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LOWER BOUNDS FOR NORMS OF PRODUCTS OF POLYNOMIALS VIA BOMBIERI INEQUALITY

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Abstract. In this paper we give a different interpretation of Bombieri's norm. This new point of view allows us to work on a problem posed by Beauzamy about the behavior of the sequence $S_n(P) = \sup_{Q_n} [PQ_n]_2$, where P is a fixed $m-$homogeneous polynomial and Q_n runs over the unit ball of the Hilbert space of $n-$homogeneous polynomials. We also study the factor problem for homogeneous polynomials defined on $\mathbb{C}^N$ and we obtain sharp inequalities whenever the number of factors is no greater than N. In particular, we prove that for the product of homogeneous polynomials on infinite dimensional complex Hilbert spaces our inequality is sharp. Finally, we use these ideas to prove that any set $\{z_k\}_{k=1}^n$ of unit vectors in a complex Hilbert space for which $\sup_{\|z\|=1} |\langle z, z_1\rangle \cdots \langle z, z_n\rangle|$ is minimum must be an orthonormal system.

1. Introduction

Let $P_1, \ldots, P_n$ be polynomials defined on $\mathbb{K}^N$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and suppose that we have a norm $\|\cdot\|$ defined on the space of polynomials. The problem of finding a constant M, depending only on the degrees of $P_1, \ldots, P_n$, such that

$$\|P_1\| \cdots \|P_n\| \leq M \|P_1 \cdots P_n\| \tag{1.1}$$

and other questions concerning inequalities for the norms of factors of a given polynomial were studied by many authors: G. Aumann [3], B. Beauzamy, E. Bombieri, P. Enflo and H. L. Montgomery [9], B. Beauzamy and P. Enflo [11], P. B. Borwein [15], D. W. Boyd [17, 18, 19], A. O. Gel′fond [24], H. Kneser [26], K. Mahler [28, 29] and I. E. Pritsker and S. Ruscheweyh [35, 36] among others.

For example, for polynomials in one complex variable endowed with the supremum norm over the unit disk, D. W. Boyd [19] proved that

$$\|P_1\| \cdots \|P_n\| \leq C_n^m \|P_1 \cdots P_n\|,$$

where the polynomial $P_1 \cdots P_n$ has degree m and the exact value of the constant C_n is

$$C_n = \exp\left(\frac{n}{\pi}\int_0^{\pi/n} \log(2\cos(t/2))dt\right).$$

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This inequality, which is an improvement of earlier results of A. O. Gel′fond [24] and K. Mahler [28], is asymptotically sharp as $m \to \infty$.

Working with multivariate polynomials and different norms related to the coefficients of the polynomials, B. Beauzamy et al. [9] and K. Mahler [29] gave estimates for the constant M in inequality (1.1). For instance, in [9], the authors defined a norm on the space of $m-$homogeneous polynomials on $\mathbb{K}^N$ by

$$[P]_2 = \left(\sum_{|\alpha|=m} \frac{\alpha!}{m!} |a_\alpha|^2 \right)^{1/2},$$

where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_N$ and P has the monomial expansion $P(z_1, \dots, z_N) = \sum_{|\alpha|=m} a_\alpha \, z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. For this norm, which is known as Bombieri's norm, they proved the following inequality: let P, Q be homogeneous polynomials of degrees m, n respectively; then

$$[P]_2 \, [Q]_2 \leq \sqrt{\frac{(m+n)!}{m!\, n!}} \; [PQ]_2.$$

Associated with inequality (1.1) we have the problem of finding

$$I(P) = \inf\{\|PQ\| : \|Q\| = 1\} \quad \text{and} \quad S(P) = \sup\{\|PQ\| : \|Q\| = 1\}.$$

This problem and other questions related to different restrictions over P and Q were studied by several authors. For example, B. Beauzamy [7], B. Beauzamy, J.-L. Frot and C. Millour [12] and B. Reznick [39] dealt with the problem of finding

$$\inf\{[PQ]_2 : [P]_2 = [Q]_2 = 1\} \quad \text{and} \quad \sup\{[PQ]_2 : [P]_2 = [Q]_2 = 1\}.$$

There are finite dimensional variations of this problem. E. Bombieri and J. Vaaler [14], J. D. Donaldson and Q. I. Rahman [21] and J.-P. Kahane [25] studied the quantities

$$\inf\{\|PQ\| : \|Q\| = 1,\ \deg(Q) = n\} \ \text{ and } \ \sup\{\|PQ\| : \|Q\| = 1,\ \deg(Q) = n\}$$

for different norms depending on the coefficients of the polynomials.

A way to compute $[PQ]_2$ using the eigenvalues of a matrix associated to homogeneous polynomials P and Q was shown by B. Beauzamy [6]. Therefore, if $\mathcal{P}(^m\mathbb{C}^N)$ denotes the space of $m-$homogeneous polynomials on $\mathbb{C}^N$, given $P \in \mathcal{P}(^m\mathbb{C}^N)$, the problem of finding the numbers

$$I_n(P) = \inf\left\{[PQ]_2 : [Q]_2 = 1,\ Q \in \mathcal{P}(^n\mathbb{C}^N)\right\}$$

and

$$S_n(P) = \sup\left\{[PQ]_2 : [Q]_2 = 1,\ Q \in \mathcal{P}(^n\mathbb{C}^N)\right\}$$

was theoretically solved. The author also showed that if $\deg(P) > 0$, then the sequence $I_n(P) \to 0$. However, it seems to be difficult to characterize the behavior of $S_n(P)$ using those techniques. For example, if $P(z) = \langle z, a\rangle^m$, then $S_n(P) = [P]_2$ for all $n \in \mathbb{N}$, but for the polynomial $P(z_1, z_2) = z_1 z_2$, $\lim_{n\to\infty} S_n(P) = 1/2 < 1/\sqrt{2} = [P]_2$. The same problem was studied independently by B. Reznick [39]. In Section 3 we will prove that $\limsup_{n\to\infty} S_n(P) = \|P\|_{\mathcal{P}}$.

Let us recall that the space of continuous $m-$homogeneous polynomials on a Banach space E, denoted by $\mathcal{P}(^mE)$, is a Banach space under the uniform norm $\|P\|_{\mathcal{P}} = \sup_{\|z\|_E=1} |P(z)|$. Considering this norm, inequality (1.1) was studied for polynomials defined on infinite dimensional Banach spaces. For instance, R. Ryan and B. Turett [38] gave bounds for the special case where the polynomials $\{P_i\}_{i=1}^n$ are continuous linear functionals on E. Moreover, C. Benítez, Y. Sarantopoulos and A. Tonge [13] proved that if P_i has degree k_i for $1 \leq i \leq n$, then inequality (1.1) holds with constant

$$M = \frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}$$

for any complex Banach space. The authors also showed an example on ℓ_1 for which the equality prevails. However, for many spaces it is possible to improve this bound. In [16], C. Boyd and R. Ryan proved that, on complex Hilbert spaces,

$$M = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}$$

is more accurate.

We will be concerned with the study of inequality (1.1) for homogeneous polynomials defined on a (finite or infinite dimensional) complex Hilbert space H. In Section 4 we will prove that

$$\|P_1\|_{\mathcal{P}} \cdots \|P_n\|_{\mathcal{P}} \leq \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}\, \|P_1 \cdots P_n\|_{\mathcal{P}}, \tag{1.2}$$

where $P_i \in \mathcal{P}(^{k_i}H)$ for $1 \leq i \leq n$. We also prove that this is a sharp inequality whenever the number of factors is no greater than $\dim(H)$. We derive sharp inequalities for products of continuous symmetric multilinear forms on complex Hilbert spaces and using complexification techniques (see [32]) we present inequalities for real Hilbert spaces.

Inequality (1.1) has been widely studied when $P_1, \ldots, P_n$ are bounded linear functionals on Hilbert spaces. In [13], the authors made the following conjecture: *given n unit vectors $\{x_k\}_{k=1}^n$ in $\mathbb{R}^n$, then*

$$\sup_{\|x\|_{\mathbb{R}^n}=1} |\langle x, x_1\rangle \cdots \langle x, x_n\rangle| \geq n^{-n/2}, \tag{1.3}$$

and equality holds if and only if $\{x_k\}_{k=1}^n$ is an orthonormal system. This conjecture was also discussed by A. E. Litvak, V. D. Milman, and G. Schechtman [27]. A positive answer for $n = 1, 2, 3, 4$ and 5 was given by A. Pappas and S. G. Révész [33]. However, the remaining cases are still open problems. See for example V. A. Anagnostopoulos and S. G. Révész [1], P. E. Frenkel [22], J. C. García-Vázquez and R. Villa [23], M. Matolcsi [30, 31] and S. G. Révész and Y. Sarantopoulos [37] for different approaches and related problems.

In the complex setting, K. Ball [4] proved a stronger result than inequality (1.3), which is known as "the complex plank problem for Hilbert spaces". A few years before K. Ball's result, J. Arias-de-Reyna [2] proved an inequality which is the complex analogue of (1.3). In his paper, the author showed that $n^{-n/2}$ is the best possible constant, because for any orthonormal system $\{z_k\}_{k=1}^n$ it follows that

$$\sup_{\|z\|_{\mathbb{C}^n}=1} |\langle z, z_1\rangle \cdots \langle z, z_n\rangle| = n^{-n/2}.$$

Unfortunately, his proof did not allow him to show if this orthogonal condition on the set $\{z_k\}_{k=1}^n$ is necessary to achieve the minimum value. In the last section, using inequality (1.2), we will be able to prove that a set $\{z_k\}_{k=1}^n$ of unit vectors in a complex Hilbert space H for which $\sup_{\|z\|_H=1} |\langle z, z_1\rangle \cdots \langle z, z_n\rangle|$ is minimum must be an orthonormal system.

2. Polynomials and Bombieri's norm

Let us fix some standard notation. From now on, α will be a multi-index, and we will denote $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\alpha! = \alpha_1! \cdots \alpha_N!$. For $z \in \mathbb{C}^N$, z^α will stand for $z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. As usual, the space of $m-$homogeneous polynomials defined on $\mathbb{C}^N$ will be denoted by $\mathcal{P}(^m\mathbb{C}^N)$. Given $P \in \mathcal{P}(^m\mathbb{C}^N)$, we will write $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$.

We will be concerned with the study of two different norms for P: the uniform norm $\|P\|_{\mathcal{P}} = \sup_{\|z\|=1} |P(z)|$ and Bombieri's norm, defined in [9] as

$$[P]_2 = \left(\sum_{|\alpha|=m} \frac{\alpha!}{m!} |a_\alpha|^2 \right)^{1/2}.$$

Canonically associated with Bombieri's norm there exists an inner product (see [10], [39]). If $P, Q \in \mathcal{P}(^m\mathbb{C}^N)$, $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ and $Q(z) = \sum_{|\alpha|=m} b_\alpha z^\alpha$, then this inner product is

$$[P, Q]_{(m)} = \sum_{|\alpha|=m} \frac{\alpha!}{m!} a_\alpha \overline{b_\alpha}.$$

The following theorem, proved by Beauzamy et al. [9], is crucial for our purposes.

Theorem 2.1 (Bombieri's inequality [9, Theorem 1.2])**.** *Let P, Q be homogeneous polynomials of degrees m, n respectively. Then*

$$[PQ]_2 \geq \sqrt{\frac{m!\, n!}{(m+n)!}}\, [P]_2 [Q]_2.$$

Corollary 2.2. *Let $P_i \in \mathcal{P}(^{k_i}\mathbb{C}^N)$ for $1 \leq i \leq n$. Then*

$$[P_1 \cdots P_n]_2 \geq \sqrt{\frac{k_1! \cdots k_n!}{(k_1 + \ldots + k_n)!}}\, [P_1]_2 \cdots [P_n]_2.$$

Proof. The proof is immediate by induction on n, since

$$[P_1 \cdots P_n P_{n+1}]_2 \geq \sqrt{\frac{(k_1 + \cdots + k_n)!\; k_{n+1}!}{(k_1 + \ldots + k_{n+1})!}}\, [P_1 \cdots P_n]_2\, [P_{n+1}]_2$$

$$\geq \sqrt{\frac{k_1! \cdots k_n!\; k_{n+1}!}{(k_1 + \ldots + k_{n+1})!}}\, [P_1]_2 \cdots [P_n]_2\, [P_{n+1}]_2. \qquad \square$$

Bombieri's inequality was also proved using differential identities based on the following property (see [10, Lemma 9] or [39]): let P and Q be homogeneous polynomials of degrees $m-1$ and m respectively. Then

$$[z_1 P, Q]_{(m)} = \frac{1}{m}\left[P, \frac{\partial Q}{\partial z_1}\right]_{(m-1)}. \tag{2.1}$$

B. Reznick [39] gave an alternative interpretation of $[P]_2$. He worked with forms of degree m in N variables and differential operators associated with them. Namely,

$$\begin{array}{c} P(x_1,\ldots,x_N) = \sum_{i_1,\ldots,i_m}^N a_{i_1,\ldots,i_m} x_{i_1}\cdots x_{i_m} \\ \Big\downarrow \\ P(D) = \sum_{i_1,\ldots,i_m}^N a_{i_1,\ldots,i_m} \frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_m}}. \end{array}$$

The author defined $\|P\|_{(R)} = (\deg(P)!)^{1/2}[P]_2$ and proved that $\|P\|_{(R)} = P(D)\overline{P}$. He also proved that $\|PQ\|_{(R)} \geq \|P\|_{(R)}\,\|Q\|_{(R)}$ (which is equivalent to Bombieri's inequality), where equality holds if and only if P and Q are unitarily disjoint. The problem of finding pairs of polynomials (P,Q) for which $[PQ]_2$ is maximum or minimum was studied in [7], [8], [12] and [39].

2.1. **A different point of view for Bombieri's norm.** This section is devoted to presenting alternative definitions of $[P]_2$ and $\|P\|_{(R)}$. From this, we will obtain a different proof of the differential identity (2.1) using a simple lemma from [34].

Given a measure space (X,μ), we will write $L^p(\mu)$ to denote the space of measurable functions $f: X\to\mathbb{C}$ such that $\int_X |f|^p\,d\mu<\infty$. From now on, Γ_N denotes the Gaussian measure defined on the Borel subsets of $\mathbb{C}^N$ by

$$\Gamma_N(\Delta) = \int_\Delta e^{-\|z\|^2}\,\frac{dz}{\pi^N},$$

where dz stands for the Lebesgue measure on $\mathbb{C}^N$.

Lemma 2.3 ([34, Lemma 2.1])**.** *Let Γ_N be the Gaussian measure defined above. Then*

$$\int_{\mathbb{C}^N} z^\alpha \overline{z}^\beta\, d\Gamma_N(z) = \delta_{\alpha\beta}\alpha!.$$

In the sequel, $L^2_m(\Gamma_N)$ stands for the closure of $\operatorname{span}\{z^\alpha\}_{|\alpha|=m}$ in $L^2(\Gamma_N)$. The previous lemma shows that if $m\neq m'$, then $L^2_m(\Gamma_N)\perp L^2_{m'}(\Gamma_N)$.

Proposition 2.4. *Given $m\in\mathbb{N}$, let $\imath : \big(\mathcal{P}(^m\mathbb{C}^N), [\,\cdot\,]_2\big)\to L^2_m(\Gamma_N)$ be defined by $\imath(P)=P$. Then $\|\imath(P)\|_{L^2(\Gamma_N)} = \sqrt{m!}\,[P]_2$.*

Proof. Note that $|P(z)|\leq \max_{\|\omega\|=1}|P(\omega)|\,\|z\|^m = \|P\|_{\mathcal{P}}\,\|z\|^m$. Thus we have

$$\int_{\mathbb{C}^N} |P(z)|^2 e^{-\|z\|^2}\,\frac{dz}{\pi^N} < \infty.$$

Consequently, the map is well defined. Since $\left\{z^\alpha/\sqrt{\alpha!}\right\}_{|\alpha|=m}$ is an orthonormal basis for the Hilbert space $L^2_m(\Gamma_N)$ and $\left\{\sqrt{m!}\,z^\alpha/\sqrt{\alpha!}\right\}_{|\alpha|=m}$ is an orthonormal basis for the Hilbert space $(\mathcal{P}(^m\mathbb{C}^N), [\,\cdot\,]_2)$, we conclude that $\|\imath(P)\|_{L^2(\Gamma_N)} = \sqrt{m!}\,[P]_2$ for all $P \in \mathcal{P}(^m\mathbb{C}^N)$. $\square$

Remark 2.5. Recall from [39] that $\|P\|_{(R)} = (\deg(P)!)^{1/2}[P]_2$. Thus, given any polynomial $P \in \mathcal{P}(^m\mathbb{C}^N)$, we have $\|P\|_{(R)} = \|P\|_{L^2(\Gamma_N)}$. We can restate Bombieri's inequality in terms of the $L^2(\Gamma_N)$ norm: given $P_i \in \mathcal{P}(^{k_i}\mathbb{C}^N)$ for $1 \le i \le n$, then it follows that

$$\|P_1 \cdots P_n\|_{L^2(\Gamma_N)} \ge \|P_1\|_{L^2(\Gamma_N)} \cdots \|P_n\|_{L^2(\Gamma_N)}.$$

Now, we are able to give a different proof of (2.1). The nature of this new point of view is more analytical than the previous ones.

Lemma 2.2 in [34] gives an integral representation formula for entire functions which are in $L^p(\Gamma_N)$ for some $p > 1$. We only need the following particular case:

Lemma 2.6. *Let $P \in \mathcal{P}(^m\mathbb{C}^N)$. Then for every $z \in \mathbb{C}^N$,*

$$P(z) = \int_{\mathbb{C}^N} e^{\langle z,\omega\rangle} P(\omega)\, d\Gamma_N(\omega).$$

We can use this integral formula to compute the directional derivatives of an $m-$homogeneous polynomial.

Proposition 2.7. *Let $P \in \mathcal{P}(^m\mathbb{C}^N)$ and $v \in \mathbb{C}^N$. Then we have*

$$\frac{\partial P}{\partial v}(z) = \int_{\mathbb{C}^N} e^{\langle z,\omega\rangle}\ \langle v,\omega\rangle\, P(\omega)\, d\Gamma_N(\omega).$$

Proof. Using Lemma 2.6 and the dominated convergence theorem,

$$\frac{\partial P}{\partial v}(z) = \lim_{h\to 0} \int_{\mathbb{C}^N} \frac{e^{h\langle v,\omega\rangle} - 1}{h}\ e^{\langle z,\omega\rangle}\, P(\omega)\, d\Gamma_N(\omega)$$

$$= \int_{\mathbb{C}^N} e^{\langle z,\omega\rangle}\ \langle v,\omega\rangle\, P(\omega)\, d\Gamma_N(\omega). \qquad \square$$

Note that, in general, $\langle v,\omega\rangle\, P(\omega) \notin L^2_{m-1}(\Gamma_N)$ because it is not a holomorphic function. Since $\{L^2_k(\Gamma_N)\}_{k\in\mathbb{N}}$ are closed subspaces of $L^2(\Gamma_N)$, we have a sequence of orthogonal projections $\pi_k : L^2(\Gamma_N) \to L^2_k(\Gamma_N)$. By Proposition 2.4, we know that $\imath : \big(\mathcal{P}(^m\mathbb{C}^N), [\,\cdot\,]_2\big) \to L^2_m(\Gamma_N)$ is an isomorphism for all $m \in \mathbb{N}$, and since

$$\frac{\partial P}{\partial v}(z) = \int_{\mathbb{C}^N} e^{\langle z,\omega\rangle}\ \frac{\partial P}{\partial v}(\omega)\, d\Gamma_N(\omega),$$

we can deduce that $\pi_{m-1}\left(\langle v,\,\cdot\,\rangle P\right) = \dfrac{\partial P}{\partial v}$.

Proposition 2.8 ([10, Lemma 9])**.** *If $P \in \mathcal{P}(^m\mathbb{C}^N)$ and $Q \in \mathcal{P}(^{m-1}\mathbb{C}^N)$, then*

$$[\langle \cdot , v\rangle Q, P]_{(m)} = \frac{1}{m}\left[Q, \frac{\partial P}{\partial v}\right]_{(m-1)}.$$

Proof.

$$\begin{aligned}
\frac{1}{m}\left[Q, \frac{\partial P}{\partial v}\right]_{(m-1)} &= \frac{1}{m}\,\frac{1}{(m-1)!}\left\langle Q, \frac{\partial P}{\partial v}\right\rangle_{L^2_{m-1}(\Gamma_N)} \\
&= \frac{1}{m!}\,\langle Q, \pi_{m-1}\left(\langle v, \cdot\ \rangle\, P\right)\rangle_{L^2_{m-1}(\Gamma_N)} \\
&= \frac{1}{m!}\,\langle Q, \langle v, \cdot\ \rangle\, P\rangle_{L^2(\Gamma_N)} \\
&= \frac{1}{m!}\int_{\mathbb{C}^N} Q(\omega)\,\overline{\langle v, \omega\rangle\, P(\omega)}\, d\Gamma_N(\omega) \\
&= \frac{1}{m!}\int_{\mathbb{C}^N} \langle \omega, v\rangle\, Q(\omega)\,\overline{P(\omega)}\, d\Gamma_N(\omega) \\
&= [\langle \cdot , v\rangle\, Q, P]_{(m)}\,. \qquad \square
\end{aligned}$$

In the following sections we present some applications of this interpretation of Bombieri's norm.

3. A limit problem for $S_n(P)$

In this section we use the relation between Bombieri's norm and $\|\cdot\|_{L^2(\Gamma_N)}$, proved in Proposition 2.4, to work on a problem originally posed by B. Beauzamy in [6].

Letting $P \in \mathcal{P}(^m\mathbb{C}^N)$, we will study the behavior of the sequence of real numbers $S_n(P)$, defined by

$$S_n(P) = \sup\left\{[PQ]_2 : [Q]_2 = 1,\ Q \in \mathcal{P}(^n\mathbb{C}^N)\right\}.$$

Our point of view of Bombieri's norm allows us to compute $\|\cdot\|_{L^2(\Gamma_N)}$ instead of $[\cdot]_2$. The main advantage of this is that we will be able to use classical tools from integration theory.

In the sequel, σ_N denotes the normalized Lebesgue measure on $\partial B_1(\mathbb{C}^N)$, the unit sphere of $\mathbb{C}^N$. To shorten notation, we let ∂B_1 and $L^p_{\partial B_1}$ stand for $\partial B_1(\mathbb{C}^N)$ and $L^p(\partial B_1(\mathbb{C}^N), \sigma_N)$ respectively.

Remark 3.1. Since for homogeneous polynomials we have a link between $[\cdot]_2$ and $\|\cdot\|_{L^2(\Gamma_N)}$, we want to find a way to relate Bombieri's norm of a polynomial (or its powers) and the different values of its $L^r_{\partial B_1}$ norms. With this we aim to use that

$$\lim_{r\to\infty} \|P\|_{L^r_{\partial B_1}} = \|P\|_{L^\infty_{\partial B_1}} = \sup_{z\in\partial B_1} |P(z)| = \|P\|_{\mathcal{P}}.$$

In the rest of this section we need to compute the following integral for different values of $k \in \mathbb{N}$:

$$\int_0^\infty t^{2k}\, e^{-t^2}\, 2t\, dt = \int_0^\infty t^k\, e^{-t}\, dt = k!.$$

Lemma 3.2. *Suppose that $P \in \mathcal{P}(^m\mathbb{C}^N)$ and $r \in \mathbb{N}$. Then*

$$[P^r]_2^2 = \binom{N-1+mr}{N-1} \|P\|^{2r}_{L^{2r}_{\partial B_1}}.$$

Proof. According to Proposition 2.4, since $P^r \in \mathcal{P}(^{rm}\mathbb{C}^N)$ we have

$$[P^r]_2^2 = \frac{1}{(rm)!} \int_{\mathbb{C}^N} |P^r(w)|^2 d\Gamma_N(w).$$

Introducing polar coordinates, we find

$$\begin{aligned}
[P^r]_2^2 &= \frac{|\partial B_1|}{(rm)!} \int_0^\infty \int_{\partial B_1} \rho^{2N-1} \rho^{2mr} e^{-\rho^2} |P(\theta)|^{2r}\, \frac{d\rho\, d\sigma_N(\theta)}{\pi^N} \\
&= \frac{2}{(N-1)!\,(rm)!} \int_0^\infty \int_{\partial B_1} \rho^{2N-1} \rho^{2mr} e^{-\rho^2} |P(\theta)|^{2r}\, d\rho\, d\sigma_N(\theta) \\
&= \left(\frac{1}{(N-1)!\,(rm)!} \int_0^\infty \rho^{2(N-1+mr)} e^{-\rho^2}\, 2\rho\, d\rho \right) \left(\int_{\partial B_1} |P(\theta)|^{2r} d\sigma_N(\theta) \right) \\
&= \frac{(N-1+mr)!}{(N-1)!\,(rm)!} \|P\|^{2r}_{L^{2r}_{\partial B_1}} = \binom{N-1+mr}{N-1} \|P\|^{2r}_{L^{2r}_{\partial B_1}},
\end{aligned}$$

which is the desired conclusion. □

Our main result in this section is the following.

Theorem 3.3. *Let $P \in \mathcal{P}(^m\mathbb{C}^N)$. Then $\limsup_{n\to\infty} S_n(P) = \|P\|_{\mathcal{P}}$.*

Proof. By Proposition 2.4, given $Q \in \mathcal{P}({}^n\mathbb{C}^N)$ we have

$$\left[P\,\frac{Q}{[Q]_2}\right]_2^2 = \frac{[PQ]_2^2}{[Q]_2^2} = \frac{n!}{(m+n)!}\frac{\displaystyle\int_{\mathbb{C}^N}|P(w)Q(w)|^2\,d\Gamma_N(w)}{\displaystyle\int_{\mathbb{C}^N}|Q(w)|^2\,d\Gamma_N(w)}$$

$$= \frac{n!}{(n+m)!}\frac{\displaystyle\int_0^\infty\int_{\partial B_1}\rho^{2N-1}\rho^{2(n+m)}e^{-\rho^2}|P(\theta)\,Q(\theta)|^2\,d\rho\,d\sigma_N(\theta)}{\displaystyle\int_0^\infty\int_{\partial B_1}\rho^{2N-1}\rho^{2n}e^{-\rho^2}|Q(\theta)|^2\,d\rho\,d\sigma_N(\theta)}$$

$$= \frac{n!\,(N-1+m+n)!}{(m+n)!\,(N-1+n)!}\frac{\displaystyle\int_{\partial B_1}|P(\theta)|^2\,|Q(\theta)|^2\,d\sigma_N(\theta)}{\displaystyle\int_{\partial B_1}|Q(\theta)|^2\,d\sigma_N(\theta)}$$

$$\leq \frac{n!\,(N-1+m+n)!}{(m+n)!\,(N-1+n)!}\frac{\displaystyle\sup_{\zeta\in\partial B_1}|P(\zeta)|^2\int_{\partial B_1}|Q(\theta)|^2\,d\sigma_N(\theta)}{\displaystyle\int_{\partial B_1}|Q(\theta)|^2\,d\sigma_N(\theta)}$$

$$= \frac{n!\,(N-1+m+n)!}{(m+n)!\,(N-1+n)!}\,\|P\|_{\mathcal{P}}^2,$$

which is equivalent to

$$\left[P\,\frac{Q}{[Q]_2}\right]_2 \leq \sqrt{\frac{n!\,(N-1+m+n)!}{(m+n)!\,(N-1+n)!}}\,\|P\|_{\mathcal{P}}.$$

Note that

$$\frac{n!\,(N-1+m+n)!}{(m+n)!\,(N-1+n)!} = \frac{\prod_{j=1}^m(N-1+n+j)}{\prod_{j=1}^m(n+j)} = \frac{n^m+o(n^m)}{n^m+o(n^m)},$$

where $o(n^m)$, as usual, means that $\dfrac{o(n^m)}{n^m}\underset{n\to\infty}{\longrightarrow}0$. From this, we conclude that

$$\frac{n!\,(N-1+m+n)!}{(m+n)!\,(N-1+n)!}\underset{n\to\infty}{\longrightarrow}1.$$

Thus for any $\varepsilon>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that for any $n\in\mathbb{N},\ n>n_\varepsilon$, and any $Q\in\mathcal{P}({}^n\mathbb{C}^N)$, we have

$$\left[P\,\frac{Q}{[Q]_2}\right]_2 \leq (1+\varepsilon)\|P\|_{\mathcal{P}}.$$

Hence $\limsup\limits_{n\to\infty} S_n(P) \leq \|P\|_{\mathcal{P}}$.

In order to prove the converse inequality consider the sequence of polynomials $\{P^r\}_{r\in\mathbb{N}}$. According to Lemma 3.2, we have

$$\left[P\,\frac{P^r}{[P^r]_2}\right]_2^2=\frac{[P^{r+1}]_2^2}{[P^r]_2^2}=\frac{\binom{N-1+m(r+1)}{N-1}\|P\|^{2(r+1)}_{L^{2(r+1)}_{\partial B_1}}}{\binom{N-1+mr}{N-1}\|P\|^{2r}_{L^{2r}_{\partial B_1}}}.$$

First, note that

$$\frac{\binom{N-1+m(r+1)}{N-1}}{\binom{N-1+mr}{N-1}}=\frac{\prod_{j=1}^m(N-1+mr+j)}{\prod_{j=1}^m(mr+j)}=\frac{(mr)^m+o(r^m)}{(mr)^m+o(r^m)}\underset{r\to\infty}{\longrightarrow}1.$$

Also, since for any probability space (X,μ) and any measurable function $f:X\to\mathbb{C}$ it follows that $\|f\|_{L^p(\mu)}\le\|f\|_{L^q(\mu)}$ whenever $p\le q$,

$$\frac{\|P\|^{2(r+1)}_{L^{2(r+1)}_{\partial B_1}}}{\|P\|^{2r}_{L^{2r}_{\partial B_1}}}=\|P\|^2_{L^{2(r+1)}_{\partial B_1}}\left(\frac{\|P\|_{L^{2(r+1)}_{\partial B_1}}}{\|P\|_{L^{2r}_{\partial B_1}}}\right)^{2r}\ge\|P\|^2_{L^{2(r+1)}_{\partial B_1}}\underset{r\to\infty}{\longrightarrow}\|P\|^2_{L^\infty_{\partial B_1}}.$$

Therefore,

$$\limsup_{r\to\infty}\left[P\,\frac{P^r}{[P^r]_2}\right]_2^2\ge\|P\|^2_{L^\infty_{\partial B_1}}=\|P\|^2_{\mathcal{P}},$$

and we conclude that $\limsup_{n\to\infty}S_n\ge\|P\|_{\mathcal{P}}$. □

4. Lower bounds for products of polynomials on Hilbert spaces

We begin this section by recalling that a continuous $m-$homogeneous polynomial from a Banach space E to the scalar field $\mathbb{K}$ is a mapping $P:E\to\mathbb{K}$ for which there exists a (unique) continuous symmetric $m-$linear form $\check{P}:E^m\to\mathbb{K}$ such that $P(z)=\check{P}(z,\dots,z)$ for all $z\in E$. The space of continuous symmetric m-linear forms $L:E^m\to\mathbb{K}$, denoted by $\mathcal{L}^s({}^mE)$, is a Banach space under the norm $\|L\|_{\mathcal{L}^s}=\sup_{\|z_j\|_E=1}|L(z_1,\dots,z_m)|$. Moreover, the mapping $P\mapsto\check{P}$ is an isomorphism between the Banach spaces $(\mathcal{P}({}^mE),\|\cdot\|_{\mathcal{P}})$ and $(\mathcal{L}^s({}^mE),\|\cdot\|_{\mathcal{L}^s})$. For a thorough treatment we refer the reader to [20].

Now we will turn our attention to studying inequality (1.1) for polynomials defined on Hilbert spaces. Here and subsequently, H stands for an infinite dimensional complex Hilbert space and $\{e_i\}_{i=1}^\infty$ denotes a fixed orthonormal basis for H.

Given $P_i\in\mathcal{P}({}^{k_i}H)$ for $1\le i\le n$, we are interested in finding the optimum constant M, depending only on $k_1,\dots,k_n$, for which the inequality

$$\|P_1\|_{\mathcal{P}}\cdots\|P_n\|_{\mathcal{P}}\le M\|P_1\cdots P_n\|_{\mathcal{P}}$$

holds.

The following theorem presents an inequality for norms of products of polynomials on $\mathbb{C}^N$. The value of the proposed constant is sharp for $n\le N$.

Theorem 4.1. *Let $P_i\in\mathcal{P}({}^{k_i}\mathbb{C}^N)$ for $i=1,\dots,n$. Then*

$$\|P_1\|_{\mathcal{P}}\cdots\|P_n\|_{\mathcal{P}}\le\sqrt{\frac{(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1}\cdots k_n^{k_n}}}\;\|P_1\cdots P_n\|_{\mathcal{P}}.$$

Proof. According to Lemma 3.2, we have

$$[(P_1 \cdots P_n)^r]_2^2 = \binom{N-1+r\sum_{i=1}^n k_i}{N-1} \|P_1 \cdots P_n\|^{2r}_{L^{2r}_{\partial B_1}}.$$

Also, for $1 \le i \le n$,

$$[P_i^r]_2^2 = \binom{N-1+rk_i}{N-1} \|P_i\|^{2r}_{L^{2r}_{\partial B_1}}.$$

Applying Bombieri's inequality, we obtain

$$[P_1^r]_2 \cdots [P_n^r]_2 \le \sqrt{\frac{(r\sum_{i=1}^n k_i)!}{(rk_1)! \cdots (rk_n)!}}\ [(P_1 \cdots P_n)^r]_2,$$

or equivalently

$$\begin{aligned} \prod_{i=1}^n \|P_i\|_{L^{2r}_{\partial B_1}} &= \sqrt[2r]{\frac{\prod_{i=1}^n [P_i^r]_2^2}{\prod_{i=1}^n \binom{N-1+rk_i}{N-1}}} \\ &\le \sqrt[2r]{\frac{(r\sum_{i=1}^n k_i)!\ [(P_1 \cdots P_n)^r]_2^2}{\prod_{i=1}^n (rk_i)!\ \binom{N-1+rk_i}{N-1}}} \\ &= \sqrt[2r]{\frac{(r\sum_{i=1}^n k_i)! \binom{N-1+r\sum_{i=1}^n k_i}{N-1}}{\prod_{i=1}^n (rk_i)!\ \binom{N-1+rk_i}{N-1}}}\ \|P_1 \cdots P_n\|_{L^{2r}_{\partial B_1}} \\ &= \sqrt[2r]{\frac{((N-1)!)^{n-1}\ (N-1+r\sum_{i=1}^n k_i)!}{\prod_{i=1}^n (N-1+rk_i)!}}\ \|P_1 \cdots P_n\|_{L^{2r}_{\partial B_1}}. \end{aligned} \tag{4.1}$$

It is well known that given a sequence of positive real numbers $\{a_r\}_{r\in\mathbb{N}}$, if $\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = L$, then $\lim_{r\to\infty} \sqrt[r]{a_r} = L$. For $s = \sum_{i=1}^n k_i$, let us choose

$$a_r = \sqrt{\frac{((N-1)!)^{n-1}\ (N-1+rs)!}{\prod_{i=1}^n (N-1+rk_i)!}}.$$

For this sequence we may compute

$$\begin{aligned} \frac{a_{r+1}}{a_r} &= \sqrt{\frac{(N-1+(r+1)s)!\ \prod_{i=1}^n (N-1+rk_i)!}{(N-1+rs)!\ \prod_{i=1}^n (N-1+(r+1)k_i)!}} \\ &= \sqrt{\frac{\prod_{j=1}^s (N-1+rs+j)}{\prod_{i=1}^n \left(\prod_{j=1}^{k_i} (N-1+rk_i+j)\right)}} \\ &= \sqrt{\frac{(rs)^s + o(r^s)}{(\prod_{i=1}^n (rk_i)^{k_i}) + o(r^s)}} = \sqrt{\frac{r^s s^s + o(r^s)}{r^s \left(\prod_{i=1}^n k_i^{k_i}\right) + o(r^s)}}. \end{aligned}$$

From this we deduce that

$$\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = \sqrt{\frac{(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1}\cdots k_n^{k_n}}}.$$

Hence

$$\lim_{r\to\infty} \sqrt[r]{a_r} = \lim_{r\to\infty} \sqrt[2r]{\frac{((N-1)!)^{n-1}\ (N-1+r\sum_{i=1}^n k_i)!}{\prod_{i=1}^n (N-1+rk_i)!}}$$

$$= \sqrt{\frac{(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1}\cdots k_n^{k_n}}}.$$

Because of Remark 3.1, letting $r\to\infty$ in inequality (4.1) we can assert that

$$\|P_1\|_{\mathcal{P}}\cdots\|P_n\|_{\mathcal{P}} \le \sqrt{\frac{(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1}\cdots k_n^{k_n}}}\ \|P_1\cdots P_n\|_{\mathcal{P}}. \qquad \square$$

Remark 4.2. The bound proved above is independent of the number of variables. If $n \le N$, then the inequality is sharp. Given any set of natural numbers $\{k_1,\dots,k_n\}$, if we define the polynomials $P_i(z) = z_i^{k_i}$ for $1 \le i \le n$, then

$$\|P_1\|_{\mathcal{P}}\cdots\|P_n\|_{\mathcal{P}} = \sqrt{\frac{(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1}\cdots k_n^{k_n}}}\ \|P_1\cdots P_n\|_{\mathcal{P}}.$$

Clearly, $\|P_i\|_{\mathcal{P}} = 1$ for all $1 \le i \le n$. On the other hand,

$$\sup_{\|z\|=1} |z_1^{k_1}\cdots z_n^{k_n}| = \max\left\{ x_1^{k_1}\cdots x_n^{k_n} : \{x_i\}_{i=1}^n \subset \mathbb{R}_{\ge 0}\ \wedge\ \sum_{i=1}^n x_i^2 = 1 \right\}.$$

Applying Lagrange multipliers, the maximum of this function is attained at a point x whose coordinates are $x_i = \sqrt{k_i/(k_1+\cdots+k_n)}$ for $1 \le i \le n$ and $x_i = 0$ for $n < i \le N$. Therefore

$$\|P_1\cdots P_n\|_{\mathcal{P}} = \sqrt{\frac{k_1^{k_1}\cdots k_n^{k_n}}{(k_1+\cdots+k_n)^{(k_1+\cdots+k_n)}}}.$$

In order to extend this result to the infinite dimensional setting we need the following lemma.

Lemma 4.3. *Let $Q \in \mathcal{P}(^m H)$ and let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for H. If we define the sequence of polynomials*

$$\widetilde{Q_l} : \mathbb{C}^l \to \mathbb{C},$$

$$\widetilde{Q_l}(z_1,\dots,z_l) = Q\left(\sum_{i=1}^l z_i e_i\right),$$

then $\|\widetilde{Q_l}\|_{\mathcal{P}} \underset{l\to\infty}{\longrightarrow} \|Q\|_{\mathcal{P}}$.

Proof. It is clear that $\|\widetilde{Q_l}\|_{\mathcal{P}} \leq \|Q\|_{\mathcal{P}}$ for all $l \in \mathbb{N}$. Let us prove the converse inequality. Given any $\varepsilon > 0$, choose a unit vector $z \in H$ such that $|Q(z)| > \|Q\|_{\mathcal{P}} - \varepsilon$ and let $z_i = \langle z, e_i \rangle$ for $i \geq 1$. Since $\sum_{i=1}^{\infty} |z_i|^2 = \|z\|^2 = 1$, we have

$$\|\widetilde{Q_l}\|_{\mathcal{P}} \geq \left|\widetilde{Q_l}(z_1, \ldots, z_l)\right| = \left| Q\left(\sum_{i=1}^{l} \langle z, e_i\rangle\, e_i\right)\right|.$$

By continuity, since $\lim_{l\to\infty} \sum_{i=1}^{l} \langle z, e_i\rangle\, e_i = z$,

$$\lim_{l\to\infty} \left| Q\left(\sum_{i=1}^{l} \langle z, e_i\rangle\, e_i\right)\right| = |Q(z)| > \|Q\|_{\mathcal{P}} - \varepsilon.$$

Hence, there exists $l_0 \in \mathbb{N}$ such that

$$\|Q\|_{\mathcal{P}} - \frac{\varepsilon}{2} < \|\widetilde{Q_l}\|_{\mathcal{P}} \leq \|Q\|_{\mathcal{P}}$$

for all $l \geq l_0$. □

We are now ready to prove a sharp lower bound for the norm of the product of continuous homogeneous polynomials defined on a complex Hilbert space H.

Theorem 4.4. *Let $P_i \in \mathcal{P}(^{k_i}H)$ for $1 \leq i \leq n$. Then*

$$\|P_1\|_{\mathcal{P}} \cdots \|P_n\|_{\mathcal{P}} \leq \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}\; \|P_1 \cdots P_n\|_{\mathcal{P}},$$

and this inequality is sharp.

Proof. As we did in Lemma 4.3, for $l \in \mathbb{N}$ and $1 \leq i \leq n$, let us define

$$\widetilde{P_{l,i}} : \mathbb{C}^l \to \mathbb{C},$$

$$\widetilde{P_{l,i}}\,(z_1, \ldots, z_l) = P_i\left(\sum_{j=1}^{l} z_j e_j\right).$$

From Theorem 4.1 we know that

$$\|\widetilde{P_{l,1}}\|_{\mathcal{P}} \cdots \|\widetilde{P_{l,n}}\|_{\mathcal{P}} \leq \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}\; \|\widetilde{P_{l,1}} \cdots \widetilde{P_{l,n}}\|_{\mathcal{P}}.$$

We may now let $l \to \infty$, and the result follows by Lemma 4.3.

The ideas used in Remark 4.2 allow us to see that given $\{k_i\}_{i=1}^{n} \subset \mathbb{N}$, if we define the polynomials $P_i(z) = \langle z, e_i\rangle^{k_i}$ for $1 \leq i \leq n$, then

$$\|P_1\|_{\mathcal{P}} \cdots \|P_n\|_{\mathcal{P}} = \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}\; \|P_1 \cdots P_n\|_{\mathcal{P}}.$$

Thus this bound is sharp. □

It is worth pointing out that working on Hilbert spaces the isomorphism $P \mapsto \check{P}$ is an isometry between $\mathcal{P}(^mH)$ and $\mathcal{L}^s(^mH)$ (see for instance [5] and [20]). We may use this fact to derive an inequality for continuous symmetric m-linear mappings from H^m to $\mathbb{C}$.

Given $\Phi_j \in \mathcal{L}^s({}^{k_j}H)$ for $1 \le j \le n$, set $k = \sum_{j=1}^n k_j$. Let us define the symmetrized product $(\Phi_1 \cdots \Phi_n)_s(z_1, \dots, z_k)$ of $\{\Phi_j\}_{j=1}^n$ by

$$\frac{1}{k!} \sum_{\tau \in \mathcal{G}_k} \Phi_1(z_{\tau(1)}, \dots, z_{\tau(k_1)}) \Phi_2(z_{\tau(k_1+1)}, \dots, z_{\tau(k_1+k_2)}) \cdots \Phi_n(z_{\tau(k-k_n+1)}, \dots, z_{\tau(k)}),$$

where $\mathcal{G}_k$ denotes the permutation group of k elements.

Corollary 4.5. *Let* $\Phi_j \in \mathcal{L}^s({}^{k_j}H)$ *for* $1 \le j \le n$. *Then*

$$\|\Phi_1\|_{\mathcal{L}^s} \cdots \|\Phi_n\|_{\mathcal{L}^s} \le \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}} \, \|(\Phi_1 \cdots \Phi_n)_s\|_{\mathcal{L}^s},$$

and this inequality is sharp.

The first step in obtaining these sharp inequalities on complex Hilbert spaces was taken in Proposition 2.4. We do not have a similar argument for the real case. However, if K is a real Hilbert space, we can give estimates for the norm of products of polynomials or symmetric multilinear mappings using the natural complexification of K (see for instance [40, pp. 313–314] and [32] for more details).

Corollary 4.6. *Let* $P_j \in \mathcal{P}({}^{k_j}K)$ *for* $1 \le j \le n$. *Then*

$$\|P_1\|_{\mathcal{P}} \cdots \|P_n\|_{\mathcal{P}} \le 2^{[(k_1+\cdots+k_n)-2]/2} \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}} \, \|P_1 \cdots P_n\|_{\mathcal{P}}.$$

Proof. Let $H = K \oplus iK$ be the natural complexification of K. Given $P_j \in \mathcal{P}({}^{k_j}K)$ for $1 \le j \le n$, let $Q_j \in \mathcal{P}({}^{k_j}H)$ be the unique complex extension of P_j. It is known that (see [32, Prop. 19])

$$\|P_j\|_{\mathcal{P}} \le \|Q_j\|_{\mathcal{P}} = \sup_{\|x+iy\|_H = 1} |Q_j(x+iy)| \le 2^{(k_j-2)/2} \|P_j\|_{\mathcal{P}}.$$

Therefore

$$\|P_1\|_{\mathcal{P}} \cdots \|P_n\|_{\mathcal{P}} \le \|Q_1\|_{\mathcal{P}} \cdots \|Q_n\|_{\mathcal{P}} \le \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}} \, \|Q_1 \cdots Q_n\|_{\mathcal{P}}$$

$$\le 2^{[(k_1+\cdots+k_n)-2]/2} \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}} \, \|P_1 \cdots P_n\|_{\mathcal{P}}.$$

□

Remark 4.7. Let $\phi_j : \mathbb{R}^n \to \mathbb{R}$ be linear functionals for $1 \le j \le n \le 5$. In [33], A. Pappas and S. G. Révész proved that

$$\|\phi_1\|_{\mathcal{P}} \cdots \|\phi_n\|_{\mathcal{P}} \le n^{n/2} \, \|\phi_1 \cdots \phi_n\|_{\mathcal{P}}.$$

Consequently, it is clear that we cannot expect a sharp inequality in Corollary 4.6.

Finally, we give a real version of Corollary 4.5.

Corollary 4.8. *Let* $\Phi_j \in \mathcal{L}^s({}^{k_j}K)$ *for* $1 \le j \le n$. *Then*

$$\|\Phi_1\|_{\mathcal{L}^s} \cdots \|\Phi_n\|_{\mathcal{L}^s} \le 2^{[(k_1+\cdots+k_n)-2]/2} \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}} \, \|(\Phi_1 \cdots \Phi_n)_s\|_{\mathcal{L}^s}.$$

5. Products of bounded linear forms on complex Hilbert spaces

Given a Banach space E, let E' denote its dual space. In [13], the authors defined the nth linear polarization constant of E by

$$c_n(E) = \inf\{M > 0 : \|\phi_1\|_{E'} \cdots \|\phi_n\|_{E'} \leq M \|\phi_1 \cdots \phi_n\|_{\mathcal{P}}, \text{ for all } \phi_1, \ldots, \phi_n \in E'\}$$

and the linear polarization constant of E by $c(E) = \limsup c_n^{1/n}(E)$. Also, in [37], it was proved that given $n \in \mathbb{N}$, then infinite dimensional Hilbert spaces have the smallest nth polarization constant.

If K is a real Hilbert space it was conjectured in [13] that $c_n(K) = n^{n/2}$. As we mentioned earlier, a positive answer was given in [33] for $n \leq 5$, but the remaining cases are still open problems. For complex Hilbert spaces this conjecture is true. The following theorem is due to J. Arias-de-Reyna.

Theorem 5.1 ([2, Theorem 4]). *Let $\{z_k\}_{k=1}^n$ be unit vectors in a complex Hilbert space H. There is a unit vector $z \in H$ such that $|\langle z, z_1\rangle \cdots \langle z, z_n\rangle| \geq n^{-n/2}$.*

In the proof, the author showed that $n^{-n/2}$ is the best possible constant because for any orthonormal system $\{z_k\}_{k=1}^n$,

$$\sup_{\|z\|=1} |\langle z, z_1\rangle \cdots \langle z, z_n\rangle| = n^{-n/2}.$$

However, he did not prove that the equality holds only for orthonormal systems.

Note that for linear functionals on a complex Hilbert space the inequality obtained in Theorem 4.4 is the same one that was proved by J. Arias-de-Reyna. Since our inequality allows us to handle nonlinear factors, we can use it to prove that the orthonormality of the set $\{z_k\}_{k=1}^n$ is a necessary condition to achieve this minimum value. In the following lemma we will prove this assertion for the special case when $n = 2$. This particular result will be needed for the general case.

Lemma 5.2. *Let z_1, z_2 be unit vectors in a complex Hilbert space. If*

$$\sup_{\|z\|=1} |\langle z, z_1\rangle \langle z, z_2\rangle| = \frac{1}{2},$$

then $\langle z_1, z_2\rangle = 0$.

Proof. First, note that $|\langle z, z_1\rangle \langle z, z_2\rangle| = |\langle z, z_1\rangle \langle z, -z_2\rangle|$. Hence we can assume, without loss of generality, that $\Re\langle z_1, z_2\rangle \geq 0$.

In order to have a lower estimate for the supremum we want to compute

$$\left|\left\langle \frac{z_1 + z_2}{\|z_1 + z_2\|}, z_1 \right\rangle \left\langle \frac{z_1 + z_2}{\|z_1 + z_2\|}, z_2 \right\rangle\right| = \frac{|(1 + \langle z_2, z_1\rangle)(1 + \langle z_1, z_2\rangle)|}{\langle z_1 + z_2, z_1 + z_2\rangle}$$

$$= \frac{|1 + \langle z_1, z_2\rangle|^2}{2 + 2\Re\langle z_1, z_2\rangle}.$$

Thus we obtain

$$\sup_{\|z\|=1} |\langle z, z_1\rangle \langle z, z_2\rangle| \geq \left| \left\langle \frac{z_1+z_2}{\|z_1+z_2\|}, z_1 \right\rangle \left\langle \frac{z_1+z_2}{\|z_1+z_2\|}, z_2 \right\rangle \right|$$

$$= \frac{[1+\Re\langle z_1, z_2\rangle]^2 + [\Im\langle z_1, z_2\rangle]^2}{2+2\Re\langle z_1, z_2\rangle}$$

$$= \frac{1}{2}\left(1+\Re\langle z_1, z_2\rangle + \frac{[\Im\langle z_1, z_2\rangle]^2}{1+\Re\langle z_1, z_2\rangle}\right),$$

which is strictly greater than $1/2$ unless $\langle z_1, z_2\rangle = 0$. □

Finally, we can prove the general case.

Theorem 5.3. *Let* $\{z_k\}_{k=1}^n$ *be unit vectors in a complex Hilbert space* H. *If*

$$\sup_{\|z\|=1} |\langle z, z_1\rangle \cdots \langle z, z_n\rangle| = n^{-n/2},$$

then $\{z_k\}_{k=1}^n$ *is an orthonormal system.*

Proof. Let $P_k(z) = \langle z, z_k\rangle$ for $1 \leq k \leq n$, and $P = P_1 \cdots P_n$. We will show that $\langle z_{n-1}, z_n\rangle = 0$, but the same idea will give us $\langle z_i, z_j\rangle = 0$ for any $1 \leq i < j \leq n$. We have

$$\begin{aligned} 1 = \|P_1\|_{\mathcal{P}} \cdots \|P_n\|_{\mathcal{P}} &= \sqrt{n^n}\,\|P\|_{\mathcal{P}} \\ &\geq \sqrt{n^n}\sqrt{\frac{2^2}{n^n}}\|P_1\|_{\mathcal{P}} \cdots \|P_{n-2}\|_{\mathcal{P}}\,\|P_{n-1}P_n\|_{\mathcal{P}} \\ &= 2\,\|P_{n-1}P_n\|_{\mathcal{P}} \geq 2\sqrt{\frac{1}{2^2}}\,\|P_{n-1}\|_{\mathcal{P}}\,\|P_n\|_{\mathcal{P}} = 1. \end{aligned}$$

In particular, $\|P_{n-1}P_n\|_{\mathcal{P}} = 1/2$ and so, from Lemma 5.2, we conclude that $\langle z_{n-1}, z_n\rangle = 0$. □

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