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A PROBABILISTIC APPROACH TO POLYNOMIAL INEQUALITIES

BY

Damián Pinasco and Ignacio Zalduendo

Depto. de Matemática, Universidad Torcuato Di Tella
Miñones 2177 (C1428ATG), Buenos Aires, Argentina, and CONICET
e-mail: dpinasco@utdt.edu and nacho@utdt.edu

ABSTRACT

We define a probability measure on the space of polynomials over $\mathbb{R}^n$ in order to address questions regarding the attainment of the norm at given points and the validity of polynomial inequalities.

Using this measure, we prove that for all degrees $k \geq 3$, the probability that a k-homogeneous polynomial attains a local extremum at a vertex of the unit ball of ℓ_1^n tends to one as the dimension n increases. We also give bounds for the probability of some general polynomial inequalities.

Introduction

If $P\colon E \to \mathbb{R}$ is a k-homogeneous polynomial over a Banach space E, its norm is defined as the supremum of the values $|P(x)|$ as x ranges over the unit sphere of E. In studying if and where on the unit sphere such a supremum might be attained, it is immediately clear that unless E is a Hilbert space, different points on the sphere play different roles. Consider where a linear form on ℓ_∞^2 ($\mathbb{R}^2$ with the max norm) attains its norm. Clearly the vertexes of the sphere are the points to look at. In case of a homogeneous polynomial the importance of these points is diminished, but still one would expect a higher likelihood of norm-attainment at vertexes than at other specific points of the sphere.

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One may argue — by considering gradients of a polynomial at different points and by comparing the ℓ_2-norms of vertexes vis-a-vis other points of the unit sphere of ℓ_∞^n — that homogeneous polynomials over ℓ_∞^n are more likely to attain their norms at vertexes than at any other point, and that such a likelihood will grow as the dimension n tends to infinity. This has been partially addressed in [3] and in [6].

The quantitative study of such questions requires the use of some measure on the space of polynomials in order to refer to more or less "likely" sets of polynomials. It is by no means obvious how such a measure should be defined. If we restrict our attention to k-homogeneous polynomials in n variables, we have an $\binom{n+k-1}{k}$-dimensional space $P({}^k\mathbb{R}^n)$. This space has no preferred metric structure, and indeed, different norms considered on $\mathbb{R}^n$ give rise to different — and geometrically rather strange ([7]) — unit balls on $P({}^k\mathbb{R}^n)$.

The aim of this paper is to introduce on $P({}^k\mathbb{R}^n)$ — and on other spaces of polynomials — a probability measure, and to present a case for its naturality and "correctness".

In Section 1 we introduce our measure. In order to do so we consider $P({}^k\mathbb{R}^n)$ as a dual Hilbert space. This structure has been considered on spaces of k-homogeneous polynomials by Dwyer [4] and by Lopushansky and Zagorodnyuk [5], and is also produced by the Bombieri norm of a polynomial (see [1] and [2]). We consider standard Gaussian measure on this Hilbert space and calculate the norm of some linear functionals. We also extend this approach to the space $P_{\leq m}(\mathbb{R}^n)$ of polynomials of degree at most m.

In Section 2 we prove the following. Consider k-homogeneous polynomials over ℓ_1^n ($\mathbb{R}^n$ with the 1-norm). Then for all degrees $k \geq 3$, the probability that a k-homogeneous polynomial attains a local maximum at a vertex of the unit ball tends to one as the dimension n grows. The case of 2-homogeneous polynomials is qualitatively completely different.

Finally, in Section 3 we calculate upper and lower bounds for the probability of certain polynomial inequalities. We state and prove all our results in the real setting; clearly analogous results can be obtained in the complex case.

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1. Gaussian measure in the space of Hilbertian polynomials.

The space of k-homogeneous polynomials in n real variables, $P(^k\mathbb{R}^n)$, is a $\binom{n+k-1}{k}$-dimensional vector space. We will require a Hilbert space structure on this space, and therefore need to introduce such a structure. We follow [5].

Denote euclidean n-space with ℓ_2^n, and consider an orthonormal basis $e_1, \dots, e_n$. The k-tensor product $\bigotimes_k \ell_2^n$ is spanned by $e_{j_1} \otimes \cdots \otimes e_{j_k}$ with each $j_i = 1, \dots, n$. We denote by $[j]$ or $[i]$ the set $\{1, \dots, n\}^k$ of all such indexes. Define on $\bigotimes_k \ell_2^n$ an inner product by setting

$$\begin{aligned}
\langle v, w \rangle &= \left\langle \sum_{[j]} c_{j_1 \cdots j_k} e_{j_1} \otimes \cdots \otimes e_{j_k}, \sum_{[i]} d_{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \right\rangle \\
&= \sum_{[j]} \sum_{[i]} c_{j_1 \cdots j_k} d_{i_1 \cdots i_k} \langle e_{j_1}, e_{i_1} \rangle \cdots \langle e_{j_k}, e_{i_k} \rangle \\
&= \sum_{[j]} c_{j_1 \cdots j_k} d_{j_1 \cdots j_k}.
\end{aligned}$$

Define also the symmetrization projector $S \colon \bigotimes_k \ell_2^n \to \bigotimes_k \ell_2^n$ by

$$S(e_{j_1} \otimes \cdots \otimes e_{j_k}) = \frac{1}{k!} \sum_{\sigma} e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(k)}},$$

where σ runs through all permutations of $\{1, \dots, k\}$. We denote the image of S by $\bigotimes_{k,s} \ell_2^n$. Note that $\bigotimes_{k,s} \ell_2^n$ is spanned by the symmetric tensors

$$S(e_{j_1} \otimes \cdots \otimes e_{j_k}) = S(\underbrace{e_1 \otimes \cdots \otimes e_1}_{\alpha_1} \cdots \underbrace{e_n \otimes \cdots \otimes e_n}_{\alpha_n})$$

(which we will denote e^α), while $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$ (we will also write $\alpha! = \alpha_1! \cdots \alpha_n!$). Note that each index α corresponds to $\frac{k!}{\alpha!}$ indexes in $[j]$; we will denote the correspondence by $j \mapsto \alpha$.

We consider in $\bigotimes_{k,s}\ell_2^n$ the inner product induced by the ambient space $\bigotimes_k \ell_2^n$. Thus we have, if $j\mapsto\alpha$ and $i\mapsto\beta$,

$$\begin{aligned}\langle e^\alpha, e^\beta\rangle =&\langle S(e_{j_1}\otimes\cdots\otimes e_{j_k}), S(e_{i_1}\otimes\cdots\otimes e_{i_k})\rangle\\ =&\frac{1}{k!}\frac{1}{k!}\sum_\sigma\sum_\mu\langle e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(k)}}, e_{i_{\mu(1)}}\otimes\cdots\otimes e_{i_{\mu(k)}}\rangle\\ =&\frac{1}{k!}\frac{1}{k!}\sum_\sigma\sum_\mu\langle e_{j_{\sigma(1)}}, e_{i_{\mu(1)}}\rangle\cdots\langle e_{j_{\sigma(k)}}, e_{i_{\mu(k)}}\rangle\\ =&\frac{1}{k!}\frac{1}{k!}\sum_\sigma\sum_\mu\delta_{j_{\sigma(1)}i_{\mu(1)}}\cdots\delta_{j_{\sigma(k)}i_{\mu(k)}} = \delta_{\alpha\beta}\frac{1}{k!}\frac{1}{k!}k!\alpha! = \frac{\alpha!}{k!}\delta_{\alpha\beta},\end{aligned}$$

and in particular $\|e^\alpha\| = \sqrt{\frac{\alpha!}{k!}}$, so $\left\{\sqrt{\frac{k!}{\alpha!}}e^\alpha\right\}$ is an orthonormal basis of $\bigotimes_{k,s}\ell_2^n$.

For each $x\in\ell_2^n$, the tensor $x\otimes\cdots\otimes x$ is symmetric, so

$$\begin{aligned}x\otimes\cdots\otimes x =&\sum_{[j]} x_{j_1}\cdots x_{j_k} e_{j_1}\otimes\cdots\otimes e_{j_k} = \sum_{[j]} x_{j_1}\cdots x_{j_k} S(e_{j_1}\otimes\cdots\otimes e_{j_k})\\ =&\sum_{|\alpha|=k} x^\alpha\frac{k!}{\alpha!}e^\alpha = \sum_{|\alpha|=k} x^\alpha\sqrt{\frac{k!}{\alpha!}}\left(\sqrt{\frac{k!}{\alpha!}}e^\alpha\right),\end{aligned}$$

where we have written $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$.

Now

$$\begin{aligned}&\left\langle x\otimes\cdots\otimes x, \sum_{|\alpha|=k} a_\alpha\sqrt{\frac{\alpha!}{k!}}\left(\sqrt{\frac{k!}{\alpha!}}e^\alpha\right)\right\rangle\\ &\qquad=\left\langle \sum_{|\beta|=k} x^\beta\sqrt{\frac{k!}{\beta!}}\left(\sqrt{\frac{k!}{\beta!}}e^\beta\right), \sum_{|\alpha|=k} a_\alpha\sqrt{\frac{\alpha!}{k!}}\left(\sqrt{\frac{k!}{\alpha!}}e^\alpha\right)\right\rangle\\ &\qquad=\sum_{|\beta|=k}\sum_{|\alpha|=k} a_\alpha x^\beta\sqrt{\frac{k!}{\beta!}}\sqrt{\frac{\alpha!}{k!}}\left\langle\sqrt{\frac{k!}{\beta!}}e^\beta, \sqrt{\frac{k!}{\alpha!}}e^\alpha\right\rangle = \sum_{|\alpha|=k} a_\alpha x^\alpha.\end{aligned}$$

Thus the polynomial $P\colon\mathbb{R}^n\to\mathbb{R}$ defined by $P(x)=\sum_{|\alpha|=k} a_\alpha x^\alpha$ is identified with the linear form over $\bigotimes_{k,s}\ell_2^n$ given by the inner product multiplication $\langle\cdot, v_P\rangle$, where $v_P = \sum_{|\alpha|=k} a_\alpha\sqrt{\frac{\alpha!}{k!}}\left(\sqrt{\frac{k!}{\alpha!}}e^\alpha\right)$, and the linear form e_x over $P(^k\mathbb{R}^n)$ defined by evaluation at x is identified with the inner product against the element $x \otimes \cdots \otimes x$ of $\bigotimes_{k,s} \ell_2^n$. Note then that we have

$$\|P\| = \Big(\sum_{|\alpha|=k} a_\alpha^2 \frac{\alpha!}{k!} \Big)^{\frac{1}{2}},$$

$$\|e_x\| = \Big(\sum_{|\alpha|=k} x^{2\alpha} \frac{k!}{\alpha!} \Big)^{\frac{1}{2}} = \Big(\sum_{r=1}^{n} x_r^2 \Big)^{\frac{k}{2}} = \|x\|^k,$$

$$\text{and} \quad \langle e_x, e_y \rangle = \sum_{|\alpha|=k} x^\alpha y^\alpha \frac{k!}{\alpha!} = \Big(\sum_{r=1}^{n} x_r y_r \Big)^k = \langle x, y\rangle^k.$$

The norm defined above on $P(^k\mathbb{R}^n)$ is referred to as the Bombieri norm in [1]

Note that the space of linear forms over $P(^k\mathbb{R}^n)$ includes a variety of functions, many of which are defined by evaluation of k-linear forms or evaluation of derivatives. Indeed, if $v_1, \ldots, v_k$ are k elements of $\mathbb{R}^n$, then *evaluation at* $v_1, \ldots, v_k$ defined by

$$P \mapsto \phi(v_1, \ldots, v_k)$$

(where ϕ is the unique symmetric k-linear form such that $\phi(x, \ldots, x) = P(x)$) is a linear form, as is (taking $v_1 = \cdots = v_{k-1} = a$ and $v_k = v$)

$$P \mapsto \frac{\partial P}{\partial v}(a) = k\phi(a, \ldots, a, v).$$

We shall need to consider the norms of such forms, as well as the angles between them. For example, it is not hard to calculate that when $\|a\| = 1$, the norm of this last functional is

$$\Big\| \frac{\partial(\cdot)}{\partial v}(a) \Big\| = \|kS(a \otimes \cdots \otimes a \otimes v)\| = \big[k\|v\|^2 + (k^2 - k)\langle a, v\rangle^2 \big]^{\frac{1}{2}}.$$

We consider on $P(^k\mathbb{R}^n)$ the standard Gaussian measure W corresponding to its Hilbert space structure, i.e., the measure

$$W(A) = \frac{1}{(2\pi)^{d/2}} \int_A e^{-\|P\|^2/2}\, dP, \qquad \text{for any Borel set } A \subset P(^k\mathbb{R}^n),$$

where $d = \binom{n+k-1}{k}$, the dimension of $P(^k\mathbb{R}^n)$.

If $\varphi\colon P({}^k\mathbb{R}^n) \to \mathbb{R}$ is a linear form, then due to the rotation invariance of the measure W we may calculate

$$\begin{aligned}
W\{P :&\varphi(P) \leq a\} \\
=&W\Big\{P : \frac{\varphi(P)}{\|\varphi\|} \leq \frac{a}{\|\varphi\|}\Big\} \\
=&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a}{\|\varphi\|}} e^{-\frac{\omega_1^2}{2}}\, d\omega_1 \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-\frac{1}{2}(\omega_2^2+\cdots+\omega_d^2)}\, d\omega_2 \cdots d\omega_d \\
=&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a}{\|\varphi\|}} e^{-\frac{(\|\varphi\|\omega_1)^2}{2\|\varphi\|^2}}\, d\omega_1 = \frac{1}{\sqrt{2\pi}\|\varphi\|} \int_{-\infty}^{a} e^{-\frac{t^2}{2\|\varphi\|^2}}\, dt.
\end{aligned}$$

Thus φ is a normal random variable with mean 0 and standard deviation $\|\varphi\|$. We note also that the measure W has another form of rotation invariance: if $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear map, and $T^*\colon P({}^k\mathbb{R}^n) \to P({}^k\mathbb{R}^n)$ is $T^*(P) = P \circ T$, then T^* preserves the measure W.

We will consider the space $P_{\leq m}(\mathbb{R}^n)$ of polynomials of degree at most m as the ℓ_2-sum

$$P_{\leq m}(\mathbb{R}^n) = \bigoplus_{k=0}^{m} P({}^k\mathbb{R}^n)$$

with the product measure, which we will continue to denote W. Thus every linear form $\psi\colon P_{\leq m}(\mathbb{R}^n) \to \mathbb{R}$ may be written

$$\psi(P) = \psi\Big(\sum_{k=0}^{m} P_k\Big) = \sum_{k=0}^{m} \psi_k(P_k),$$

where P_k is the k-homogeneous part of P and ψ_k is the restriction of ψ to the space $P({}^k\mathbb{R}^n)$. The norm $\|\psi\|$ is then given by

$$\|\psi\|^2 = \sum_{k=0}^{m} \|\psi_k\|^2,$$

and ψ is a gaussian random variable with mean zero and standard deviation $\|\psi\|$.

2. Polynomials on ℓ_1^n

We consider now the problem of where a polynomial $P \in P({}^k\ell_1^n)$ attains its norm. Pérez-García and Villanueva have proved in [6] that there is a set of positive measure (independent of the dimension n) of 2-homogeneous polynomials not attaining their norms on vertexes of the unit sphere.

We will show that for degree $k \geq 3$, the probability of a k-homogeneous polynomial attaining a local extremum at a vertex tends to one as the dimension n increases. We do this by considering gradients at the vertexes and reducing the problem to a question about gaussian random variables. Let us begin then with a probabilistic lemma.

Lemma 2.1: *Let $X, Y_1, \dots, Y_{n-1}$ be independent gaussian random variables with mean zero and standard deviations $\sigma_X = 1, \sigma_{Y_i} = 1/\sqrt{k}$, and call ω_n the probability that*

$$|X| \geq \max\{|Y_1|, \dots, |Y_{n-1}|\}.$$

Then, as n increases, ω_n tends to zero slower than $1/n$ (that is, $n\omega_n \to \infty$).

Proof. Consider the distribution of $|X|$

$$F(u) = \mathrm{Prob}\{|X| \leq u\} = 2\int_0^u e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}};$$

F is a probability distribution function on $[0, \infty)$ whose density is

$$dF(u) = 2e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}}.$$

Thus F^n is also a probability distribution function on $[0, \infty)$

$$(F(u))^n = \left(2\int_0^u e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}\right)^n,$$

whose density is

$$dF^n(u) = 2n\left(2\int_0^u e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}\right)^{n-1} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}}.$$

Fixing a_n such that $F^n(a_n) = \frac{1}{2}$ (i.e., $F(a_n) = (\frac{1}{2})^{1/n}$), we see that $a_n \to \infty$.

Now if X is a normal $(0,1)$ random variable and Y an independent normal $(0, 1/\sqrt{k})$, then to find the probability of $|X| \geq |Y|$, we normalize

$$|X| \geq \left|\frac{Y}{\|Y\|}\right| \frac{1}{\sqrt{k}},$$

and must find the Gaussian area measure of

$$\Big\{(x,y)\in\mathbb{R}^2 : |x| \geq \frac{1}{\sqrt{k}}|y|\Big\},$$

which is

$$4\int_0^\infty \int_0^{\sqrt{k}x} e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}.$$

For $|X| \geq \max\{|Y_1|,\dots,|Y_{n-1}|\}$, we have

$$\omega_n = 2^n \int_0^\infty \left(\int_0^{\sqrt{k}x} e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}\right)^{n-1} e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}.$$

Multiplying by n,

$$n\omega_n = \int_0^\infty 2n\Big(2\int_0^{\sqrt{k}x} e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}\Big)^{n-1} e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}},$$

and changing variables: $u=\sqrt{k}x$, we have

$$\begin{aligned}
&= \int_0^\infty 2n\Big(2\int_0^{u} e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}\Big)^{n-1} e^{-\frac{1}{2k}u^2}\frac{du}{\sqrt{2\pi}\sqrt{k}}\\
&= \int_0^\infty 2n\Big(2\int_0^{u} e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}\Big)^{n-1} e^{-\frac{u^2}{2}}\frac{e^{\frac{k-1}{2k}u^2}}{\sqrt{k}}\frac{du}{\sqrt{2\pi}}\\
&=\frac{1}{\sqrt{k}}\int_0^\infty e^{\frac{k-1}{2k}u^2}\,dF^n(u)\\
&\geq\frac{1}{\sqrt{k}}\int_{a_n}^\infty e^{\frac{k-1}{2k}u^2}\,dF^n(u)\\
&\geq\frac{1}{\sqrt{k}}e^{\frac{k-1}{2k}{a_n}^2}\int_{a_n}^\infty dF^n(u)\\
&=\frac{1}{2\sqrt{k}}e^{\frac{k-1}{2k}{a_n}^2}\to\infty,
\end{aligned}$$

as $n\to\infty$. ∎

Before we proceed to the theorem, let us fix some notation. Consider the closed unit ball of ℓ_1^n. Its extreme points are $\{e_1,\dots,e_n,-e_1,\dots,-e_n\}$. Denote by A_j the set of k-homogeneous polynomials P such that $|P|$ has a local maximum at the vertex e_j (and thus also at $-e_j$). Then the probability that a polynomial has a local extremum at some vertex of the unit ball is the measure of the union

$$W\Big(\bigcup_{j=1}^{n} A_j\Big).$$

We have the following theorem.

THEOREM 2.2: *For all degrees $k \geq 3$, the probability that a k-homogeneous polynomial attains a local extremum at a vertex of the unit ball of ℓ_1^n tends to one as the dimension n increases.*

Proof. We must consider first the independence of the events A_j. Note that P belongs to A_j if it has a local extremum at e_j, and this happens if the gradient $\nabla P(e_j)$ points in the *correct* direction, i.e.,

$$P \in A_j \Leftrightarrow \Big|\frac{\partial P}{\partial e_j}(e_j)\Big| \geq \Big|\frac{\partial P}{\partial e_i}(e_j)\Big| \quad \text{for all } i.$$

Thus we must check the independence of the random variables

$$\frac{1}{k}\frac{\partial(\cdot)}{\partial e_i}(e_j), \quad \text{defined by } P \mapsto \phi(e_j, \dots, e_j, e_i).$$

In the notation of Section 1, these are the linear forms e^α where

$$\alpha = (\dots, \underbrace{k-1}_{j}, \dots, \underbrace{1}_{i}, \dots).$$

Note that if $k = 2$, $\alpha = (\dots, \underbrace{1}_{j}, \dots, \underbrace{1}_{i}, \dots)$, and both

$$\frac{1}{2}\frac{\partial(\cdot)}{\partial e_i}(e_j) \quad \text{and} \quad \frac{1}{2}\frac{\partial(\cdot)}{\partial e_j}(e_i)$$

are the same linear form. However, for any $k \geq 3$,

$$\frac{1}{k}\frac{\partial(\cdot)}{\partial e_i}(e_j) \text{ is } e^\alpha \text{ given by } \alpha = (\dots, \underbrace{k-1}_{j}, \dots, \underbrace{1}_{i}, \dots),$$

$$\frac{1}{k}\frac{\partial(\cdot)}{\partial e_r}(e_s) \text{ is } e^\beta \text{ given by } \beta = (\dots, \underbrace{k-1}_{s}, \dots, \underbrace{1}_{r}, \dots),$$

so

$$\Big\langle \frac{1}{k}\frac{\partial(\cdot)}{\partial e_i}(e_j), \frac{1}{k}\frac{\partial(\cdot)}{\partial e_r}(e_s) \Big\rangle = \langle e^\alpha, e^\beta \rangle = \frac{\alpha!}{k!}\delta_{\alpha\beta} = 0 \quad \text{if } \alpha \neq \beta.$$

Thus for $k \geq 3$ all the partial derivatives involved are independent random variables. Note also that $\frac{\partial(\cdot)}{\partial e_i}(e_j)$ has mean 0 and standard deviation $\|\frac{\partial(\cdot)}{\partial e_i}(e_j)\|$. These standard deviations turn out to be

$$\sigma_i = \Big\| \frac{\partial(\cdot)}{\partial e_i}(e_j) \Big\| = \begin{cases} k, & \text{if } j = i, \\ \sqrt{k}, & \text{if } j \neq i. \end{cases}$$

Call $\omega_n = W(A_j)$ the probability of attaining a local extremum at e_j (by the rotation-invariance of W, $W(A_j)$ is the same for all j). This probability can then be described as follows. Suppose $Y_1, \dots, Y_{n-1}$ and X are independent normal random variables with mean 0 and standard deviations $\sigma_{Y_i} = \sqrt{k}$ and $\sigma_X = k$ (normalizing, we may suppose $\sigma_{Y_i} = 1/\sqrt{k}$ and $\sigma_X = 1$). Then ω_n is the probability that

$$|X| \geq \max\{|Y_1|, \dots, |Y_{n-1}|\},$$

and we may apply the lemma.

We then have, by independence of the A_j^c's, for any fixed degree $k \geq 3$,

$$\begin{aligned} 1 - W\Big(\bigcup_{j=1}^n A_j\Big) &= W\Big(\bigcap_{j=1}^n A_j^c\Big) = \prod_{j=1}^n W(A_j^c) \\ &= \prod_{j=1}^n (1 - W(A_j)) = \prod_{j=1}^n (1 - \omega_n) = (1 - \omega_n)^n \\ &= \big[(1 - \omega_n)^{-\frac{1}{\omega_n}}\big]^{-n\omega_n} \to e^{-\infty} = 0, \qquad \text{as } n \to \infty. \end{aligned}$$

Thus

$$W\Big(\bigcup_{j=1}^n A_j\Big) \to 1$$

as n tends to infinity. ∎

Note that the case of degree $k = 2$ is completely different due to the lack of independence of the random variables involved.

3. Polynomial inequalities

In this section we give a formula for the measure of sets of polynomials verifying quite general polynomial inequalities. We denote by φ_i linear forms over the space of polynomials $P_{\leq m}(\mathbb{R}^n)$. The angle between the linear forms φ_1 and φ_2 will be denoted by $(\varphi_1, \varphi_2) \in (0, \pi)$. We have the following.

Theorem 3.1: *Let φ_1 and φ_2 be linear forms over $P_{\leq m}(\mathbb{R}^n)$, and $a > 0$. Then the measure of the set of polynomials*

$$\{P \in P_{\leq m}(\mathbb{R}^n) : |\varphi_1(P)| \geq |\varphi_2(P)| + a\}$$

is bounded below by

$$e^{-\frac{r^2}{2}}\Big(\frac{1}{\pi} - \frac{r}{\sqrt{2\pi}}\Big)\Bigg[\arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|} - \cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right) + \arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|} + \cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right)\Bigg],$$

and bounded above by

$$e^{-\frac{r^2}{2}}\frac{1}{\pi}\Bigg[\arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|} - \cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right) + \arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|} + \cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right)\Bigg],$$

where

$$r = \frac{a}{\|\varphi_1\| \sin(\varphi_1,\varphi_2)}.$$

Proof. In order to project the set $\{|\varphi_1| \geq |\varphi_2| + a\}$ onto $\mathbb{R}^2$ to calculate its measure, we orthonormalize $\{\varphi_1, \varphi_2\}$. Thus, let

$$\psi = \varphi_2 - c\frac{\varphi_1}{\|\varphi_1\|}, \qquad \text{where } c = \|\varphi_2\| \cos(\varphi_1,\varphi_2)$$

and note that $\|\psi\| = \|\varphi_2\| \sin(\varphi_1,\varphi_2)$. Now $|\varphi_1| \geq |\varphi_2| + a$ if and only if

$$|\varphi_1| - a \geq \left|\psi + c\frac{\varphi_1}{\|\varphi_1\|}\right|,$$

i.e.,

$$-|\varphi_1| + a \leq \psi + c\frac{\varphi_1}{\|\varphi_1\|} \leq |\varphi_1| - a,$$

which, when $\varphi_1 > 0$, is

$$-\varphi_1 - c\frac{\varphi_1}{\|\varphi_1\|} + a \leq \psi \leq \varphi_1 - c\frac{\varphi_1}{\|\varphi_1\|} - a,$$
$$-(\|\varphi_1\| + c)\frac{\varphi_1}{\|\varphi_1\|} + a \leq \psi \leq (\|\varphi_1\| - c)\frac{\varphi_1}{\|\varphi_1\|} - a,$$
$$-\frac{(\|\varphi_1\| + c)}{\|\psi\|}\frac{\varphi_1}{\|\varphi_1\|} + \frac{a}{\|\psi\|} \leq \frac{\psi}{\|\psi\|} \leq \frac{(\|\varphi_1\| - c)}{\|\psi\|}\frac{\varphi_1}{\|\varphi_1\|} - \frac{a}{\|\psi\|}.$$

Analogously, when $\varphi_1 < 0$ we have

$$\frac{(\|\varphi_1\| - c)}{\|\psi\|}\frac{\varphi_1}{\|\varphi_1\|} + \frac{a}{\|\psi\|} \leq \frac{\psi}{\|\psi\|} \leq -\frac{(\|\varphi_1\| + c)}{\|\psi\|}\frac{\varphi_1}{\|\varphi_1\|} - \frac{a}{\|\psi\|}.$$

The measure of $\{|\varphi_1| \geq |\varphi_2|+a\}$ is then the Gaussian area measure $\Gamma(A_+ \bigcup A_-)$ of $A_+ \bigcup A_-$, where

$$A_+ = \Big\{(s,t) : -\frac{(\|\varphi_1\|+c)}{\|\psi\|}s + \frac{a}{\|\psi\|} \leq t \leq \frac{(\|\varphi_1\|-c)}{\|\psi\|}s - \frac{a}{\|\psi\|}\Big\}$$

and

$$A_- = \Big\{(s,t) : \frac{(\|\varphi_1\|-c)}{\|\psi\|}s + \frac{a}{\|\psi\|} \leq t \leq -\frac{(\|\varphi_1\|+c)}{\|\psi\|}s - \frac{a}{\|\psi\|}\Big\}.$$

Since A_+ can be taken onto A_- by a rotation, $\Gamma(A_+ \bigcup A_-) = 2\Gamma(A_+)$.

We calculate first the upper bound. Let v be the vertex of A_+. Then

$$\begin{aligned} r =&\|v\| = \Big\|\Big(\frac{a}{\|\varphi_1\|}, \frac{-ac}{\|\psi\|\|\varphi_1\|}\Big)\Big\| = \Big(\frac{a^2(\|\psi\|^2+c^2)}{\|\psi\|^2\|\varphi_1\|^2}\Big)^{\frac{1}{2}} \\ =&\frac{a}{\|\psi\|\|\varphi_1\|}\left(\|\varphi_2\|^2\sin^2(\varphi_1,\varphi_2) + \|\varphi_2\|^2\cos^2(\varphi_1,\varphi_2)\right)^{\frac{1}{2}} \\ =&\frac{a}{\|\varphi_1\|\sin(\varphi_1,\varphi_2)}. \end{aligned}$$

Now, A_+ is contained in the region U which — in polar coordinates — is

$$U = \Big\{(\theta,\rho) : \arctan\Big(-\frac{(\|\varphi_1\|+c)}{\|\psi\|}\Big) \leq \theta \leq \arctan\Big(\frac{(\|\varphi_1\|-c)}{\|\psi\|}\Big), r \leq \rho < \infty\Big\}.$$

Thus

$$\begin{aligned} 2\Gamma(A_+) &\leq 2\Gamma(U) \\ &= 2\frac{1}{2\pi}\int_{-\frac{(\|\varphi_1\|+c)}{\|\psi\|}}^{\frac{(\|\varphi_1\|-c)}{\|\psi\|}}\int_r^\infty \rho e^{-\frac{\rho^2}{2}}\,d\rho\,d\theta \\ &= \frac{1}{\pi}\Big[\arctan\Big(\frac{(\|\varphi_1\|-c)}{\|\psi\|}\Big) + \arctan\Big(\frac{(\|\varphi_1\|+c)}{\|\psi\|}\Big)\Big]e^{-\frac{r^2}{2}} \\ &= e^{-\frac{r^2}{2}}\frac{1}{\pi}\Big[\arctan\Big(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|} - \cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\Big) \\ &\qquad\qquad + \arctan\Big(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|} + \cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\Big)\Big]. \end{aligned}$$

For the lower bound, consider the ray L through 0 and v; L separates A_+ into a lower region A_{+1} and an upper region A_{+2}. Rotate A_+ around 0 until L coincides with the positive s-axis. Then we can calculate the Gaussian measure of A_{+2} (here α_2 is the angle between L and upper boundary of A_{+2}):

$$\begin{aligned}
\Gamma(A_{+2}) &= \frac{1}{2\pi}\int_{A_{+2}} e^{-\frac{\|(s,t)\|^2}{2}}d\,sdt = \frac{1}{2\pi}\int_{A_{+2}-(r,0)} e^{-\frac{\|(x+r,t)\|^2}{2}}d\,xd\,t \\
&= \frac{1}{2\pi}\int_{A_{+2}-(r,0)} e^{-\frac{x^2+t^2+2xr+r^2}{2}}d\,xd\,t \\
&= \frac{1}{2\pi}\int_{A_{+2}-(r,0)} e^{-\frac{\|(x,t)\|^2}{2}}e^{-rx}e^{-\frac{r^2}{2}}d\,xd\,t \\
&= \frac{e^{-\frac{r^2}{2}}}{2\pi}\int_0^{\alpha_2}\int_0^{\infty}\rho e^{-\frac{\rho^2}{2}}e^{-r\rho\cos\theta}d\,\rho d\,\theta \\
&= \frac{e^{-\frac{r^2}{2}}}{2\pi}\int_0^{\infty}\rho e^{-\frac{\rho^2}{2}}\int_0^{\alpha_2}e^{-r\rho\cos\theta}d\,\theta d\,\rho.
\end{aligned}$$

Now, the function $f(t) = e^{-r\rho\cos\theta}$ is increasing in $(0,\pi)$, and $f(0) = e^{-r\rho}$, so

$$\int_0^{\alpha_2} e^{-r\rho\cos\theta}d\,\theta \geq \alpha_2 e^{-r\rho},$$

thus

$$\begin{aligned}
\Gamma(A_{+2}) &\geq \frac{e^{-\frac{r^2}{2}}}{2\pi}\int_0^{\infty}\alpha_2\rho e^{-\frac{\rho^2+2r\rho}{2}}d\,\rho = \frac{\alpha_2 e^{-\frac{r^2}{2}}}{2\pi}\int_0^{\infty}\rho e^{-\frac{(\rho+r)^2}{2}}e^{\frac{r^2}{2}}d\,\rho \\
&= \frac{\alpha_2}{2\pi}\int_0^{\infty}\rho e^{-\frac{(\rho+r)^2}{2}}d\,\rho \\
&= \frac{\alpha_2}{2\pi}\int_r^{\infty}(u-r)e^{-\frac{u^2}{2}}d\,u \\
&= \frac{\alpha_2}{2\pi}\left[\int_r^{\infty}ue^{-\frac{u^2}{2}}d\,u - r\int_r^{\infty}e^{-\frac{u^2}{2}}d\,u\right] \\
&= \frac{\alpha_2}{2\pi}e^{-\frac{r^2}{2}} - \frac{r\alpha_2}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_r^{\infty}e^{-\frac{u^2}{2}}d\,u \\
&\geq \frac{\alpha_2}{2\pi}e^{-\frac{r^2}{2}} - \frac{r\alpha_2 e^{-\frac{r^2}{2}}}{2\sqrt{2\pi}} \\
&= \frac{\alpha_2}{2}e^{-\frac{r^2}{2}}\Big(\frac{1}{\pi} - \frac{r}{\sqrt{2\pi}}\Big).
\end{aligned}$$

And similarly for A_{+1},

$$\Gamma(A_{+1}) \geq \frac{\alpha_1}{2}e^{-\frac{r^2}{2}}\Big(\frac{1}{\pi} - \frac{r}{\sqrt{2\pi}}\Big),$$

where α_1 is the angle between L and the lower boundary of A_{+1}. Thus, considering that

$$\alpha_1+\alpha_2=\Big[\arctan\Big(\frac{(\|\varphi_1\|-c)}{\|\psi\|}\Big)+\arctan\Big(\frac{(\|\varphi_1\|+c)}{\|\psi\|}\Big)\Big],$$

we have the lower bound

$$2\Gamma(A_+)\geq e^{-\frac{r^2}{2}}\Big(\frac{1}{\pi}-\frac{r}{\sqrt{2\pi}}\Big)\Big[\arctan\Big(\frac{(\|\varphi_1\|-c)}{\|\psi\|}\Big)+\arctan\Big(\frac{(\|\varphi_1\|+c)}{\|\psi\|}\Big)\Big],$$

which is

$$\begin{aligned}2\Gamma(A_+)\geq & e^{-\frac{r^2}{2}}\Big(\frac{1}{\pi}-\frac{r}{\sqrt{2\pi}}\Big)\\ &\times\left[\arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|}-\cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right)+\arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|}+\cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right)\right].\end{aligned}$$

This completes the proof. ∎

Note that when $a=0$, $r=0$, and we have that the measure of $\{P\in P_{\leq m}(\mathbb{R}^n):|\varphi_1(P)|\geq|\varphi_2(P)|\}$ is exactly

$$\frac{1}{\pi}\left[\arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|}-\cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right)+\arctan\left(\frac{\frac{\|\varphi_1\|}{\|\varphi_2\|}+\cos(\varphi_1,\varphi_2)}{\sin(\varphi_1,\varphi_2)}\right)\right].$$

Note also that when φ_1 and φ_2 are orthogonal, then

$$W\{|\varphi_1|\geq|\varphi_2|\}=\frac{2}{\pi}\arctan\Big(\frac{\|\varphi_1\|}{\|\varphi_2\|}\Big).$$

We also have, in general,

Corollary 3.2: *If* $\|\varphi_1\|\geq\|\varphi_2\|$, *then*

$$W\{|\varphi_1|\geq|\varphi_2|\}\geq\frac{2}{\pi}\arctan\Big(\frac{\|\varphi_1\|}{\|\varphi_2\|}\Big).$$

Proof. Fix $\lambda=\frac{\|\varphi_1\|}{\|\varphi_2\|}$, and consider the function

$$F(z)=\frac{1}{\pi}\Big[\arctan\Big(\frac{\lambda-z}{\sqrt{1-z^2}}\Big)+\arctan\Big(\frac{\lambda+z}{\sqrt{1-z^2}}\Big)\Big].$$

Then F is even, and

$$F'(z)=\frac{2\lambda z(\lambda^2-1)}{\sqrt{1-z^2}(1-z^2+(\lambda-z)^2)(1-z^2+(\lambda+z)^2)}>0$$

for $0 < z < 1$. Thus we have

$$W\{|\varphi_1| \geq |\varphi_2|\} = F(\cos(\varphi_1, \varphi_2)) \geq F(0) = \frac{2}{\pi} \arctan\Big(\frac{\|\varphi_1\|}{\|\varphi_2\|}\Big). \qquad \blacksquare$$

We end with two examples related to well-known inequalities. In both, what we see is that a certain inequality is valid for a large set of polynomials, and that the measure of this set tends to one as the degree grows. We use the notation and comments given at the end of Section 1.

Example 1: A Chebyshev-type inequality.

Say $P \in P_{\leq m}(\mathbb{R}^n)$, and write $P = \sum_{k=0}^m P_k$. Chebyshev's inequality says that $\|P_m\| \leq 2^{m-1}\|P\|$. Using our results, we obtain: for any x of norm one, any y of norm one orthogonal to x, and any $c > 1$,

$$|P_m(x)| \leq c^m \, |P(y)|$$

for a large set of polynomials P. Indeed, the measure of the complement can be calculated to be

$$W\Big\{P : \Big|P_m\Big(\frac{x}{c}\Big)\Big| > |P(y)|\Big\} = \frac{2}{\pi} \arctan\Big(\frac{1}{c^m\sqrt{m+1}}\Big),$$

by considering in Theorem 3.1

$$\begin{aligned} \varphi_1 &= \Big(0, 0 \otimes 0, \dots, 0 \otimes \cdots \otimes 0, \frac{x}{c} \otimes \cdots \otimes \frac{x}{c}\Big) \text{ and} \\ \varphi_2 &= (y, y \otimes y, \dots, y \otimes \cdots \otimes y). \end{aligned}$$

This measure tends to zero as m grows.

Example 2: A Markov-type inequality.

Say P is a polynomial of degree m. Markov's inequality says that $\sup_{\|x\|=1} \|\nabla P(x)\|_2 \leq m^2\|P\|$. Using our results, we obtain: for any x, y, and v of norm one, with $x \perp y$ and $v \neq \pm y$,

$$|\langle \nabla P(x), v \rangle| \leq m^2 \, |P(y)|$$

for a large set of polynomials P. Indeed, the measure of the complement

$$W\Big\{P : \Big|\frac{\partial P}{\partial \frac{v}{m^2}}(x)\Big| > |P(y)|\Big\},$$

can be calculated to be bounded above by

$$\frac{1}{\pi}\left[\arctan\left(\frac{\frac{\sqrt{2m^2+3m+1}}{\sqrt{6m^4+6m^3}}-\cos(v,y)}{\sin(v,y)}\right)+\arctan\left(\frac{\frac{\sqrt{2m^2+3m+1}}{\sqrt{6m^4+6m^3}}+\cos(v,y)}{\sin(v,y)}\right)\right],$$

by considering in Theorem 3.1,

$$\varphi_1=\left(\frac{v}{m^2},x\otimes\frac{v}{m^2},\dots,x\otimes\cdots\otimes x\otimes\frac{v}{m^2}\right)\text{ and}$$
$$\varphi_2=(y,y\otimes y,\dots,y\otimes\cdots\otimes y).$$

But $\frac{\sqrt{2m^2+3m+1}}{\sqrt{6m^4+6m^3}}$ tends to zero as m grows. Since the function arctan is odd, the measure tends to zero.

References

[1] B. Beauzamy, E. Bombieri, P. Enflo and H. L. Montgomery, *Products of polynomials in many variables*, Journal of Number Theory **36** (1990), 219–245.

[2] B. Beauzamy and J. Dégot, *Differential identities*, Transactions of the American Mathematical Society **347** (1995), 2607–2619.

[3] D. Carando and I. Zalduendo, *Where do homogeneous polynomials attain their norm?*, Publicationes Mathematicae Debrecen **63** (2003), 19–28.

[4] T. A. W. Dwyer III, *Partial differential equations in Fischer–Fock spaces for the Hilbert–Schmidt holomorphy type*, American Mathematical Society. Bulletin **77** (1971), 725–730.

[5] O. Lopushansky and A. Zagorodnyuk, *Hilbert spaces of analytic functions of infinitely many variables*, Annales Polonici Mathematici **81** (2003), 111–122.

[6] D. Pérez-García and I. Villanueva, *Where do homogeneous polynomials on ℓ_1^n attain their norm?*, Journal of Approximation Theory **127** (2004), 124–133.

[7] R. Ryan and B. Turett, *Geometry of spaces of polynomials*, Journal of Mathematical Analysis and Applications **221** (1998), 698–711.