanglement entropy. In this paper we take an "experimental" attitude and check these inequalities in all (admittedly few) known exact calculations of entanglement entropy in QFT. So far, the entanglement entropy can be calculated exactly mostly for free fields in simple geometries and these are the cases we have checked. We use either numerical evaluation or analytical tools. Up to what we have checked, we could not find a counterexample to these inequalities in the QFT examples.

Then we check them for minimal areas representing holographic entanglement entropies. We find that even though the inequalities are remarkably satisfied in several non trivial cases, there are counterexamples. We distinguish two types of counterexamples, the ones in which the minimal areas in the correlation matrix experiment a phase transition as a function of the shape of the region, and the ones in which they do not. These phase transitions are related to the fact that the boundary theory is evaluated in a large N and large coupling constant limit, and are not expected to occur in the entanglement entropy for ordinary field theories with finite number of degrees of freedom. The counterexamples without phase transition violate the "infinitesimal" inequalities obtained from (7) for infinitesimally displaced regions. The infinitesimal inequalities involve an infinite series of inequalities for the function derivatives of arbitrarily high order. Remarkably, we show that the conditional positivity of the entropy can be integrated from the infinitesimal inequalities to yield the finite inequalities in regions of analyticity for the correlators. Hence, when the infinitesimal inequalities hold in the neighborhood of a fixed region V , they automatically hold for all the domain of analyticity. These domains of analyticity are broken by phase transitions. Our results give a negative answer for the general validity of conditional positivity for the holographic entanglement entropy.

The theme of minimal surfaces in AdS establishes a natural connection with Wilson loop operators. The Maldacena correspondence gives a geometric prescription for the correlators of Wilson loops in a nontrivial interacting theory, the large N limit of $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills gauge theories in four dimensions [22]. It is then interesting to look at the way the eminently quantum mechanical positivity property (2) is realized for Wilson loops in this

context. The expectation values of the Wilson loops in Euclidean space in the large t'Hooft coupling limit $\lambda \gg 1$ are given by the Maldacena prescription [22]

$$\langle W(C) \rangle = \exp(-\sqrt{\lambda} \mathcal{A}(C)), \quad (11)$$

where C is the loop, and $\mathcal{A}(C)$ is the minimal area of a bulk two-dimensional surface whose boundary coincides with C . Because the Wilson loops are operators, we have a reflection positivity relation analogous to (4),

$$\det \left(\{e^{-\sqrt{\lambda} \mathcal{A}(C_i \bar{C}_j)}\}_{i,j=1\dots m} \right) \geq 0. \quad (12)$$

Here C_i , $i = 1\dots m$ is a collection of loops included in the half-space of positive Euclidean time, and \bar{C}_j is the Euclidean time-reflected loop corresponding to C_j . The vector tangent to the loop indicating the circulation direction gets multiplied by $-T$, where T is the time reflection matrix.

Since the inequality (12) have to be valid only in the large λ limit, it is not difficult to see that the only relevant information it provides is given by the inequality for $m = 2$, which is the only one homogeneous in λ . This writes, analogously to (10),

$$2\mathcal{A}(C_1 \bar{C}_2) \geq \mathcal{A}(C_1 \bar{C}_1) + \mathcal{A}(C_2 \bar{C}_2). \quad (13)$$

This linear reflection positivity inequality for the logarithm of the expectation value of the Wilson loops is valid in any gauge theory independently of the large N , large λ limit, and was discussed for example in [23]. It implies for example that the behavior of the Wilson loop is bounded between a perimeter and an area law. A generalization inspired in strong subadditivity has been discussed in [24] for the case of Wilson loops which live in the same two-dimensional plane in the AdS-CFT context. On the geometrical side, the reason for the validity of (13) is again the triangle inequality of the minimal areas, the same reason as in the case of holographic entanglement entropy [14].

By analogy with the case of the entanglement entropy, we also investigate infinite divisibility for Wilson loops in AdS-CFT. The infinite divisibility property for a correlator implies that arbitrary fractional powers of correlators also satisfy the positivity inequalities. In the present case it would mean we can take any $\lambda > 0$ in (4) and the inequalities are still valid (this of course does not mean that the corresponding exponentials are the expectation values of Wilson loops for the theory with arbitrary λ). Again, Taylor expanding in λ we get, analogously to (7),

$$\det(\{\mathcal{A}(C_i \bar{C}_{j+1}) + \mathcal{A}(C_{i+1} \bar{C}_j) - \mathcal{A}(C_i \bar{C}_j) - \mathcal{A}(C_{i+1} \bar{C}_{j+1})\}_{i,j=1,\dots,m-1}) \geq 0. \quad (14)$$

These inequalities are homogeneous polynomials in the area. This is why the factors of λ drop out from (14). They are also independent of ultraviolet divergences. Using essentially the same calculations used for the entanglement entropy (most of the known solutions can in fact be interpreted either in terms of Wilson loops or entanglement entropy) we find these inequalities do not hold in general for the Wilson loops. Curiously, we find they do hold for several examples of minimal surfaces depending only in one parameter.

A situation analogous to the one above for the Wilson loops arises in the classical limit of the expectation values of functions of the coordinate and momentum (without involving \hbar) in ordinary quantum mechanics. The triangle inequality for the classical action¹ is all what is

¹The triangle inequality is immediate for a real Euclidean action, containing at most second derivatives of the fields. The possible failure of the triangle inequalities for higher derivative theories due to boundary terms is then related to the failure of these theories to be unitary.

needed in order to satisfy the reflection positivity inequalities at the leading exponential order in \hbar . However, this weakening reduction from an infinite series of inequalities for any m to just the $m = 2$ case in the semi classical limit does not hold for expectation values of other kind of operators. For these, as we show in this paper, the full power of quantum mechanics positivity survives the classical limit.

Based on the relation between large N limits and classical limits [25], we show that the correlation matrices for some specific operators in large N theories satisfy infinite divisibility in addition to ordinary positivity. It is well known that infinite divisibility identifies the possible probability distributions which arise as central limits in probability theory. That is, it characterizes the probability distributions of sums of an infinite series of independent random variables [26]. The result discussed in this paper about large N theories is in fact an analogous one in the QFT context.

Concerning the Wilson loops, we establish they do not belong to the class of operators which are shown to be infinite divisible in the large N limit (Section 3). This agrees with our explicit checks for minimal areas. In fact, in a lattice model, the Wilson loop operators are “classical” operators in the sense of [25], and then they are not expected to be infinite divisible.

The plan of the paper is as follows. In the next Section we introduce the property of infinite divisibility for positive definite matrices. We discuss two basic examples, the central limit of probability distributions and exponential operators in free field theories. In Section 3 we show that some exponential operators are infinite divisible in the semiclassical limit of quantum mechanics and extend this result to large N theories. This is done with the help of the interpretation of the large N limit as a classical one. In the classical limit language the infinite divisible operators are just the exponentials of classical operators divided by $\sqrt{\hbar}$. In section 4 we prove conditional positivity property spreads on the domain of analyticity of correlation functions. In section 5 we analyze known exact examples of entanglement entropy in QFT and check conditional positivity. In section 6 we check the geometrical inequalities (7) for exact solutions of minimal surfaces in AdS, with interpretation either as Wilson loops or entanglement entropy. Finally, we discuss the results and present the conclusions.

2 Infinite divisibility, the central limit and free fields

The infinite divisible property is inspired by the Schur theorem for positive definite matrices: If $\{a_{ij}\}$ is positive definite (it is hermitian and all its eigenvalues are positive) then the matrix of the integer powers of the elements $\{(a_{ij})^N\}$ is also positive definite. A positive definite matrix $\{a_{ij}\} = \{c_{ij}^N\}$ which is equal to the entry-wise N -power of another positive definite matrix $\{c_{ij}\}$ for any N (here $\{c_{ij}\}$ may depend on N) is called infinite divisible [27]. In QFT, the infinite divisibility of a correlator, means that its fractional powers also satisfy the positivity conditions a correlator ought to satisfy. Then, these fractional powers can be thought as correlators for operators in another theory, non necessarily the original one.

It is not difficult to show that the positive definite matrix $\{a_{ij}\}$, $i, j = 1 \dots m$, is infinite divisible if and only if the matrix of the logarithms of its entries [28]

$$b_{ij} = \log(a_{ij}) \tag{15}$$

is conditionally positive, meaning that

$$x_i^* b_{ij} x_j \geq 0, \quad \sum_i x_i = 0. \tag{16}$$

That is, it is positive for any vector x_i satisfying $\sum_i x_i = 0$. In the language of QFT b_{ij} is almost a correlator, since it satisfies the positivity inequalities except for a one dimensional subspace. Equivalently, b_{ij} is conditionally positive if there is a vector ξ_i such that

$$b_{ij} + \xi_i + \xi_j \quad (17)$$

is positive definite. Another equivalent condition is that the matrix [28]

$$b_{i,j} + b_{i+1,j+1} - b_{i+1,j} - b_{i,j+1} \quad i, j = 1 \dots m - 1 \quad (18)$$

is positive definite.

An example of correlators which are powers of other correlators is given by a QFT which is the tensor product of identical independent sectors named by $1, \dots, N$. Choosing the correlators of product operators

$$\langle 0 | \mathcal{O}_i^{(1)\dagger} \dots \mathcal{O}_i^{(N)\dagger} \times \mathcal{O}_j^{(1)} \dots \mathcal{O}_j^{(N)} | 0 \rangle = \langle 0 | \mathcal{O}_i^{(1)\dagger} \mathcal{O}_j^{(1)} | 0 \rangle \dots \langle 0 | \mathcal{O}_i^{(N)\dagger} \mathcal{O}_j^{(N)} | 0 \rangle. \quad (19)$$

Most correlator matrices in QFT are not infinite divisible. In principle, this property should be valid only for special type of theories and special types of operators in these theories. Eq. (19) suggests that in order to have infinite divisibility one has to consider a theory in the limit of a large number of degrees of freedom, and some special kind of operators. We show in the next section this is the case of some exponential operators in large N theories.

Infinite divisibility is also related, and for the same reason which operates in large N theories, to the central limit of probability theory [26]. Take the probability distribution for a variable $y = \frac{1}{\sqrt{N}}(x_1 + x_2 + \dots + x_N)$ which is proportional to the sum of N independent variables x_i , with the same distribution $P(x)$. Then the probability distribution $Q(y)$ for y is the N -fold convolution

$$Q(y) = \int dx_1 \dots dx_N \delta \left(y - \frac{1}{\sqrt{N}}(x_1 + \dots + x_N) \right) P(x_1) \dots P(x_N). \quad (20)$$

The expectation value for the exponential $G_Q(k) = \langle e^{iky} \rangle$ (called the characteristic function of the distribution) is

$$G_Q(k) = \int dy e^{iky} Q(y) = \langle e^{i \frac{k}{\sqrt{N}} x} \rangle^N = G_P(k/\sqrt{N})^N. \quad (21)$$

Then, the infinite divisible case for the characteristic functions corresponds exactly to probability distributions which arise from an infinite sum of variables. Under the condition that the variance of x is finite, the limit of the sum of infinitely many variables can only be the Gaussian distribution, which of course has infinite divisible characteristic function. This is the central limit theorem. Different central limit distributions are also possible when the variance is non finite.

Notice that the characteristic functions satisfy positivity relations which are similar to the ones corresponding to two point correlators in QFT:

$$\lambda_i^* G_Q(k_i - k_j) \lambda_j = \int dx Q(x) (\lambda_i e^{-i \frac{k_i}{\sqrt{N}} x})^* (\lambda_j e^{-i \frac{k_j}{\sqrt{N}} x}) \geq 0. \quad (22)$$

Thus, $G_Q(k_i - k_j)$ are positive definite matrices, and (21) means these write as entry-wise N powers of other positive definite matrices.

Gaussian distributions are analogous to free fields, and the proof of infinite divisibility for the expectation value of exponential operators is direct. We have for a free field $\phi(x)$

$$\langle 0 | e^{i \int dx k(x) \phi(x)} | 0 \rangle = e^{-\frac{1}{2} \int dx dy k(x) g(x-y) k(y)}, \quad (23)$$

where $g(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$ is the two point function, and $k(x)$ is an arbitrary function. Then

$$\langle 0 | e^{i \int dx k(x) \phi(x)} | 0 \rangle = \langle 0 | e^{i \int dx \frac{k(x)}{\sqrt{N}} \phi(x)} | 0 \rangle^N = \langle 0 | e^{i \int dx k(x) \frac{1}{\sqrt{N}} (\phi_1(x) + \dots + \phi_N(x))} | 0 \rangle. \quad (24)$$

The first equation shows the expectation value of an exponential operator as a power of the expectation value of a different exponential operator in the same theory, and the second equality shows it as an expectation value in a different theory, which is one of N independent free fields $\phi_1(x), \dots, \phi_N(x)$, with the same two point function.

Any exponential of a free field gives an operator with infinite divisible correlators. Another example is the Wilson loop operator $e^{ie \oint dx^\mu A_\mu}$ for the free electromagnetic field. Notice that the two point function of the free field $g(x-y)$ has to be only conditionally positive, and this allows for operators which are not strictly speaking quantum fields. This is the case of a massless scalar field in two dimensions, whose two point function $\langle 0 | \phi(x) \phi(y) | 0 \rangle \sim -\log|x-y|$ cannot be positive for all range of coordinates x, y . In section 6 below we show $-\log|x-y|$ is a conditionally positive correlation function.

In the large N limit of interacting theories [29] a result on infinite divisibility analogous to the one for free fields depends on choosing adequate operators that test only Gaussian fields fluctuations in leading approximation.

3 Infinite divisibility in the semiclassical and large N limits

The large N limits can be understood in the same terms as the classical limit of quantum mechanics [25]. Then we first look at infinite divisibility in the semi classical limit.

Consider evaluating the expectation value

$$\langle e^{\frac{F[q(t)]}{\sqrt{\hbar}}} \rangle = \frac{\int \mathcal{D}q(t) e^{-\frac{S[q(t)]}{\hbar}} e^{\frac{F[q(t)]}{\sqrt{\hbar}}}}{\int \mathcal{D}q(t) e^{-\frac{S[q(t)]}{\hbar}}}, \quad (25)$$

where the path integral is written in Euclidean time. Let the minimum of the action be attained for some value $q(t) = q_0$. In the $\hbar \rightarrow 0$ limit the action can be expanded around the classical minimum. Writing the deviations from the classical value as $x(t) = q(t) - q_0$ we have

$$S = S_0 + \frac{1}{2} \int dt_1 dt_2 x(t_1) \left. \frac{\delta^2 S[q(t)]}{\delta q(t_1) \delta q(t_2)} \right|_{q(t)=q_0} x(t_2) + \mathcal{O}(x(t)^3). \quad (26)$$

Because of positivity of the quadratic kernel $\kappa(t_1, t_2) = \left. \frac{\delta^2 S[q(t)]}{\delta q(t_1) \delta q(t_2)} \right|_{q(t)=q_0}$, the most relevant part of the functional integral in the limit of small \hbar comes from values of the fluctuations $x(t) \sim \sqrt{\hbar}$. In a similar way we can expand

$$\frac{F[q(t)]}{\sqrt{\hbar}} = \frac{F[q_0]}{\sqrt{\hbar}} + \int dt_1 \left. \frac{\delta F[q(t)]}{\delta q(t_1)} \right|_{q(t)=q_0} \frac{x(t_1)}{\sqrt{\hbar}} + \mathcal{O}\left(\frac{x(t)^2}{\sqrt{\hbar}}\right). \quad (27)$$

Therefore higher order deviations from the classical point in the action and the operator $F[q(t)]$ are suppressed for small \hbar . In this limit, the operator will behave as an exponential of an operator linear in a Gaussian field, and we expect infinite divisibility. We have

$$\langle e^{\frac{F[q(t)]}{\sqrt{\hbar}}} \rangle = e^{\frac{F[q_0]}{\sqrt{\hbar}}} e^{\frac{1}{2} \int dt_1 dt_2 J(t_1) (\kappa^{-1})(t_1, t_2) J(t_2)} \left(1 + \mathcal{O}(\sqrt{\hbar}) \right), \quad (28)$$

where $J(t) = \left. \frac{\delta F[q]}{\delta q(t)} \right|_{q=q_0}$.

As in the free field case discussed in the preceding section, here, infinite divisibility in the correlator matrices follows rightly from reflection positivity. Given a collection of functionals $F_i[q(t)]$ with support on the positive time $t > 0$ region, this later requires the positivity of the matrix (to leading order in \hbar)

$$\langle e^{\frac{F_i[q(-t)]^*}{\sqrt{\hbar}}} e^{\frac{F_j[q(t)]}{\sqrt{\hbar}}} \rangle = e^{\frac{F_i[q_0]^*}{\sqrt{\hbar}}} e^{\frac{F_j[q_0]}{\sqrt{\hbar}}} e^{\frac{1}{2} \int dt_1 dt_2 (J_i^*(-t_1) + J_j(t_1)) (\kappa^{-1})(t_1, t_2) (J_i^*(-t_1) + J_j(t_1))}. \quad (29)$$

The positivity of the matrix in (29) is equivalent to the positivity of the last, \hbar independent, factor on the right hand side. This is because a matrix with entries $y_i^* M_{ij} y_j$, $y_i \neq 0$, (no sum of the indices) is positive definite if and only if M_{ij} is also positive definite.

Since this relation has to be valid for any functional F_i , it is immediate that arbitrary positive real powers (exponent $r > 0$) of the matrix elements in (29),

$$\langle e^{\frac{F_i[q(-t)]^*}{\sqrt{\hbar}}} e^{\frac{F_j[q(t)]}{\sqrt{\hbar}}} \rangle^r = e^{\frac{r F_i[q_0]^*}{\sqrt{\hbar}}} e^{\frac{r F_j[q_0]}{\sqrt{\hbar}}} e^{\frac{1}{2} \int dt_1 dt_2 (\sqrt{r} J_i^*(-t_1) + \sqrt{r} J_j(t_1)) (\kappa^{-1})(t_1, t_2) (\sqrt{r} J_i^*(-t_1) + \sqrt{r} J_j(t_1))}, \quad (30)$$

satisfy the same reflection positive inequalities by taking different functionals F_i .

Using functional integrals in phase space, one can also show in the same fashion the infinite divisibility of operators of the form $e^{\frac{F[q,p]}{\sqrt{\hbar}}}$, provided the functional $F[q,p]$ is well behaved in the $\hbar \rightarrow 0$ limit, and that $F[q,p]$ is such that the functional integral converges. The operators such as $F[q,p]$ which in the limit $\hbar \rightarrow 0$ can be expressed as a function in phase space (technically, they have a well defined limit of expectation values in any coherent state) are called classical operators in [25]. Thus, the infinite divisible operators $e^{\frac{F[q,p]}{\sqrt{\hbar}}}$ are not classical operators. They are half way between classical operators of the form $e^{F[q,p]}$ and coherent operators of the form $e^{\frac{F[q,p]}{\hbar}}$. In the first case (no dependence on \hbar) the matrix of expectation values at leading order will be $(e^{F_i[q_0,p_0]})^* e^{F_j[q_0,p_0]}$ which has rank one. Thus, the question of infinite divisibility depends on the subleading terms, and the answer will be almost always negative. The vacuum expectation values of operators of the form $e^{\frac{F[q,p]}{\hbar}}$ produce much greater excitations in the $\hbar \rightarrow 0$ limit, and test terms in the action which are of higher order than the Gaussian term. Thus, in general there is also no infinite divisibility for these operators.

An important caveat arises if the classical limit of the action has degenerate minima. This does not affect the uniqueness of the quantum vacuum if the theory has finite energy instantons. In order to exemplify the potential problems which can appear, let us think in a simplistic model given by the following ordinary integral in several classical real variables $x \equiv (x_\alpha)$, $\alpha = 1 \dots M$,

$$\langle e^{\frac{F_i[x]}{\sqrt{\hbar}}} e^{\frac{F_j[x]^*}{\sqrt{\hbar}}} \rangle = \frac{\int d^M x e^{-\frac{S[x]}{\hbar}} e^{\frac{F_i[x]}{\sqrt{\hbar}}} e^{\frac{F_j[x]^*}{\sqrt{\hbar}}}}{\int d^M x e^{-\frac{S[x]}{\hbar}}}. \quad (31)$$

Let the minima of the action be attained at $x = y^{(k)}$, $k = 1 \dots p$. We have, up to order $\sqrt{\hbar}$ corrections,

$$\langle e^{\frac{F_i[x]}{\sqrt{\hbar}}} e^{\frac{F_j[x]^*}{\sqrt{\hbar}}} \rangle = \frac{1}{p} \left(\sum_{k=1}^p e^{\frac{F_i[y^{(k)}] + F_j[y^{(k)}]^*}{\sqrt{\hbar}}} e^{\frac{1}{2} (\partial_\alpha F_i[y^{(k)}] + \partial_\alpha F_j[y^{(k)}]^*) \kappa(y^{(k)})^{\alpha\beta} (\partial_\beta F_i[y^{(k)}] + \partial_\beta F_j[y^{(k)}]^*)} \right), \quad (32)$$

where the contravariant metric is $(\kappa^{\alpha\beta}(x)) = (\kappa_{\alpha\beta}(x))^{-1} = (\partial^2 S(x)/\partial x^\alpha \partial x^\beta)^{-1}$.

The result (32) is a superposition of the contributions of the different minima. While a sum of positive definite matrices is positive definite, the sum of infinite divisible matrices is not infinite divisible in general. Then, in general, infinite divisibility will be lost in the degenerate case.

However, we can think in two possible situations where the matrix (32) is still infinite divisible in the leading order in \hbar . A first possibility is that the values of $F_i[y^{(k)}]$ suppress all vacua except one, having maximal real part for one of the $y^{(k)}$. A second, more interesting possibility is that we choose a symmetric condition for the operators, where $F_i[y^{(k)}]$ is independent of k . A natural value would be $F_i[y^{(k)}] = 0$, and the correlators are finite and independent of \hbar in leading approximation.

In order to see how this last case can be made infinite divisible, let us think the degeneracy is a natural one, related to a symmetry of the action under some group G . That is, the action of the group elements g , $x' = g(x)$, leaves the action $S[x]$ invariant $S[x'] = S[x]$. The different vacua are permuted by the action of the group elements, $g(y^{(k)}) = y^{(gk)}$. The covariant metric matrices satisfy

$$\kappa(y^{(k)}) = D(g, k)^T \kappa(y^{(gk)}) D(g, k), \quad (33)$$

$$D(g, k)^\alpha{}_\beta = \left. \frac{\partial g^\alpha(x)}{\partial x^\beta} \right|_{x=y^{(k)}}. \quad (34)$$

In equation (33) we have used the fact that linear derivatives of the action at the minimum vanish. We have $D(g_2, g_1 k) D(g_1, k) = D(g_2 g_1, k)$. If $g(y^{(k)}) = y^{(k)}$, g belongs to the isotropy subgroup $I(k)$ of $y^{(k)}$, and $D(g, k)^T \kappa(y^{(k)}) D(g, k) = \kappa(y^{(k)})$. All the isotropy subgroups are isomorphic to each other. In fact, since all we are interested in is the linear structure around the different vacua, we can enlarge the symmetry by adding to the isotropy groups all the matrices which leave the metric invariant.

One can think the space of the vacuum solutions as a base manifold and the tangent space to the whole coordinate space at the vacuum solutions as a fiber on each $y^{(k)}$. A parallel transport from point $y^{(k_1)}$ to $y^{(k_2)}$ which preserves the metric is given by $D(g, k_1)$ for any g mapping $y^{(k_1)}$ to $y^{(k_2)}$. This is not unique in general, and several assignments of group elements $g_{k_1 k_2}$ to pairs of points is possible. All the assignments for a fixed pair of points differ by arbitrary elements of the isotropy group on each of the points. However, given two global assignments of group elements to links $k_i k_j$, it is possible that one cannot be converted to the other by transformations with local isotropy group elements.

Then, in order to make (32) infinite divisible we make the following choice. Choose one base point $y^{(1)}$, and an element of the group $g^{(k)}$ for each $y^{(k)}$ such that $y^{(k)} = g^{(k)}(y^{(1)})$. This $g^{(k)}$ gives place to a connection. Then choose the function $F_i[x]$ such that its gradients transform covariantly

$$\partial_\alpha F_i[y^{(k)}] = D(g^{(k)}, y^{(1)})^\beta{}_\alpha \partial_\beta F_i[y^{(1)}]. \quad (35)$$

Then, if we chose all the functions $F_i[x]$ to satisfy this covariance with the same connection all the terms in the sum in (32) become equal to each other, and the correlators are infinite divisible, while this is not the case for correlation matrices involving functions F_i satisfying (35) but for different connections. A particular case leading to infinite divisible matrices is given by the class of operators symmetric under the group operation $F_i(gx) = F_i(x)$.

3.1 Infinite divisibility in large N theories

According to the general scheme of [25] the large N limits can be thought as classical limits. This is done by adequately choosing a set of classical variables in the large N theory, and identifying the small parameter χ , which plays a role analogous to of \hbar in the classical limit, with some inverse power of N . The expectation values are then computed with a path integral over the classical variables, with a weight given by $e^{-\frac{S_{cl}}{\chi}}$, S_{cl} being a function of the classical variables, regular in the $\chi \rightarrow 0$ limit. Then, our prescription for the infinite divisible operators is very clear, these are the exponentials of the classical operators divided by $\sqrt{\chi}$. While the computation of the action in terms of the classical variables, or evaluating its minimum, can be very difficult in specific problems, we do not need this information here. We only need to identify the classical operators. This has been done already in [25] for a large class of models.

The general scheme (for more details see [25]) starts by defining a coherent group G and a unitary representation G_χ for each χ . The group G depends on the problem. Choosing a base state $|0\rangle_\chi$, the coherent states are defined as the states $|u\rangle = u|0\rangle_\chi$ for any $u \in G_\chi$. These form an overcomplete basis of the Hilbert space. The importance of the coherent states is that their overlap in the $\chi \rightarrow 0$ limit is exponentially small, and they can be used to write down a path integral. The classical operators can be defined as the ones which have a well defined limit of

$$\lim_{\chi \rightarrow 0} \frac{\langle u | A | u' \rangle_\chi}{\langle u | u' \rangle_\chi}. \quad (36)$$

The interest to us here is that a generating set of the classical operators is obtained from the Lie algebra of the coherence group. If Λ belongs to the Lie algebra of G_χ , then $(\chi\Lambda)$ is a classical operator. The classical operators are generated by these ones. From this, natural candidates to infinite divisible operators are of the form $e^{i\sqrt{\chi}\Lambda}$.

The example of the classical limit can be useful here. The coherence group is given by operators of the form

$$u = e^{\frac{i}{\hbar}(ap+bq+c)}. \quad (37)$$

The operators generated by the Lie algebra elements times \hbar are just the algebra of polynomials in p, q . As explained above, we find the operators $e^{iF(p,q)/\sqrt{\hbar}}$ are infinite divisible.

Then, in order to find out which operators are infinite divisible in each model we only need to know the lie algebra of the coherence group.

For example, in vector models with a $O(N)$ symmetry, the basic coordinates and momenta are $x_i(\alpha), p_j(\beta)$, where i, j are $O(N)$ indices and α, β are extra indices (i.e. position coordinates in QFT). They are rescaled so they satisfy

$$[x_i(\alpha), p_j(\beta)] = \frac{i}{N} \delta_{ij} \delta_{\alpha\beta}. \quad (38)$$

From the analogy with the classical limit, it follows $\chi = 1/N$. The classical operators are again functions of x_i and p_j as in the true classical limit, but here these functions are essentially required to be symmetric under $O(N)$, and this is why they are functions of the following basic

polynomials [25]

$$A(\alpha, \beta) = \frac{1}{2} \sum_{i=1}^N x_i(\alpha) x_i(\beta), \quad (39)$$

$$B(\alpha, \beta) = \frac{1}{2} \sum_{i=1}^N (x_i(\alpha) p_i(\beta) + p_i(\beta) x_i(\alpha)), \quad (40)$$

$$C(\alpha, \beta) = \frac{1}{2} \sum_{i=1}^N p_i(\alpha) p_i(\beta). \quad (41)$$

Therefore, from the comparison with the classical limit, we conclude that the infinite divisible operators have the form

$$e^{\sqrt{N}F(A(\alpha,\beta),B(\alpha,\beta),C(\alpha,\beta))}, \quad (42)$$

where F does not depend on N .

Another example is given by large N $U(N)$ -gauge theories. Following [25], consider a lattice model with link variables V^α which are $N \times N$ unitary matrices, and α labels a oriented link on the lattice. The conjugate momentum E^α is a hermitian matrix, and the commutation relations write

$$[E_{ij}^\alpha, V_{kl}^\beta] = \frac{1}{2N} \delta^{\alpha\beta} \delta_{kj} V_{il}^\alpha, \quad (43)$$

$$[E_{ij}^\alpha, E_{kl}^\beta] = \frac{1}{2N} \delta^{\alpha\beta} (\delta_{kj} E_{il}^\alpha - \delta_{il} E_{kj}^\alpha). \quad (44)$$

The Kogut-Susskind Hamiltonian is invariant under local $U(N)$ transformations, and is given by

$$H = N^2 \tilde{\text{tr}} \left(\lambda \sum_{\alpha} (E^\alpha)^2 - \lambda^{-1} \sum_p (V^{\partial p} + V^{\partial \bar{p}}) \right). \quad (45)$$

Here the t'Hooft coupling constant $\lambda = g^2 N$ is chosen fixed such that the theory has a meaningful large N limit. The normalized trace is $\tilde{\text{tr}} = (N^{-1}) \text{tr}$, p and \bar{p} represent a lattice plaquette with the two possible orientations, and V^Γ for a lattice path Γ , is the ordered product of the link variables along the path.

In this case the Lie algebra of the coherence group is infinite dimensional, and the elements depend on arbitrary closed loops Γ on the lattice. It is generated by the operators

$$i N^2 \tilde{\text{tr}} V^\Gamma, \quad i N^2 (\tilde{\text{tr}} E^\alpha V^\Gamma + V^\Gamma E^\alpha). \quad (46)$$

Notice that the Wilson loop operators are precisely of the form $\tilde{\text{tr}} V^\Gamma$, which are traces of ordered products of the link variables along Γ . The small parameter is here $\chi = N^{-2}$ [25]. Then, our candidate for infinite divisible operators include

$$\exp(i N \tilde{\text{tr}} V^\Gamma) = \exp(i \text{tr} V^\Gamma). \quad (47)$$

Thus, the Wilson loop is a classical operator (which is the reason it has a N independent expectation value (11) in the large N limit), and typically will not be infinite divisible. This is what we find in specific holographic examples below. Because of the Ryu-Takayanagi entanglement entropy is a close relative to the Wilson loop, we may not expect infinite divisibility in a general case of the holographic entanglement entropy either.

It is interesting to notice that in general the classical operators are expected to have factorisable expectation values $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$. This is the case in AdS-CFT for two Wilson loops which are far apart, but not when they are close to each other. The non vanishing of the connected correlator can only be due to the presence of degenerate classical vacua. Because of the spatial dependence of the correlators, there should be an infinite number of non translational invariant classical vacua.

4 Positivity and analyticity

In this section we analyze the interplay between analyticity and positivity properties. The positivity inequalities in QFT have different formulations. The inequalities coming directly from positivity of the scalar product in real time (2) always involve the product of operators at coinciding points, and the distributional character of the correlation functions appears in a fundamental way. We are using here the reflection positivity inequalities in Euclidean time (where, on the other hand, the minimal area prescription holds in the AdS-CFT formulation) which do not have this problem².

The conditional positivity of the entropies is then equivalent to (7). The analysis of this section can also be rephrased in terms of the Wilson loops. For definiteness we will refer here to the entropy. According to the general results of section 2, all the matricial inequalities (7) can be summarized as a single relation expressing the conditional positivity of the entropy (minimal area) kernel:

$$- \int \mathcal{D}V_1 \mathcal{D}V_2 F[V_1] S(V_1 \bar{V}_2) F[V_2]^* \geq 0, \quad \int \mathcal{D}V F[V] = 0, \quad (48)$$

for any functional on regions $F[V]$ with support on the positive time half-space. The measure in region space is not relevant in (48) since it can be absorbed in F . Eq. (7) follows from this one by considering $F[V]$ as a sum of localized delta functionals in the different V_i . Eq. (48) follows from (7) by taking a discrete approximation to $F[V]$ as a sum of delta functions.

In the following, we consider a set of regions described by a finite number of parameters which live in an open set $D \in R^q$. Let us start with the single parameter case $q = 1$. Eq. (48) then writes

$$- \int dx dy \phi(x) S(x, y) \phi(y)^* \geq 0, \quad \int dx \phi(x) = 0, \quad (49)$$

where $\phi(x)$ is any test function with support in D , and we have written $S(x, y) = S(V_1 \bar{V}_2)$, for the regions V_1 and V_2 described by one parameter, x and y respectively (notice y is the parameter corresponding to V_2 , not to \bar{V}_2). In order to fulfill the second condition in (49) we take $\phi(x) = \partial_x \varphi(x)$. Then, integrating by parts we have

$$\int dx dy \varphi(x) f_1(x, y) \varphi(y)^* \geq 0, \quad (50)$$

where $f_1(x, y) = -\partial_x \partial_y S(x, y)$. Therefore, the conditional positivity of S becomes the positivity of f_1 .

²There is also a real time formulation of the reflection positivity inequalities which does not involve products of operators at coinciding points. These ‘‘wedge reflection positivity’’ inequalities [3, 30] write in the present context as (4), but the regions V_i are spatial regions included inside the right wedge $x^1 \geq |x^0|$ in Minkowski space, and the reflection operation is the wedge reflection, $x^{0'} = -x^0$, $x^{1'} = -x^1$, $x^{j'} = x^j$, $j = 2 \dots d$.

Now, we want to see the implications of (50) for coordinates in a very small region in the neighborhood of a point x . We assume $f_1(x, y)$ is real analytic around a point (x, x) . In this neighborhood we write any coordinate as $y = x + \epsilon$. We use again the discrete version of (50) involving only a finite number M of points, which follows taking φ as a sum of localized delta-like wave packets. We have

$$\sum_{\alpha, \beta=1}^M \lambda_\alpha \lambda_\beta^* f_1(x + \epsilon_\alpha, x + \epsilon_\beta) = \sum_{\alpha, \beta=1}^M \lambda_\alpha \lambda_\beta^* \sum_{m, n=0}^{\infty} f_1^{m, n}(x, x) \frac{(\epsilon_\alpha)^m}{m!} \frac{(\epsilon_\beta)^n}{n!} \geq 0, \quad (51)$$

for any M , λ_α and ϵ_α with $\alpha = 1, \dots, M$. Given a fixed M , we choose $\epsilon_\alpha = \epsilon \hat{\epsilon}_\alpha$, with all the $\hat{\epsilon}_\alpha$ different. We will later take the limit $\epsilon \rightarrow 0$. Then, the matrix $(\epsilon_\alpha)^n/n!$ for $\alpha = 1 \dots M$, and $n = 0 \dots M - 1$, is invertible. This follows from the Vandermonde determinant. Hence, for a fixed ϵ and $\hat{\epsilon}_\alpha$ we can choose the λ_α such that $v_m = \sum_{\alpha=1}^M \lambda_\alpha (\epsilon_\alpha)^m / m!$ for $m = 0 \dots M - 1$ is any chosen eigenvector of $\{f_1^{m, n}(x, x)\}_{m, n=0}^{M-1}$, with unit norm. Lets call β the corresponding eigenvalue. Since the λ_α are at most of order $\epsilon^{-(M-1)}$, the sum $u_m = \sum_{\alpha=1}^M \lambda_\alpha (\epsilon_\alpha)^m / m!$ for any $m \geq M$ is then of order ϵ at least. We then have

$$\begin{aligned} \sum_{\alpha, \beta=1}^M \lambda_\alpha \lambda_\beta^* \sum_{m, n=0}^{\infty} f_1^{m, n}(x, x) \frac{(\epsilon_\alpha)^m}{m!} \frac{(\epsilon_\beta)^n}{n!} &= \beta + \sum_{m=0}^{M-1} \sum_{n=M}^{\infty} f_1^{m, n}(x, x) v_m u_n^* + \\ \sum_{m=M}^{\infty} \sum_{n=0}^{M-1} f_1^{m, n}(x, x) u_m v_n^* + \sum_{m=M}^{\infty} \sum_{n=M}^{\infty} f_1^{m, n}(x, x) u_m u_n^* &= \beta + \mathcal{O}(\epsilon) \geq 0. \end{aligned} \quad (52)$$

Taking the limit $\epsilon \rightarrow 0$, this gives $\beta \geq 0$. This is equivalent to say that the matrix $f_1^{m, n}(x, x)$, $m, n = 0, \dots, M - 1$ is positive definite. Therefore, we obtain the infinitesimal inequalities

$$\det(\{f_1^{m, n}(x, x)\}_{m, n=0}^{M-1}) \geq 0 \quad (53)$$

for all M . Hence, the kernel $f_1^{m, n}(x, x)$ is positive definite in the space of all m, n .

Now, for an analytic function, these infinitesimal inequalities imply the ones for finite values of the displacements from the point x . Writing (51) as

$$\sum_{m, n=0}^{\infty} f_1^{m, n}(x, x) \left(\sum_{\alpha} \frac{(\epsilon_\alpha)^m}{m!} \lambda_\alpha \right) \left(\sum_{\beta=1}^M \frac{(\epsilon_\beta)^n}{n!} \lambda_\beta \right)^* \geq 0, \quad (54)$$

we see this inequality holds automatically once the infinitesimal inequalities are valid.

The equivalence between the infinitesimal and finite inequalities in the convergence neighborhood of the Taylor series have a very important consequence. Once the infinitesimal inequalities hold at a point x they automatically hold at any other point in a neighborhood, and then the infinitesimal inequalities hold in any point inside the convergence radius around x . Regions of validity of the infinitesimal inequalities can then be extended in the same way analyticity domains are extended. The infinitesimal inequalities have the same domain as the analyticity domain. The finite inequalities hold (at least) provided the points in the correlator matrix are contained in the convergence radius of the Taylor expansion around some point in this domain. Since correlation functions are usually analytic in QFT, the positivity turns out to be quite a local property.

From (53) we see the infinitesimal inequalities contain derivatives of the function f_1 of all orders. Specifying for the case of a translational invariant entropy which only depends on the Euclidean time variable x , $S(x, y) = S(x + y)$, the inequalities write

$$f_M(x) = \det(\{-S^{m+n+2}(x)\}_{m,n=0}^{M-1}) \geq 0. \quad (55)$$

More explicitly the first three inequalities read

$$f_1(x) = -S''(x) \geq 0, \quad (56)$$

$$f_2(x) = S(x)''S(x)'''' - (S(x)''')^2 \geq 0, \quad (57)$$

$$f_3(x) = (S^{(4)}(x))^3 + S^{(2)}(x)(S^{(5)}(x))^2 + (S^{(3)}(x))^2S^{(6)}(x) - S^{(4)}(x)(2S^{(3)}(x)S^{(5)}(x) + S^{(2)}(x)S^{(6)}(x)) \geq 0. \quad (58)$$

Writing $f_0(x) = 1$ we have a simple recurrence rule for the successive inequalities,

$$f_{n+1}(x)f_{n-1}(x) = f_n''f_n(x) - (f_n'(x))^2, \quad (59)$$

$$f_n(x) \geq 0, \quad n = 0, 1, 2, \dots \quad (60)$$

All these inequalities are clearly independent. It is not difficult to find solutions of $f_1(x) > 0$ which are not solutions of $f_2(x) > 0$ (i.e. $S(x) = \tanh(x)$).

Perhaps it is interesting to see in a simple example how subtle the inequalities (59) can turn for f_n with large n . Take for example the function $1/x$. Being a correlation function in some QFT, gives place to positive definite correlation matrices. Then taking $f_1(x) = 1/x$, $x > 0$, all the inequalities are satisfied. Now consider writing $f_1(x) = 1/x - a$ for some positive a . This is not any more a positive function since for large enough x it changes sign. However, for small x the positive definite correlator dominates and one can expect the inequalities are satisfied. A simple calculation shows that $f_n(x)$ changes sign at $x = \frac{1}{na}$. Thus, for any x there is a negative $f_n(x)$ for large enough n . This is in accordance with the above result on the local character of the inequalities for analytic functions: The non positivity of $f_1(x)$ can be detected with the infinitesimal inequalities at any x , no matter how small. This also shows that the f_n for large n are sensitive to smaller disturbances. On the other hand, if we consider $f_1(x) = 1/x + a$ with positive a , this is a positive definite function. It then follows that the short distance leading term $f_1(x) \sim 1/x$ has to be a positive definite function by itself. This shows that in checking the inequalities, it is also useful to check positivity for the leading short or long distance approximations.

When the region depends on more than one variable a similar analysis can be done. Let us write collectively these variables as $x \equiv (x^\mu)$, $\mu = 1, \dots, q$. In a similar way as for the one dimensional case, the conditional positivity of $-S(x, y)$ implies the positivity of the vectorial kernel given by the derivatives of the entropy, $f_{\sigma\delta}(x, y) = -\partial_\sigma^x \partial_\delta^y S(x, y)$. Expanding this in power series around a point x we get

$$\sum_{\alpha, \beta=1}^M \lambda_\alpha^\sigma \lambda_\beta^{\delta*} f_{\sigma\delta}(x + \epsilon_\alpha, x + \epsilon_\beta) = \sum_{\alpha, \beta=1}^M \lambda_\alpha^\sigma \lambda_\beta^{\delta*} \sum_{(\mu)(\nu)}^\infty f_{\sigma\delta}^{(\mu),(\nu)}(x, x) \frac{(\epsilon_\alpha^1)^{\mu_1}}{\mu_1!} \dots \frac{(\epsilon_\alpha^q)^{\mu_q}}{\mu_q!} \frac{(\epsilon_\beta^1)^{\nu_1}}{\nu_1!} \dots \frac{(\epsilon_\beta^q)^{\nu_q}}{\nu_q!} \geq 0, \quad (61)$$

where we have used the multi-index notation $(\mu) = (\mu_1, \mu_2, \dots, \mu_q)$, with $\mu_i = 0, 1, 2, \dots$, for $i = 1 \dots q$. The strategy is the same as for the one dimensional case. First, we choose the maximum derivative order $p = \max(\mu_1 + \mu_2 + \dots + \mu_q)$ which will appear in the infinitesimal inequality. This maximum number of derivatives includes a number of different derivative types (μ) given

by the combinatorial $\binom{p+q}{q}$ [31]. Then, we choose the case of $M = \binom{p+q}{q}$, and the λ_α^σ such that

$$v^{\sigma(\mu)} = \sum_{\alpha=1}^M \lambda_\alpha^\sigma \frac{(\epsilon_\alpha^1)^{\mu_1}}{\mu_1!} \dots \frac{(\epsilon_\alpha^q)^{\mu_q}}{\mu_q!} \quad (62)$$

is a normalized eigenvector of $f_{\sigma\delta}^{(\mu),(\nu)}(x,x)$ (in this matrix the derivative types (μ) and (ν) belong to the first M types). This can be done because it is always possible to choose ϵ_α^i such that the $M \times M$ multidimensional Vandermonde matrix $(\epsilon_\alpha^1)^{\mu_1} \dots (\epsilon_\alpha^q)^{\mu_q}$ is non singular [32]. Then, the same reasoning as in (52) leads us to conclude that all the eigenvalues of $f_{\sigma\delta}^{(\mu),(\nu)}(x,x)$ (a $qM \times qM$ matrix) are positive, and this matrix is positive definite. These are the differential inequalities in the multidimensional case. Again, the differential inequalities imply the finite inequalities in the domain of convergence of the Taylor series. This in turn leads to an automatic extension of the validity of the differential inequalities to all the domain of analyticity.

In order to clarify the discussion, let us be more explicit and look at the example of two variables, $q = 2$. For each $p = 0, 1, 2, \dots$ we have that a square matrix of order $q \binom{p+q}{q} = (p+2)(p+1)$ in the derivatives of $S(x,y)$ is positive definite. However, there may be repeated lines and columns in $f_{\sigma\delta}^{(\mu),(\nu)}(x,x)$ because of the derivatives in σ, δ . Since a submatrix of a positive definite matrix is also positive definite, we can eliminate these redundant lines. The first case $p = 0$ involves only the type $(\mu) = (0,0)$, and contains, due to the σ, δ derivatives, only first derivatives on each of the entries of $S(x,y)$,

$$- \begin{pmatrix} S^{(1,0),(1,0)}(x,x) & S^{(1,0),(0,1)}(x,x) \\ S^{(0,1),(1,0)}(x,x) & S^{(0,1),(0,1)}(x,x) \end{pmatrix}. \quad (63)$$

This is a consequence of the linear inequality (10), and always holds for a minimal surface which is the true minimum, not only an extremum of the area functional. The second infinitesimal inequality $p = 1$ involves the types $(\mu) = (0,0)$, $(\mu) = (1,0)$ and $(\mu) = (0,1)$, and contains one or two derivatives in each entry

$$- \begin{pmatrix} S^{(1,0),(1,0)}(x,x) & S^{(1,0),(0,1)}(x,x) & S^{(1,0),(1,1)}(x,x) & S^{(1,0),(2,0)}(x,x) & S^{(1,0),(0,2)}(x,x) \\ S^{(0,1),(1,0)}(x,x) & S^{(0,1),(0,1)}(x,x) & S^{(0,1),(1,1)}(x,x) & S^{(0,1),(2,0)}(x,x) & S^{(0,1),(0,2)}(x,x) \\ S^{(1,1),(1,0)}(x,x) & S^{(1,1),(0,1)}(x,x) & S^{(1,1),(1,1)}(x,x) & S^{(1,1),(2,0)}(x,x) & S^{(1,1),(0,2)}(x,x) \\ S^{(2,0),(1,0)}(x,x) & S^{(2,0),(0,1)}(x,x) & S^{(2,0),(1,1)}(x,x) & S^{(2,0),(2,0)}(x,x) & S^{(2,0),(0,2)}(x,x) \\ S^{(0,2),(1,0)}(x,x) & S^{(0,2),(0,1)}(x,x) & S^{(0,2),(1,1)}(x,x) & S^{(0,2),(2,0)}(x,x) & S^{(0,2),(0,2)}(x,x) \end{pmatrix}. \quad (64)$$

We recognize the upper 2×2 submatrix is just the matrix (63), and the positive definiteness of (64) entails the one of (63). However, in the positivity of (64), three more inequalities are added. Hence, it is not enough to check the positivity of the determinant but to check the one of all the eigenvalues is necessary.

More generally, the analysis of this section can be generalized to the case of infinitesimal inequalities for correlator matrices (or conditional positive entropy matrices as we have focused the discussion here), of different operators $\mathcal{O}_i(x_i)$ evaluated in the neighborhood of different points x_i (there may be also different number of parameters determining each x_i). The infinitesimal inequalities imply the finite ones inside the radius of convergence of the Taylor expansions. This provides a further extension of the inequalities, which propagates inside the multidimensional analyticity domain of the correlator matrices. Basically, once positivity holds at some

neighborhood of a multipoint x_i , the only obstacles for further expansion have to be sourced by singularities which break up analyticity.

5 Conditional positivity and entanglement entropy in QFT

In this Section we check conditional positivity of (minus) the entanglement entropy in QFT using numerical and analytical methods. The available exact results for the entropy in QFT are not many, and involve mainly free fields in low dimensions and simple geometries [33]. The numerical tests require high precision, typically around 15 – 20 good digits for the $f_{\mathfrak{S}}$ defined in the previous Section. This leaves us with a rather restricted scenario, both from the analytical and numerical point of view, and precludes the use of approximations such as the ones obtained by putting models in a lattice.

The studied examples are consistent with conditional positivity of $-S(V)$. However, by the above mentioned reasons we are not able to distinguish if this is due to the simple geometries used, or the free character of the models. In this sense, we have to mention two facts. The first one is that the counterexamples found for infinitesimal inequalities on the minimal surfaces in the next Section occur for more complicated geometries for which there are no pure QFT calculations of the entanglement entropy. These involve the entropies of two disjoint circular regions with different radius. The one parameter functions which are our best QFT cases also seem to be infinite divisible in the minimal surface case (of course, with the difference that only conformal examples are given by the minimal surfaces, while here we have also massive examples).

The second commentary is that the case of free fields is up to a certain extent special since in this case the Renyi entropies are also found to satisfy the infinite divisible inequalities while this does not hold for some interacting cases we review below. There is also an indication that the traces $\text{tr} \rho_V^n$ with non-integer index n might be positive. If this “fractionality” property is correct, extending the positivity to non-integer n , it would naturally lead to infinite divisibility of the exponentials of the entropy in free fields.

It is possible to suspect that the infinite divisibility in the free case could be a more general property holding also for discrete systems. However, this is not the case. There is a natural generalization of the reflection positive inequalities applicable to discrete systems [3]. We have applied it for free discrete fermions (though we are not presenting details of this calculation in this paper), and have found counterexamples to the inequalities for some numerical examples using random density matrices in low dimensional Hilbert spaces (though infinite divisibility holds for the large majority of random matrices, specially as the dimensions are increased). Hence, if infinite divisibility holds for free QFT, the continuum limit has to be an important ingredient.

5.1 One interval in two dimensions: conformal and large mass limits

The entanglement entropy for an interval of length L in two dimensions and any CFT is given by [5, 6]

$$S(L) = \frac{c}{3} \log(L/\epsilon), \quad (65)$$

where $c > 0$ is the Virasoro central charge, and ϵ a short distance cutoff. The logarithm is proportional to the correlation function of massless scalar fields $\langle \phi(L)\phi(0) \rangle \sim -\log(L)$, hence

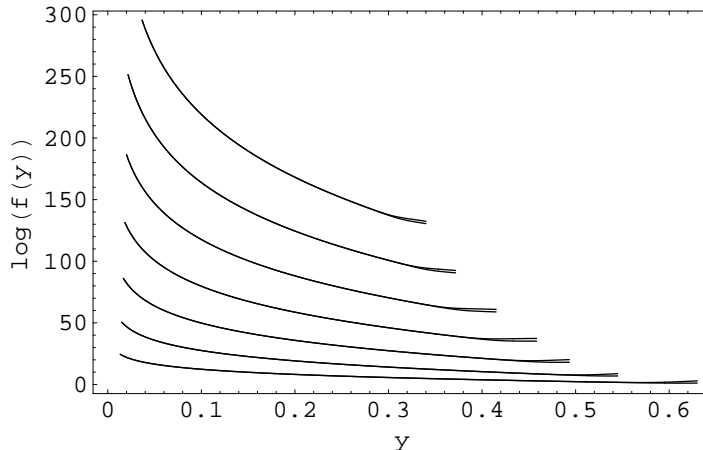


Figure 3: The logarithm of the functions f_i , $i = 1, \dots, 8$ (from bottom to top) of the parameter $y = mL$ for the fermionic entropy $S(y)$, computed with a series expansion around the origin up to y^{16} and y^{18} . The bifurcation between these two approximations at the end of the curves indicates the limit of the region where the series approximation can be trusted. The logarithm is real showing these functions are positive.

it is a conditionally positive function. We give a direct proof of this in the next Section.

On the opposite extreme to the conformal case, the long distance, large mass leading term for the entropy of an interval in a massive (interacting) theory is also known to have a general expression given by

$$S(L) = -\frac{1}{8} \sum_i K_0(2m_i L) + \text{const}, \quad (66)$$

where the sum is over the spectrum of massive particles of the theory with masses m_i , and $K_0(x)$ is the Bessel function. This formula has been shown to hold for generic integrable models [7, 34], and argued to be valid for any massive two dimensional QFT [35]. For free fields a direct calculation is in [36, 37]. Again, we find this entropy is proportional to a correlator. This is the one of massive free scalar fields in two dimensions. Hence, it is conditionally positive. In fact, this contribution is positive (rather than conditionally positive) since the massive scalar is a well defined field operator.

5.2 One interval in two dimensions: massive free fields

Still in $(1+1)$ dimensions, we now study the case of massive scalar (S) and Dirac (D) free fields in a one interval set. The entropy function $S(y)$, with $y = mL$, is known exactly and given by [36, 37] (see also [33])

$$S_D(y) = \int_0^\infty db \frac{\pi}{\sinh(\pi b)^2} S_b^D(y), \quad (67)$$

$$S_S(y) = \int_0^\infty db \frac{\pi}{\cosh(\pi b)^2} S_b^S(y), \quad (68)$$

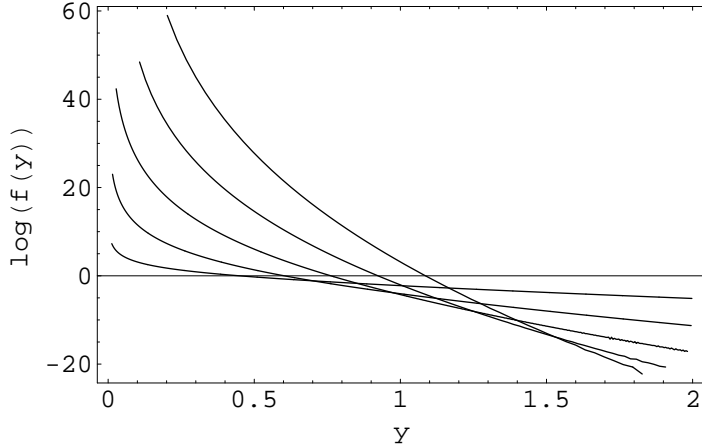


Figure 4: The logarithm of the functions f_1, f_2, f_3, f_4, f_5 (from smaller to higher slope) of the parameter $y = mL$ for the fermionic function S_{ia} in the case $a = 1/3$. The logarithm is real showing these functions are positive. Similar results hold for different a and for the scalar field.

where

$$S_b^{(D,S)}(y) = - \int_y^\infty dt t (u_b^{(D,S)}(t))^2 \log(t/y), \quad (69)$$

$$u_{(D,S)}'' + \frac{1}{y} u_{(D,S)}' = \frac{u_{(D,S)}}{1 + u_{(D,S)}^2} (u_{(D,S)}')^2 + u_{(D,S)} (1 + u_{(D,S)}^2) - \frac{4b^2}{y^2} \frac{u_{(D,S)}}{1 + u_{(D,S)}^2}. \quad (70)$$

Equation (70) is a non linear ordinary differential equation of the Painlevé type. The difference between the fermionic and scalar cases is exclusively due to the boundary conditions on this equation,

$$u_D(y) \rightarrow \frac{2}{\pi} \sinh(b\pi) K_{i2b}(y) \quad \text{as } y \rightarrow \infty, \quad (71)$$

$$u_S(y) \rightarrow \frac{2}{\pi} \cosh(b\pi) K_{i2b}(y) \quad \text{as } y \rightarrow \infty. \quad (72)$$

The functions $-S_b(y)$ are not conditionally positive, since their large distance approximation $K_{i2b}(y)$ does not satisfy the inequalities. However, our numerical tests for the entropies in (67) and (68) using random interval sizes are compatible with conditional positivity. It is not easy to attain excellent numerical precision for these integrals except for the case of the Dirac field where the solution for the differential equation can be obtained by a series expansion around the origin in terms of powers of y and $\log(y)$ [33]. The figure (3) shows the functions f_i , $i = 1, \dots, 8$, calculated from the fermionic entropy in a region around the origin for the series expansion up to orders y^{16} and y^{18} . These functions are positive, what is compatible with the conditional positivity of $-S$. For the scalar case, the expansion for small y starts as $S(y) \sim \frac{1}{3} \log(y) + \frac{1}{2} \log(\log(y))$, which also satisfies the inequalities.

The Renyi entropies are obtained from an analytic extension of S_b for imaginary values of

b [33],

$$S_n^S(y) = -\frac{1}{n-1} \sum_{k=-(n-2)/2}^{(n-2)/2} S_{i\frac{k}{n}}^S(y), \quad (73)$$

$$S_n^D(y) = -\frac{1}{n-1} \sum_{k=-(n-1)/2}^{(n-1)/2} S_{i\frac{k}{n}}^D(y). \quad (74)$$

We checked numerically the differential inequalities (59) for the analytically continued functions S_{ia} , both for scalar and Dirac fields, and for several values of $a \in (-1/2, 1/2)$. In all cases we find these inequalities are satisfied. This points to conditional positivity of these functions and of the Renyi entropies $-S_n$. The figure (4) shows a particular case $a = 1/3$. As shown in the figure, all the curves f_i look very similar. This may indicate the conditional positivity in this case could be a consequence of the interplay between the recurrence rules (59) for the inequalities and some recurrence relation on the Painlevé differential equation (70). As explained in the next section, a proof of conditional positivity for these functions would also follow from the evaluation of the Laplace antitransform of S_{ia} . Unfortunately we have not been able to compute it.

The same rules (67), (68), (73) and (74) relates the entropy and Renyi entropies for free fields of any region and space-time dimensions, perhaps pointing to the generality of this relation between conditional positivity for the entropies, Renyi entropies, and fractionality in free theories.

It is interesting to note that the exponentials of the Painlevé-related functions S_{ia} and the entropies, would then belong to the class of positive definite infinite divisible functions. Infinite divisible functions which are also probability distributions have been classified in relation to the central limit theorem in probability theory [26].

5.3 Multiple intervals for a free massless fermion in two dimensions

Our next example is given by the massless fermion in two dimensions [36]. This is, so far, the only known complete entropy function. The entanglement entropy corresponding to a set V formed by p disjoint intervals (a_i, b_i) on a spatial line, with $a_i < b_i < a_{i+1}$, is

$$S((a_1, b_1) \dots (a_p, b_p)) = \frac{1}{3} \left(\sum_{i,j} \log |a_i - b_j| - \sum_{i < j} \log |a_i - a_j| - \sum_{i < j} \log |b_i - b_j| - p \log \epsilon \right). \quad (75)$$

From this entropy function, we find infinite divisibility for the matrices $(e^{-\lambda S(V_i \bar{V}_j)})$. This follows from reflection positivity, since we can write for any $\lambda > 0$

$$\tilde{\mathcal{C}}^p e^{-\lambda S((a_1, b_1) \dots (a_p, b_p))} = \langle 0 | : e^{i\sqrt{\frac{2\pi\lambda}{3}}\phi(a_1)} :: e^{-i\sqrt{\frac{2\pi\lambda}{3}}\phi(b_1)} : \dots : e^{i\sqrt{\frac{2\pi\lambda}{3}}\phi(a_p)} :: e^{-i\sqrt{\frac{2\pi\lambda}{3}}\phi(b_p)} : | 0 \rangle, \quad (76)$$

in terms of vertex (exponential) operators constructed with a free massless scalar field $\phi(x)$ [36]. The right hand side gives the left hand one since we have

$$\langle 0 | e^{i \int dx f(x) \phi(x)} | 0 \rangle = e^{\frac{1}{8\pi} \int dx dy f(x) \log |x-y| f(y)}. \quad (77)$$

For general free fields the Renyi entropies are given in terms of exponentials of operators which are quadratic in the free fields. In this particular case, due to the bosonization (76),

they can also be written in terms of exponentials of operators linear in a free scalar field. The infinite divisibility property coincides with the one of these exponential operators as discussed in Section 2³.

In this example, the Renyi entropies S_n are proportional to the entropy, $S_n = \frac{1}{2}(\frac{n+1}{n})S$, and hence $-S_n$ is conditionally positive. It is interesting to note that in the massive case the Renyi entropies can also be mapped to correlators of exponentials of a scalar field [36]. However, this belongs to the interacting Sine-Gordon theory. Therefore, the infinite divisibility of the two point function discussed in the previous subsection corresponds to the one of correlators of these exponential operators in the Sine Gordon model.

5.4 Two intervals in a CFT

Formula (75) does not hold for general conformal field theories. Even if no exact result for the entropy is known for more than on interval in interacting models⁴ there are several interesting exact results for the Renyi entropies of integer n and two intervals for compactified scalars or the Ising model [39, 40, 41]. Most importantly, the exact results for the Renyi entropies allow us to show that in general the $\text{tr}\rho^n$ are not infinite divisible in QFT [3]. An approximation expansion for the entropy in these models are found in [42]. However, these approximations seem not to be suitable to test infinite divisibility.

Conformal invariance implies the entropies of two intervals can be written as

$$e^{-(n-1)S_n} = k^2(x(a_2 - b_1)(b_2 - a_1))^{-\frac{c}{6}(n-\frac{1}{n})} F_n(x), \quad (78)$$

where $F_n(x) = F_n(1-x)$, $F(0) = 1$, is a function of the cross ratio $x = \frac{(b_1-a_1)(b_2-a_2)}{(a_2-a_1)(b_2-b_1)}$, k is a constant, and c is the Virasoro central charge. The functions $F_n(x)$ for the known examples are given by expressions involving theta functions and their inverses. A simple example is S_2 for the critical Ising model which is given in terms of algebraic functions [40]

$$F_2(x) = \frac{1}{\sqrt{2}} \left[\left(\frac{(1+\sqrt{x})(1+\sqrt{1-x})}{2} \right)^{1/2} + x^{1/4} + (x(1-x))^{1/4} + (1-x)^{1/4} \right]^{1/2}. \quad (79)$$

We have tested the infinite divisibility inequalities numerically and find that this Renyi entropy is not conditional positive. However, as expected, the non-linear inequalities (4) hold, what is consistent with $\text{tr}\rho_V^n$ being given by a vacuum expectation value of a twisting operator.

We do not know if the infinite divisibility is recovered in the so far unknown limit $n \rightarrow 1$ for the entanglement entropy. Since the Renyi entropies do not satisfy the inequalities, this case would be a very important test.

5.5 Long distance limit for two regions

In general, for free theories and two sets A and B which are far apart we have the mutual information is proportional to the squared of the Dirac or scalar field correlators [33]. Thus,

³The hypothetical theories with extensive mutual information considered in [38] also have infinite divisible $e^{-\lambda S}$ given by correlation functions of exponentials of free fields.

⁴We call a theory free if the correlations functions of the fields which are local with respect to the partitions of the space considered to evaluate the entanglement entropy obey the Wick's theorem. There are cases where the localized fields do not commute spatially with the free fields, such as the Ising model. This model has different Renyi entropies than the free fermion.

this gives conditional positivity when only the distance d between A and B are changed but not the shapes of the sets. For the case of two intervals in a massive Dirac field in two dimensions we know also the dependence on the shape for large separating distances [43],

$$I(A, B) \sim \frac{1}{6}m^2(a_+b_- + a_-b_+)K_0^2(md) + \frac{1}{6}m^2(a_+b_+ + a_-b_-)K_1^2(md), \quad (80)$$

where a_+ , b_+ , a_- and b_- are the projections on the null coordinates of the intervals A and B , which are not assumed to lie in a single spatial line but can also be boosted to each other. The positivity of this quantity is proved by noting that the relevant matrix

$$\begin{pmatrix} K_1^2(md) & K_0^2(md) \\ K_0^2(md) & K_1^2(md) \end{pmatrix} \quad (81)$$

is the Hadamard square of the correlation function of a massive Dirac field $i\gamma^0\gamma^1\langle\Psi(x)\Psi(y)^\dagger\rangle$. This last correlator satisfies the Minkowski space analog of reflection positivity [30].

5.6 Angular sectors in 2+1 dimension

In (2+1) dimensions, we study the infinite divisibility property in the entropy of planar angular sectors of a fixed side length L , for free scalar and Dirac fields. We place the angular sectors with common vertex in a plane x_0, x_1 . We have for the entropy

$$S(\theta) = (2 + \theta)\frac{L}{\epsilon} + s(\theta)\log(\epsilon) + \text{finite}. \quad (82)$$

The only divergent term in the entropy which does not cancel in the conditional positive inequalities is the logarithmic divergent contribution of the vertex angles. This has to satisfy the inequalities independently of the other finite contributions. The problem has then only one parameter.

The checks require to solve with enough precision a coupled set of non linear differential equations and integrate the results on one parameter to obtain the logarithmic coefficient of the entropy [44]. Following [44], this is done expanding the involved functions and differential equations in Taylor series around the value $x = \pi$ up to order $(x - \pi)^{14}$. (for details see [44]). Again, as in the one dimensional case, up to what we have checked, we found the entropy satisfies the conditional positivity inequalities (59), as functions of the angle θ of the angular sector, giving support to infinite divisibility for the exponential of the entropy. The figure (5) shows the functions f_n up to $n = 5$ (6×6 determinants), obtained from $s(\theta)$ corresponding to a massless scalar. For the fermion field we find completely analogous results.

5.7 Dimensional reduction for free fields

In higher dimensions, for free fields, some universal terms in the entanglement entropy can be obtained via dimensional reduction from results calculated in lower dimensions [33]. Let us consider sets in $k + d$ spatial dimensions of the form $V = B \times X$, where B is a box on the first k coordinates x_1, \dots, x_k , of sides lengths R_i , $i = 1, \dots, k$, and X is a set in d dimensions. The entropy of V in the limit of large R_i is extensive in the sides R_i . Thus, we can compactify the directions x_i , with $i = 1, \dots, k$, by imposing periodic boundary conditions $x_i \equiv x_i + R_i$, without changing the result of the leading extensive term. In the limit of large R_i , the entanglement

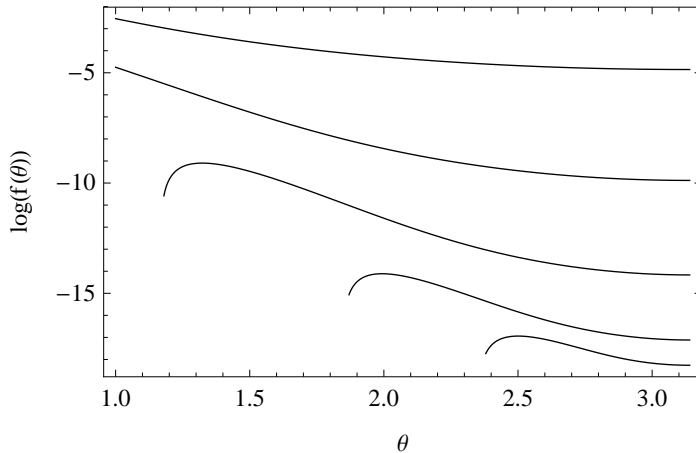


Figure 5: The logarithm of the functions f_1, f_2, f_3, f_4, f_5 (from top to bottom) obtained from the angular entropy function given by the coefficient of the logarithmic term $s(\theta)$ of a plane angular sector for a massless scalar field. We evaluate these functions of the angle θ as an expansion around $\theta = \pi$ up to order $(\pi - \theta)^{14}$ for $s(\theta)$. These functions are positive in a neighborhood of $\theta = \pi$. For small values of θ the Taylor expansion for f_n does not give a good approximation, and this is the reason the last three functions go down steeply at some θ .

entropy $S(V)$ is written in terms of $S(X, m)$, the lower dimensional free entropy function for mass m . We have [33]

$$S(V) = \frac{k \mathcal{A}}{2^k \pi^{k/2} \Gamma(k/2 + 1)} \int_0^\infty dp p^{k-1} S(X, \sqrt{m^2 + p^2}), \quad (83)$$

where the transversal area $\mathcal{A} = \prod_{i=1}^k R_i$. The universal terms in $S(V)$ will then come from the ones of X after this integration over the mass. From (83), it is evident that if $S(X, m)$ is conditionally positive, this is inherited by $S(V)$ for these special type of regions. For example, the conditional positivity of the entropy of an interval in one spatial dimension is inherited by the entropy of an infinite strip in two dimensions.

6 Some checks of the geometric inequalities for minimal surfaces

In this section we check the inequalities which arise from infinite divisibility of $e^{-\lambda S(V)}$, using the holographic prescription (Ryu-Takayanagi ansatz) for the entropy in terms of minimal areas in the bulk AdS space. We find counterexamples for the most complex geometries available, which depend on two parameters. Remarkable, the one parameter functions analyzed turn out to be conditionally positive.

6.1 Long strips

Consider the case where V is a rectangular strip of width R and length L , in the limit of large L , in three Euclidean dimensions. The cases of different space-time dimension have an analogous

treatment. Because the result has to be extensive in L , and because of conformal invariance, we have

$$S(V) = c_0 \frac{L}{\epsilon} - c_1 \frac{L}{R}, \quad (84)$$

for some constants $c_0 > 0$, c_1 , and a distance cutoff ϵ .

In order to obtain the positivity inequalities for single component regions of this kind, we position different regions in the $x^0 \geq 0$ space, with the long sides lying parallel to the x^3 direction. Further, we require one of the vertices to lie at the origin $x = 0$ (see figure 6) and all strips to have the same length L and orientation. Hence, all regions have a common boundary line. Because of the cancellation of the boundary lines passing on the point $x = 0$ belonging to the regions and their reflected counterparts, the area is a two point function depending on the coordinates $\hat{x} = (x^0, x^1)$ of the free boundary line of each region. We now prove the conditional positivity of minus the function (84), where $R = |\hat{x} - \hat{y}|$. Notice the conditional positivity condition does not take into account the cutoff dependent term in (84).

We can expect the positivity of R^{-1} , and hence the conditional positivity of $-S(V)$, because R^{-1} is a field correlator in two dimensions. This requires $c_1 \geq 0$ only. The same happens for any power $R^{-\nu}$ with $\nu \geq 0$, since $R^{-\nu}$ are field correlators for any $\nu > 0$: these are two point correlators of fields with different conformal weight in two dimensional CFT. It is interesting to consider also the case for different ν since higher inverse powers of R appear in other brane loops functions [22] and the holographic entanglement entropy for multidimensional strips, depending on the strip dimension and the space-time dimension.

It is instructive to go through the explicit proof in this two dimensional case. We are going to show that the correlators $R^{-\nu}$ are positive for all $\nu > 0$. This is equivalent to $-\log(R)$ being conditionally positive. This writes

$$- \int d^2x d^2y f(x) f(y)^* \log((x^0 + y^0)^2 + (x^1 - y^1)^2) \geq 0, \quad (85)$$

conditioned to $\int d^2x f(x) = 0$. Writing the correlators as a Laplace-Fourier transform

$$\log((x^0 + y^0)^2 + (x^1 - y^1)^2) = \int dk_0 dk_1 e^{ik_1(x^1 - y^1) - k_0(x^0 + y^0)} g(k_0, k_1), \quad (86)$$

and replacing this in (85) we have

$$\int dk_0 dk_1 |\hat{f}(k)|^2 g(k_0, k_1) \geq 0, \quad (87)$$

with

$$\hat{f}(k) = \int d^2x e^{-k_0 x^0 + i k_1 x^1} f(x), \quad \hat{f}(0) = 0. \quad (88)$$

This last Laplace transform makes sense since the condition for reflection positivity is $f(x) = 0$ for $x^0 < 0$. Then, all the infinite series of inequalities are summarized as $g(k_0, k_1) \geq 0$ for $k \neq 0$. The explicit evaluation of the Laplace antitransform of the logarithm confirms this expectation

$$g(k_0, k_1) = \frac{\delta(k_0 - |k_1|)}{|k_1|}. \quad (89)$$

For loops living in more dimensions, the proof can be done in each case using the same trick. One important difference with respect to the two dimensional case appears. The correlator R^{-1}

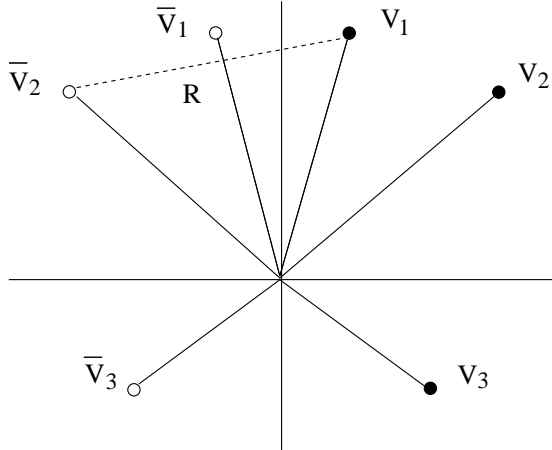


Figure 6: Three rectangular regions V_1 , V_2 and V_3 , and their time reflected images, with common side at the origin. Only the short sides of the rectangles are shown. The loop $V_1\bar{V}_2$ is equivalent to a rectangular loop with side R , in the limit of large longitudinal size $L \gg R$.

is positive, and then, because of Hadamard property, all the correlators R^{-k} are still positive, with $k > 0$ integer. However, for $d > 2$ the $\log(R)$ is not conditionally positive anymore. This means the functions $R^{-\nu}$ for $\nu \leq c$ for some $c < 1$ are not positive and do not constitute correlators of any QFT.

For the infinite divisibility of R^{-1} in any dimension, we can argue as follows. The free electromagnetic Wilson loops for rectangular regions with a long side L , which are also infinitely divisible, behave as $\langle W(C) \rangle \sim \text{const } e^{cLR^{-1}}$. This shows infinite divisibility for strips in the general case. It also shows this is not a very strong test, since mainly conformal invariance is involved. However, in terms of the minimal areas the infinite divisible property is rather peculiar. It is easy to check it does not hold for ordinary geodesics in AdS, but only for geodesics starting and ending at spatial infinity.

6.2 Angular sectors

Now consider loops which bound a plane angular sector of angle θ . The area as a function of the angle is [45]

$$\mathcal{A}(C) = c_1 + c_2(2 + \theta)\frac{L}{\epsilon} - g(\theta) \log(L/\epsilon). \quad (90)$$

The function $g(\theta)$ is given in parametric form in terms of a variable $x \in (0, \infty)$ as

$$\theta = 2x\sqrt{1+x^2} \int_0^\infty \frac{dz}{(z^2+x^2)\sqrt{(z^2+x^2+1)(z^2+2x^2+1)}}, \quad (91)$$

$$g = \int_0^\infty dz \left(1 - \frac{\sqrt{z^2+x^2+1}}{\sqrt{z^2+2x^2+1}} \right). \quad (92)$$

This gives $g(\theta)$ for θ between $\theta = 0$ ($x \rightarrow \infty$) and $\theta = \pi$ ($x \rightarrow 0$). We can extend it to $\theta \in [0, 2\pi]$ by using $g(2\pi - \theta) = g(\theta)$.

We consider angular sectors on the plane (x^0, x^1) , of the same size L with common vertex at $x = 0$, and common side along the x^1 axes. Then, the first two terms in (90) are not relevant

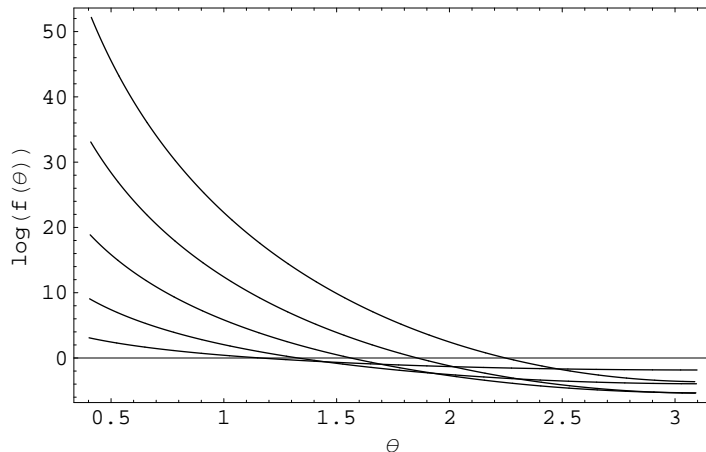


Figure 7: The logarithm of the functions $f_i(\theta)$, with $i = 1$ to $i = 5$ (the curves with higher slope correspond to larger i). These functions are all positive (the logarithms are real). They are symmetric around $\theta = \pi$, $f_i(\theta) = f_i(2\pi - \theta)$.

for the inequalities. These write

$$\int_0^\pi d\theta_1 \int_0^\pi d\theta_2 u(\theta_1)g(\theta_1 + \theta_2)u(\theta_2)^* \geq 0, \quad (93)$$

for arbitrary $u(\theta)$ with support in $\theta \in [0, \pi]$, and $\int d\theta u(\theta) = 0$.

If we could write $g(\theta)$ as a Laplace transform (perhaps for a distributional or discrete function $\hat{g}(p)$),

$$g(\theta) = \int dp e^{-p\theta} \hat{g}(p), \quad (94)$$

we could simplify the inequalities as $\hat{g}(p) \geq 0$ using the same trick as in the previous section. Unfortunately at present we cannot understand whether an expression like (94) is valid. We also wonder whether the positivity of $g(\theta)$ can be understood as the one of a correlator for a theory in a circle. Thus, we can not offer a complete analytical check of these inequalities here.

However, we have checked numerically the functions $f_n(\theta)$ of the derivatives of $g(\theta)$ which are predicted to be positive by the inequalities (59). Up to what we have checked, these functions are indeed positive (see figure 7). We have also checked the positivity of the determinants using up to 6 random angles entropy matrices.

6.3 Coplanar circles

We consider two coplanar circles of radius R_1 and R_2 , at a distance $h = h_1 + h_2$, where h_1 and h_2 are the distances of the circles to the plane $x^0 = 0$. This case contains objects determined by two parameters, (h, R) . The area for two parallel coaxial circles (rather than coplanar) was calculated in [46] and [47]. This case can be transformed conformally to the one of the two coplanar circles [48]. Taking out a divergent piece proportional to the circles perimeter, the relevant part of the area is conformally invariant. It is then given as a function of the cross ratio $z = \frac{|x_1 - x_4||x_2 - x_3|}{|x_1 - x_2||x_3 - x_4|}$ of four points. We choose x_1 and x_4 to be the points in the two circles which are at a greatest distance to each other, and x_2 and x_3 the ones at the minimal distance.

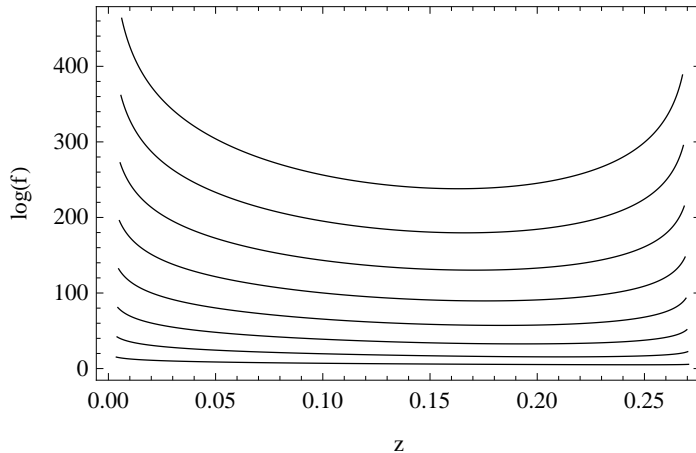


Figure 8: The logarithm of the functions $f_i(z)$ for $i = 1 \dots 8$ (from bottom to top) corresponding to coplanar circles of equal radius (here $R_1 = R_2 = 1$). They are all positive in the range $z \in (0, z_0)$, corresponding to connected minimal surfaces.

Then the cross ratio is $z = h \frac{(2R_1 + 2R_2 + h)}{4R_1 R_2} \in (0, \infty)$. We have a parametric expression for the area [47]

$$\mathcal{A} = -4\pi \frac{\alpha}{\sqrt{\alpha - 1}} \int_0^{\pi/2} \frac{dt}{(1 + \alpha \sin^2(t) + \sqrt{1 + \alpha \sin^2(t)})} = -4\pi \frac{E(-\alpha) - K(-\alpha)}{\sqrt{\alpha - 1}}, \quad (95)$$

$$\alpha = \frac{1 + 2x^2 + \sqrt{1 + 4x^2}}{2x^2}, \quad (96)$$

where E and K are the complete elliptic integrals of the first kind. The parameter x is related to the geometry of the configuration by the following equation involving the cross ratio z

$$\begin{aligned} \frac{1}{2} \log \left(1 + 2z + 2\sqrt{z^2 + z} \right) &= x \int_0^{\arccos\left(\frac{\sqrt{4x^2 + 1} - 1}{2x}\right)} d\phi \frac{\sin^2(\phi)}{\sqrt{\cos^2(\phi) - x^2 \sin^4(\phi)}} \\ &= \frac{\sqrt{2}x \left(K \left(-\frac{1 + 2x^2 + \sqrt{1 + 4x^2}}{2x^2} \right) - \Pi \left(-\frac{1 + \sqrt{1 + 4x^2}}{2x^2} \middle| -\frac{1 + 2x^2 + \sqrt{1 + 4x^2}}{2x^2} \right) \right)}{\sqrt{\sqrt{1 + 4x^2} - 1}}, \end{aligned} \quad (97)$$

where $\Pi(x|y)$ is the complete elliptic integral of the third kind.

We are interested in the solution which describes a surface connecting the two loops. For each $z \leq z_0 = 0.27288553\dots$ there are two solutions of equation (97) for x , but only one gives the minimal area. This corresponds to the solution with the greater value of x (having $x \geq 0.58110028\dots$). The other solution corresponds to loops having the same orientation, which is not the case appearing in the reflection positivity inequalities (and it does not satisfy these inequalities).

Considering the case of circles of equal radius $R_1 = R_2$ we have that the entropy is a one dimensional function and we evaluate the corresponding functions f_n . These are shown in figure (8). This one parameter case seems to lead to an infinite divisible function.

Then, we consider the full two parameter case, and evaluate the eigenvalues of the matrices in the derivatives discussed in Section 4 with respect to both parameters, radius R and distance h to $x^0 = 0$ plane. The first non trivial matrix (64) has 5 eigenvalues which we evaluate at

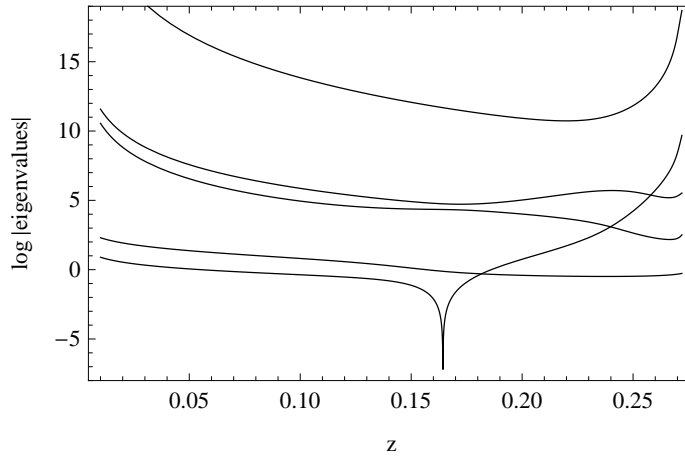


Figure 9: The logarithm of the absolute value of the eigenvalues of the matrix of infinitesimal inequalities of order 5 as a function of z (there are global factors depending on R which we fix by setting $R_1 = R_2 = 1$). One of the eigenvalues becomes negative for $z \gtrsim 0.165$.

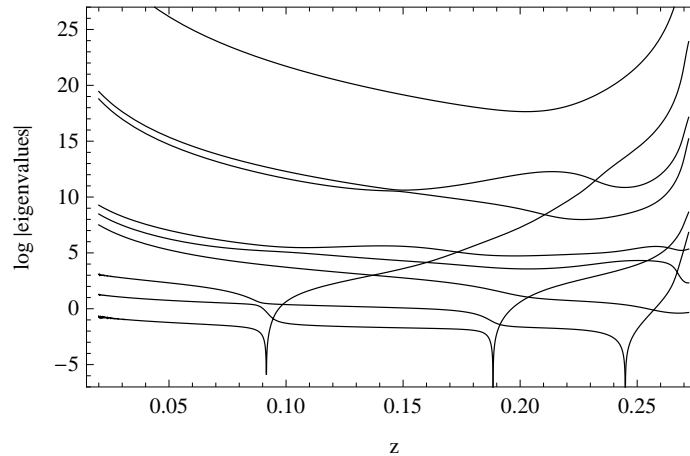


Figure 10: The logarithm of the absolute value of the eigenvalues of the matrix of infinitesimal inequalities of order 9 as a function of z (there are global factors depending on R which we fix by setting $R_1 = R_2 = 1$). There are negative eigenvalues for $z \gtrsim 0.09$.

the point of equal radius $R_1 = R_2$. We plot these eigenvalues as a function of z in figure (9). We see this has a negative eigenvalue for $z \gtrsim 0.165$. Figure (10) shows the eigenvalues of the matrices at the next order which consists of 9×9 matrices. Negative eigenvalues appear now for $z \gtrsim 0.09$. According to the general theory, since the functions are definitely not conditionally positive in this case, the matrices of higher rank should be able to detect negative eigenvalues for z as small as we want.

The same analysis can be carried out for the case of two parallel concentric circular Wilson loops. The function of the cross ratio is the same, but this is given as $z = \frac{(h_1+h_2)^2+(R_1-R_2)^2}{4R_1R_2} \in (0, \infty)$ in terms of the radius R_1 and R_2 of the circles, and their distances h_1 and h_2 to the $x^0 = 0$ plane (which in this case is assumed parallel to the circles). The results are completely analogous to the ones above. In fact it can be seen that a conformal transformation commuting with the time reflection operation maps the positivity relations in one case and the other. Hence, we have infinite divisibility for the one parameter case of circles with equal radius, but not for the full two parameter function.

6.4 Two spheres in four dimensions

The case of two spheres can be treated similarly. Again the relevant part of the minimal area is conformally invariant and depends on a cross ratio $z = h \frac{(2R_1+2R_2+h)}{4R_1R_2} \in (0, \infty)$. The formula for the area can be obtained from the result for two concentric shells [21] by a conformal transformation. We have for the mutual information of the two spheres A and B ,

$$\begin{aligned} \log(1 + 2z + 2\sqrt{z + z^2}) &= \frac{\cos^2(t_0)}{\sin^3(t_0)} \int_0^{\sin^2(t_0)} dy \frac{y}{\sqrt{(1-y)((1-y)^2 - y^3 \cos^4(t_0)/\sin^6(t_0))}}, \\ I(A, B) &= \lim_{\epsilon \rightarrow 0} \left[\frac{R_1^2 + R_2^2}{2\epsilon^2} - \frac{1}{2} \log \left(\frac{R_1 R_2}{\epsilon^2} \right) - \frac{1}{2} - \log(2) \right. \\ &\quad \left. - \int_{\frac{\epsilon}{R_1}}^{t_0} dt \frac{\cos^4(t)}{\sin^3(t) \sqrt{\cos^4(t) - \cos^4(t_0) \frac{\sin^6(t)}{\sin^6(t_0)}}} - \int_{\frac{\epsilon}{R_2}}^{t_0} dt \frac{\cos^4(t)}{\sin^3(t) \sqrt{\cos^4(t) - \cos^4(t_0) \frac{\sin^6(t)}{\sin^6(t_0)}}} \right]. \end{aligned} \quad (98)$$

The connected minimal surface exists for $z \lesssim 0.0965$.

We obtain results very similar to the ones of the previous subsection. The one parameter function for equal radius $R_1 = R_2$ gives place to (apparently) a conditional positive function. When the spheres are also allowed to vary in size the conditional positivity is lost, and we find negative eigenvalues for the matrices of multidimensional infinitesimal inequalities. We do not present the relevant figures here since they are in all respect similar to the ones of the last subsection for circular regions.

6.5 Multicomponent strips and the topology of the surfaces

Now we consider regions formed by an arbitrary set of disjoint parallel strips. The analysis of infinite divisibility will display a curious phenomena, related to the Gross-Ooguri phase transition [49].

We take all long edges on the regions parallel to the x^1 direction, with size L . Let us call $a_1 \dots a_n$ to the lines on the boundary with one orientation and $b_1 \dots b_n$ to the lines with the opposite orientation, where n is the number of connected components. In the large L limit the relevant piece of the minimal surface will be some choice of all the possible surfaces formed by

the union of the minimal surfaces connecting just two lines, one of type a and one of type b . This can then be written as $(a_1, b_{\sigma(1)}, a_2, b_{\sigma(2)}, \dots, a_n, b_{\sigma(n)})$, with $\sigma(i)$, the permutation of $1 \dots n$, producing the minimal area. This minimal area is just the sum $\mathcal{A}(a_1, b_{\sigma(1)}) + \dots + \mathcal{A}(a_n, b_{\sigma(n)})$.

Let us then think in a correlator involving m of these multicomponent Wilson loops V_i , $i = 1 \dots m$. We are interested in the matrix $\mathcal{A}(V_i \bar{V}_j)$. Let V_i have $2n_i$ boundary lines. Then the minimal area of $V_i \bar{V}_j$ contains a group of $2q_{ij} \leq 2n_i$ surfaces which cross the $x^0 = 0$ plane and are connected to a group of $2q_{ij}$ lines in V_i and $2q_{ij}$ lines in \bar{V}_j . There is also a group of $2n_i - 2q_{ij}$ lines in V_i which are connected among themselves, decoupled from \bar{V}_j . Then, suppose that in any of the correlators involving V_i in the matrix, the same group of lines of V_i are connected among themselves. The contribution of these lines is necessarily the same in all the correlators involving V_i in the correlator matrix in question. Let us call this contribution ξ_i . Now suppose the same happens for all the regions V_k , that is, all the minimal surfaces in the correlator matrix elements $V_k \bar{V}_j$ which contain V_k always has the same subset of the lines of V_k which are connected among themselves. Then we write

$$\mathcal{A}(V_i \bar{V}_j) = \hat{\mathcal{A}}(V_i \bar{V}_j) + \xi_i + \xi_j, \quad (99)$$

where $\hat{\mathcal{A}}(V_i \bar{V}_j)$ contains the contribution from the surfaces which cross the $x^0 = 0$ plane. Eq. (17) then tells that $-\mathcal{A}(V_i \bar{V}_j)$ is conditionally positive if and only if $-\hat{\mathcal{A}}(V_i \bar{V}_j)$ is conditionally positive. In this situation, the number of surfaces which cross x^0 must be the same in all correlator in the matrix, and hence $q_{ij} = q$ is independent of i, j . In this case the problem is simplified, we can forget about the disconnected pieces and think all lines are connected across $x^0 = 0$ by the minimal surface, and all loops have the same number of lines $2q$.

Once we have disposed off the disconnected pieces, we prove infinite divisibility under the following conditions: For all i, j we require there is a smooth transformation $V_{ij}(t)$, for $t \in [0, 1]$, such that $V_{ij}(0) = V_i$, $V_{ij}(1) = V_j$. This requires in particular that all the regions have the same topology. Further, we ask these transformations to form a group under concatenation of mappings, $V_{ij} \circ V_{jk} = V_{ik}$. And finally, but most importantly, we require that the sequence of minimal surfaces corresponding to $V_{ii'}(t) \bar{V}_{jj'}(t)$, going from the minimal surface of $V_i \bar{V}_j$ to the one of $V_{i'} \bar{V}_{j'}$, to be smooth. That is, the trajectory of minimal surfaces does not have to cross any phase transitions in the middle. In particular, the minimal surfaces on the correlator matrix must belong the same homotopy class of minimal surfaces.

It is immediate that this condition implies for the kind of loops we are considering here that the minimal surfaces connect only homologous lines. More clearly, let us name the lines in V_i as $a_1^{(i)}, \dots, a_q^{(i)}$ and $b_1^{(i)}, \dots, b_q^{(i)}$. The conditions above imply we can choose the names of the lines such that in the minimal surface of $V_i \bar{V}_j$ the line $a_k^{(i)}$ is connected with $\bar{a}_k^{(j)}$, and $b_k^{(i)}$ with $\bar{b}_k^{(j)}$, for all i, j, k . With this naming of lines, all the minimal surfaces of the correlator matrix connect the k^{th} lines of type a , and b among themselves. Then we can write

$$-\bar{\mathcal{A}}(V_i \bar{V}_j) = -\sum_{k=1}^q (\mathcal{A}(a_k^{(i)}, \bar{a}_k^{(j)}) + \mathcal{A}(b_k^{(i)}, \bar{b}_k^{(j)})). \quad (100)$$

This is a sum of conditionally positive matrices and then conditional positive.

The infinite divisibility cannot be proved in the case where the surfaces involved in the correlator matrix are separated by phase transitions, and some numerical examples show that generically it does not hold in these cases.

Then, we can think that for more general cases than the one studied here the condition of infinite divisibility, except for the disconnected components, requires the minimal surfaces in

the correlator matrix to lie in the same sector in the sense precised above. Notice that this does not exclude any topology for the minimal surfaces involved in the correlator. Generally, the conditions can always be achieved if the loops participating in the correlation matrix lie in some sufficiently small neighborhood of a given loop V . However, even in this cases without phase transitions we have already found some counterexamples of the inequalities.

7 Final comments

Motivated by the positivity relations satisfied by the integer n index Renyi entropies, and a natural conjecture about the path integral representation for the $n \rightarrow 1$ limit, we have checked whether the exponentials of the entanglement entropy give place to infinitely divisible correlation functions. We could not find counterexamples of the inequalities (7) among the few known exact QFT results. This is not so for the minimal areas of the holographic entanglement entropies where we found counterexamples for the case of two spheres or circles of different radius. In this case, the two parameter infinitesimal inequalities fail. The relation between analyticity and positivity then implies that the inequalities have to fail in the whole domain of holomorphy of the two variable correlator. We have also found phase transitions are a source of violations of the inequalities for multicomponent regions.

A question which remains open is the status of the infinite divisible inequalities for the entropy in QFT. In the light of the results for minimal surfaces, the evidence in favor of the inequalities in QFT is quite weak. On one hand, most of the discussed cases refer to entanglement entropies for free theories, which have there own special properties and relations to the Renyi entropies. On the other, for the simple geometries analyzed, the minimal areas also obey the inequalities. The entanglement entropy for two spheres is not known exactly for a QFT yet. Of course, a positive answer for the infinite divisible property of the entanglement entropy in QFT would mean the holographic entropy ansatz has to be modified for some geometries. Perhaps the full solution of the entanglement entropy for a simple geometry such as the case of two intervals in general CFT would be enough to settle this question.

Another interesting question is the direct mechanism (either in the context of QFT or a purely geometric one) behind the validity of the inequalities in the one-parameter functions discussed in this paper. Indeed, we have found evidence indicating that several non trivial functions are infinite divisible. This includes exponentials of the entropies of one interval and one angular sector for free scalar and fermions, and the minimal areas corresponding to angular sectors, and pairs of circles and spheres of equal radius. One possible explanation could reside in the interplay between the positivity recurrence relations and some unknown differential recurrence relations for these functions. On the minimal surfaces these examples of infinite divisibility may be hinting at the validity for the small angle path integral representation of the entropy in these geometries. It would be particularly interesting to understand and generalize this enlargement of cases beyond the single sphere and the plane (where there exists a rotational conformal symmetry). It would also be very relevant to find a reason for infinite divisibility in these geometries from the interpretation of the minimal areas in terms of Wilson loops.

It has been discussed in the literature that the small angle path integral representation is a weakness in Fursaev's proof of Ryu-Takayanagi proposal for holographic entanglement entropy [10, 16]. We have further found it seems to involve contradiction for certain geometries. On the QFT side the assumptions on which the proof is based lead to infinite divisibility, but the holographic result is in general not infinite divisible. However, for some geometries this contradiction is absent. One interesting geometry which deserves future investigation is the

cylinder. If the small angle hypothesis is valid for the cylinder it would provide the missing step for Solodukhin's formula for the logarithmically divergent term in this case.

The infinite series of inequalities corresponding to the infinite divisibility of certain operators in the semiclassical and large N limits restore the full strength of Hilbert space positivity in these theories. In this sense, it would be interesting to understand the translation of the correlators of these type of operators in AdS-CFT context.

Non zero connected correlators for Wilson loops in the large N limit can exist due to the presence of degenerate vacua for the classical action in loop space. These correlators must then be purely classical in origin, arising from the superposition of factorisable expectation values in each vacuum. These type of correlations must entail no quantum entanglement, and must not violate Bell inequalities. Other peculiar properties of entanglement in these theories have been recently discussed [50].

Acknowledgments

This work was supported by CONICET, Universidad Nacional de Cuyo, and CNEA, Argentina. H.C was benefitted by discussions with Rob Myers, Misha Smolkin, and the participants of the Workshop "Quantum information in quantum gravity and condensed matter physics" held at the Aspen Center for Physics.

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