Implicit definition of the quaternary discriminator

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Abstract. Let $\mathbf{A}$ be an algebra. A function $f : A^n \to A$ is *implicitly definable* by a system of term equations $\bigwedge t_i(x_1, ..., x_n, z) = s_i(x_1, ..., x_n, z)$ if f is the only n-ary operation on A making the identities $t_i(\vec{x}, f(\vec{x})) \approx s_i(\vec{x}, f(\vec{x}))$ hold in $\mathbf{A}$. Let $\mathcal{K}$ be a class of non trivial algebras. We prove that the quaternary discriminator is implicitly definable on every member of $\mathcal{K}$ (via the same system) iff $\mathcal{K}$ is contained in the class of relatively simple members of some relatively semisimple quasivariety with equationally definable relative principal congruences. As an application we obtain a characterization of the relatively permutable members of such type of quasivarieties. Furthermore, we prove that every algebra in such a quasivariety has a unique relatively permutable extension.

1. Introduction

Let $\mathbf{A}$ be an algebra and let $t_i(x_1, ..., x_n, z), s_i(x_1, ..., x_n, z)$, $i = 1, ..., k$, be terms such that the system of equations

$$\begin{array}{rcl} t_1(x_1, ..., x_n, z) & = & s_1(x_1, ..., x_n, z) \\ & \vdots & \\ t_k(x_1, ..., x_n, z) & = & s_k(x_1, ..., x_n, z) \end{array}$$

has a unique solution $z \in A$ whenever we fix the values $x_1, ..., x_n \in A$. One such system on an algebra $\mathbf{A}$ *implicitly defines* a function $f : A^n \to A$ by letting $f(a_1, ..., a_n)$ be the unique $b \in A$ such that

$$t_i(a_1, ..., a_n, b) = s_i(a_1, ..., a_n, b), \text{ for } i = 1, ..., k.$$

For example, if $\mathbf{G} = (G, ., e)$ is a group, then the system

$$x_1.z = e$$

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implicitly defines the inverse operation on G and if $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a Boolean lattice, then the system

$$x_1 \vee z = 1$$
$$x_1 \wedge z = 0$$

defines the complement operation on L.

Given a set A, we use d^A to denote the *quaternary discriminator* on A, that is the function:

$$d^A(x, y, z, w) = \begin{cases} z & \text{if} \quad x = y \\ w & \text{if} \quad x \neq y \end{cases}$$

The quaternary discriminator, also known as the switching function, is equivalent to the usual ternary discriminator in the sense that each one is a term of the other [2].

In this paper we prove that the quaternary discriminator is implicitly definable on every member of a class $\mathcal{K}$ of non trivial algebras (via the same system) iff $\mathcal{K}$ is contained in the class of relatively simple members of some relatively semisimple quasivariety with equationally definable relative principal congruences. As an application we obtain a characterization of the relatively permutable members of such type of quasivarieties. Furthermore, we prove that every algebra in such a quasivariety has a unique relatively permutable extension.

2. Notation and some results

Let $\mathcal{K}$ be a class of algebras. As usual, let $I(\mathcal{K})$, $S(\mathcal{K})$, $P(\mathcal{K})$, $P_S(\mathcal{K})$, $P_u(\mathcal{K})$ and $H(\mathcal{K})$ denote the classes of isomorphic images, subalgebras, direct products, subdirect products, ultraproducts and homomorphic images of elements of $\mathcal{K}$. We write $S_{\leq 4}(\mathcal{K})$ to denote the class of subalgebras of elements of $\mathcal{K}$ which are generated by a set of four or less elements. Let $V(\mathcal{K})$ denote the variety generated by $\mathcal{K}$ and $Q(\mathcal{K})$ the quasivariety generated by $\mathcal{K}$. We write $\mathcal{K}^+$ to denote the class $\mathcal{K} \cup \{\text{trivial algebras}\}$ and $\mathcal{K}^-$ to denote the class $\mathcal{K} - \{\text{trivial algebras}\}$. If $\mathcal{V}$ is a variety we use $\mathcal{V}_{FSI}$ (resp. $\mathcal{V}_{SI}$) to denote the class of finitely subdirectly irreducible (resp. subdirectly irreducible) members of $\mathcal{V}$.

Let $\mathbf{A}$ be an algebra. Let $\nabla^{\mathbf{A}}$ be the universal congruence on $\mathbf{A}$ and $\Delta^{\mathbf{A}}$ the trivial congruence on $\mathbf{A}$. By $Con(\mathbf{A})$ we denote the congruence lattice of $\mathbf{A}$. If $\theta \in Con(\mathbf{A})$ and $S \subseteq A$, we use $\theta\mid_S$ to denote the relation $\theta \cap (S \times S)$. If Σ is contained in $Con(\mathbf{A})$, let Σ^+ denote $\Sigma \cup \{\nabla^{\mathbf{A}}\}$. We define $\Sigma(\mathbf{A}, \mathcal{K}) = \{\theta \in Con(\mathbf{A}) : \mathbf{A}/\theta \in I(\mathcal{K})\}$.

Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$. A *relative congruence* on $\mathbf{A}$ is a $\theta \in Con(\mathbf{A})$ satisfying $\mathbf{A}/\theta \in \mathcal{Q}$. It is well known that the relative congruences on $\mathbf{A}$, with the partial order of inclusion, form an algebraic lattice in which arbitrary meets coincide with intersection. We use $Con_{\mathcal{Q}}(\mathbf{A})$ to denote the lattice of relative congruences of $\mathbf{A}$. The join operation on $Con_{\mathcal{Q}}(\mathbf{A})$ is denoted by the symbol $\sqcup$. If $a, b \in A$, let $\theta_{\mathcal{Q}}^{\mathbf{A}}(a, b)$ denote the *relative principal congruence* generated by (a, b), that is, the least relative congruence containing the pair (a, b). If $\vec{a}, \vec{b} \in A^n$, we write $\theta_{\mathcal{Q}}^{\mathbf{A}}(\vec{a}, \vec{b})$ to denote the congruence $\theta_{\mathcal{Q}}^{\mathbf{A}}(a_1, b_1) \sqcup ... \sqcup \theta_{\mathcal{Q}}^{\mathbf{A}}(a_n, b_n)$. An important property of $Con_{\mathcal{Q}}(\mathbf{A})$ is that its compact elements are precisely the ones of the form $\theta_{\mathcal{Q}}^{\mathbf{A}}(\vec{a}, \vec{b})$, with $\vec{a}, \vec{b} \in A^n$, $n \geq 1$. We use $MaxCon_{\mathcal{Q}}(\mathbf{A})$ to denote the set of maximal elements of the poset $Con_{\mathcal{Q}}(\mathbf{A}) - \{\nabla^{\mathbf{A}}\}$ and $CMICon_{\mathcal{Q}}(\mathbf{A})$ (resp. $MICon_{\mathcal{Q}}(\mathbf{A})$, $MPCon_{\mathcal{Q}}(\mathbf{A})$) denotes the set of completely meet irreducible (resp. meet irreducible, meet prime) elements of $Con_{\mathcal{Q}}(\mathbf{A})$.

If $p_i = p_i(x_1, ..., x_n)$, $i = 1, ..., m$ are terms, then $\vec{p}$ denotes the m-tuple $(p_1, ..., p_m)$. If $\vec{a} \in A^n$, then $\vec{p}^{\mathbf{A}}(\vec{a})$ denotes $(p_1^{\mathbf{A}}(\vec{a}), ..., p_m^{\mathbf{A}}(\vec{a}))$.

$\mathcal{Q}$ has *equationally definable relative principal congruences* (*EDRPC* for short) if there exist terms r_i, s_i, $i = 1, ..., n$, such that $\theta_{\mathcal{Q}}^{\mathbf{A}}(a, b) = \{(c, d) : \vec{r}^{\mathbf{A}}(a, b, c, d) = \vec{s}^{\mathbf{A}}(a, b, c, d)\}$, for any $\mathbf{A} \in \mathcal{Q}$.

$\mathcal{Q}$ has *equationally definable principal meets* (*EDPM* for short) if there exist terms r_i, s_i, $i = 1, ..., n$, such that
$\theta_{\mathcal{Q}}^{\mathbf{A}}(a, b) \cap \theta_{\mathcal{Q}}^{\mathbf{A}}(c, d) = \theta_{\mathcal{Q}}^{\mathbf{A}}(\vec{r}^{\mathbf{A}}(a, b, c, d), \vec{s}^{\mathbf{A}}(a, b, c, d))$, for any $\mathbf{A} \in \mathcal{Q}$.

$\mathcal{Q}$ is *relatively congruence distributive* if $Con_{\mathcal{Q}}(\mathbf{A})$ is distributive, for any $\mathbf{A} \in \mathcal{Q}$. We use $\mathcal{Q}_{RFSI}$ (resp. $\mathcal{Q}_{RSI}$, $\mathcal{Q}_{RS}$) to denote the class of all relatively finitely subdirectly irreducible (resp. subdirectly irreducible, simple) members of $\mathcal{Q}$. We note that trivial algebras do not belong to $\mathcal{Q}_{FRSI}$. $\mathcal{Q}$ is called *relatively semisimple* if every member of $\mathcal{Q}$ is isomorphic to a subdirect product of relatively simple members of $\mathcal{Q}$.

The following lemma shows that implicitly definable functions have natural algebraic properties.

Lemma 1. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}_i$, $i \in I$ *be algebras. Let* S *be the system* $\bigwedge_{i=1}^{m} p_i(\vec{x}, z) = q_i(\vec{x}, z)$ *and suppose* S *implicitly defines functions* $f : A^n \to A$ *on* $\mathbf{A}$, $g : B^n \to B$ *on* $\mathbf{B}$ *and* $f_i : A_i^n \to A_i$, *on each* $\mathbf{A}_i$.

(1) *If* F *is a homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$, *then* F *is a homomorphism from* $(\mathbf{A}, f)$ *to* $(\mathbf{B}, g)$.
(2) *If* $\mathbf{A}$ *is a subalgebra of* $\mathbf{B}$, *then* $f = g$ *restricted to* A^n.
(3) S *implicitly defines the function* $(f_i)_{i \in I}$ *on* $\Pi\{\mathbf{A}_i : i \in I\}$.

Proof. Routine. □

We conclude the section with a result characterizing the quasivarieties in which no non trivial algebra has a trivial subalgebra. First a basic lemma.

Lemma 2. *Let* $\mathcal{Q}$ *be a quasivariety and let* X *be a set of variables. Let* $\mathbf{F}$ *be the* $\mathcal{Q}$*-free algebra generated by* X. *Let* $r, s, r_1, ..., r_m, s_1, ..., s_m \in \mathbf{F}$, *the following are equivalent*

(1) $(r, s) \in \theta_{\mathcal{Q}}^{\mathbf{F}}(\vec{r}, \vec{s})$.
(2) $\mathcal{Q} \vDash \vec{r} = \vec{s} \to r = s$.

Proof. Routine. □

Proposition 3. *Let $\mathcal{Q}$ be a quasivariety. The following are equivalent*

(1) *No non trivial algebra in $\mathcal{Q}$ has a trivial subalgebra.*
(2) *No non trivial algebra in $\mathcal{Q}_{RSI}$ has a trivial subalgebra.*
(3) *There exist unary terms $0_1(w), ..., 0_n(w), 1_1(w), ..., 1_n(w)$ such that* $\mathcal{Q} \models \vec{0}(w) = \vec{1}(w) \to x = y$.
(4) *There exist unary terms $0_1(w), ..., 0_n(w), 1_1(w), ..., 1_n(w)$ such that* $\nabla^{\mathbf{A}} = \theta^{\mathbf{A}}_{\mathcal{Q}}(\vec{0}(a), \vec{1}(a))$, *for every $\mathbf{A} \in \mathcal{Q}$ and $a \in A$.*
(5) *There exist unary terms $0_1(w), ..., 0_n(w), 1_1(w), ..., 1_n(w)$ such that* $\nabla^{\mathbf{F}} = \theta^{\mathbf{F}}_{\mathcal{Q}}(\vec{0}, \vec{1})$, *where $\mathbf{F}$ is the $\mathcal{Q}$-free algebra freely generated by $\{w, x\}$.*
(6) $\nabla^{\mathbf{A}}$ *is a compact element of $Con_{\mathcal{Q}}(\mathbf{A})$, for every $\mathbf{A} \in \mathcal{Q}$.*
(7) $\nabla^{\mathbf{F}}$ *is a compact element of $Con_{\mathcal{Q}}(\mathbf{F})$, where $\mathbf{F}$ is a $\mathcal{Q}$-free algebra generated by some infinite set X of free generators.*
If the language of $\mathcal{Q}$ has a constant symbol, then the terms of 3., 4. and 5. can be chosen to be closed terms.

Proof. (1)⇒(2) Obvious.

(2)⇒(3) Since
$\mathcal{Q}_{RSI} \models \left(\bigwedge_{t,s \text{ unary terms}} t(w) = s(w)\right) \to x = y$
we have that
$\mathcal{Q} \models \left(\bigwedge_{t,s \text{ unary terms}} t(w) = s(w)\right) \to x = y.$
Thus a compactness argument produces the terms $0_1, ..., 0_n, 1_1, ..., 1_n$.

(3)⇒(4) Easy.

(4)⇒(5) Trivial.

(5)⇒(6) By Lemma 2 we obtain that
$\mathcal{Q} \models \vec{0}(w) = \vec{1}(w) \to w = x$
So (3) holds and hence we obtain (4) which clearly implies (6).

(6)⇒(7). Obvious.

(7)⇒(1) Suppose $\nabla^{\mathbf{F}} = \theta^{\mathbf{F}}_{\mathcal{Q}}(\vec{r}, \vec{s})$, with $r_i = r_i(x_1, ..., x_n)$ and $s_i = s_i(x_1, ..., x_n)$, $i = 1, ..., m$. Take $x, y \in X - \{x_1, ..., x_n\}$. By Lemma 2 we have that $\mathcal{Q} \models \vec{r} = \vec{s} \to x = y$, which implies (1). □

For the case in which $\mathcal{Q}$ is a variety, the equivalence (1)⇔(6) was proved in [6], the other implications have been proved in [11, Lemma 5.2] and [9].

3. Two topologies on $Con(\mathbf{A})$

In this section we introduce two natural topologies on the lattice of congruences of an algebra. Our treatment is an improvement of the one used in [8] to give a lattice theoretic proof of Jónsson‘s Lemma.

For an algebra $\mathbf{A}$ and $x, y \in A$, define

$e(x,y) = \{\theta \in Con(\mathbf{A}) : (x,y) \in \theta\}$
$d(x,y) = \{\theta \in Con(\mathbf{A}) : (x,y) \notin \theta\}$.
We use τ_e to denote the topology on $Con(\mathbf{A})$ generated by $\{e(x,y) : x,y \in A\}$ and τ_{ed} to denote the topology on $Con(\mathbf{A})$ generated by
$\{e(x,y) : x,y \in A\} \cup \{d(x,y) : x,y \in A\}$.
In most of the sheaf representations of algebras as algebras of continuous sections, the spectrum topology is the restriction of one of these two topologies to the spectrum (see for example [4] and [13]).

We say that a set $S \subseteq A^2$ *minorizes* $\Sigma \subseteq Con(\mathbf{A})$ when for each $\theta \in \Sigma$ there exists $(x,y) \in S$ such that $(x,y) \in \theta$. An immediate consequence of Alexander's Lemma is the following

Lemma 4. *For $\Sigma \subseteq Con(\mathbf{A})$, the following are equivalent*
(1) *Σ is τ_e-compact.*
(2) *If S minorizes Σ, then there exists a finite subset of S which minorizes Σ.*

Corollary 5. *If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\Sigma \subseteq Con(\mathbf{A})$ is τ_e-compact, then $\{\theta\mid_B : \theta \in \Sigma\} \subseteq Con(\mathbf{B})$ is τ_e-compact.*

Lemma 6 ([11, Lemma 2.1])**.** *Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$. If $\theta \in MPCon_{\mathcal{Q}}(\mathbf{A})$, then for each τ_e-compact set $\Sigma \subseteq Con_{\mathcal{Q}}(\mathbf{A})$ we have that*
$\theta \supseteq \bigcap \Sigma$ implies $\theta \supseteq \delta$, for some $\delta \in \Sigma$.
That is to say the meet prime elements of $Con_{\mathcal{Q}}(\mathbf{A})$ are completely meet prime with respect to intersections of τ_e-compact sets.

Corollary 7. *If $Con_{\mathcal{Q}}(\mathbf{A})$ is distributive and $\Sigma \subseteq MaxCon_{\mathcal{Q}}(\mathbf{A})$ is τ_e-compact and $\bigcap \Sigma = \Delta^{\mathbf{A}}$, then*
$\Sigma = MaxCon_{\mathcal{Q}}(\mathbf{A}) = MICon_{\mathcal{Q}}(\mathbf{A}) = MPCon_{\mathcal{Q}}(\mathbf{A}) = CMICon_{\mathcal{Q}}(\mathbf{A})$.

Proof. Since $Con_{\mathcal{Q}}(\mathbf{A})$ is distributive, $MICon_{\mathcal{Q}}(\mathbf{A}) = MPCon_{\mathcal{Q}}(\mathbf{A})$. Let $\theta \in MPCon_{\mathcal{Q}}(\mathbf{A})$. Since $\bigcap \Sigma = \Delta^{\mathbf{A}} \subseteq \theta$, Lemma 6 says that there exists $\delta \in \Sigma$ such that $\delta \subseteq \theta$. But δ is in $MaxCon_{\mathcal{Q}}(\mathbf{A})$, hence $\delta = \theta$. Thus $MPCon_{\mathcal{Q}}(\mathbf{A}) \subseteq \Sigma$, which implies the other inclusions. □

For $\Sigma \subseteq Con(\mathbf{A})$ and $\mathcal{U}$ an ultrafilter on Σ, define
$\theta^{\Sigma}_{\mathcal{U}} = \{(x,y) : e(x,y) \cap \Sigma \in \mathcal{U}\}$.
It is easy to check that $\theta^{\Sigma}_{\mathcal{U}} \in Con(\mathbf{A})$. Also we note that
- $(x,y) \notin \theta^{\Sigma}_{\mathcal{U}}$ iff $d(x,y) \cap \Sigma \in \mathcal{U}$.
- If $\theta \in \Sigma$ and $\mathcal{U} = \{\Gamma \subseteq \Sigma : \theta \in \Gamma\}$, then $\theta^{\Sigma}_{\mathcal{U}} = \theta$.

Theorem 8. *$(Con(\mathbf{A}), \tau_{ed})$ is a Boolean space. For $\Sigma \subseteq Con(\mathbf{A})$, the closure of Σ in $(Con(A), \tau_{ed})$ is given by*
$\bar{\Sigma} = \{\theta^{\Sigma}_{\mathcal{U}} : \mathcal{U}$ is an ultrafilter on $\Sigma\}$.

Proof. It is easy to check that $(Con(\mathbf{A}), \tau_{ed})$ is Hausdorff and zero dimensional. Assume that $(Con(\mathbf{A}), \tau_{ed})$ is not compact. Then we have

$Con(\mathbf{A}) = \bigcup_{i\in I} e(x_i, y_i) \cup \bigcup_{j\in J} d(z_j, w_j)$
and
$Con(\mathbf{A}) \neq \bigcup_{i\in I_0} e(x_i, y_i) \cup \bigcup_{j\in J_0} d(z_j, w_j)$
whenever $I_0 \subseteq I$ and $J_0 \subseteq J$ are finite. Note that
$\bigcup_{i\in I_0} e(x_i, y_i) \not\supseteq \bigcap_{j\in J_0} e(z_j, w_j)$
whenever $I_0 \subseteq I$ and $J_0 \subseteq J$ are finite. By the prime ideal theorem there is an ultrafilter $\mathcal{U}$ on $Con(\mathbf{A})$ satisfying
$e(x_i, y_i) \notin \mathcal{U}$, for each $i \in I$
$e(z_j, w_j) \in \mathcal{U}$, for each $j \in J$.
Hence we have
$\theta_{\mathcal{U}}^{Con(\mathbf{A})} \notin \bigcup_{i\in I} e(x_i, y_i) \cup \bigcup_{j\in J} d(z_j, w_j)$,
which is an absurd.

To see the last affirmation of the theorem, we note that for $\Sigma \subseteq Con(A)$, the following are equivalent

(1) $\theta \in \bar{\Sigma}$.

(2) The family $\{e(x, y) \cap \Sigma : (x, y) \in \theta\} \cup \{d(x, y) \cap \Sigma : (x, y) \notin \theta\}$ has the finite intersection property.

But it is easy to prove that (2) is equivalent to

(3) $\theta = \theta_{\mathcal{U}}^{\Sigma}$ for some ultrafilter $\mathcal{U}$,

and this concludes the proof of the theorem. □

Corollary 9. *If $\mathcal{M}$ is a universal class, then $\Sigma(\mathbf{A}, \mathcal{M}) = \{\theta \in Con(A) : \mathbf{A}/\theta \in \mathcal{M}\}$ is τ_{ed}-compact, for any algebra A.*

Proof. Let $\mathcal{U}$ be an ultrafilter on $\Sigma(\mathbf{A}, \mathcal{M})$. Note that $\theta_{\mathcal{U}}^{\Sigma(\mathbf{A},\mathcal{M})}$ is the kernel of the canonical map $\mathbf{A} \to \prod\{\mathbf{A}/\theta : \theta \in \Sigma(\mathbf{A}, \mathcal{M})\}/\mathcal{U}$, which says that $\mathbf{A}/\theta_{\mathcal{U}}^{\Sigma(\mathbf{A},\mathcal{M})} \in ISP_u\{\mathbf{A}/\theta : \theta \in \Sigma(\mathbf{A}, \mathcal{M})\} \subseteq \mathcal{M}$ and hence $\theta_{\mathcal{U}}^{\Sigma(\mathbf{A},\mathcal{M})} \in \Sigma(\mathbf{A}, \mathcal{M})$. □

The combination of Corollary 9 and Lemma 6 is a useful tool which we shall use several times in the sequel.

Corollary 10 ([5, Lemma 1.5])**.** *Let $\mathcal{M}$ be a class of algebras. Then $Q(\mathcal{M})_{RFSI} \subseteq ISP_u(\mathcal{M})$.*

Proof. Suppose $\mathbf{A} \in Q(\mathcal{M})_{RFSI}$. Since $Q(\mathcal{M}) = IP_S(ISP_u(\mathcal{M}))$ we have that $\bigcap \Sigma(A, ISP_u(\mathcal{M})) = \Delta^{\mathbf{A}} \in MPCon_{Q(\mathcal{M})}(A)$. Since $\Sigma(A, ISP_u(\mathcal{M}))$ is τ_e-compact (Corollary 9), Lemma 6 says that $\Delta^{\mathbf{A}} \in \Sigma(A, ISP_u(\mathcal{M}))$, hence $\mathbf{A} \in ISP_u(\mathcal{M})$. □

Corollary 11. *(Jónsson‘s Lemma): Let $\mathcal{K}$ be such that $V(\mathcal{K})$ is congruence distributive. Then $V(\mathcal{K})_{FSI} \subseteq HSP_u(\mathcal{K})$.*

Proof. Let $\mathbf{A} \in SP(\mathcal{K})$, and suppose $\mathbf{A}/\gamma$ is finitely subdirectly irreducible. We shall see that $\mathbf{A}/\gamma \in HSP_u(\mathcal{K})$. Since $\gamma \in MPCon(A)$ and $\bigcap \Sigma(A, ISP_u(\mathcal{K})) = \Delta^{\mathbf{A}}$ we can apply a similar argument as in the above corollary. □

Lemma 12. *If $\mathcal{Q}$ is a relative congruence distributive quasivariety and $\mathcal{M}$ is a universal class such that $\mathcal{M} \subseteq (\mathcal{Q}_{RFSI})^+$, then $Q(\mathcal{M}) = IP_S(\mathcal{M})$ is a relative congruence distributive quasivariety such that $Q(\mathcal{M})_{RFSI} = \mathcal{M}^-$.*

Proof. It follows from Corollary 1.4 of [5] and Corollary 10. □

We conclude the section with a result linking the concept of τ_{ed}-compactness with that of EDPM

Proposition 13. *Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$. If $Con_{\mathcal{Q}}(\mathbf{A})$ is distributive, then the following are equivalent*

(1) *$MICon_{\mathcal{Q}}(\mathbf{A})^+$ is τ_{ed}-compact.*
(2) *For any $x, y, z, w \in A$, we have that $\theta^{\mathbf{A}}_{\mathcal{Q}}(x, y) \cap \theta^{\mathbf{A}}_{\mathcal{Q}}(z, w)$ is a compact element of the lattice $Con_{\mathcal{Q}}(\mathbf{A})$.*

Proof. (1)⇒(2) Note that the sets
$e(x, y) \cap MICon_{\mathcal{Q}}(\mathbf{A})^+$
$e(z, w) \cap MICon_{\mathcal{Q}}(\mathbf{A})^+$
$d(u, v) \cap MICon_{\mathcal{Q}}(\mathbf{A})^+$, with $(u, v) \in \theta^{\mathbf{A}}_{\mathcal{Q}}(x, y) \cap \theta^{\mathbf{A}}_{\mathcal{Q}}(z, w)$
form a covering of $MICon_{\mathcal{Q}}(\mathbf{A})^+$. Thus the τ_{ed}-compactness of $MICon_{\mathcal{Q}}(\mathbf{A})^+$ says that there are $(u_1, v_1), ..., (u_k, v_k) \in \theta^{\mathbf{A}}_{\mathcal{Q}}(x, y) \cap \theta^{\mathbf{A}}_{\mathcal{Q}}(z, w)$ such that
$e(x, y) \cap MICon_{\mathcal{Q}}(\mathbf{A})^+$
$e(z, w) \cap MICon_{\mathcal{Q}}(\mathbf{A})^+$
$d(u_i, v_i) \cap MICon_{\mathcal{Q}}(\mathbf{A})^+$, $i = 1, ..., k$
is a covering of $MICon_{\mathcal{Q}}(\mathbf{A})^+$ and the reader can check that
$\theta^{\mathbf{A}}_{\mathcal{Q}}(x, y) \cap \theta^{\mathbf{A}}_{\mathcal{Q}}(z, w) = \theta^{\mathbf{A}}_{\mathcal{Q}}(u_1, v_1) \sqcup ... \sqcup \theta^{\mathbf{A}}_{\mathcal{Q}}(u_k, v_k)$.

(2)⇒(1) We prove the case in which $\nabla^{\mathbf{A}}$ is not a compact element of $Con_{\mathcal{Q}}(\mathbf{A})$. Let $\mathbf{L}$ be the sublattice of $Con_{\mathcal{Q}}(\mathbf{A})$ consisting of all compact elements of $Con_{\mathcal{Q}}(\mathbf{A})$ together with $\nabla^{\mathbf{A}}$. Let $Id(\mathbf{L})$ be the lattice of all ideals of $\mathbf{L}$. We note that the map $\theta \to \{\delta \in L - \{\nabla^{\mathbf{A}}\} : \delta \subseteq \theta\}$ is an isomorphism from $Con_{\mathcal{Q}}(\mathbf{A})$ to $Id(\mathbf{L})$ which identifies $MICon_{\mathcal{Q}}(\mathbf{A})^+$ with $\{I \in Id(\mathbf{L}) : I$ is prime$\}$. Since this isomorphism connects the topology τ_{ed} relativized to $MICon_{\mathcal{Q}}(\mathbf{A})^+$ with the Priestley topology on $\{I \in Id(\mathbf{L}) : I$ is prime$\}$, we have that $MICon_{\mathcal{Q}}(\mathbf{A})^+$ is τ_{ed}-compact.

The remaining case is similar. □

Corollary 14. *A relatively congruence distributive quasivariety $\mathcal{Q}$ has EDPM iff for every $\mathbf{A} \in \mathcal{Q}$, the set $MICon_{\mathcal{Q}}(\mathbf{A})^+$ is τ_{ed}-compact.*

Proof. (⇒) Since $(\mathcal{Q}_{RFSI})^+$ is universal, we can apply Corollary 9. (⇐) It follows from (ii)⇒(i) of [5, Thm 2.3] and the above proposition. □

4. Main theorem and applications

We observe that if $\mathcal{Q}$ is a relatively congruence distributive quasivariety, then for $\theta \in Con_{\mathcal{Q}}(A)$, with $\mathbf{A} \in \mathcal{Q}$, the congruence

$\theta^* = \bigcap\{\gamma \in MICon_{\mathcal{Q}}(A) : \theta \nsubseteq \gamma\}$
is the pseudo-complement of θ in the lattice $Con_{\mathcal{Q}}(A)$, i.e. θ^* is the greatest $\delta \in Con_{\mathcal{Q}}(A)$ satisfying
$\theta \cap \delta = \Delta^{\mathbf{A}}$
(use that $MICon_{\mathcal{Q}}(A) = MPCon_{\mathcal{Q}}(A)$).

Theorem 15. *Let $\mathcal{K}$ be any class of non trivial algebras. The following are equivalent*

(1) *There are terms p_i, q_i, $i = 1, ..., n$ such that the system*
$\vec{p}(x_1, x_2, x_3, x_4, z) = \vec{q}(x_1, x_2, x_3, x_4, z)$
implicitly defines the quaternary discriminator, for every $\mathbf{A} \in \mathcal{K}$.
(2) $SP_uS_{\leq 4}(\mathcal{K}) \subseteq (\mathcal{Q}_{RS})^+$ *for some relatively congruence distributive quasivariety $\mathcal{Q}$.*
(3) $Q(\mathcal{K})$ *has EDRPC and* $Q(\mathcal{K})_{RS} = ISP_u(\mathcal{K})^-$.
(4) $\mathcal{K} \subseteq \mathcal{Q}_{RS}$ *for some quasivariety $\mathcal{Q}$ with EDRPC.*
(5) *There are terms p_i, q_i, $i = 1, ..., n$, such that*
$\mathcal{K} \vDash \vec{p}(x, y, z, w) = \vec{q}(x, y, z, w) \leftrightarrow (x = y \to z = w)$.
(6) *For every trivial satisfiable open formula $O(x_1, ..., x_m)$ there are terms p_i, q_i, $i = 1, ..., n$, such that*
$\mathcal{K} \vDash \vec{p}(x_1, ..., x_m) = \vec{q}(x_1, ..., x_m) \leftrightarrow O(x_1, ..., x_m)$.

If $\mathcal{K}$ generates a locally finite quasivariety, 2. can be replaced by

$\tilde{2}$. $S_{\leq 4}(\mathcal{K}) \subseteq (\mathcal{Q}_{RS})^+$ *for some relative congruence distributive quasivariety $\mathcal{Q}$.*

If $Q(\mathcal{K})$ satisfies some of the equivalent conditions of Proposition 3, 6. can be replaced by

$\tilde{6}$. *For every open formula $O(x_1, ..., x_m)$ there are terms p_i, q_i, $i = 1, ..., n$, such that*
$\mathcal{K} \vDash \vec{p}(x_1, ..., x_m) = \vec{q}(x_1, ..., x_m) \leftrightarrow O(x_1, ..., x_m)$.

Proof. (1)⇒(2) Note that
$\mathcal{K} \vDash \vec{p}(x, y, z, w, z) = \vec{q}(x, y, z, w, z) \leftrightarrow (x = y \vee z = w)$
and hence
$ISP_u(\mathcal{K}) \vDash \vec{p}(x, y, z, w, z) = \vec{q}(x, y, z, w, z) \leftrightarrow (x = y \vee z = w)$.
Thus Lemma 10 says that
$Q(\mathcal{K})_{RFSI} \vDash \vec{p}(x, y, z, w, z) = \vec{q}(x, y, z, w, z) \leftrightarrow (x = y \vee z = w)$
which by Theorem 2.3 of [5] implies that $Q(\mathcal{K})$ is relatively congruence distributive. By Lemma 1 $ISP_u(\mathcal{K})^- \subseteq Q(\mathcal{K})_{RS}$.

(2)⇒(3) By Lemma 12 we have that $\mathcal{R} = Q(ISP_uS_{\leq 4}(\mathcal{K}))$ is relatively congruence distributive and $\mathcal{R}_{RFSI} = ISP_uS_{\leq 4}(\mathcal{K})^-$. Since $ISP_uS_{\leq 4}(\mathcal{K}) \subseteq (\mathcal{Q}_{RS})^+$ we have that $\mathcal{R}$ is relatively semisimple. Since $(\mathcal{R}_{RS})^+$ is universal, Corollary 9 says that $MaxCon_{\mathcal{R}}(\mathbf{B})^+$ is τ_{ed}-compact, for any $\mathbf{B} \in \mathcal{R}$.
First we shall prove that for $\mathbf{B} \in \mathcal{R}$ and $a, b \in B$,
(*) $\gamma \supseteq \theta^{\mathbf{B}}_{\mathcal{R}}(a, b)^*$ iff $\gamma \in d(a, b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})$, for any $\gamma \in MaxCon_{\mathcal{R}}(\mathbf{B})$.

The if-part is obvious. Suppose that $\gamma \supseteq \theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^*$. Since $\theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^* = \bigcap d(a,b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$ and $d(a,b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$ is τ_{ed}-compact, Lemma 6 says that there is $\theta \in d(a,b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$ such that $\gamma \supseteq \theta$. By the maximality of θ we have that $\gamma = \theta$ and so $\gamma \in d(a,b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})$.

Next we prove that

(**) $\theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^*$ is the complement of $\theta^{\mathbf{B}}_{\mathcal{R}}(a,b)$ in the distributive lattice $Con_{\mathcal{R}}(\mathbf{B})$.

Suppose that $\theta^{\mathbf{B}}_{\mathcal{R}}(a,b) \sqcup \theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^* \neq \nabla^{\mathbf{B}}$. Since $\mathcal{R}$ is relatively semisimple, there is $\gamma \in MaxCon_{\mathcal{R}}(\mathbf{B})$ satisfying $\theta^{\mathbf{B}}_{\mathcal{R}}(a,b) \sqcup \theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^* \subseteq \gamma$. So we have that $(a,b) \in \gamma$ and by (*) we obtain that $(a,b) \notin \gamma$, which is absurd.

We shall prove that

(***) $\theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^* \cap \theta^{\mathbf{B}}_{\mathcal{R}}(c,d)$ is a compact element of $Con_{\mathcal{R}}(\mathbf{B})$.

To prove this, first we shall prove that the sets

$d(a,b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$

$e(c,d) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$

$d(u,v) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$, with $(u,v) \in \theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^* \cap \theta^{\mathbf{B}}_{\mathcal{R}}(c,d)$

form a covering of $MaxCon_{\mathcal{R}}(\mathbf{B})^+$. Note that $\nabla^{\mathbf{B}} \in e(c,d) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$. Let $\gamma \in MaxCon_{\mathcal{R}}(\mathbf{B})$. Suppose that $\gamma \notin d(u,v) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$, for every $(u,v) \in \theta^{\mathbf{B}}_{\mathcal{R}}(a,b) \cap \theta^{\mathbf{B}}_{\mathcal{R}}(c,d)^*$. Thus $\gamma \supseteq \theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^* \cap \theta^{\mathbf{B}}_{\mathcal{R}}(c,d)$ and hence $\gamma \supseteq \theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^*$ or $\gamma \supseteq \theta^{\mathbf{B}}_{\mathcal{R}}(c,d)$, which by (*) says that either $\gamma \in d(a,b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$ or $\gamma \in e(c,d) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$. Thus the τ_{ed}-compactness of $MaxCon_{\mathcal{R}}(\mathbf{B})^+$ says that there are $(u_1,v_1),...,(u_k,v_k)$ such that

$d(a,b) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$

$e(c,d) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$

$d(u_i,v_i) \cap MaxCon_{\mathcal{R}}(\mathbf{B})^+$, $i = 1,...,k$

is a covering of $MaxCon_{\mathcal{R}}(\mathbf{B})^+$ and the reader can check that

$\theta^{\mathbf{B}}_{\mathcal{R}}(a,b)^* \cap \theta^{\mathbf{B}}_{\mathcal{R}}(c,d) = \theta^{\mathbf{B}}_{\mathcal{R}}(u_1,v_1) \sqcup ... \sqcup \theta^{\mathbf{B}}_{\mathcal{R}}(u_k,v_k)$.

Let $\mathbf{F}$ be the $\mathcal{R}$-free algebra freely generated by x,y,z,w. Let r_i, s_i, $i = 1,...,m$, be terms such that

$\theta^{\mathbf{F}}_{\mathcal{R}}(x,y)^* \cap \theta^{\mathbf{B}}_{\mathcal{R}}(z,w) = \theta^{\mathbf{F}}_{\mathcal{R}}(r_1,s_1) \sqcup ... \sqcup \theta^{\mathbf{F}}_{\mathcal{R}}(r_m,s_m)$.

By (*) we have that for every $\gamma \in MaxCon_{\mathcal{R}}(\mathbf{F})$, the following are equivalent

(i) $x/\gamma \neq y/\gamma$ or $z/\gamma = w/\gamma$.

(ii) $\vec{r}^{\mathbf{F}/\gamma}(x/\gamma, y/\gamma, z/\gamma, w/\gamma) = \vec{s}^{\mathbf{F}/\gamma}(x/\gamma, y/\gamma, z/\gamma, w/\gamma)$.

Thus for every $\mathbf{B} \in \mathcal{R}_{RS} = ISP_uS_4(\mathcal{K})^-$ we have that

$\mathbf{B} \vDash \vec{r}(x,y,z,w) = \vec{s}(x,y,z,w) \leftrightarrow (x \neq y \vee z = w)$

and hence

(****) $ISP_u(\mathcal{K}) \vDash \vec{r}(x,y,z,w) = \vec{s}(x,y,z,w) \leftrightarrow (x \neq y \vee z = w)$.

Since $S_{\leq 4}(\mathcal{K}) \subseteq \mathcal{R}_{RS}$ and $\mathcal{R}$ has EDPM Theorem 2.3 of [5] says that there are terms p_i, q_i, $i = 1,...,n$, such that

$S_{\leq 4}(\mathcal{K}) \vDash \vec{p}(x,y,z,w) = \vec{q}(x,y,z,w) \leftrightarrow (x = y \vee z = w)$.

Thus

$ISP_u(\mathcal{K}) \vDash \vec{p}(x,y,z,w) = \vec{q}(x,y,z,w) \leftrightarrow (x = y \vee z = w)$.

By Lemma 10

$Q(\mathcal{K})_{RFSI} \vDash \vec{p}(x,y,z,w,z) = \vec{q}(x,y,z,w,z) \leftrightarrow (x = y \vee z = w)$

which combining Corollary 2.4 of [5] with Theorem 2.3 of [5] says that $Q(\mathcal{K})$ is relatively congruence distributive and $Q(\mathcal{K})_{RFSI} = ISP_u(\mathcal{K})^-$. Finally we note that for $\mathbf{B} \in Q(\mathcal{K})$ and $a, b, c, d \in B$, we have

$\mathbf{B} \vDash \vec{r}(a,b,c,d) = \vec{s}(a,b,c,d)$

iff

$(r_i(a,b,c,d), s_i(a,b,c,d)) \in \gamma$, for every $\gamma \in MICon_{Q(\mathcal{K})}(\mathbf{B})$, $i = 1, ..., m$

iff (by (****))

$(a,b) \in \gamma$ implies $(c,d) \in \gamma$, for every $\gamma \in MICon_{Q(\mathcal{K})}(\mathbf{B})$

iff

$(c,d) \in \theta^{\mathbf{B}}_{Q(\mathcal{K})}(a,b)$

which says that $Q(\mathcal{K})$ has EDRPC via the terms r_i, s_i, $i = 1, ..., m$. Thus (****) implies that $ISP_u(\mathcal{K})^- \subseteq Q(\mathcal{K})_{RS}$ and hence

$Q(\mathcal{K})_{RS} = ISP_u(\mathcal{K})^- = Q(\mathcal{K})_{RFSI}$.

(3)⇒(4) Trivial.

(4)⇒(5) Suppose $\vec{p}(x,y,z,w) = \vec{q}(x,y,z,w)$ defines $(z,w) \in \theta_{\mathcal{Q}}(x,y)$ in $\mathcal{Q}$. It is clear that

$\mathcal{K} \vDash \vec{p}(x,y,z,w) = \vec{q}(x,y,z,w) \leftrightarrow (x = y \to z = w)$.

(5)⇒(6) By [7, Remark 2.4] we have that there are terms r_i, s_i, $i = 1, ..., m$, such that

$\mathcal{K} \vDash \vec{r}(x,y,z,w) = \vec{s}(x,y,z,w) \leftrightarrow (x = y \vee z = w)$.

Thus we have that (6) holds for every open formula $O(x_1, ..., x_m)$ of the form $\bigvee_{j=1}^{l} t_j = k_j$. Since every trivial satisfiable open formula is equivalent to a conjunction of formulas of the form

$u_1 = v_1 \to \Big(u_2 = v_2 \to \Big(... \to \Big(u_k = v_k \to \bigvee_{j=1}^{l} t_j = k_j\Big) ...\Big)\Big)$

with $k, l \geq 1$, an inductive argument proves (6).

(6)⇒(1) Let p_i, q_i, $i = 1, ..., n$ and r_i, s_i, $i = 1, ..., m$, be such that

$\mathcal{K} \vDash \vec{p}(x,y,z,w) = \vec{q}(x,y,z,w) \leftrightarrow (x = y \to z = w)$

$\mathcal{K} \vDash \vec{r}(x,y,z,w) = \vec{s}(x,y,z,w) \leftrightarrow (x = y \vee z = w)$.

Note that the system

$\vec{p}(x_1,x_2,z,x_3) = \vec{q}(x_1,x_2,z,x_3) \wedge \vec{r}(x_1,x_2,z,x_4) = \vec{s}(x_1,x_2,z,x_4)$

witnesses the fact that the discriminator is implicitly definable in every member of $\mathcal{K}$.

Finally we shall prove that (6)⇒($\tilde{6}$), when $Q(\mathcal{K})$ satisfies some of the equivalent conditions of Proposition 3. Let $\vec{0}(w)$ and $\vec{1}(w)$ be terms such that

$Q(\mathcal{K}) \vDash \vec{0}(w) = \vec{1}(w) \to x = y$.

Let $O(x_1, ..., x_m)$ be an open formula. First suppose $m \geq 1$. Note that

$\mathcal{K} \vDash \Big(\vec{0}(x_1) = \vec{1}(x_1) \vee O(x_1, ..., x_m)\Big) \leftrightarrow O(x_1, ..., x_m)$

(remember that $\mathcal{K}$ has no trivial algebra). Since $\vec{0}(x_1) = \vec{1}(x_1) \vee O(x_1, ..., x_m)$ is trivial satisfiable, we can apply (6) to obtain ($\tilde{6}$). For the case $m = 0$, the language need to have a constant and hence we can make a similar argument. □

Corollary 16. *Let $\mathcal{Q}$ be a quasivariety. The following are equivalent*

(1) *$\mathcal{Q}$ has EDPM and $\mathcal{Q}_{RFSI} = \mathcal{Q}_{RS}$.*
(2) *$\mathcal{Q}$ is relatively semisimple and has EDRPC.*
(3) *$\mathcal{Q}$ has EDPM and $\theta^{\mathbf{A}}_{\mathcal{Q}}(x, y)$ is a complemented element of $Con_{\mathcal{Q}}(\mathbf{A})$, for every $\mathbf{A} \in \mathcal{Q}$, and $x, y \in A$.*
(4) *$\mathcal{Q} = Q(\mathcal{K})$, for some class $\mathcal{K}$ satisfying some of the equivalent conditions of Theorem 15.*

We observe that if we consider an infinite field $\mathbf{F}$ as a model of the language of rings with identity, then the quaternary discriminator is not implicitly definable in $\mathbf{F}$. This is because if $(\tilde{6})$ of Theorem 15 holds for $\mathcal{K} = \{\mathbf{F}\}$, then there exist terms p_i, q_i, $i = 1, ..., n$, such that

$\mathbf{F} \vDash (\bigwedge_{i=1}^{n} p_i(x) = q_i(x)) \leftrightarrow x \neq 0$

which is impossible. In contrast, it is well known that in every finite field the discriminator is given by a term.

To give another application of Theorem 15, consider a relatively subdirectly irreducible algebra $\mathbf{A}$ of a quasivariety $\mathcal{Q}$ with EDRPC. Let $a, b \in A$ be such that $\theta^{\mathbf{A}}_{\mathcal{Q}}(a, b)$ is a monolith of $Con_{\mathcal{Q}}(\mathbf{A})$. Let $\mathcal{L}_e$ be the language of $\mathcal{Q}$ expanded by adding two new distinct constants c_a and c_b. Let $\mathbf{A}_e = (\mathbf{A}, a, b)$. We shall prove that the quaternary discriminator is implicitly definable in $\mathbf{A}_e$. Let $\mathcal{Q}_e = Q(ISP_u(\mathbf{A}_e))$. We observe that $\theta^{\mathbf{B}}_{\mathcal{Q}}(c^{\mathbf{B}}_a, c^{\mathbf{B}}_b)$ is a monolith of every non trivial algebra $\mathbf{B} \in ISP_u(\mathbf{A}_e)$. Also note that every homomorphism between members of $ISP_u(\mathbf{A}_e)$ must be injective. Thus

$(\mathcal{Q}_e)_{RSI} = (\mathcal{Q}_e)_{RS} = ISP_u(\mathbf{A}_e)$.

Let

$\mathcal{G} = \{(\mathbf{B}, a, b) : a, b \in B \text{ and } \mathbf{B} \in \mathcal{Q}\}$.

Note that $\mathcal{G}$ is a relatively congruence distributive quasivariety (since so is $\mathcal{Q}$) and that

$\mathcal{G}_{RSI} = \{(\mathbf{B}, a, b) : a, b \in B \text{ and } \mathbf{B} \in \mathcal{Q}_{RSI}\}$

Also we note that

$(\mathcal{Q}_e)_{RSI} \subseteq \mathcal{G}_{RSI}$

which by Lemma 12 says that $\mathcal{Q}_e$ is relatively congruence distributive. Thus (2)⇒(1) of Theorem 15 implies that the quaternary discriminator is implicitly definable on $\mathbf{A}_e$

Relatively permutable members

Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$. We say that $\mathbf{A}$ is *relatively permutable* if for any $\theta, \delta \in Con_{\mathcal{Q}}(\mathbf{A})$ we have that $\theta \circ \delta = \delta \circ \theta = \theta \sqcup \delta$.

Theorem 17. *Let $\mathcal{Q}$ be a relatively semisimple quasivariety with EDRPC. Let r_i, s_i, $i = 1, ..., n$, be quaternary terms such that $\vec{r}(x, y, z, w) = \vec{s}(x, y, z, w)$ defines $(z, w) \in \theta_{\mathcal{Q}}(x, y)$ in $\mathcal{Q}$ and let p_i, q_i, $i = 1, ..., m$, be quaternary terms such that $\theta_{\mathcal{Q}}(x, y) \cap \theta_{\mathcal{Q}}(z, w) = \theta_{\mathcal{Q}}(\vec{p}(x, y, z, w), \vec{q}(x, y, z, w))$, in every member of $\mathcal{Q}$. Then for every $\mathbf{A} \in \mathcal{Q}$, the following are equivalent*

(1) *$\mathbf{A}$ is relatively permutable.*

(2) $\mathbf{A} \models \forall xyzw \exists! u \quad \vec{r}(x,y,u,z) = \vec{s}(x,y,u,z) \wedge \vec{p}(x,y,u,w) = \vec{q}(x,y,u,w)$.
(3) $\mathbf{A} \models \forall xyzw \exists u \quad \vec{r}(x,y,u,z) = \vec{s}(x,y,u,z) \wedge \vec{p}(x,y,u,w) = \vec{q}(x,y,u,w)$.
(4) $\mathbf{A}$ *is isomorphic to a Boolean product whose factors belongs to* $(\mathcal{Q}_{RS})^+$.
(5) $\mathbf{A}$ *is isomorphic to the algebra of global sections of a sheaf whose stalks belongs to* $(\mathcal{Q}_{RS})^+$.
(6) $\theta^{\mathbf{A}}_{\mathcal{Q}}(a,b) \circ \theta^{\mathbf{A}}_{\mathcal{Q}}(a,b)^* = \nabla^{\mathbf{A}}$, *for every* $a, b \in A$.
Moreover, if $\mathbf{A}$ *is relatively permutable, then*
(7) $Con_{\mathcal{Q}}(\mathbf{A}) = Con(\mathbf{A}, N^{\mathbf{A}})$, *where* $N^{\mathbf{A}}$ *is defined by* $N^{\mathbf{A}}(a,b,c,d) =$ *only* u *such that* $\vec{r}(a,b,u,c) = \vec{s}(a,b,u,c) \wedge \vec{p}(a,b,u,d) = \vec{q}(a,b,u,d)$.
(8) *If* $\theta \in Con(\mathbf{A}, N^{\mathbf{A}})$, *then* $\mathbf{A}/\theta$ *is relatively permutable and* $(\mathbf{A}, N^{\mathbf{A}})/\theta = (\mathbf{A}/\theta, N^{\mathbf{A}/\theta})$.

Proof. The implications (1)⇒(6), (2)⇒(3) are trivial. It is well known that Boolean products are specialized sheaves and hence (4)⇒(5) follows.

Next, for $\mathbf{A} \in \mathcal{Q}$, and $a,b,c,d \in A$, note that u is a solution of the system
$\vec{r}(a,b,u,c) = \vec{s}(a,b,u,c) \wedge \vec{p}(a,b,u,d) = \vec{q}(a,b,u,d)$
iff
$(u,c) \in \theta^{\mathbf{A}}_{\mathcal{Q}}(a,b)$ and $(u,d) \in \theta^{\mathbf{A}}_{\mathcal{Q}}(a,b)^*$.
Furthermore, such a u is unique. Thus we have that (2), (3) and (6) are equivalent.

Also note that every member of $\mathcal{Q}_{RS}$ satisfies (2). Since $S\,(\mathcal{Q}_{RS})^+ \subseteq (\mathcal{Q}_{RS})^+$ and sheaves preserve sentences of the form $\forall \exists! \bigwedge p = q$, when the algebra of global sections is a subdirect product of the stalks [12], we have that (5)⇒(2) holds.

In what follows, for an algebra $\mathbf{A}$ satisfying (2), let $N^{\mathbf{A}}$ denote the function
$N^{\mathbf{A}}(a,b,c,d) =$ only u such that $\vec{r}(a,b,u,c) = \vec{s}(a,b,u,c) \wedge \vec{p}(a,b,u,d) = \vec{q}(a,b,u,d)$.
Note that if $\mathbf{A}$ is relatively permutable, then $N^{\mathbf{A}} = d^{\mathbf{A}}$.

Suppose (6) holds. We shall use some terminology of [11]. Note that condition (6) says that $Con_{\mathcal{Q}}(\mathbf{A})$ is a factorial projective. Since the closure operator on $A \times A$:
$S \to \bigcap\{\theta \in Con_{\mathcal{Q}}(\mathbf{A}) : S \subseteq \theta\}$
is algebraic we have that $Con_{\mathcal{Q}}(\mathbf{A})$ is a congruencially algebraic projective. Thus (4)⇒(1) of [11, Th 6.2] implies that $Con_{\mathcal{Q}}(\mathbf{A})$ is a locally Boolean projective. By [11, Th 6.4] we have that $MaxCon_{\mathcal{Q}}(\mathbf{A})$ is a locally Boolean spectra and hence the canonical embedding $\mathbf{A} \to \Pi_{\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})} \mathbf{A}/\gamma$ is a locally Boolean representation. It is well known that if we add a trivial factor, then we obtain a Boolean representation, i.e. the embedding $\mathbf{A} \to \Pi_{\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A}) \cup \{\nabla^{\mathbf{A}}\}} \mathbf{A}/\gamma$ is a Boolean representation. Note that, in particular, we have proved (6)⇒(4). By (2) of [11, Th 5.3] we have that there is a quaternary operation $c : A^4 \to A$ such that
$\rho : (\mathbf{A}, c) \to \Pi_{\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})} (\mathbf{A}/\gamma, d^{\mathbf{A}/\gamma})$

is an irredundant fully expanded subdirect representation. Since $d^{\mathbf{A}/\gamma} = N^{\mathbf{A}/\gamma}$, for $\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})$, we have that $c = N^{\mathbf{A}}$. As ρ is fully expanded we have that $Con(\mathbf{A}, N^{\mathbf{A}}) \subseteq Con_{\mathcal{Q}}(\mathbf{A})$. Since ρ is an embedding of the extended language, it follows that $MaxCon_{\mathcal{Q}}(\mathbf{A}) \subseteq Con(\mathbf{A}, N^{\mathbf{A}})$, which implies that $Con_{\mathcal{Q}}(\mathbf{A}) = Con(\mathbf{A}, N^{\mathbf{A}})$. But the algebras $(\mathbf{A}/\gamma, d^{\mathbf{A}/\gamma})$, with $\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})$ generate a discriminator variety and hence $(\mathbf{A}, N^{\mathbf{A}})$ is congruence permutable. Thus we have proved (6)⇒(1) and (7).

We leave to the reader the proof of (8). □

Let $\mathcal{Q}$ be a relatively semisimple quasivariety with EDRPC and let $\mathcal{L}$ be the language of $\mathcal{Q}$. Let $\mathcal{L}_N$ be the language $\mathcal{L}$ expanded by adding the new 4-ary function symbol N. Define

$\mathcal{P}_N = \{(\mathbf{A}, N^{\mathbf{A}}) : \mathbf{A} \text{ is relatively permutable}\}$

We note that $\mathcal{P}_N$ is a quasivariety since it can be axiomatized by a set of quasiidentities axiomatizing $\mathcal{Q}$ plus the identities

$\vec{r}(x, y, N(x, y, z, w), z) \approx \vec{s}(x, y, N(x, y, z, w), z)$

$\vec{p}(x, y, N(x, y, z, w), w) \approx \vec{q}(x, y, N(x, y, z, w), w)$

(use (3)⇒(1) of Theorem 17). But by (8) of Theorem 17, $\mathcal{P}_N$ is closed under quotients which says that it is a variety. Also note that (7) of Theorem 17 implies that

$(\mathcal{P}_N)_{SI} = (\mathcal{P}_N)_S = \{(\mathbf{A}, d^A) : \mathbf{A} \in \mathcal{Q}_{RS}\}$

and hence $\mathcal{P}_N$ is a discriminator variety. One can think of N as a 'missing operation' in the relatively permutable algebras in $\mathcal{Q}$, that once added makes the class of these algebras into a variety.

If $\mathbf{A} \in \mathcal{Q}$ and $\mathbf{E} \in \mathcal{P}_N$, we say that $\mathbf{E}$ is a *relatively permutable extension* of $\mathbf{A}$ if $\mathbf{A}$ is a subalgebra of the reduct of $\mathbf{E}$ to the language of $\mathcal{Q}$ and $\mathbf{E}$ is generated by A. Of course, since every algebra in $\mathcal{Q}$ is embeddable in a direct product of relatively simple algebras, every $\mathbf{A} \in \mathcal{Q}$ has at least a relatively permutable extension. Moreover, every $\mathbf{A} \in \mathcal{Q}$ has a *free relatively permutable extension*, i.e. a relatively permutable extension $\mathbf{E}$ with the property that for every onto homomorphism $f : \mathbf{A} \to \mathbf{B}$, and every relatively permutable extension $\mathbf{G}$ of $\mathbf{B}$, there exists a unique homomorphism $\bar{f} : \mathbf{E} \to \mathbf{G}$ such that $\bar{f} \mid_A = f$. This is because we can take $\mathbf{E}$ to be the $\mathcal{L}_N$-reduct of the $\mathcal{R}$-free algebra generated by the empty set of free generators, where $\mathcal{R}$ is the variety defined as follows

- the language of $\mathcal{R}$ is $\mathcal{L}_N$ expanded by adding a new constant c_a for each $a \in A$.

- $\mathcal{R}$ is axiomatized by a set of axioms for $\mathcal{P}_N$ plus the identities

$F(c_{a_1}, ..., c_{a_n}) = c_{F^{\mathbf{A}}(a_1,...,a_n)}$, $F \in \mathcal{L}$, $a_1, ..., a_n \in A$.

If $\mathbf{E}_1$ and $\mathbf{E}_2$ are two relatively permutable extensions of $\mathbf{A}$, then we say that $\mathbf{E}_1$ and $\mathbf{E}_2$ are *equivalent* if there is an isomorphism $f : \mathbf{E}_1 \to \mathbf{E}_2$ such that $f \mid_A$ is the identity on A.

Theorem 18. *Let $\mathcal{Q}$ be a relatively semisimple quasivariety with EDRPC. Every $\mathbf{A} \in \mathcal{Q}$ has, modulo equivalence, exactly one relatively permutable extension. Moreover if $\mathbf{E}$ is the relatively permutable extension of $\mathbf{A}$, then the map*

$$\begin{array}{ccc} Con(\mathbf{E}) & \to & Con_{\mathcal{Q}}(\mathbf{A}) \\ \theta & \to & \theta \mid_A \end{array}$$

is a lattice isomorphism.

Proof. First note that

(*) If $\mathbf{E}$ is a relatively permutable extension of $\mathbf{A}$ and $\mathbf{A}$ is relatively permutable, then $\mathbf{E} = (\mathbf{A}, N^{\mathbf{A}})$.

Next, suppose $\mathbf{E}$ is a relatively permutable extension of $\mathbf{A}$ and let $\mathbf{E}_r$ be the reduct of $\mathbf{E}$ to the language of $\mathcal{Q}$. If $\theta \in Con(\mathbf{E})$, we shall use $\mathbf{A}/\theta$ to denote the subalgebra of $\mathbf{E}_r/\theta$ whose universe is the set $\{a/\theta : a \in A\}$. Note that

(**) $\mathbf{E}/\theta$ is a relatively permutable extension of $\mathbf{A}/\theta$.

We shall prove that

(***) $\gamma \mid_A \in MaxCon_{\mathcal{Q}}(\mathbf{A})$, for every $\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r)$.

Since $\mathbf{A}/\gamma \mid_A$ is embeddable in $\mathbf{E}_r/\gamma \in (\mathcal{Q}_{RS})^+$, we have that $\gamma \mid_A \in MaxCon_{\mathcal{Q}}(\mathbf{A})^+$. Suppose $\gamma \mid_A = \nabla^{\mathbf{A}}$. By (**) we have that $\mathbf{E}/\gamma$ is a relatively permutable extension of a trivial algebra, which implies that $\mathbf{E}/\gamma$ is trivial since $\mathcal{P}_N \vDash N(x,x,x,x) = x$. But this is absurd since $\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r)$ and hence we have that $\gamma \mid_A \in MaxCon_{\mathcal{Q}}(\mathbf{A})$, concluding the proof of (***).

Next we prove that

(****) For every $\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})$ there is exactly one $\lambda \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r)$ satisfying $\gamma = \lambda \mid_A$.

Since $MaxCon_{\mathcal{Q}}(\mathbf{E}_r)^+$ is τ_e-compact (Corollary 9), we have that $\{\lambda \mid_A : \lambda \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r)^+\} \subseteq MaxCon_{\mathcal{Q}}(\mathbf{A})$ is τ_e-compact (Corollary 5), which by Corollary 7 says that $\{\lambda \mid_A : \lambda \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r)^+\} = MaxCon_{\mathcal{Q}}(\mathbf{A})^+$ and hence we have proved the existence part of (****). For $\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})$, let

$$\theta = \bigcap\{\lambda \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r) : \gamma = \lambda \mid_A\}.$$

Since $\theta \in Con(\mathbf{E})$ and $\gamma = \theta \mid_A$, (**) implies that $\mathbf{E}/\theta$ is a relatively permutable extension of $\mathbf{A}/\theta \cong \mathbf{A}/\gamma$. Since $\mathbf{A}/\gamma$ is relatively simple, it is relatively permutable, which by (*) says that $\mathbf{E}/\theta \cong (\mathbf{A}/\gamma, N^{\mathbf{A}/\gamma})$. Thus $\theta \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r)$. This proves the uniqueness part of (****).

For $\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})$, let γ_e be the only $\lambda \in MaxCon_{\mathcal{Q}}(\mathbf{E}_r)$ satisfying $\gamma = \lambda \mid_A$. Note that the map

$$\begin{array}{ccc} MaxCon_{\mathcal{Q}}(\mathbf{A}) & \to & MaxCon_{\mathcal{Q}}(\mathbf{E}_r) \\ \gamma & \to & \gamma_e \end{array}$$

is bijective. Also note that

$$\begin{array}{ccc} A/\gamma & \to & E/\gamma_e \\ a/\gamma & \to & a/\gamma_e \end{array}$$

is an isomorphism between $(\mathbf{A}/\gamma, N^{\mathbf{A}/\gamma})$ and $\mathbf{E}/\gamma_e$. Thus for two $\mathcal{L}_N$-terms $t = t(x_1, ..., x_n)$ and $s = s(x_1, ..., x_n)$ and elements $a_1, ..., a_n, b_1, ..., b_n \in A$ we have have that

$$t^{\mathbf{E}}(a_1, ..., a_n) = s^{\mathbf{E}}(b_1, ..., b_n)$$

iff

$$t^{\mathbf{E}}(a_1, ..., a_n)/\gamma_e = s^{\mathbf{E}}(b_1, ..., b_n)/\gamma_e,\ \forall\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})$$

iff

$$t^{\mathbf{E}/\gamma_e}(a_1/\gamma_e, ..., a_n/\gamma_e) = s^{\mathbf{E}/\gamma_e}(b_1/\gamma_e, ..., b_n/\gamma_e),\ \forall\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A})$$

iff

$$t^{(\mathbf{A}/\gamma, N^{\mathbf{A}/\gamma})}(a_1/\gamma, ..., a_n/\gamma) = s^{(\mathbf{A}/\gamma, N^{\mathbf{A}/\gamma})}(b_1/\gamma, ..., b_n/\gamma),\ \forall\gamma \in MaxCon_{\mathcal{Q}}(\mathbf{A}). \blacksquare$$

We note that this last property depends only from $\mathbf{A}$ which says that the relatively permutable extension $\mathbf{E}$ is uniquely determined by $\mathbf{A}$.

Next we shall prove that

$$\begin{array}{ccc} Con(\mathbf{E}) & \to & Con_{\mathcal{Q}}(\mathbf{A}) \\ \theta & \to & \theta\mid_A \end{array}$$

is a lattice isomorphism. Since $(\bigcap_i \gamma_i)\mid_A = \bigcap_i \gamma_i\mid_A$ and every member of $Con_{\mathcal{Q}}(\mathbf{A})$ is intersection of a subset of $MaxCon_{\mathcal{Q}}(\mathbf{A})$, (****) says that the above map is onto. To see that this map is injective, suppose that $\theta, \delta \in Con(\mathbf{E})$ are such that $\theta\mid_A = \delta\mid_A$ and $\theta \neq \delta$. W. l. o. g. we can suppose that $\delta \subseteq \theta$. By (**) we have that $\mathbf{E}/\delta$ is a relatively permutable extension of $\mathbf{A}/\delta$. Note that $\theta/\delta \in Con(\mathbf{E}/\delta) - \{\Delta^{\mathbf{E}/\delta}\}$ and $(\theta/\delta)\mid_{A/\delta} = \Delta^{\mathbf{A}/\delta}$. Thus we can suppose that $\delta = \Delta^{\mathbf{E}}$. Since $\mathbf{E}/\theta$ is a relatively permutable extension of $\mathbf{A}/\theta \cong \mathbf{A}$, there is an isomorphism

$$g : \mathbf{E}/\theta \to \mathbf{E}$$

such that $g(a/\theta) = a$, for every $a \in A$. Note that $g \circ \pi_\theta$ is an isomorphism such that $g \circ \pi_\theta\mid_A = id_A$ and $g \circ \pi_\theta \neq id_E$. But this is an absurd since we could take $\mathbf{E}$ to be the free relatively permutable extension of $\mathbf{A}$. $\square$

In the case in which $\mathcal{Q}$ is the variety of bounded distributive lattices the free relatively permutable extensions are the free Boolean extensions [1].

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