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VARIATIONAL REDUCTION OF LAGRANGIAN SYSTEMS WITH GENERAL CONSTRAINTS

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Abstract. In this paper we present an alternative procedure for reducing, in the Lagrangian formalism, the equations of motion of first order constrained mechanical systems with symmetry. The procedure involves two principal connections: one of them is used to define the reduced degrees of freedom and the other one to decompose variations into horizontal and vertical components. On the one hand, we show that this new procedure is particularly useful when the configuration space is a trivial principal bundle over the symmetry group, which is the case of many interesting examples. On the other hand, based on that procedure, we extend in a natural way the variational reduction methods to the Lagrangian systems with higher order constraints. Examples are discussed in order to illustrate the involved theorethical constructions.

1. **Introduction.** In Reference [7], the Lagrangian reduction of *generalized nonholonomic systems* (GNHS) was studied. Let us roughly describe how such a procedure works.

Let the triple

$$(L, C_K, C_V)$$

be a GNHS[1] in the restricted sense of Ref. [7]. Thus, L denotes a Lagrangian function on a configuration manifold Q, $C_K \subset TQ$ a submanifold defining kinematic constraints, and $C_V \subset TQ$ a distribution defining constraints in variations (or equivalently, defining the subspace where constraint forces live). The case of standard nonholonomic systems is obtained when $C_K = C_V$. If $dimC_V = v$, the equations of motion of the triple are given by the kinematic constraint equations, defined by C_K, together with a system of v second order ordinary differential equations (ODE), defined by L and C_V. In this discussion, we will not take into account the kinematic constraints, because they can be studied in a completely separated way.

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[1]See [2, 8, 10, 27] for basic definitions and examples.

Suppose that the system (L, C_K, C_V) is invariant under the action on Q of a Lie group G, with Lie algebra $\mathfrak{g}$, and that

$$Q \to \mathcal{X} = Q/G$$

is a principal fiber bundle. Then, fixing a principal connection

$$A : TQ \to \mathfrak{g},$$

the above mentioned system of v second order ODEs, whose unknown is a curve living inside Q, can be transformed into

- a set of v lower order ODEs whose unknown is a curve inside $T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$: the *reduced equations*,[2]
- and the so-called *reconstruction equations*, which we will not discuss here.

(As usual, $\widetilde{\mathfrak{g}}$ denotes the associated adjoint bundle.) The connection A is simply used to identify TQ/G and $T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$ *via* its related Atiyah isomorphism. In addition, if we choose A such that

$$C_V = (C_V \cap \mathcal{H}) \oplus (C_V \cap \mathcal{V}), \tag{1}$$

being $\mathcal{H}$ and $\mathcal{V}$ the horizontal and vertical subspaces (defined by A), and writing $v_{\mathcal{H}} = dim\,(C_V \cap \mathcal{H})$ and $v_{\mathcal{V}} = dim\,(C_V \cap \mathcal{V})$, the reduced equations decompose, in turn, into two parts:

- $v_{\mathcal{H}}$ second order ODEs
- plus $v_{\mathcal{V}}$ first order ODEs.

They are similar to the equations appearing in Ref. [14] (see also [3]), where Lagrangian reduction of standard nonholonomic systems was studied. Following the last reference, such equations were called *horizontal and vertical* ***generalized*** *Lagrange-d'Alembert-Poincaré equations* in [7]. They are defined by a function

$$l : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to \mathbb{R},$$

the *reduced Lagrangian*, the quotient C_V/G, the *reduced variations*, and by the connection A.

It is worth mentioning that A only depends on C_V (and not on C_K), because it is constructed in such a way that variations can be decomposed into independent horizontal and vertical terms [see Eq. (1)].

One difficulty with this reduction procedure is that, in order to give, for a concrete system, an explicit expression of its reduced equations, one has to calculate (beside the curvature B of A) several covariant derivatives of the reduced Lagrangian l, even when $Q \to \mathcal{X}$ is a trivial bundle. These calculations use to be too laborious. On the other hand, this procedure can not be applied to mechanical systems subjected to more general constraints, as the *higher order constrained systems* (HOCS) (see [9] and [10]). The latter are given by triples (L, C_K, C_V) where

$$C_K \subset T^{(k)}Q \quad \text{and} \quad C_V \subset T^{(l)}Q \times TQ$$

are submanifolds[3] such that, for each element $\zeta \in T_q^{(l)}Q$, the subset

$$C_V(\zeta) \equiv C_V \cap (\{\zeta\} \times T_q Q),$$

[2]The order of some of these equations is 1. This is why they are called *reduced equations*. One says that the symmetry *reduces* the order of some equations, i.e. the symmetry enable us to partially *integrate* the original system ODEs.

[3]By $T^{(r)}Q$ we are denoting the r-th order tangent bundle of Q (see [16, 17] for a review).

naturally identified with a subset of T_qQ, is a subspace. Then, by using a connection, we can not make a decomposition of $C_V(\zeta)$ into horizontal and vertical parts. We would need something depending on ζ.

In this paper, we shall consider an alternative procedure for Lagrangian reduction that uses two connection-like objects instead of only one. One connection is used to identify TQ/G and $T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$, and the other connection to decompose the reduced equations into independent horizontal and vertical terms. This allows us:

1. to easily write, for each concrete system where $Q \to \mathcal{X}$ is a trivial bundle, an explicit expression of the reduced equations (also useful in the case of standard nonholonomic systems);
2. to develop a reduction procedure for HOCSs.

A similar idea has been implemented for the variational reduction of discrete (generalized) nonholonomic systems (see Ref. [18]), where also two connection-like objects are used.

It is worth mentioning that we shall restrict ourself to the Lagrangian formalism only. Another approach to the problem of reducing constrained systems with symmetry, but in the symplectic setting of the Hamiltonian formalism, can be found in Ref [6]. In a forthcoming paper, we shall study the Hamiltonian counterpart of the variational reduction developed here (extending the results of [12] to HOCSs), comparing such a variational approach with the symplectic one of Ref. [6].

The organization of the paper is as follows. In Section 2 we recall the usual variational reduction procedure for holonomic, standard nonholonomic and generalized nonholonomic systems, and we present the alternative procedure mentioned above. In Section 3 we study both procedures (the usual and the alternative) on trivial principal bundles, illustrating how they work on a concrete example. We show (at least in that example) that the alternative procedure is particularly useful when trivial bundles are involved, in the sense that the derivation of reduced equations is much simpler for the alternative procedure than for the usual one. Finally, in Section 4, we extend the alternative variational reduction for higher order constrained systems (HOCS). This drives us to the definition of a connection-like object

$$A : T^{(l)}Q \times_Q TQ \to \mathfrak{g},$$

whose properties and related Atiyah-like isomorphism are studied. The case of trivial bundles is again analyzed carefully. At the end of the section we show how the procedure works on a particular HOCS.

We assume that the reader is familiar with basic concepts of Differential Geometry (see [4, 25, 29]) as well as the ideas of Lagrangian systems with symmetry in the context of the Geometric Mechanics (see [1, 28]).

2. **Reduction of GNHSs.** In this section we shall review the main results of [7] and then we shall formulate, based on such results, a new reduction procedure for generalized nonholonomic systems (GNHS).

2.1. **GNHSs with symmetry.** Motivated by mechanical systems such as rubber wheels and certain servomechanisms, where d'Alembert's principle is typically violated, it was defined and studied in Refs. [2, 8, 10, 27] a class of dynamical systems that include the mentioned ones and encode, in our opinion, their main features. We recall that definition below.

Definition 2.1. Given a manifold Q, let us consider the triples (L, C_K, C_V) with

$$L : TQ \to \mathbb{R}, \quad C_K, C_V \subset TQ,$$

where C_K is a submanifold of TQ and C_V is a distribution on Q. We shall refer to them as **generalized nonholonomic systems (GNHS)**, with Lagrangian function L, **kinematic constraints** C_K and **variational constraints** C_V. The elements of C_V will be called **virtual displacements**. We shall say that $\gamma : [t_1, t_2] \to Q$ is a **trajectory** of (L, C_K, C_V) if $\gamma'(t) \in C_K$, and for all infinitesimal variations[4] $\delta\gamma$, such that $\delta\gamma(t) \in C_V$, we have

$$\int_{t_1}^{t_2} \langle dL(\gamma'(t)), \kappa(\delta\gamma'(t))\rangle \, dt = 0.$$

By $\gamma' : [t_1, t_2] \to TQ$ we are denoting the *velocity* of γ, defined as

$$\gamma'(t) = \frac{d}{dt}\gamma(t) = \gamma_*\left(d/dt|_t\right) \in T_{\gamma(t)}Q;$$

and by κ the *canonical involution* $\kappa : TTQ \to TTQ$ (see [19, 35]).

Remark 1. Triples above presented actually constitute a subclass of those defined in [8] (identified there as the $l = 0$ subclass). Here, we choose the above definition since in this section we are going to focus on such triples only. The rest of the systems appearing in [8] (the $l = 1$ subclass) will be studied in the last section of the paper, within a more general setting: the higher order constrained systems.

For a physical interpretation, applications and examples, see Refs. [8, 10, 21, 27, 30].

Suppose that a Lie group G acts on Q, with (left) action $\rho : G \times Q \to Q$, and let us consider its lifted action given by

$$\widetilde{\rho} : G \times TQ \to TQ : (g, v) \mapsto (\rho_g)_*(v).$$

Definition 2.2. We shall say the triple (L, C_K, C_V) is G-**invariant** if, for all $g \in G$,

a.: $L \circ (\rho_g)_* = L$,
b.: $(\rho_g)_*(C_K) = C_K$ and $(\rho_g)_*(C_V) = C_V$,
with $\rho_g : Q \to Q : q \mapsto \rho(g, q)$.

A similar definition can be formulated for right actions.

From the canonical projection $p : TQ \twoheadrightarrow TQ/G$ we can define the **reduced Lagrangian** $l : TQ/G \to \mathbb{R}$ by the formula

$$l \circ p = L, \tag{2}$$

[4] Recall that, given a curve $\gamma\colon [t_1, t_2] \to Q$, an infinitesimal variation of γ is a curve $\delta\gamma\colon [t_1, t_2] \to TQ$ satisfying
a.: $\delta\gamma(t) \in T_{\gamma(t)}Q$, $\forall t \in [t_1, t_2]$;
b.: $\delta\gamma(t_1), \delta\gamma(t_2)$ belongs to the null distribution on Q.

and the **reduced constraints**

$$\mathfrak{C}_K = p(C_K) = C_K/G \quad \text{and} \quad \mathfrak{C}_V = p(C_V) = C_V/G. \tag{3}$$

We shall assume that TQ/G is a manifold and p a submersion.

2.2. **Usual variational reduction procedure.** The aim of this subsection is to write the equations of motion of (L, C_K, C_V) in terms of the *reduced data* l, $\mathfrak{C}_K$ and $\mathfrak{C}_V$. More precisely, by introducing an appropriate principal connection, we are going to separate the reduced virtual displacements $\mathfrak{C}_V$ into horizontal and vertical components, and construct the *horizontal and vertical generalized Lagrange–d'Alembert–Poincaré equations* derived in [7].

2.2.1. *Generalized nonholonomic connection.* From now on, we will write $\mathcal{X} = Q/G$, and assume that the canonical projection $\pi : Q \twoheadrightarrow \mathcal{X}$ is a principal fiber bundle with structure group G. In particular, we will assume that ρ is a free action. Let us denote the vertical distribution by $\mathcal{V}$, that is, $\mathcal{V} = \ker(\pi_*) \subset TQ$.[5]

Now, we shall construct a principal connection related to (L, C_K, C_V) and the group G. We shall proceed in several steps.

1. Fix a G-invariant metric on Q. If L is simple, we can chose the metric which defines its kinetic term. Let us assume that this is the case.
2. Consider the intersection
$$\mathcal{S} = C_V \cap \mathcal{V}, \tag{4}$$
and write
$$C_V = \mathcal{T} \oplus \mathcal{S} \quad \text{and} \quad \mathcal{V} = \mathcal{S} \oplus \mathcal{U}, \tag{5}$$
where $\mathcal{T}$ and $\mathcal{U}$ are the orthogonal complements of $\mathcal{S}$ in C_V and $\mathcal{V}$, respectively.
3. Consider the orthogonal complement of $C_V + \mathcal{V}$ in TQ. Let us call it $\mathcal{R}$.
   Assume that $\mathcal{T}$ and $\mathcal{R}$ are C^∞-distributions.
4. Define the principal connection form $A^\bullet : TQ \to \mathfrak{g}$ (where $\mathfrak{g}$ the Lie algebra of G), which we shall call the **generalized nonholonomic connection**, with horizontal distribution $\mathcal{H}^\bullet = \mathcal{R} \oplus \mathcal{T}$.
   Summing up, we have the Whitney sum
$$TQ = \mathcal{H}^\bullet \oplus \mathcal{V} \quad \text{with} \quad \mathcal{H}^\bullet = \mathcal{R} \oplus \mathcal{T} \quad \text{and} \quad \mathcal{V} = \mathcal{S} \oplus \mathcal{U}. \tag{6}$$

Remark 2. It is worth mentioning that the generalized nonholonomic connection is just the nonholonomic connection of Ref. [14], but related to C_V instead of C_K. Then, $A^\bullet$ only depends on L (since we are assuming that L is simple and that the chosen metric is the one related to its kinetic term) and the variational constraints C_V **only**. The kinematic constraints given by C_K are not involved in the construction of the connection $A^\bullet$.

Remark 3. Some sums in (6) are not necessarily orthogonal.

Since
$$\mathcal{T} = C_V \cap \mathcal{H}^\bullet,$$
it follows from (5) and (4) that
$$C_V = \mathcal{T} \oplus \mathcal{S} = (C_V \cap \mathcal{H}^\bullet) \oplus (C_V \cap \mathcal{V}).$$

[5]The fact that ρ is free implies that $dim\mathcal{V} = dimG$.

On the other hand, all the (generalized) distributions $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ are G-invariant. Therefore we can write

$$TQ/G = \mathcal{R}/G \oplus \mathcal{T}/G \oplus \mathcal{S}/G \oplus \mathcal{U}/G,$$

and in particular [see the first parts of (3) and (5)]

$$\mathfrak{C}_V = \mathcal{T}/G \oplus \mathcal{S}/G. \tag{7}$$

2.2.2. *Decomposition of the reduced virtual displacements.* Let $\widetilde{\mathfrak{g}} = Q \times_G \mathfrak{g}$ be the associated adjoint bundle over $\mathcal{X} = Q/G$. The elements of $\widetilde{\mathfrak{g}}$ will be denoted as equivalence classes $[q, \eta]$, with $q \in Q$ and $\eta \in \mathfrak{g}$.

Given a principal connection A (not necessarily the generalized nonholonomic one $A^\bullet$), we can construct a fiber bundle isomorphism (see Ref. [13])

$$\alpha_A : TQ/G \to T\mathcal{X} \oplus \widetilde{\mathfrak{g}},$$

known as Atiyah isomorphism, such that, for all $q \in Q$ and $v \in T_qQ$,

$$[v] \mapsto \pi_* (v) \oplus [q, A (v)] \, .$$

Here, $[v] = p (v) \in TQ/G$. Denoting by a the map

$$a : TQ \to \widetilde{\mathfrak{g}} \ : \ v \mapsto [q, A (v)] \, , \tag{8}$$

we have

$$\alpha_A \circ p (v) = \pi_* (v) \oplus a (v) \, , \quad \forall v \in TQ. \tag{9}$$

Remark 4. Since α_A is a linear bundle isomorphism, we have in particular that, for each $q \in Q$, spaces $(TQ/G)_{\pi(q)}$ and $T_{\pi(q)}\mathcal{X} \oplus \widetilde{\mathfrak{g}}_{\pi(q)}$ have the same dimension. Moreover, it can be shown that the map

$$\alpha_A \circ p : TQ \to T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$$

defines a linear isomorphism when restricted to each T_qQ.

When there is no risk of confusion, we shall identify the fiber bundles TQ/G and $T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$ *via* the map α_A. For instance, we shall see the reduced data $\mathfrak{C}_K$ and $\mathfrak{C}_V$ as subsets of $T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$, i.e. we shall identify $\mathfrak{C}_K$ and $\mathfrak{C}_V$ with $\alpha_A \circ p (C_K)$ and $\alpha_A \circ p (C_V)$, respectively. If $A = A^\bullet$, in terms of the identification $\alpha_{A^\bullet}$ we have

$$\mathcal{H}^\bullet/G = \alpha_{A^\bullet} (\mathcal{H}^\bullet/G) = \pi_* (\mathcal{H}^\bullet) = T\mathcal{X}$$

and

$$\mathcal{V}/G = \alpha_{A^\bullet} (\mathcal{V}/G) = a^\bullet (\mathcal{V}) = \tilde{\mathfrak{g}}.$$

As a consequence,

Proposition 1. $\mathfrak{C}_V^\bullet = \alpha_{A^\bullet} \circ p (C_V)$ *can be decomposed as*

$$\mathfrak{C}_V^\bullet = \mathfrak{C}_V^{hor} \oplus \mathfrak{C}_V^{ver},$$

where [*see Eq.* (7)]

$$\mathfrak{C}_V^{hor} = \pi_* (C_V) = T\mathcal{X} \cap \mathfrak{C}_V^\bullet \tag{10}$$

and

$$\mathfrak{C}_V^{ver} = a^\bullet (C_V) = \tilde{\mathfrak{g}} \cap \mathfrak{C}_V^\bullet. \tag{11}$$

2.2.3. *The* ***generalized Lagrange–d'Alembert–Poincaré*** *equations.* As we have said at the beginning of this section, we want to find the equations of motion of a GNHS (L, C_K, C_V) in terms of its reduced data l, $\mathfrak{C}_K$ and $\mathfrak{C}_V$. By definition, a curve $\gamma : [t_1, t_2] \to Q$ is a trajectory of that system only if

$$\int_{t_1}^{t_2} \langle dL\left(\gamma'\left(t\right)\right), \kappa\left(\delta\gamma'\left(t\right)\right)\rangle \, dt = 0 \tag{12}$$

for all $\delta\gamma$ such that $\delta\gamma\left(t\right) \in C_V$. Since $L = l \circ p$ [see (2)], then

$$\langle dL\left(\gamma'\left(t\right)\right), \kappa\left(\delta\gamma'\left(t\right)\right)\rangle = \langle p^*\left(dl\right)\left(\gamma'\left(t\right)\right), \kappa\left(\delta\gamma'\left(t\right)\right)\rangle$$

$$= \langle dl\left(p\left(\gamma'\left(t\right)\right)\right), p_* \circ \kappa\left(\delta\gamma'\left(t\right)\right)\rangle .$$

Using the identification of TQ/G and $T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$ given by some α_A (again, we are not assuming at this point that $A = A^\bullet$), let us denote the composition

$$l \circ \alpha_A : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to \mathbb{R}$$

simply as l. Then,

$$\langle dL\left(\gamma'\left(t\right)\right), \kappa\left(\delta\gamma'\left(t\right)\right)\rangle = \langle dl\left(\alpha_A \circ p\left(\gamma'\left(t\right)\right)\right), \left(\alpha_A \circ p\right)_* \circ \kappa\left(\delta\gamma'\left(t\right)\right)\rangle .$$

Following the notation of [13], given a curve γ and a variation $\delta\gamma$, let us write

$$\pi \circ \gamma = x, \quad \alpha_A \circ p \circ \gamma' = \mu = \dot{x} \oplus \overline{v} \quad \text{and} \quad \alpha_A \circ p \circ \delta\gamma = \delta x \oplus \overline{\eta},$$

where

$$\dot{x} = \pi_* \circ \gamma', \quad \overline{v} = a \circ \gamma', \quad \delta x = \pi_* \circ \delta\gamma \quad \text{and} \quad \overline{\eta} = a \circ \delta\gamma. \tag{13}$$

In these terms, it was shown in [7] that, omitting dependence on time,

$$\langle dL\left(\gamma'\right), \kappa\left(\delta\gamma'\right)\rangle = \left\langle \frac{\partial l}{\partial x}\left(\mu\right), \delta x \right\rangle + \left\langle \frac{\partial l}{\partial \dot{x}}\left(\mu\right), \frac{D}{Dt}\delta x \right\rangle \tag{14}$$

$$+ \left\langle \frac{\partial l}{\partial \overline{v}}\left(\mu\right), \frac{D}{Dt}\overline{\eta} + [\overline{v}, \overline{\eta}] - \widetilde{B}\left(\dot{x}, \delta x\right) \right\rangle ,$$

where

$$\widetilde{B} : T\mathcal{X} \times_{\mathcal{X}} T\mathcal{X} \to \widetilde{\mathfrak{g}} \; : \; \left(\pi_*\left(u_q\right), \pi_*\left(v_q\right)\right) \mapsto [q, B\left(u_q, v_q\right)] \tag{15}$$

is the *reduced curvature* of A, $B : TQ \times_Q TQ \to \mathfrak{g}$ is its curvature,

$$\frac{\partial l}{\partial \dot{x}} : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to T^*\mathcal{X} \quad \text{and} \quad \frac{\partial l}{\partial \overline{v}} : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}^* \tag{16}$$

are the first and second components of the fiber derivative

$$\mathbb{F}l : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to T^*\mathcal{X} \oplus \widetilde{\mathfrak{g}}^* \tag{17}$$

of l, and

$$\frac{\partial l}{\partial x} : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to T^*\mathcal{X}$$

is its base derivative (see [13] for more details). It follows from equality (14), integrating by parts, that (12) holds if and only if

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial l}{\partial x} - \left\langle \frac{\partial l}{\partial \overline{v}}, i_{\dot{x}}\tilde{B} \right\rangle, \delta x \right\rangle + \left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \overline{v}} + \mathrm{ad}^*_{\overline{v}}\frac{\partial l}{\partial \overline{v}}, \overline{\eta} \right\rangle = 0. \tag{18}$$

Remark 5. So far, we have been working with a left action. For a right action, we only have to change the sign of the Lie bracket $[\overline{v},\overline{\eta}]$ appearing in (14). Accordingly, Eq. (18) translates to

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}+\frac{\partial l}{\partial x}-\left\langle \frac{\partial l}{\partial \overline{v}},i_{\dot{x}}\tilde{B}\right\rangle ,\delta x\right\rangle +\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \overline{v}}-\mathrm{ad}_{\overline{v}}^{*}\frac{\partial l}{\partial \overline{v}},\overline{\eta}\right\rangle =0.$$

If we now assume that $A=A^{\bullet}$, using (10), (11) and (13), it follows that, when $\delta\gamma\left(t\right)$ varies inside $\left.C_V\right|_{\gamma(t)}$, reduced variations $\delta x\left(t\right)$ and $\overline{\eta}\left(t\right)$ vary independently inside

$$\left.\mathfrak{C}_V^{\mathrm{hor}}\right|_{x(t)}\qquad\text{and}\qquad\left.\mathfrak{C}_V^{\mathrm{ver}}\right|_{x(t)},$$

respectively. This enable us to decompose Eq. (18) into two parts, as we describe in the next result (see Ref. [7] for a proof).

Theorem 2.3. *Let (L,C_K,C_V) be a GNHS and G a Lie group acting on Q. Suppose that the system is G-invariant, $\pi:Q\twoheadrightarrow\mathcal{X}=Q/G$ is a principal fiber bundle and $A^{\bullet}:TQ\rightarrow\mathfrak{g}$ is the generalized nonholonomic connection of the system. Fix a curve $\gamma:[t_1,t_2]\rightarrow Q$. Then, γ is a trajectory of (L,C_K,C_V) if and only if the curve*

$$\mu:[t_1,t_2]\rightarrow T\mathcal{X}\oplus\tilde{\mathfrak{g}},$$

given by

$$\mu\left(t\right)=\dot{x}\left(t\right)\oplus\overline{v}\left(t\right)=\alpha_{A^{\bullet}}\circ p\left(\gamma'\left(t\right)\right),$$

satisfies

$$\mu\left(t\right)\in\mathfrak{C}_K^{\bullet},$$

the ***Horizontal Generalized Lagrange–d'Alembert–Poincaré Equations***

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}\left(\mu\left(t\right)\right)+\frac{\partial l}{\partial x}\left(\mu\left(t\right)\right)-\left\langle \frac{\partial l}{\partial \overline{v}}\left(\mu\left(t\right)\right),i_{\dot{x}(t)}\tilde{B}\right\rangle ,\delta x\left(t\right)\right\rangle =0 \tag{19}$$

and ***Vertical Generalized Lagrange–d'Alembert–Poincaré Equations***

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \overline{v}}\left(\mu\left(t\right)\right)+\mathrm{ad}_{\overline{v}}^{*}\frac{\partial l}{\partial \overline{v}}\left(\mu\left(t\right)\right),\overline{\eta}\left(t\right)\right\rangle =0, \tag{20}$$

for all curves

$$\delta x:[t_1,t_2]\rightarrow T\mathcal{X}\quad\text{ and }\quad\overline{\eta}:[t_1,t_2]\rightarrow\tilde{\mathfrak{g}}$$

satisfying

$$\delta x\left(t\right)\in\left.\mathfrak{C}_V^{hor}\right|_{x(t)}\quad\text{ and }\quad\overline{\eta}\left(t\right)\in\left.\mathfrak{C}_V^{ver}\right|_{x(t)}.$$

Remark 6. For a right action (see Remark 5) we simply must change the sign of $\mathrm{ad}_{\overline{v}}^{*}\frac{\partial l}{\partial \overline{v}}$ in the vertical equation.

Note that, given a concrete mechanical system, in order to find an explicit expression of the equations (19) and (20), we have to calculate several covariant derivatives of l (beside calculating the curvature B of $A^{\bullet}$). This can be very laborious, even in the case in which $Q\rightarrow\mathcal{X}$ is a trivial bundle, because the generalized nonholonomic connection is not, in general, the related trivial principal connection.

2.3. **The alternative procedure.** In this subsection we shall develop the first contribution of this paper: an alternative approach to variational reduction which solve the above mentioned issue.

Given an arbitrary principal connection A, we have shown in the previous subsection that Eq. (12) holds if and only if Eq. (18) does. Moreover, we have the following result.

Proposition 2. *Let (L, C_K, C_V) be a GNHS and G a Lie group acting on Q. Suppose that the system is G-invariant and that $\pi : Q \twoheadrightarrow \mathcal{X} = Q/G$ is a principal fiber bundle. Fix an arbitrary principal connection $A : TQ \to \mathfrak{g}$ and a curve $\gamma : [t_1, t_2] \to Q$. Then, γ is a trajectory of (L, C_K, C_V) if and only if the curve*

$$\mu : [t_1, t_2] \to T\mathcal{X} \oplus \tilde{\mathfrak{g}},$$

given by

$$\mu(t) = \dot{x}(t) \oplus \bar{v}(t) = \alpha_A \circ p(\gamma'(t)),$$

satisfies $\mu(t) \in \mathfrak{C}_K$ and

$$\begin{aligned} &\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(\mu(t)) + \frac{\partial l}{\partial x}(\mu(t)) - \left\langle \frac{\partial l}{\partial \bar{v}}(\mu(t)), i_{\dot{x}(t)}\tilde{B} \right\rangle, \delta x(t) \right\rangle \\ &+ \left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(\mu(t)) + \mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial \bar{v}}(\mu(t)), \bar{\eta}(t) \right\rangle = 0, \end{aligned} \tag{21}$$

for all curves

$$\delta x : [t_1, t_2] \to T\mathcal{X} \quad \text{and} \quad \bar{\eta} : [t_1, t_2] \to \tilde{\mathfrak{g}}$$

satisfying

$$\delta x(t) \oplus \bar{\eta}(t) \in \mathfrak{C}_V|_{x(t)}.$$

To prove the above proposition, we need the next Lemma, which is a slightly modification of the Lemma 10 that appears in Ref. [7].

Lemma 2.4. *Under the conditions of last proposition, fix an arbitrary principal connection $A : TQ \to \mathfrak{g}$, a curve $\gamma : [t_1, t_2] \to Q$ and consider its projection*

$$x : [t_1, t_2] \to \mathcal{X} \quad : \quad t \mapsto x(t) = \pi(\gamma(t)).$$

Given curves $\delta x : [t_1, t_2] \to T\mathcal{X}$ and $\bar{\eta} : [t_1, t_2] \to \tilde{\mathfrak{g}}$, we have that

$$\delta x(t) \oplus \bar{\eta}(t) \in \mathfrak{C}_V|_{x(t)} \tag{22}$$

if and only if there exists a curve $\delta\gamma : [t_1, t_2] \to TQ$ satisfying

$$\pi_*(\delta\gamma(t)) = \delta x(t) \quad \text{and} \quad a(\delta\gamma(t)) = \bar{\eta}(t), \tag{23}$$

and such that $\delta\gamma(t) \in C_V|_{\gamma(t)}$.

Proof. See the proof of Lemma 4.6, for the $l = 0$ case. □

If $Q \to \mathcal{X}$ is a trivial bundle, we can take A as the related trivial connection. Then, the curvature and the reduced curvature are 0, and Eq. (21) reduces to

$$\begin{aligned} &\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(\mu(t)) + \frac{\partial l}{\partial x}(\mu(t)), \delta x(t) \right\rangle + \\ &+ \left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(\mu(t)) + \mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial \bar{v}}(\mu(t)), \bar{\eta}(t) \right\rangle = 0. \end{aligned}$$

Also, the involved derivatives are easy to calculate. This will be studied later, in Section 3. The problem is that variations δx and $\bar{\eta}$ are not independent. (We only know that their sum must be an element of $\mathfrak{C}_V$.) Let us work on that.

2.3.1. *Using two connections.* Given a triple (L, C_V, C_K) with symmetry group G, let us consider an arbitrary principal connection A and the generalized nonholonomic one $A^\bullet$, with related horizontal spaces $\mathcal{H}$ and $\mathcal{H}^\bullet$ respectively. Consider also the isomorphisms

$$\alpha_A, \alpha_{A^\bullet} : TQ/G \to T\mathcal{X} \oplus \widetilde{\mathfrak{g}},$$

and write

$$\alpha_A \circ p(v) = \pi_*(v) \oplus a(v),$$

$$\alpha_{A^\bullet} \circ p(v) = \pi_*(v) \oplus a^\bullet(v),$$

$$\mathfrak{C}_{V,K} = \alpha_A \circ p(C_{V,K}) \quad \text{and} \quad \mathfrak{C}^\bullet_{V,K} = \alpha_{A^\bullet} \circ p(C_{V,K}).$$

As we saw in Proposition 1, for $\mathfrak{C}^\bullet_V$ we have the decomposition

$$\mathfrak{C}^\bullet_V = \mathfrak{C}^{\text{hor}}_V \oplus \mathfrak{C}^{\text{ver}}_V = (\mathfrak{C}^\bullet_V \cap T\mathcal{X}) \oplus (\mathfrak{C}^\bullet_V \cap \widetilde{\mathfrak{g}}).$$

At this point, in order to avoid any confusion, we need to change the notation for reduced variations. Given a curve $\delta\gamma$, we shall write

$$\alpha_A \circ p(\delta\gamma(t)) = \pi_*(\delta\gamma(t)) \oplus a(\delta\gamma(t)) = \delta x(t) \oplus \bar{\eta}(t) \tag{24}$$

and

$$\alpha_{A^\bullet} \circ p(\delta\gamma(t)) = \pi_*(\delta\gamma(t)) \oplus a^\bullet(\delta\gamma(t)) = \delta x^\bullet(t) \oplus \bar{\eta}^\bullet(t). \tag{25}$$

Of course, if $\delta\gamma$ is inside C_V, then

$$\delta x(t) \oplus \bar{\eta}(t) \in \mathfrak{C}_V \quad \text{and} \quad \delta x^\bullet(t) \in \mathfrak{C}^{\text{hor}}_V,\ \bar{\eta}^\bullet(t) \in \mathfrak{C}^{\text{ver}}_V.$$

Let us study the relationship between variations $\delta x^\bullet$ and $\bar{\eta}^\bullet$ with variations δx and $\bar{\eta}$. It is clear that $\delta x(t) = \delta x^\bullet(t)$. From (24) and (25), it easily follows that

$$\delta x(t) \oplus \bar{\eta}(t) = \alpha_A \circ (\alpha_{A^\bullet})^{-1} [\delta x^\bullet(t) \oplus \bar{\eta}^\bullet(t)].$$

In terms of the canonical projections and inclusions

$$P_{T\mathcal{X}} : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to T\mathcal{X}, \qquad P_{\widetilde{\mathfrak{g}}} : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}, \tag{26}$$

and

$$I_{T\mathcal{X}} : T\mathcal{X} \to T\mathcal{X} \oplus \widetilde{\mathfrak{g}}, \qquad I_{\widetilde{\mathfrak{g}}} : \widetilde{\mathfrak{g}} \to T\mathcal{X} \oplus \widetilde{\mathfrak{g}}, \tag{27}$$

we have that

$$\delta x(t) = P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{T\mathcal{X}}(\delta x^\bullet(t)) + P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{\widetilde{\mathfrak{g}}}(\bar{\eta}^\bullet(t)) \tag{28}$$

and

$$\bar{\eta}(t) = P_{\widetilde{\mathfrak{g}}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{T\mathcal{X}}(\delta x^\bullet(t)) + P_{\widetilde{\mathfrak{g}}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{\widetilde{\mathfrak{g}}}(\bar{\eta}^\bullet(t)). \tag{29}$$

Lemma 2.5. *For the projections and inclusions given above, we have the identities*

$$P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{T\mathcal{X}} = id_{T\mathcal{X}}, \qquad P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{\widetilde{\mathfrak{g}}} = 0$$

and

$$P_{\widetilde{\mathfrak{g}}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{\widetilde{\mathfrak{g}}} = id_{\widetilde{\mathfrak{g}}}.$$

Proof. Given $u \in T\mathcal{X}$, if

$$(\alpha_{A^\bullet})^{-1}(u \oplus 0) = p(v)$$

for some $v \in TQ$, then $\pi_*(v) = u$. On the other hand,

$$\alpha_A \circ p(v) = \pi_*(v) \oplus \zeta,$$

i.e. $\alpha_A \circ p(v) = u \oplus \zeta$ for some $\zeta \in \widetilde{\mathfrak{g}}$. Therefore,

$$\begin{aligned} P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{T\mathcal{X}}(u) &= P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1}(u \oplus 0) \\ &= P_{T\mathcal{X}} \circ \alpha_A \circ p(v) \\ &= P_{T\mathcal{X}}(u \oplus \zeta) = u. \end{aligned}$$

For the second identity, given $\zeta \in \widetilde{\mathfrak{g}}$, if $(\alpha_{A^\bullet})^{-1}(0 \oplus \zeta) = p(v)$ for some $v \in TQ$, then $\pi_*(v) = 0$, and accordingly

$$\alpha_A \circ p(v) = \pi_*(v) \oplus \xi = 0 \oplus \xi$$

for some $\xi \in \widetilde{\mathfrak{g}}$. Consequently,

$$\begin{aligned} P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{\widetilde{\mathfrak{g}}}(\zeta) &= P_{T\mathcal{X}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1}(0 \oplus \zeta) \\ &= P_{T\mathcal{X}} \circ \alpha_A \circ p(v) \\ &= P_{T\mathcal{X}}(0 \oplus \xi) = 0. \end{aligned}$$

Let us show now the last identity. Given $[q, \eta] \in \widetilde{\mathfrak{g}}$, if

$$(\alpha_{A^\bullet})^{-1}(0 \oplus [q, \eta]) = p(v_q),$$

then $v_q = (X_\eta)_q$, being X_η the fundamental vector field associated to η. This implies that

$$\alpha_A \circ p(v_q) = \alpha_A \circ p\left((X_\eta)_q\right) = 0 \oplus [q, \eta],$$

from which last identity easily follows. □

Using the last lemma and defining

$$\varphi = P_{\widetilde{\mathfrak{g}}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{T\mathcal{X}} : T\mathcal{X} \to \widetilde{\mathfrak{g}},$$

Eqs. (28) and (29) tell us that

$$\delta x(t) = \delta x^\bullet(t) \quad \text{and} \quad \bar{\eta}(t) = \varphi(\delta x^\bullet(t)) + \bar{\eta}^\bullet(t).$$

Note that $\varphi = 0$ when $A = A^\bullet$.

2.3.2. *The alternative reduced equations.* The above relation enable us to write variations δx and $\bar{\eta}$, appearing in Eq. (21), and such that $\delta x(t) \oplus \bar{\eta}(t) \in \mathfrak{C}_V$, in terms of independent variations $\delta x^\bullet$ and $\bar{\eta}^\bullet$ living inside $\mathfrak{C}_V^{\mathrm{hor}}$ and $\mathfrak{C}_V^{\mathrm{ver}}$, respectively. This gives rise to a new set of variationally reduced equations.

Theorem 2.6. *Let (L, C_K, C_V) be a GNHS and G a Lie group acting on Q. Suppose that the system is G-invariant, $\pi : Q \twoheadrightarrow \mathcal{X} = Q/G$ is a principal fiber bundle and $A^\bullet$ is its generalized nonholonomic connection. Fix an arbitrary principal connection A and a curve $\gamma : [t_1, t_2] \to Q$. Then, γ is a trajectory of (L, C_K, C_V) if and only if the curve*

$$\mu : [t_1, t_2] \to T\mathcal{X} \oplus \tilde{\mathfrak{g}},$$

given by

$$\mu(t) = \dot{x}(t) \oplus \bar{v}(t) = \alpha_A \circ p(\gamma'(t)),$$

satisfies

$$\mu(t) \in \mathfrak{C}_K,$$

$$\begin{aligned}&\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(\mu(t)) + \frac{\partial l}{\partial x}(\mu(t)) - \left\langle \frac{\partial l}{\partial \bar{v}}(\mu(t)), i_{\dot{x}(t)}\tilde{B} \right\rangle, \delta x^\bullet(t) \right\rangle \\ &+ \left\langle \varphi^* \left(-\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(\mu(t)) + \mathrm{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}(\mu(t)) \right), \delta x^\bullet(t) \right\rangle = 0,\end{aligned} \tag{30}$$

and

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(\mu(t)) + \mathrm{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}(\mu(t)), \bar{\eta}^\bullet(t) \right\rangle = 0, \tag{31}$$

for all curves

$$\delta x^\bullet : [t_1, t_2] \to T\mathcal{X} \quad \textit{and} \quad \bar{\eta}^\bullet : [t_1, t_2] \to \tilde{\mathfrak{g}}$$

satisfying

$$\delta x^\bullet(t) \in \left.\mathfrak{C}_V^{hor}\right|_{x(t)} \quad \textit{and} \quad \bar{\eta}^\bullet(t) \in \left.\mathfrak{C}_V^{ver}\right|_{x(t)}.$$

Remark 7. Note that the variables x, $\dot{x}$ and $\bar{v}$, the submanifold $\mathfrak{C}_K$ and the curvature B are related to A, while the variations $\delta x^\bullet$ and $\bar{\eta}^\bullet$, and the distributions $\mathfrak{C}_V^{\mathrm{hor}}$ and $\mathfrak{C}_V^{\mathrm{ver}}$, are related to $A^\bullet$.

Remark 8. For a right action, recall that we have to change the sign of $\mathrm{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}$ (Remark 5).

Although the Eqs. (30) and (31) seem to be more complicated than Eqs. (19) and (20), we shall see in the next section that, for trivial principal bundles, calculations that drive us to concrete expressions of the equations of motion can be drastically simplified.

3. **The case of trivial bundles.** The purpose of the present section is to study the form that the reduced equations of a GNHS with symmetry G adopt, for the usual and alternative procedures, in the cases in which the configuration space Q of the system is a trivial principal bundle with structure group G. At the end of the section we shall compare, in the context of trivial principal bundles, how both procedures work on a concrete example.

Notation. Let $\mathcal{X}$ be a manifold, G a Lie group, and define $Q = \mathcal{X} \times G$. Consider the left and right actions

$$G \times Q \to Q : (g, (x, h)) \mapsto (x, L_g h) = (x, g\,h) \tag{32}$$

and

$$Q \times G \to Q : ((x, h), g) \mapsto (x, R_g h) = (x, h\,g). \tag{33}$$

They make $\pi : Q \to \mathcal{X} : (x, h) \mapsto x$ into the left and right trivial principal fiber bundles with base $\mathcal{X}$ and structure group G. For the lifted actions we shall use the notation

$$G \times TQ \to TQ : \left(g, \left(x, h, \dot{x}, \dot{h}\right)\right) \mapsto \left(x, g\,h, \dot{x}, g\,\dot{h}\right)$$

and

$$TQ \times G \to TQ : \left(\left(x, h, \dot{x}, \dot{h}\right), g\right) \mapsto \left(x, h\,g, \dot{x}, \dot{h}\,g\right).$$

As usual, we shall see $h^{-1}\,\dot{h}$ and $\dot{h}\,h^{-1}$ as elements in the Lie algebra $\mathfrak{g}$ of G.

3.1. **Left actions.**

3.1.1. *Connections.* Given a manifold Q and a Lie group G, with Lie algebra $\mathfrak{g}$, recall that a principal connection $A : TQ \to \mathfrak{g}$, for a principal fiber bundle constructed with a **left** action ρ, must satisfy

$$A\left(\left(\rho_g\right)_*(v)\right) = Ad_g\left(A(v)\right), \quad \forall g \in G,\ \forall v \in TQ,$$

and

$$A\left(X_\eta\right) = \eta, \qquad \forall\, \eta \in \mathfrak{g},$$

where $Ad_g : G \to G$ is the adjoint action of g and X_η is the fundamental vector field of η related to the action ρ.

Suppose that $Q = \mathcal{X} \times G$ and that the action ρ is given by (32). With above notation, last equations tell us that

$$A\left(x, g\,h, \dot{x}, g\,\dot{h}\right) = Ad_g\left(A\left(x, h, \dot{x}, \dot{h}\right)\right)$$

and

$$A\left(x, e, 0, h^{-1}\,\dot{h}\right) = h^{-1}\,\dot{h}.$$

As usual, e denotes the unit element of G. Then,

$$\begin{aligned} A\left(x, h, \dot{x}, \dot{h}\right) &= A\left(x, h, \dot{x}, 0\right) + A\left(x, h, 0, \dot{h}\right) \\ &= Ad_h\left(A\left(x, e, \dot{x}, 0\right)\right) + Ad_h\left(A\left(x, e, 0, h^{-1}\,\dot{h}\right)\right) \\ &= Ad_h\left(\mathcal{A}(x)\,\dot{x}\right) + \dot{h}\,h^{-1}, \end{aligned}$$

where $\mathcal{A}$ is a $\mathfrak{g}$-valued 1-form on $\mathcal{X}$, i.e. $\mathcal{A} : \mathcal{X} \to T^*\mathcal{X} \otimes \mathfrak{g}$, and it is given by $\mathcal{A}(x)\,\dot{x} = A(x, e, \dot{x}, 0)$. Note that, an element $(x, e, \dot{x}, \xi)$ is horizontal if and only if

$$\mathcal{A}(x)\,\dot{x} + \xi = 0. \tag{34}$$

We shall say that the connection A is the **trivial** one if $\mathcal{A}(x) = 0$ for all x.

3.1.2. *The isomorphism α_A and the reduced curvature.* For a trivial bundle $Q = \mathcal{X} \times G$, the left adjoint bundle $\widetilde{\mathfrak{g}}$ can be identified with $\mathcal{X} \times \mathfrak{g}$ by the map

$$\widetilde{\mathfrak{g}} \to \mathcal{X} \times \mathfrak{g} : [(x,h), \xi] \longmapsto (x, Ad_{h^{-1}}\xi),$$

with inverse

$$(x,\xi) \longmapsto [(x,e), \xi].$$

This identification is well-defined because the action of G on $Q \times \mathfrak{g}$ that defines $\widetilde{\mathfrak{g}}$ is given by

$$(g; ((x,h), \xi)) \mapsto ((x, g\,h), Ad_g \xi).$$

Using such an identification, α_A can be seen as the map $\alpha_A : TQ/G \to T\mathcal{X} \times \mathfrak{g}$ such that [recall Eq. (9)]

$$\alpha_A \circ p\left(x,h,\dot{x},\dot{h}\right) = (x,\dot{x}) \oplus \left(x, \mathcal{A}(x)\ \dot{x} + h^{-1}\,\dot{h}\right). \tag{35}$$

In terms of $\mathcal{A}$, it can be shown that

$$B((x,e,\dot{x},0), (x,e,\delta x,0)) = d\mathcal{A}((x,\dot{x}), (x,\delta x)) - [\mathcal{A}(x)\ \dot{x}, \mathcal{A}(x)\ \delta x].$$

Accordingly, from the very definition of $\widetilde{B}$ [see (15)], and identifying $\widetilde{\mathfrak{g}}$ and $\mathcal{X} \times \mathfrak{g}$, we have that

$$\begin{aligned}\widetilde{B}((x,\dot{x}), (x,\delta x)) &= (x, B((x,e,\dot{x},0), (x,e,\delta x,0))) \\ &= (x, d\mathcal{A}((x,\dot{x}), (x,\delta x)) - [\mathcal{A}(x)\ \dot{x}, \mathcal{A}(x)\ \delta x]).\end{aligned} \tag{36}$$

3.1.3. *The usual and alternative reduced equations.* Lets go back to Eqs. (30) and (31), where the horizontal and vertical alternative reduced equations for a GNHS are given. Recall that all the covariant derivatives and also the curvature B, that appear in these equations, are related to an arbitrarily given connection A, while the map φ is defined by A and the connection $A^{\bullet}$. So, the usual reduced equations are obtained from (30) and (31) by taking $A = A^{\bullet}$.

For a trivial principal bundle, and under natural identifications, we can replace

$$\frac{\partial l}{\partial \overline{v}} : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}^*,$$

which is the second component of the fiber derivative

$$\mathbb{F}l : T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to T^*\mathcal{X} \oplus \widetilde{\mathfrak{g}}^*$$

[see (16) and (17)], by a partial derivative in a vector space, and replace the covariant derivative

$$\frac{D}{Dt}\frac{\partial l}{\partial \overline{v}}$$

by a standard derivative of a vector with respect to t. Also, we can give a more concrete expression for the base derivative $\partial l / \partial x$. Let us see how to do that.

Covariant derivatives on $\widetilde{\mathfrak{g}}$ and its dual. The covariant derivative of a curve on $\widetilde{\mathfrak{g}}$ is given by

$$\frac{D}{Dt}\left[\left(x\left(t\right),h\left(t\right)\right),\xi\left(t\right)\right]=\Big[\left(x\left(t\right),h\left(t\right)\right),\dot{\xi}\left(t\right)-\Big[A\Big(x\left(t\right),h\left(t\right),\dot{x}\left(t\right),\dot{h}\left(t\right)\Big),\xi\left(t\right)\Big]\Big].$$

This defines the usual affine connection ∇^{A} of $\widetilde{\mathfrak{g}}$, related to the principal connection A. If the curve is of the form $\left[\left(x\left(t\right),e\right),\xi\left(t\right)\right]$, then

$$\frac{D}{Dt}\left[\left(x\left(t\right),e\right),\xi\left(t\right)\right]=\Big[\left(x\left(t\right),e\right),\dot{\xi}\left(t\right)-\left[\mathcal{A}\left(x\left(t\right)\right)\,\dot{x}\left(t\right),\xi\left(t\right)\right]\Big],$$

and as a consequence, identifying $\widetilde{\mathfrak{g}}$ and $\mathcal{X}\times\mathfrak{g}$ as above,

$$\begin{aligned}\frac{D}{Dt}\left(x\left(t\right),\xi\left(t\right)\right)&=\Big(x\left(t\right),\dot{\xi}\left(t\right)-\left[\mathcal{A}\left(x\left(t\right)\right)\,\dot{x}\left(t\right),\xi\left(t\right)\right]\Big)\\&=\Big(x\left(t\right),\dot{\xi}\left(t\right)-ad_{\mathcal{A}\left(x\left(t\right)\right)\,\dot{x}\left(t\right)}\,\xi\left(t\right)\Big).\end{aligned}\tag{37}$$

From that, it easily follows that the covariant derivative of a curve

$$\left(x\left(t\right),\alpha\left(t\right)\right)\in\mathcal{X}\times\mathfrak{g}^{*}\simeq\widetilde{\mathfrak{g}}^{*}$$

is

$$\frac{D}{Dt}\left(x\left(t\right),\alpha\left(t\right)\right)=\Big(x\left(t\right),\dot{\alpha}\left(t\right)+ad^{*}_{\mathcal{A}\left(x\left(t\right)\right)\,\dot{x}\left(t\right)}\,\alpha\left(t\right)\Big).\tag{38}$$

Now, consider $\partial l/\partial\bar{v}$. It takes values inside $\mathcal{X}\times\mathfrak{g}^{*}$. So, for each point $(x,\dot{x},\xi)$, such a derivative is essentially a partial derivative of l w.r.t. ξ. More precisely,

$$\frac{\partial l}{\partial\bar{v}}\left(x,\dot{x},\xi\right)=\left(x,\frac{\partial l}{\partial\xi}\left(x,\dot{x},\xi\right)\right).\tag{39}$$

Accordingly, from Eq. (38),

$$\begin{aligned}&\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}\left(x\left(t\right),\dot{x}\left(t\right),\xi\left(t\right)\right)=\\&=\left(x\left(t\right),\frac{d}{dt}\frac{\partial l}{\partial\xi}\left(x\left(t\right),\dot{x}\left(t\right),\xi\left(t\right)\right)+ad^{*}_{\mathcal{A}\left(x\left(t\right)\right)\,\dot{x}\left(t\right)}\frac{\partial l}{\partial\xi}\left(x\left(t\right),\dot{x}\left(t\right),\xi\left(t\right)\right)\right).\end{aligned}$$

Moreover, seeing $\dfrac{D}{Dt}\dfrac{\partial l}{\partial\bar{v}}$ as an element of $\mathfrak{g}^{*}$, and omitting dependence on $\left(x\left(t\right),\dot{x}\left(t\right),\xi\left(t\right)\right)$,

$$\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}=\frac{d}{dt}\frac{\partial l}{\partial\xi}+ad^{*}_{\mathcal{A}(x)\,\dot{x}}\frac{\partial l}{\partial\xi}.\tag{40}$$

Let us also say that, using (39), we have that

$$\mathrm{ad}^{*}_{\bar{v}}\frac{\partial l}{\partial\bar{v}}=\mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial\xi}.\tag{41}$$

Base derivative of l. The base derivative

$$\frac{\partial l}{\partial x}: T\mathcal{X} \oplus \widetilde{\mathfrak{g}} \to T^*\mathcal{X}$$

is defined by an affine connection ∇ on $T\mathcal{X} \oplus \widetilde{\mathfrak{g}}$ given as a sum $\nabla = \nabla_{\mathcal{X}} \oplus \nabla^A$, being $\nabla_{\mathcal{X}}$ an affine connection on $\mathcal{X}$. More precisely[6]

$$\left\langle \frac{\partial l}{\partial x}(x,\, y, \xi), \delta x \right\rangle = \left.\frac{d}{ds} l\left(x(s),\, y(s), \xi(s)\right)\right|_{s=0},$$

with $(x(s),\, y(s), \xi(s))$ a horizontal curve w.r.t. ∇ such that

$$(x(0), y(0), \xi(0)) = (x,\, y, \xi) \qquad and \qquad \left.\frac{d}{ds} x(s)\right|_{s=0} = \delta x. \tag{42}$$

It can be shown that

$$\left\langle \frac{\partial l}{\partial x}(x,\, y, \xi), \delta x \right\rangle = \left\langle \frac{\partial^c l}{\partial x}(\xi)(x,\, y), \delta x \right\rangle + \left\langle \frac{\partial l}{\partial \xi}(x,\, y, \xi), ad_{\mathcal{A}(x)\,\delta x}\, \xi \right\rangle, \tag{43}$$

where

$$\frac{\partial^c l}{\partial x}(\xi): T\mathcal{X} \to T^*\mathcal{X}$$

is the base derivative of l w.r.t. connection $\nabla_{\mathcal{X}}$ for ξ fixed. The second term of (43) can be derived by using the fact that, if $(x(s),\, y(s), \xi(s))$ is horizontal w.r.t. $\nabla_{\mathcal{X}} \oplus \nabla^A$, then [using (37) and (42)]

$$\begin{aligned}\left.\frac{d}{ds}\xi(s)\right|_{s=0} &= \left[\mathcal{A}(x(0))\ \left.\frac{d}{ds}x(s)\right|_{s=0}, \xi(0)\right] = \left[\mathcal{A}(x)\ \delta x, \xi\right] \\ &= ad_{\mathcal{A}(x)\,\delta x}\, \xi.\end{aligned}$$

For brevity, we shall write (43) as

$$\left\langle \frac{\partial l}{\partial x}, \delta x \right\rangle = \left\langle \frac{\partial^c l}{\partial x}, \delta x \right\rangle + \left\langle \frac{\partial l}{\partial \xi}, ad_{\mathcal{A}(x)\,\delta x}\, \xi \right\rangle. \tag{44}$$

The map φ for the trivial connection. Let us calculate the map $\varphi: T\mathcal{X} \to \widetilde{\mathfrak{g}}$ when A is the trivial connection, i.e. when the related 1-form $\mathcal{A}$ is the null one. Let us denote $\mathcal{A}^\bullet$ the 1-form defining the generalized nonholonomic connection $A^\bullet$. We shall show that

$$\varphi(x, \dot{x}) = (x, -\mathcal{A}^\bullet(x)\ \dot{x}). \tag{45}$$

Using the Eq. (35), we know that

$$\alpha_{A^\bullet} \circ p\left(x, h, \dot{x}, \dot{h}\right) = (x, \dot{x}) \oplus \left(x, \mathcal{A}^\bullet(x)\ \dot{x} + h^{-1}\,\dot{h}\right)$$

and

$$\alpha_A \circ p\left(x, h, \dot{x}, \dot{h}\right) = (x, \dot{x}) \oplus \left(x, h^{-1}\,\dot{h}\right).$$

[6]To indicate an element of $T_x\mathcal{X}$, we are writing y instead of $\dot{x}$ to avoid any possible confusion.

In particular,

$$\alpha_{A^\bullet} \circ p\,(x,h,\dot{x},0) = (x,\dot{x}) \oplus (x, \mathcal{A}^\bullet\,(x)\;\dot{x})\,,$$

$$\alpha_{A^\bullet} \circ p\left(x,h,0,\dot{h}\right) = 0 \oplus \left(x, h^{-1}\,\dot{h}\right),$$

$$\alpha_A \circ p\,(x,h,\dot{x},0) = (x,\dot{x}) \oplus 0,$$

and

$$\alpha_A \circ p\left(x,h,0,\dot{h}\right) = 0 \oplus \left(x, h^{-1}\,\dot{h}\right).$$

As a consequence,

$$\begin{aligned}
\alpha_{A^\bullet}^{-1}\,((x,\dot{x}) \oplus 0) &= (\alpha_{A^\bullet})^{-1}\left[(x,\dot{x}) \oplus (x, \mathcal{A}^\bullet\,(x)\;\dot{x} - \mathcal{A}^\bullet\,(x)\;\dot{x})\right] \\
&= (\alpha_{A^\bullet})^{-1}\left[(x,\dot{x}) \oplus (x, \mathcal{A}^\bullet\,(x)\;\dot{x})\right] \\
&+ (\alpha_{A^\bullet})^{-1}\left[0 \oplus (x, -\mathcal{A}^\bullet\,(x)\;\dot{x})\right] \\
&= p\,(x,h,\dot{x},0) + p\,(x,h,0,-h\,\mathcal{A}^\bullet\,(x)\;\dot{x})\,,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_A \circ (\alpha_{A^\bullet})^{-1}\,((x,\dot{x}) \oplus 0) &= \left[(x,\dot{x}) \oplus 0\right] + \left[0 \oplus (x, -\mathcal{A}^\bullet\,(x)\;\dot{x})\right] \\
&= (x,\dot{x}) \oplus (x, -\mathcal{A}^\bullet\,(x)\;\dot{x})\,.
\end{aligned}$$

Finally, since by definition

$$\varphi = P_{\tilde{\mathfrak{g}}} \circ \alpha_A \circ (\alpha_{A^\bullet})^{-1} \circ I_{T\mathcal{X}},$$

and

$$I_{T\mathcal{X}}\,(x,\dot{x}) = (x,\dot{x}) \oplus 0,$$

we have that

$$\varphi\,(x,\dot{x}) = P_{\tilde{\mathfrak{g}}}\left[(x,\dot{x}) \oplus (x, -\mathcal{A}^\bullet\,(x)\;\dot{x})\right] = (x, -\mathcal{A}^\bullet\,(x)\;\dot{x})\,,$$

as we wanted to show.

In the following we shall write the (usual and alternative) reduced equations in the general case and in some useful particular situations.
Case 1: General case. Based on above results, the alternative horizontal and vertical reduced equations for a trivial principal bundle are [using Eqs. (40), (41) and (44)]

$$\begin{aligned}
&\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial^c l}{\partial x} + \varphi^*\left(-\frac{d}{dt}\frac{\partial l}{\partial \xi} - ad^*_{\mathcal{A}(x)\,\dot{x}}\frac{\partial l}{\partial \xi} + \mathrm{ad}^*_\xi\frac{\partial l}{\partial \xi}\right), \delta x^\bullet\right\rangle - \\
&- \left\langle \frac{\partial l}{\partial \xi}, \tilde{B}\,(\dot{x}, \delta x^\bullet) - ad_{\mathcal{A}(x)\,\delta x}\,\xi\right\rangle = 0
\end{aligned}$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} - ad^*_{\mathcal{A}(x)\,\dot{x}}\frac{\partial l}{\partial \xi} + \mathrm{ad}^*_\xi\frac{\partial l}{\partial \xi}, \eta^\bullet\right\rangle = 0,$$

where $\eta^{\bullet}$ is seen as a curve on $\mathfrak{g}$, φ as a map $\varphi : T\mathcal{X} \to \mathfrak{g}$, and $\tilde{B}$ is given by Eq. (36). In the usual case, i.e. $A = A^{\bullet}$ (and accordingly $\varphi = 0$), the equations have the form

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial^c l}{\partial x}, \delta x \right\rangle - \left\langle \frac{\partial l}{\partial \xi}, \tilde{B}\left(\dot{x}, \delta x\right) - ad_{\mathcal{A}(x)\,\delta x}\,\xi \right\rangle = 0$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} - ad^{*}_{\mathcal{A}(x)\,\dot{x}}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}, \eta \right\rangle = 0.$$

Remark 9. In the above expression we are omitting the dot "$\bullet$", since we have only one connection and we do not need to make any distinction (as in Section 2.2.3).

Case 2: Choosing A as the trivial connection. For the alternative procedure, we can take $\mathcal{A} = 0$, what implies that $\tilde{B} = 0$. As a consequence, the equations read

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial^c l}{\partial x} + \varphi^{*}\left(-\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}\right), \delta x^{\bullet} \right\rangle = 0$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}, \eta^{\bullet} \right\rangle = 0.$$

Using in addition the Eq. (45), the horizontal equation can be written

$$\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial^c l}{\partial x}, \delta x^{\bullet} \right\rangle - \left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}, \mathcal{A}^{\bullet}\left(x\right)\, \delta x^{\bullet} \right\rangle = 0.$$

This can not be done for the usual procedure, because $A^{\bullet}$ need not to coincide with the trivial connection.

Case 3: $T\mathcal{X}$ is a trivial bundle and A is again the trivial connection. If $T\mathcal{X}$ is trivial, then $\partial l/\partial \dot{x}$ can be seen as a partial derivative in a linear space. In addition, if we choose $\nabla_{\mathcal{X}}$ as the trivial affine connection, then

$$\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}$$

is a standard derivative of a vector with respect to t, and $\partial^c l/\partial x$ is also a standard partial derivative: $\partial l/\partial x$. That is to say, the reduced equations translate to

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial l}{\partial x} + \varphi^{*}\left(-\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}\right), \delta x^{\bullet} \right\rangle = 0, \tag{46}$$

or using Eq. (51)

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial l}{\partial x}, \delta x^{\bullet} \right\rangle - \left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}, \mathcal{A}^{\bullet}\left(x\right)\, \delta x^{\bullet} \right\rangle = 0, \tag{47}$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}, \eta^{\bullet} \right\rangle = 0 \tag{48}$$

in the alternative case, and to

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial l}{\partial x}, \delta x \right\rangle - \left\langle \frac{\partial l}{\partial \xi}, \tilde{B}\left(\dot{x}, \delta x\right) - ad_{\mathcal{A}(x)\,\delta x}\,\xi \right\rangle = 0 \tag{49}$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} - ad^{*}_{\mathcal{A}(x)\,\dot{x}}\frac{\partial l}{\partial \xi} + \mathrm{ad}^{*}_{\xi}\frac{\partial l}{\partial \xi}, \eta \right\rangle = 0 \tag{50}$$

in the usual one (see Remark 9 for notation).

3.2. **Right actions.** Let us reproduce the results of the last subsection for the case of right actions. If we have again a manifold Q and a Lie group G, with Lie algebra $\mathfrak{g}$, a principal connection $A : TQ \to \mathfrak{g}$ related to a **right** action ρ of G on Q must satisfy

$$A\left(\left(\rho_g\right)_*(v)\right) = Ad_{g^{-1}}\left(A(v)\right), \quad \forall g \in G,\ \forall v \in TQ,$$

and

$$A\left(X_\eta\right) = \eta, \qquad \forall\, \eta \in \mathfrak{g}.$$

If $Q = \mathcal{X} \times G$ and ρ is given by (33), using the notation presented at the beginning of the section, last equations say that

$$A\left(x, h\,g, \dot{x}, \dot{h}\,g\right) = Ad_{g^{-1}}\left(A\left(x, h, \dot{x}, \dot{h}\right)\right)$$

and

$$A\left(x, e, 0, h^{-1}\,\dot{h}\right) = h^{-1}\,\dot{h}.$$

Then,

$$A\left(x, h, \dot{x}, \dot{h}\right) = Ad_{h^{-1}}\left(\mathcal{A}(x)\ \dot{x}\right) + h^{-1}\,\dot{h},$$

where, as for the left action (32), $\mathcal{A}$ is a $\mathfrak{g}$-valued 1-form on $\mathcal{X}$ such that $\mathcal{A}(x)\ \dot{x} = A(x, e, \dot{x}, 0)$. Again, we shall say that the connection A is the **trivial** one if $\mathcal{A} = 0$.

Now, we shall list, without proof, the results obtained in the previous section, but for the right action (33).

- Since the action of G on $Q \times \mathfrak{g}$ defining the right adjoint bundle $\widetilde{\mathfrak{g}}$ is

  $$\left(\left(\left(x, h\right), \xi\right), g\right) \mapsto \left(\left(x, h\,g\right), Ad_{g^{-1}}\xi\right),$$

  the identification between $\widetilde{\mathfrak{g}}$ and $\mathcal{X} \times \mathfrak{g}$ is now given by

  $$\widetilde{\mathfrak{g}} \to \mathcal{X} \times \mathfrak{g} : \left[\left(x, h\right), \xi\right] \longmapsto \left(x, Ad_h\xi\right),$$

  with inverse

  $$\left(x, \xi\right) \longmapsto \left[\left(x, e\right), \xi\right].$$

- Identifying $\widetilde{\mathfrak{g}}$ and $\mathcal{X} \times \mathfrak{g}$, the map α_A is defined by [compare to (35)]

  $$\alpha_A \circ p\left(x, h, \dot{x}, \dot{h}\right) = \left(x, \dot{x}\right) \oplus \left(x, \mathcal{A}(x)\ \dot{x} + \dot{h}\,h^{-1}\right),$$

  and the reduced curvature $\widetilde{B}$ by [compare to (36)]

  $$\widetilde{B}\left(\left(x, \dot{x}\right), \left(x, \delta x\right)\right) = \left(x, d\mathcal{A}\left(\left(x, \dot{x}\right), \left(x, \delta x\right)\right) + \left[\mathcal{A}(x)\ \dot{x}, \mathcal{A}(x)\ \delta x\right]\right).$$

- The covariant derivative $\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}$ [see (40)] is given by

  $$\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}} = \frac{d}{dt}\frac{\partial l}{\partial \xi} - ad^*_{\mathcal{A}(x)\,\dot{x}}\,\frac{\partial l}{\partial \xi},$$

  and $\frac{\partial l}{\partial x}$ by [compare to (44)]

  $$\left\langle \frac{\partial l}{\partial x}, \delta x \right\rangle = \left\langle \frac{\partial^c l}{\partial x}, \delta x \right\rangle - \left\langle \frac{\partial l}{\partial \xi}, ad_{\mathcal{A}(x)\,\delta x}\,\xi \right\rangle.$$

- The map φ, when A is the trivial connection, again satisfies the identity

  $$\varphi\left(x, \dot{x}\right) = \left(x, -\mathcal{A}^\bullet(x)\ \dot{x}\right). \tag{51}$$

- Using above results, taking A as the trivial connection and assuming that $T\mathcal{X}$ is trivial (i.e. the *Case 3* of the last subsection), the reduced equations are, in essence, (47) and (48) for the alternative procedure, and (49) and (50) for the usual one, but changing the signs of
$$\mathrm{ad}_\xi^* \frac{\partial l}{\partial \xi}, \quad ad^*_{\mathcal{A}(x)\,\dot{x}} \frac{\partial l}{\partial \xi} \quad \text{and} \quad ad_{\mathcal{A}(x)\,\delta x}\,\xi.$$
(Recall Remark 5.)

3.3. **Ball rolling on a plane.** Let us compare how the usual and the alternative procedures work. To do that, it is enough to consider a standard nonholonomic system, as a ball rolling on a horizontal plane. The configuration space of the system is $Q = \mathbb{R}^2 \times SO(3)$. We shall denote $(\mathbf{a}, R)$ its points. The Lagrangian is

$$L\left(\mathbf{a}, R, \dot{\mathbf{a}}, \dot{R}\right) = -\frac{1}{4}\, I\, tr\left[\left(\dot{R}\, R^{-1}\right)^2\right] + \frac{1}{2}\, m\, \dot{\mathbf{a}}^2,$$

where I and m are the moment of inertia w.r.t. the center of gravity and the mass of the ball, respectively. The rolling constraints are

$$\dot{\mathbf{a}} = r\, \dot{R}\, R^{-1}\mathbf{E}_3,$$

with r the radius of the ball and $\mathbf{E}_3 = (0, 0, 1)$. These are the kinematic constraints. Since we are dealing with a standard nonholonomic system, the variational constraints coincide with the kinematic ones, i.e. they are given by

$$\delta\mathbf{a} = r\, \delta R\, R^{-1}\mathbf{E}_3.$$

Note that we are identifying the vectors of the form $(x, y, 0) \in \mathbb{R}^3$ with vectors of $\mathbb{R}^2$. It is easy to see that the group $G = SO(3)$ and the right action

$$\rho : Q \times G \to Q : ((\mathbf{a}, R), M) \longmapsto (\mathbf{a}, R\, M) \tag{52}$$

define a symmetry for the system. Regarding this action, the manifold Q and the Lie group G define a trivial fiber bundle with base $\mathcal{X} = \mathbb{R}^2$. As a consequence, we have the identification

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{so}(3)} = \mathcal{X} \times \mathfrak{so}(3) = \mathbb{R}^2 \times \mathfrak{so}(3).$$

We shall also write

$$T\mathcal{X} = \mathbb{R}^2 \times \mathbb{R}^2.$$

If we take A as the trivial connection, i.e.

$$A\left(\mathbf{a}, R, \dot{\mathbf{a}}, \dot{R}\right) = R^{-1}\, \dot{R},$$

then we are in the *Case 3* (see the end of Section 3.1.3) for a right action. Let us write the reduced equations of the system for both the alternative and the usual procedures.

The generalized nonholonomic connection, which in this case it is just a nonholonomic connection, is given by (see [11])

$$A^{\bullet}\left(\mathbf{a}, R, \dot{\mathbf{a}}, \dot{R}\right) = R^{-1}\, \dot{R} - \frac{1}{r}\, R^{-1}\left[\hat{\mathbf{E}}_3, \hat{\dot{\mathbf{a}}}\right] R,$$

where we are using the map

$$\hat{} : \mathbb{R}^3 \to \mathfrak{so}(3) \tag{53}$$

such that

$$\mathbf{x}=\left(\mathbf{x^1},\mathbf{x^2},\mathbf{x^3}\right)\longmapsto\hat{\mathbf{x}}=\begin{bmatrix}0 & -x^3 & x^2\\ x^3 & 0 & -x^1\\ -x^2 & x^1 & 0\end{bmatrix}.$$

It can be shown that

$$\widehat{\mathbf{x}\times\mathbf{y}}=[\hat{\mathbf{x}},\hat{\mathbf{y}}],\quad \hat{\mathbf{x}}\mathbf{y}=\mathbf{x}\times\mathbf{y},\quad \widehat{R\mathbf{x}}=R\hat{\mathbf{x}}R^{-1},\quad \langle\mathbf{x},\mathbf{y}\rangle=-\frac{1}{2}\operatorname{tr}\left(\hat{\mathbf{x}}\hat{\mathbf{y}}\right),$$

$\forall x,y\in\mathbb{R}^3,\ R\in\mathrm{SO}(3)$. The symbols $\times$ and $\langle\cdot,\cdot\rangle$ indicate the cross product and the euclidean inner product in $\mathbb{R}^3$ respectively.

The isomorphisms

$$\alpha_A,\alpha_{A^\bullet}:\frac{TQ}{G}\to\mathbb{R}^2\times\mathbb{R}^2\times\mathfrak{so}(3)$$

are given by

$$\alpha_A\circ p\left(\mathbf{a},R,\dot{\mathbf{a}},\dot{R}\right)=\left(\mathbf{a},\dot{\mathbf{a}},\dot{R}\,R^{-1}\right)$$

and

$$\alpha_{A^\bullet}\circ p\left(\mathbf{a},R,\dot{\mathbf{a}},\dot{R}\right)=\left(\mathbf{a},\dot{\mathbf{a}},\dot{R}\,R^{-1}-\frac{1}{r}\left[\hat{\mathbf{E}}_3,\hat{\dot{\mathbf{a}}}\right]\right).$$

Identifying $\mathbb{R}^3$ and $\mathfrak{so}(3)$ through (53), $\mathcal{A}^\bullet$ is given by

$$\mathcal{A}^\bullet(\mathbf{a})\ \dot{\mathbf{a}}=-\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}.$$

Thus

$$\varphi:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2\times\mathfrak{so}(3):(\mathbf{a},\dot{\mathbf{a}})\longmapsto\left(\mathbf{a},\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}\right).$$

On the other hand, identifying $\mathbb{R}^n$ and its dual (for $n=2,3$),

$$\varphi^*:\mathbb{R}^2\times\mathfrak{so}(3)\to\mathbb{R}^2\times\mathbb{R}^2:(\mathbf{a},\xi)\longmapsto\left(\mathbf{a},\frac{1}{r}\,\xi\times\mathbf{E}_3\right).\tag{54}$$

The reduced Lagrangian related to A is

$$l\left(\mathbf{a},\dot{\mathbf{a}},\xi\right)=\frac{1}{2}\,I\,\langle\xi,\xi\rangle+\frac{1}{2}\,m\,\dot{\mathbf{a}}^2,$$

too much simpler than that related to $A^\bullet$:

$$l\left(\mathbf{a},\dot{\mathbf{a}},\xi\right)=\frac{1}{2}\,I\,\left\langle\xi+\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}},\xi+\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}\right\rangle+\frac{1}{2}\,m\,\dot{\mathbf{a}}^2.$$

In order to emphasize the presence of different connections, we are going to write the last Lagrangian as

$$l^\bullet\left(\mathbf{a},\dot{\mathbf{a}},\xi^\bullet\right)=\frac{1}{2}\,I\,\left\langle\xi^\bullet+\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}},\xi^\bullet+\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}\right\rangle+\frac{1}{2}\,m\,\dot{\mathbf{a}}^2.$$

The relationship between ξ and $\xi^\bullet$ is given by

$$\xi^\bullet=\xi-\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}.$$

The partial derivatives of these Lagrangians are

$$\frac{\partial l}{\partial\mathbf{a}}=\frac{\partial l^\bullet}{\partial\mathbf{a}}=0,$$

$$\frac{\partial l}{\partial \dot{\mathbf{a}}} = m\,\dot{\mathbf{a}},$$

$$\frac{\partial l^{\bullet}}{\partial \dot{\mathbf{a}}} = \left(m + \frac{I}{r^2}\right)\dot{\mathbf{a}} + \frac{I}{r}\,\xi^{\bullet} \times \mathbf{E}_3,$$

$$\frac{\partial l}{\partial \xi} = I\,\xi,$$

$$\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}} = I\,\xi^{\bullet} + \frac{I}{r}\,\mathbf{E}_3 \times \dot{\mathbf{a}}.$$

The reduced curvature of A is 0 and that of $A^{\bullet}$ is

$$\widetilde{B}^{\bullet}(\mathbf{a})(\dot{\mathbf{a}}, \delta\mathbf{a}) = \left(\mathbf{a}, \frac{1}{r^2}\,\dot{\mathbf{a}} \times \delta\mathbf{a}\right).$$

The horizontal and vertical reduced variations, w.r.t. to $A^{\bullet}$, live inside

$$\mathfrak{C}_V^{\bullet} \cap T\mathcal{X} = \mathbb{R}^2 \quad \text{and} \quad \mathfrak{C}_V^{\bullet} \cap \widetilde{\mathfrak{g}} = \langle \mathbf{E}_3 \rangle,$$

respectively.

According to the results of the last Section, the reduced equations (30) and (31) corresponding to the alternative procedure, for connections A and $A^{\bullet}$, are

$$-\frac{d}{dt}\frac{\partial l}{\partial \dot{\mathbf{a}}} + \frac{\partial l}{\partial \mathbf{a}} + \varphi^{*}\left(-\frac{d}{dt}\frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial \xi} \times \xi\right) = 0$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial \xi} \times \xi, \mathbf{E}_3 \right\rangle = 0.$$

Using the above expressions for the partial derivatives of l we have

$$-\frac{d}{dt}\frac{\partial l}{\partial \dot{\mathbf{a}}} + \frac{\partial l}{\partial \mathbf{a}} = -m\,\ddot{\mathbf{a}}$$

and

$$-\frac{d}{dt}\frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial \xi} \times \xi = -I\,\dot{\xi}.$$

And using the expression for φ^{*} given in (54), we arrive at the equations

$$m\,\ddot{\mathbf{a}} + \frac{I}{r}\,\dot{\xi} \times \mathbf{E}_3 = 0 \tag{55}$$

and

$$\left\langle \dot{\xi}, \mathbf{E}_3 \right\rangle = 0. \tag{56}$$

The reduced constraint equations say

$$\xi \times \mathbf{E}_3 = \frac{1}{r}\,\dot{\mathbf{a}}, \tag{57}$$

what is equivalent to

$$\xi = \lambda\,\mathbf{E}_3 + \frac{1}{r}\,\mathbf{E}_3 \times \dot{\mathbf{a}}, \tag{58}$$

where λ is a new variable. Combining (55)-(58) it follows that

$$\ddot{\mathbf{a}} = 0, \qquad \dot{\lambda} = 0.$$

Now, let us write the Equations (19) and (20) corresponding to the usual procedure, using the connection $A^{\bullet}$. Again, according to the results of the last Section, such equations are

$$-\frac{d}{dt}\frac{\partial l^{\bullet}}{\partial \dot{\mathbf{a}}}+\frac{\partial l^{\bullet}}{\partial \mathbf{a}}=\mathfrak{p}\left(\frac{1}{r^2}\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}\times\dot{\mathbf{a}}\right)-\frac{1}{r}\left(\xi^{\bullet}\times\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}\right)\times\mathbf{E}_3$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}+\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}\times\left(\xi^{\bullet}+\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}\right),\mathbf{E}_3\right\rangle=0,$$

where $\mathfrak{p}:\mathbb{R}^3\to\mathbb{R}^2$ is the projection $(x,y,z)\mapsto(x,y)$. On the one hand,

$$-\frac{d}{dt}\frac{\partial l^{\bullet}}{\partial \dot{\mathbf{a}}}+\frac{\partial l^{\bullet}}{\partial \mathbf{a}}=-\left(m+\frac{I}{r^2}\right)\ddot{\mathbf{a}}-\frac{I}{r}\,\dot{\xi}^{\bullet}\times\mathbf{E}_3$$

and

$$-\frac{d}{dt}\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}+\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}\times\left(\xi^{\bullet}+\frac{1}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}\right)=-I\,\dot{\xi}^{\bullet}-\frac{I}{r}\,\mathbf{E}_3\times\ddot{\mathbf{a}},$$

where we have used that

$$\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}=I\,\xi^{\bullet}+\frac{I}{r}\,\mathbf{E}_3\times\dot{\mathbf{a}}.$$

On the other hand,

$$\begin{aligned}\frac{1}{r^2}\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}\times\dot{\mathbf{a}}&=\left(\frac{I}{r^2}\,\xi^{\bullet}+\frac{I}{r^3}\,\mathbf{E}_3\times\dot{\mathbf{a}}\right)\times\dot{\mathbf{a}}\\ &=\frac{I}{r^2}\,\xi^{\bullet}\times\dot{\mathbf{a}}+\frac{I}{r^3}\,(\mathbf{E}_3\times\dot{\mathbf{a}})\times\dot{\mathbf{a}}.\end{aligned}$$

Accordingly,

$$\mathfrak{p}\left(\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}\times\dot{\mathbf{a}}\right)=\mathfrak{p}\left(\xi^{\bullet}\times\dot{\mathbf{a}}\right),$$

and

$$\frac{1}{r}\left(\xi^{\bullet}\times\frac{\partial l^{\bullet}}{\partial \xi^{\bullet}}\right)\times\mathbf{E}_3=\frac{I}{r^2}\,\left(\xi^{\bullet}\times(\mathbf{E}_3\times\dot{\mathbf{a}})\right)\times\mathbf{E}_3.$$

With all that, we arrive at the equations

$$-\left(m+\frac{I}{r^2}\right)\ddot{\mathbf{a}}-\frac{I}{r}\,\dot{\xi}^{\bullet}\times\mathbf{E}_3=\frac{I}{r^2}\,\left[\mathfrak{p}\left(\xi^{\bullet}\times\dot{\mathbf{a}}\right)-\left(\xi^{\bullet}\times(\mathbf{E}_3\times\dot{\mathbf{a}})\right)\times\mathbf{E}_3\right]$$

and

$$\left\langle\dot{\xi}^{\bullet},\mathbf{E}_3\right\rangle=0.$$

They are clearly more complicated than (55) and (56), and they were harder to be derived. Working out a little bit the right member of the first equation, we have

$$\left(\xi^{\bullet}\times(\mathbf{E}_3\times\dot{\mathbf{a}})\right)\times\mathbf{E}_3=(\mathbf{E}_3\times\dot{\mathbf{a}})\,\langle\xi^{\bullet},\mathbf{E}_3\rangle=\langle\xi^{\bullet},\mathbf{E}_3\rangle\,\mathbf{E}_3\times\dot{\mathbf{a}}.$$

Also, writing $\xi^{\bullet}=\langle\xi^{\bullet},\mathbf{E}_3\rangle\,\mathbf{E}_3+\xi^{\bullet}_{\perp}$,

$$\mathfrak{p}\left(\xi^{\bullet}\times\dot{\mathbf{a}}\right)=\mathfrak{p}\left[\langle\xi^{\bullet},\mathbf{E}_3\rangle\,\mathbf{E}_3\times\dot{\mathbf{a}}+\xi^{\bullet}_{\perp}\times\dot{\mathbf{a}}\right]=\langle\xi^{\bullet},\mathbf{E}_3\rangle\,\mathbf{E}_3\times\dot{\mathbf{a}},$$

since $\langle\xi^{\bullet},\mathbf{E}_3\rangle\,\mathbf{E}_3\times\dot{\mathbf{a}}$ lives in the horizontal plane and $\xi^{\bullet}_{\perp}\times\dot{\mathbf{a}}$ lives in the vertical axis. This transforms the equations into

$$\left(m+\frac{I}{r^2}\right)\ddot{\mathbf{a}}+\frac{I}{r}\,\dot{\xi}^{\bullet}\times\mathbf{E}_3=0\qquad\textit{and}\qquad\left\langle\dot{\xi}^{\bullet},\mathbf{E}_3\right\rangle=0,$$

which coincides with (55) and (56) when we make the change of variables

$$\xi^{\bullet}=\xi-\frac{1}{r}\,\mathbf{E}_{3}\times\dot{\mathbf{a}},$$

as it must be. Imposing the rolling constraint, which says that

$$\xi^{\bullet}=\lambda\,\mathbf{E}_{3},$$

we finally arrive at the equations

$$\ddot{\mathbf{a}}=0,\qquad\dot{\lambda}=0.$$

4. **Reduction of HOCSs.** The aim of this section is to extend the results of the Section 2, for GNHSs, to the case of **higher order constrained systems** (HOCSs). So, in the first place, we shall recall the definition of HOCSs as presented in Ref. [8], then, given a Lie group G, we shall say what we mean by a G-invariant HOCS, and finally we shall develop for the later a reduction procedure that generalize that presented in Section 2.3. At the end of the section, a simple example will be studied.

To do all that, we need to introduce some basic concepts and notation on higher order tangent bundles.

4.1. **Bundles $T^{(k)}Q$.** For $k\geq 0$, let us denote by $T^{(k)}Q$ the *k-th order tangent bundle* of Q [17, 16], given by a fiber bundle $\tau_{Q}^{(k)}:T^{(k)}Q\twoheadrightarrow Q$ such that, for each $q\in Q$, the fiber $T_{q}^{(k)}Q$ is a set of equivalence classes $[\gamma]^{(k)}$ of curves $\gamma:(-\varepsilon,\varepsilon)\rightarrow Q$ satisfying $\gamma(0)=q$. The equivalence relation says that $\gamma_{1}\backsim\gamma_{2}$ *iff*, for every chart (U,φ) containing q, the equations

$$\left.\frac{d^{s}}{dt^{s}}\right|_{t=0}(\varphi\circ\gamma_{1})=\left.\frac{d^{s}}{dt^{s}}\right|_{t=0}(\varphi\circ\gamma_{2}),\quad for\quad s=0,...,k,$$

are fulfilled. Of course, $T^{(0)}Q=Q$ and $T^{(1)}Q=TQ$. Accordingly, $\tau_{Q}^{(0)}=id_{Q}$ (the identity map) and $\tau_{Q}^{(1)}=\tau_{Q}$ (the canonical projection of TQ onto Q).

Given a curve $\gamma:[t_{1},t_{2}]\rightarrow Q$, its *$k$-lift* is the curve

$$\gamma^{(k)}:(t_{1},t_{2})\rightarrow T^{(k)}Q\quad:\quad t\mapsto[\gamma_{t}]^{(k)},$$

being $\gamma_{t}:(-\varepsilon_{t},\varepsilon_{t})\rightarrow Q$ such that

$$\gamma_{t}(s)=\gamma(s+t)\qquad and\qquad\varepsilon_{t}=\min\{t-t_{1},t_{2}-t\}.$$

The 1-lift of γ is precisely its velocity $\gamma':(t_{1},t_{2})\rightarrow TQ$.

Consider now a map $f:N\rightarrow M$. Its *k-lift*

$$f^{(k)}:T^{(k)}N\rightarrow T^{(k)}M$$

is given by

$$f^{(k)}\left([\gamma]^{(k)}\right)=[f\circ\gamma]^{(k)}.$$

Of course, $f^{(1)}=f_{*}$.

4.2. HOCS with symmetry.

Definition 4.1. Given a manifold Q, let us consider the triples (L, C_K, C_V) with

$$L : TQ \to \mathbb{R}, \quad C_K \subset T^{(k)}Q, \quad C_V \subset T^{(l)}Q \times_Q TQ,$$

with $k, l \geq 0$, being C_K a submanifold and C_V such that, for every $q \in Q$ and $\zeta \in T_q^{(l)}Q$, the subset

$$C_V(\zeta) = C_V \cap (\{\zeta\} \times T_q Q) \subset \{\zeta\} \times T_q Q,$$

seen as a subset of $T_q Q$, is void or a linear subspace. We shall refer to them as **Lagrangian systems with higher order constraints**, or simply, **higher order constrained systems (HOCS)**, with Lagrangian function L, **kinematic constraints** C_K and **variational constraints** C_V, whose elements will be called **virtual displacements**; and we shall say that $\gamma : [t_1, t_2] \to Q$ is a **trajectory** of (L, C_K, C_V) if:

1. $\gamma^{(k)}(t) \in C_K$, $\forall\, t \in (t_1, t_2)$;
2. for all variations $\delta\gamma$ such that $\left(\gamma^{(l)}(t), \delta\gamma(t)\right) \in C_V$, $\forall\, t \in (t_1, t_2)$, we have

$$\int_{t_1}^{t_2} \langle dL(\gamma'(t)), \kappa(\delta\gamma'(t))\rangle \; dt = 0. \tag{59}$$

For each $q \in Q$ and $\zeta \in T_q^{(l)}Q$, consider the annihilator

$$F_V(\zeta) = (C_V(\zeta))^o \subset T_q^* Q,$$

whenever $C_V(\zeta)$ is non void. Such subspaces give rise to a subset $F_V \subset T^{(l)}Q \times_Q T^*Q$ that we shall call the space of **constraint forces**.

For physical interpretation and applications, see [9, 10, 21, 22, 23, 24, 31].

Definition 4.2. Let (L, C_K, C_V) be a HOCS. Suppose a group G acts on Q, with action $\rho : G \times Q \to Q$, in such a way that

a.: $L \circ (\rho_g)_* = L$,

b.: $\rho_g^{(k)}(C_K) = C_K$,

c.: for each $q \in Q$ and $\zeta \in T_q^{(l)}Q$

$$(\rho_g)_*(C_V(\zeta)) = C_V\left(\rho_g^{(l)}(\zeta)\right),$$

for all $g \in G$, with $\rho_g : Q \to Q : q \mapsto \rho(g, q)$.

In such a case, we shall say (L, C_K, C_V) is a G**-invariant** triple.

Consider the canonical projections

$$p_n : T^{(n)}Q \twoheadrightarrow T^{(n)}Q \big/ G$$

related to actions

$$\rho_n : G \times T^{(n)}Q \to T^{(n)}Q \quad : \quad (g, \zeta) \mapsto \rho_g^{(n)}(\zeta).$$

For $n = 1$ we shall write $p_1 = p$. In terms of them we can define the **reduced Lagrangian** $l : TQ/G \to \mathbb{R}$ by the formula

$$l \circ p = L, \tag{60}$$

and the **reduced constraints** $\mathfrak{C}_K \subset T^{(k)}Q \big/ G$ as

$$\mathfrak{C}_K = p_k(C_K) = C_K/G$$

and

$$\mathfrak{C}_V \subset T^{(l)}Q\Big/ G \times_{Q/G} TQ/ G$$

through the subspaces

$$\mathfrak{C}_V \left(p_l \left(\zeta\right)\right) = p \left(C_V \left(\zeta\right)\right) = C_V \left(\zeta\right) / G. \tag{61}$$

We shall assume that all quotients are manifolds and related projections are surjective submersions. In the following, we shall write an analogue of the Lagrange-d'Alembert-Poincaré (LDP) equations for the triple (L, C_K, C_V), in terms of the reduced data l, $\mathfrak{C}_K$ and $\mathfrak{C}_V$. As for the alternative reduction developed in the previous sections, we shall use two connection-like objects: one of them (a genuine principal connection) will be used to defined the reduced coordinates; the other object will be used to decompose the variations into horizontal and vertical components.

4.3. **A reduction procedure.** Let us consider a G-invariant HOCS (L, C_K, C_V) with configuration space Q. Again, we will write $\mathcal{X} = Q/G$, and assume that the canonical projection $\pi : Q \twoheadrightarrow \mathcal{X}$ is a principal fiber bundle with structure group G. Following the same reasoning as in the Section 2.3, and fixing an arbitrary principal connection A, it can be shown that a curve $\gamma : [t_1, t_2] \to Q$ is a trajectory of (L, C_K, C_V) if and only if

$$p_k \left(\gamma^{(k)} \left(t\right)\right) \in \mathfrak{C}_K$$

and the curve $\mu : [t_1, t_2] \to T\mathcal{X} \oplus \tilde{\mathfrak{g}}$, given by

$$\mu \left(t\right) = \dot{x} \left(t\right) \oplus \bar{v} \left(t\right) = \alpha_A \circ p \left(\gamma' \left(t\right)\right),$$

satisfies

$$\begin{aligned} &\left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}} \left(\mu \left(t\right)\right) + \frac{\partial l}{\partial x} \left(\mu \left(t\right)\right) - \left\langle \frac{\partial l}{\partial \bar{v}} \left(\mu \left(t\right)\right), i_{\dot{x}(t)}\tilde{B} \right\rangle, \delta x \left(t\right) \right\rangle \\ &+ \left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}} \left(\mu \left(t\right)\right) + \mathsf{ad}^*_{\bar{v}}\frac{\partial l}{\partial \bar{v}} \left(\mu \left(t\right)\right), \bar{\eta} \left(t\right) \right\rangle = 0, \end{aligned} \tag{62}$$

for all curves

$$\delta x : [t_1, t_2] \to T\mathcal{X} \quad \text{and} \quad \bar{\eta} : [t_1, t_2] \to \tilde{\mathfrak{g}}$$

such that

$$\delta x \left(t\right) \oplus \bar{\eta} \left(t\right) \in \mathfrak{C}_V \left(p_l \left(\gamma^l \left(t\right)\right)\right).$$

We want to decompose the Eqs. (62) into horizontal and vertical parts. In order to do that we need to decompose each subspace $\mathfrak{C}_V \left(p_l \left(\gamma^l \left(t\right)\right)\right)$. Since these subspaces depend not only on $x \in \mathcal{X}$ but on the points of $\left(T^{(l)}Q/ G\right)_x$, a standard connection it is not useful in this case. We need a more general object.

4.3.1. *The l-connections.*

Definition 4.3. Given $l \in \mathbb{N} \cup \{0\}$, an $\mathbf{l}-$**connection** on the principal fiber bundle $\pi : Q \twoheadrightarrow \mathcal{X}$ is a map

$$A : T^{(l)}Q \times_Q TQ \to \mathfrak{g}$$

such that, $\forall\, q \in Q$ and $\forall\, \zeta \in T^{(l)}_q Q$, it is a linear transformation when restricted to $\{\zeta\} \times T_q Q$, and $\forall\, v \in T_q Q$, $\forall\, g \in G$ and $\forall\, \eta \in \mathfrak{g}$

$$A \left(\zeta, X_\eta \left(q\right)\right) = \eta \quad \text{and} \quad A \left(\rho^{(l)}_g \left(\zeta\right), \rho_{g*} \left(v\right)\right) = Ad_g \left[A \left(\zeta, v\right)\right]. \tag{63}$$

Remark 10. Note that, when $l = 0$, and identifying $Q \times_Q TQ$ with TQ, we have a genuine principal connection.

The following proposition is easy to prove.

Proposition 3. *Giving an l-connection is the same as giving an assignment of a linear subspace $\mathcal{H}(\zeta) \subset T_qQ$ to each $q \in Q$ and $\zeta \in T_q^{(l)}Q$, such that*

- $T_qQ = \mathcal{H}(\zeta) \oplus \mathcal{V}(\zeta)$, *where* $\mathcal{V}(\zeta) = \mathcal{V}_q$*: the vertical subspace at* q*;*
- $\mathcal{H}\left(\rho_g^{(l)}(\zeta)\right) = (\rho_g)_* [\mathcal{H}(\zeta)]$, $\forall\, g \in G$*;*
- *and the subspaces $\mathcal{H}(\zeta)$, which we shall call **horizontal subspaces**, depend differentiably on q and ζ.*

Given A, the horizontal spaces $\mathcal{H}(\zeta)$ are defined by equality

$$\mathcal{H}(\zeta) = \{v \in T_qQ : A(\zeta, v) = 0\},$$

and given horizontal spaces $\mathcal{H}(\zeta)$, the l-connection A is defined by the formula

$$A(\zeta, v) = \eta,$$

where $v - X_\eta(q) \in \mathcal{H}(\zeta)$.

Related to an l-connection we have a map

$$\alpha_A : T^{(l)}Q \big/ G \times_{\mathcal{X}} TQ/G \to T\mathcal{X} \oplus \widetilde{\mathfrak{g}}, \tag{64}$$

similar to the Atiyah isomorphism of a principal connection, defined in the following way:

1. Take a class of $T^{(l)}Q\big/ G$ and a class of TQ/G, both of them based on the same point $x \in \mathcal{X}$.
2. Consider representatives $\zeta \in T^{(l)}Q$ and $v \in TQ$ of each one of these classes, based on the same point $q \in Q$ such that $\pi(q) = x$ (this is always possible).
3. Define α_A on the classes given in **1** as $\pi_*(v) \oplus [q, A(\zeta, v)]$, i.e.

$$\alpha_A([\zeta], [v]) = \pi_*(v) \oplus [q, A(\zeta, v)].$$

It is well-defined because, if ζ' and v' are representatives satisfying **2**, based on a point q', then there exists (only one) $g \in G$ such that $\zeta' = \rho_g^{(l)}(\zeta)$ and $v' = \rho_{g*}(v)$: the unique element g of G such that $q' = \rho_g(q)$. (Recall that action ρ is free.) Accordingly, from the second part of (63),

$$\alpha_A([\zeta'], [v']) = \alpha_A([\zeta], [v]).$$

We are writing, as usual, $[\zeta] = p_l(\zeta)$ and $[v] = p(v)$.

Denoting by a the map

$$a : T^{(l)}Q \times_Q TQ \to \widetilde{\mathfrak{g}} \ : \ (\zeta, v) \mapsto [q, A(\zeta, v)],$$

we have

$$\alpha_A([\zeta], [v]) = \pi_*(v) \oplus a(\zeta, v), \quad \forall\, v \in TQ.$$

For each $\zeta \in T_q^{(l)}Q$, it can be shown that

$$\begin{aligned} \alpha_A^{[\zeta]} \quad &: \quad (TQ/G)_{\pi(q)} \to T_{\pi(q)}\mathcal{X} \oplus \widetilde{\mathfrak{g}}_{\pi(q)} \\ &: \quad [v] \mapsto \alpha_A([\zeta], [v]) = \pi_*(v) \oplus a(\zeta, v) \end{aligned} \tag{65}$$

defines a linear isomorphism. In fact, spaces $(TQ/G)_{\pi(q)}$ and $T_{\pi(q)}\mathcal{X} \oplus \widetilde{\mathfrak{g}}_{\pi(q)}$ have the same dimension (see Remark 4), and

$$\pi_* (v) \oplus a(\zeta, v) = 0$$

if and only if $v \in \mathcal{V}_q \cap \mathcal{H}(\zeta) = \{0\}$, what ensure that $\alpha_A^{[\zeta]} \circ p$ is injective. Then, $\alpha_A^{[\zeta]}$ is injective and, as a consequence, a linear isomorphism.

From the last discussion it also follows that

$$\alpha_A^{[\zeta]} \circ p : T_q Q \to T_{\pi(q)}\mathcal{X} \oplus \widetilde{\mathfrak{g}}_{\pi(q)}$$

defines a linear isomorphism (see Remark 4 again).

Remark 11. Let us clarify the definition of each $\alpha_A^{[\zeta]}$. Consider a class $c \in T^{(l)}Q/G$, based on the point $x \in \mathcal{X}$. We shall define

$$\alpha_A^c : (TQ/G)_x \to T_x\mathcal{X} \oplus \widetilde{\mathfrak{g}}_x$$

as follows:

1. Fix a representative $\zeta \in T^{(l)}Q$ of c based on $q \in Q$. Of course, $\pi(q) = x$.
2. Consider a class of TQ/G based on x.
3. Fix a representative $v \in TQ$ based on q.
4. Define α_A^c on the class given in **2** as $\pi_*(v) \oplus a(\zeta, v)$.

If we fix another representative ζ' of c, based on q', and another representative v' of the class given in **2**, also based on q', from the discussion on α_A, it follows that

$$\pi_* (v) \oplus a(\zeta, v) = \pi_* (v') \oplus a(\zeta', v').$$

Then, each α_A^c is well-defined.

In terms of the maps

$$a_\zeta : T_q Q \to \widetilde{\mathfrak{g}}_{\pi(q)} : v \mapsto a(\zeta, v), \tag{66}$$

defined for each $\zeta \in T_q^{(l)}Q$, we can write

$$\alpha_A^{[\zeta]}([v]) = \pi_*(v) \oplus a_\zeta(v),$$

and we shall do it from now on.

4.3.2. *The higher order connection.* In order to develop a reduction procedure for HOCS with symmetry we shall introduce, related to each G-invariant triple HOCS, a particular l-connection: the *higher order connection*. This will enable us to separate the reduced virtual displacements $\mathfrak{C}_V$ into horizontal and vertical components. The construction of such an object will be done in several steps (compare with those appearing in 2.2.1).

1. Fix a G-invariant metric on Q. We shall assume that L is simple, and that we chose the metric which defines its kinetic term.
2. For each $q \in Q$ and $\zeta \in T_q^{(l)}Q$, consider the intersection

   $$\mathcal{S}(\zeta) = C_V(\zeta) \cap \mathcal{V}(\zeta),$$

   and write

   $$C_V(\zeta) = \mathcal{T}(\zeta) \oplus \mathcal{S}(\zeta) \quad \text{and} \quad \mathcal{V}(\zeta) = \mathcal{S}(\zeta) \oplus \mathcal{U}(\zeta),$$

   where $\mathcal{T}(\zeta)$ and $\mathcal{U}(\zeta)$ are the orthogonal complements of $\mathcal{S}(\zeta)$ in $C_V(\zeta)$ and $\mathcal{V}(\zeta)$, respectively.

3. Consider the orthogonal complement of $C_V(\zeta)+\mathcal{V}(\zeta)$ in T_qQ. Let us call it $\mathcal{R}(\zeta)$.

   We shall assume that the spaces $\mathcal{R}(\zeta)\oplus\mathcal{T}(\zeta)$ depend differentiably on q and ζ.
4. Define the l-connection $A^\bullet : T^{(l)}Q\times TQ\to\mathfrak{g}$, which we shall call the **higher order connection**, with horizontal subspaces (see Proposition 3)

$$\mathcal{H}^\bullet(\zeta)=\mathcal{R}(\zeta)\oplus\mathcal{T}(\zeta).$$

   In other words, given $v\in T_qQ$, define

$$A^\bullet(\zeta,v)=\eta$$

   if $v-X_\eta(q)\in\mathcal{H}^\bullet(\zeta)$.

It is easy to show that $A^\bullet$ is effectively an l-connection. In particular,

$$T_qQ=\mathcal{H}^\bullet(\zeta)\oplus\mathcal{V}(\zeta).$$

Note that

$$\mathcal{T}(\zeta)=C_V(\zeta)\cap\mathcal{H}^\bullet(\zeta).$$

Thus,

$$C_V(\zeta)=[C_V(\zeta)\cap\mathcal{H}^\bullet(\zeta)]\oplus[C_V(\zeta)\cap\mathcal{V}(\zeta)]. \tag{67}$$

Using the isomorphisms

$$\begin{aligned}\alpha^{[\zeta]}_{A^\bullet} \quad &: \quad (TQ/G)_{\pi(q)}\to T_{\pi(q)}\mathcal{X}\oplus\widetilde{\mathfrak{g}}_{\pi(q)}\\ &: \quad [v]\mapsto\pi_*(v)\oplus[q,A^\bullet(\zeta,v)],\end{aligned}$$

and the Eqs. (64) and (65), we have

$$\mathcal{H}^\bullet(\zeta)/G\simeq\alpha^{[\zeta]}_{A^\bullet}(\mathcal{H}^\bullet(\zeta)/G)=\pi_*(\mathcal{H}^\bullet(\zeta))=T_{\pi(q)}\mathcal{X}, \tag{68}$$

and [see (66)]

$$\mathcal{V}(\zeta)/G\simeq\alpha^{[\zeta]}_{A^\bullet}(\mathcal{V}(\zeta)/G)=a^\bullet_\zeta(\mathcal{V}(\zeta))=\tilde{\mathfrak{g}}_{\pi(q)}. \tag{69}$$

Accordingly, combining (67), (68) and (69), the next result is immediate.

Proposition 4. *Since $\mathfrak{C}^\bullet_V([\zeta])\simeq\alpha^{[\zeta]}_{A^\bullet}\circ p(C_V(\zeta))$, we have*

$$\mathfrak{C}^\bullet_V([\zeta])=\mathfrak{C}^{hor}_V([\zeta])\oplus\mathfrak{C}^{ver}_V([\zeta])$$

with

$$\mathfrak{C}^{hor}_V([\zeta])\simeq\pi_*(C_V(\zeta))=T_{\pi(q)}\mathcal{X}\cap\mathfrak{C}^\bullet_V([\zeta])$$

and

$$\mathfrak{C}^{ver}_V([\zeta])\simeq a^\bullet_\zeta(C_V(\zeta))=\tilde{\mathfrak{g}}_{\pi(q)}\cap\mathfrak{C}^\bullet_V([\zeta]).$$

4.3.3. *The maps $\varphi^{[\zeta]}$.* Given a G-invariant triple (L, C_K, C_V), consider the related higher order connection $A^{\bullet} : T^{(l)}Q \times_Q TQ \to \mathfrak{g}$ and fix an arbitrary principal connection $A : TQ \to \mathfrak{g}$. Given an infinitesimal variation $\delta\gamma$ on the curve γ, let us write

$$\alpha_A \circ p\left(\delta\gamma\left(t\right)\right) = \pi_*\left(\delta\gamma\left(t\right)\right) \oplus a\left(\delta\gamma\left(t\right)\right) = \delta x\left(t\right) \oplus \bar{\eta}\left(t\right) \tag{70}$$

as before, and

$$\alpha_{A^{\bullet}}^{[\zeta]} \circ p\left(\delta\gamma\left(t\right)\right) = \pi_*\left(\delta\gamma\left(t\right)\right) \oplus a_{\zeta}^{\bullet}\left(\delta\gamma\left(t\right)\right) = \delta x^{\bullet}\left(t\right) \oplus \bar{\eta}^{\bullet}\left(t\right), \tag{71}$$

if $\zeta = \gamma^{(l)}(t)$. It is clear that, if $\delta\gamma(t) \in C_V\left(\gamma^{(l)}(t)\right)$, then

$$\delta x\left(t\right) \oplus \bar{\eta}\left(t\right) \in \mathfrak{C}_V\left(\left[\gamma^{(l)}\left(t\right)\right]\right),$$

and

$$\delta x^{\bullet}\left(t\right) \in \mathfrak{C}_V^{\mathrm{hor}}\left(\left[\gamma^{(l)}\left(t\right)\right]\right) \quad \text{and} \quad \bar{\eta}^{\bullet}\left(t\right) \in \mathfrak{C}_V^{\mathrm{ver}}\left(\left[\gamma^{(l)}\left(t\right)\right]\right).$$

Because of Proposition 4, all reduced variations inside $\mathfrak{C}_V$ can be written in terms of independent variations $\delta x^{\bullet} \in \mathfrak{C}_V^{\mathrm{hor}}$ and $\bar{\eta}^{\bullet} \in \mathfrak{C}_V^{\mathrm{ver}}$. Let us derive, as in the Section 2.3.1, an expression of variations δx and $\bar{\eta}$ in terms of $\delta x^{\bullet}$ and $\bar{\eta}^{\bullet}$.

From (70) and (71), it is clear that, for $\zeta = \gamma^{(l)}(t)$,

$$\delta x\left(t\right) \oplus \bar{\eta}\left(t\right) = \alpha_A \circ \left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left[\delta x^{\bullet}\left(t\right) \oplus \bar{\eta}^{\bullet}\left(t\right)\right].$$

Then, using the canonical projection $P_{T\mathcal{X}}$ and inclusion $I_{T\mathcal{X}}$ [see (26) and (27)], we have that

$$\begin{aligned} \delta x\left(t\right) &= P_{T\mathcal{X}} \circ \alpha_A\left[\left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left(I_{T\mathcal{X}}\left(\delta x^{\bullet}\left(t\right)\right)\right)\right] \\ &\quad + P_{T\mathcal{X}} \circ \alpha_A\left[\left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left(I_{\widetilde{\mathfrak{g}}}\left(\bar{\eta}^{\bullet}\left(t\right)\right)\right)\right] \end{aligned} \tag{72}$$

and

$$\begin{aligned} \bar{\eta}\left(t\right) &= P_{\widetilde{\mathfrak{g}}} \circ \alpha_A\left[\left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left(I_{T\mathcal{X}}\left(\delta x^{\bullet}\left(t\right)\right)\right)\right] \\ &\quad + P_{\widetilde{\mathfrak{g}}} \circ \alpha_A\left[\left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left(I_{\widetilde{\mathfrak{g}}}\left(\bar{\eta}^{\bullet}\left(t\right)\right)\right)\right]. \end{aligned} \tag{73}$$

Repeating the steps in the proof of the Lemma 2.5, we can demonstrate the next result.

Lemma 4.4. *Given $q \in Q$ and $\zeta \in T_q^{(l)}Q$, the identities*

$$P_{T\mathcal{X}} \circ \alpha_A\left[\left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left(I_{T\mathcal{X}}\left(u\right)\right)\right] = u, \qquad P_{T\mathcal{X}} \circ \alpha_A\left[\left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left(I_{\widetilde{\mathfrak{g}}}\left(\eta\right)\right)\right] = 0,$$

and

$$P_{\widetilde{\mathfrak{g}}} \circ \alpha_A\left[\left(\alpha_{A^{\bullet}}^{[\zeta]}\right)^{-1}\left(I_{\widetilde{\mathfrak{g}}}\left(\eta\right)\right)\right] = \eta$$

hold for all $u \in T_{\pi(q)}\mathcal{X}$ and $\eta \in \widetilde{\mathfrak{g}}_{\pi(q)}$.

So, defining

$$\varphi^{[\zeta]} : T_{\pi(q)}\mathcal{X} \to \widetilde{\mathfrak{g}}_{\pi(q)}$$

such that

$$\varphi^{[\zeta]}(u) = P_{\widetilde{\mathfrak{g}}} \circ \alpha_A \left[\left(\alpha_{A^\bullet}^{[\zeta]} \right)^{-1} (I_{T\mathcal{X}}(u)) \right],$$

the Eqs. (72) and (73) tell us that

$$\delta x(t) = \delta x^\bullet(t) \quad \text{and} \quad \bar{\eta}(t) = \varphi^{[\zeta]}(\delta x^\bullet(t)) + \bar{\eta}^\bullet(t).$$

Note that the maps $\varphi^{[\zeta]}$ define another one

$$\varphi : T^{(l)}Q \big/ G \times_{\mathcal{X}} T\mathcal{X} \to \widetilde{\mathfrak{g}},$$

given by

$$\varphi([\zeta], u) = \varphi^{[\zeta]}(u), \quad \forall q \in Q,\ \zeta \in T_q^{(l)}Q,\ u \in T_{\pi(q)}\mathcal{X}.$$

4.3.4. *The higher order Lagrange–d'Alembert–Poincaré equations.* We shall finally derive a set of equations that describe the dynamics of a G-invariant HOCS on Q, in terms of their corresponding reduced variables on $T\mathcal{X}$ and $\widetilde{\mathfrak{g}}$. So, fix again a G-invariant triple (L, C_K, C_V). Let $A^\bullet : T^{(l)}Q \times TQ \to \mathfrak{g}$ be its higher order connection, and fix an arbitrary principal connection $A : TQ \to \mathfrak{g}$. Since we shall use the maps $\varphi^{[\zeta]}$, we need an expression of $[\zeta]$ in terms of reduced variables. We also need to express the kinematic constraint equations, defined by the submanifold C_K, in terms of such variables.

From connection A, we can define for each $n \geq 1$ the bundle isomorphism[7] (see Ref. [13])

$$\alpha_A^{(n)} :\ T^{(n)}Q \big/ G \to T^{(n)}\mathcal{X} \oplus n\widetilde{\mathfrak{g}},$$

such that, given a curve $\gamma : [t_1, t_2] \to Q$,

$$\alpha_A^{(n)}\left(\left[\gamma^{(n)}(t) \right] \right) = \left([\pi \circ \gamma]^{(n)}(t) ; \oplus_{i=1}^n \frac{D^{i-1}}{Dt^{i-1}} a(\gamma'(t)) \right).$$

Coming back to the notation introduced in the Section 2.2.3 [see Eq. (13)], since

$$\pi(\gamma(t)) = x(t) \quad \text{and} \quad a(\gamma'(t)) = \bar{v}(t),$$

then

$$\alpha_A^{(n)}\left(\left[\gamma^{(n)}(t) \right] \right) = \left(x^{(n)}(t) ; \oplus_{i=1}^n \frac{D^{i-1}}{Dt^{i-1}} \bar{v}(t) \right).$$

Using the map $\alpha_A^{(n)}$ for $n = k$, we shall identify $\mathfrak{C}_K = p_k(C_K)$ with

$$\alpha_A^{(k)} \circ p_k(C_K).$$

Then, $\gamma^{(k)}(t) \in C_K$ if and only if

$$\left(x^{(k)}(t) ; \oplus_{i=1}^k \frac{D^{i-1}}{Dt^{i-1}} \bar{v}(t) \right) \in \mathfrak{C}_K.$$

On the other hand, using $\alpha_A^{(l)}$ to identify $T^{(l)}Q \big/ G$ and $T^{(l)}\mathcal{X} \oplus l\widetilde{\mathfrak{g}}$, we have that $\delta\gamma(t) \in C_V\left(\gamma^{(l)}(t)\right)$ if and only if

$$\delta x(t) \oplus \bar{\eta}(t) \in \mathfrak{C}_V(\mathfrak{c}(t)),$$

[7] By $n\widetilde{\mathfrak{g}}$ we are denoting the Whitney sum of n copies of $\widetilde{\mathfrak{g}}$.

being

$$\mathfrak{c}(t)=\left(x^{(l)}(t)\,;\oplus_{i=1}^{l}\frac{D^{i-1}}{Dt^{i-1}}\bar{v}(t)\right).$$

Now, we are in conditions to write down the wanted equations.

Theorem 4.5. *Let (L,C_K,C_V) be a HOCS and G a Lie group acting on Q. Suppose that the system is G-invariant, $\pi:Q\twoheadrightarrow\mathcal{X}=Q/G$ is a principal fiber bundle, and $A^{\bullet}$ is its higher order connection. Fix an arbitrary principal connection A and a curve $\gamma:[t_1,t_2]\to Q$. Then, γ is a trajectory of (L,C_K,C_V) if and only if the curve*

$$\mu:[t_1,t_2]\to T\mathcal{X}\oplus\tilde{\mathfrak{g}},$$

given by

$$\mu(t)=\dot{x}(t)\oplus\bar{v}(t)=\alpha_A\circ p(\gamma'(t)),$$

satisfies

$$\left(x^{(k)}(t)\,;\oplus_{i=1}^{k}\frac{D^{i-1}}{Dt^{i-1}}\bar{v}(t)\right)\in\mathfrak{C}_K,$$

the higher order LDP horizontal equations

$$\begin{aligned}&\left\langle-\frac{D}{Dt}\frac{\partial l}{\partial\dot{x}}(\mu(t))+\frac{\partial l}{\partial x}(\mu(t))-\left\langle\frac{\partial l}{\partial\bar{v}}(\mu(t)),i_{\dot{x}(t)}\tilde{B}\right\rangle,\delta x^{\bullet}(t)\right\rangle\\&+\left\langle\left(\varphi^{\mathfrak{c}(t)}\right)^{*}\left(-\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}(\mu(t))+\mathsf{ad}_{\bar{v}}^{*}\frac{\partial l}{\partial\bar{v}}(\mu(t))\right),\delta x^{\bullet}(t)\right\rangle=0,\end{aligned}\tag{74}$$

and ***the higher order LDP vertical equations***

$$\left\langle-\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}(\mu(t))+\mathsf{ad}_{\bar{v}}^{*}\frac{\partial l}{\partial\bar{v}}(\mu(t)),\bar{\eta}^{\bullet}(t)\right\rangle=0,\tag{75}$$

for all curves

$$\delta x^{\bullet}:[t_1,t_2]\to T\mathcal{X}\qquad\text{and}\qquad\bar{\eta}^{\bullet}:[t_1,t_2]\to\tilde{\mathfrak{g}}$$

satisfying

$$\delta x^{\bullet}(t)\in\mathfrak{C}_V^{hor}(\mathfrak{c}(t))\qquad\text{and}\qquad\bar{\eta}^{\bullet}(t)\in\mathfrak{C}_V^{ver}(\mathfrak{c}(t)),$$

where

$$\mathfrak{c}(t)=\left(x^{(l)}(t)\,;\oplus_{i=1}^{l}\frac{D^{i-1}}{Dt^{i-1}}\bar{v}(t)\right).$$

Remark 12. The variables x, $\dot{x}$ and $\bar{v}$, submanifold $\mathfrak{C}_K$, curvature B and curve $\mathfrak{c}(t)$ are related to A, while the variations $\delta x^{\bullet}$ and $\bar{\eta}^{\bullet}$, and subspaces $\mathfrak{C}_V^{\mathrm{hor}}(\mathfrak{c}(t))$ and $\mathfrak{C}_V^{\mathrm{ver}}(\mathfrak{c}(t))$, are related to $A^{\bullet}$.

Remark 13. Again, for a right action we have to change the sign of $\mathsf{ad}_{\bar{v}}^{*}\frac{\partial l}{\partial\bar{v}}$.

To prove the above theorem, it is enough to show the following result, which generalizes the Lemma 2.4 to l-connections.

Lemma 4.6. *Under the conditions and notation of last Theorem, consider a curve $\gamma:[t_1,t_2]\to Q$. Given $\delta x:[t_1,t_2]\to T\mathcal{X}$ and $\bar{\eta}:[t_1,t_2]\to\tilde{\mathfrak{g}}$, we have that*

$$\delta x(t)\in\mathfrak{C}_V^{hor}\left(\left[\gamma^{(l)}(t)\right]\right)\qquad\text{and}\qquad\bar{\eta}(t)\in\mathfrak{C}_V^{ver}\left(\left[\gamma^{(l)}(t)\right]\right)\tag{76}$$

if and only if there exists $\delta\gamma:[t_1,t_2]\to TQ$ satisfying

$$\pi_*(\delta\gamma(t))=\delta x(t)\qquad\text{and}\qquad a^{\bullet}_{\gamma^{(l)}(t)}(\delta\gamma(t))=\bar{\eta}(t),$$

and such that $\delta\gamma(t)\in C_V\left(\left[\gamma^{(l)}(t)\right]\right)$.

Proof. Given a curve $\delta\gamma$ such that $\delta\gamma(t) \in C_V\left(\left[\gamma^{(l)}(t)\right]\right)$, it follows from the Proposition 4 that the curves $\delta x(t) = \pi_*(\delta\gamma(t))$ and $\bar{\eta}(t) = a^{\bullet}_{\gamma^{(l)}(t)}(\delta\gamma(t))$ satisfy (76). Let us show the converse. Consider curves δx and $\bar{\eta}$ fulfilling (76), and let $\eta : [t_1, t_2] \to \mathfrak{g}$ be the unique curve such that

$$\bar{\eta}(t) = [\gamma(t), \eta(t)].$$

Define

$$\delta\gamma^v(t) = X_{\eta(t)}(\gamma(t)).$$

Taking into account that for each $q \in Q$ and $\zeta \in T^{(l)}_q Q$ the map $\pi_{*,q} : T_q Q \to T_{\pi(q)}\mathcal{X}$ gives rise to a linear isomorphism when restricted to $\mathcal{H}(\zeta)$, define

$$\delta\gamma^h(t) = \left(\pi_{*,\gamma(t)}|_{\mathcal{H}\left(\gamma^{(l)}(t)\right)}\right)^{-1}(\delta x(t)).$$

By construction, $\delta\gamma^v(t) \in \mathcal{V}_{\gamma(t)}$ and $\delta\gamma^h(t) \in \mathcal{H}\left(\gamma^{(l)}(t)\right)$. Then, the sum $\delta\gamma(t) = \delta\gamma^h(t) + \delta\gamma^v(t)$ gives rise to a curve $\delta\gamma : [t_1, t_2] \to TQ$ such that

$$\pi_*(\delta\gamma(t)) = \pi_*\left(\delta\gamma^h(t)\right) = \delta x(t)$$

and

$$\begin{aligned} a^{\bullet}_{\gamma^{(l)}(t)}(\delta\gamma(t)) &= \left[\gamma(t), A^{\bullet}\left(\gamma^{(l)}(t), \delta\gamma(t)\right)\right] = \left[\gamma(t), A^{\bullet}\left(\gamma^{(l)}(t), \delta\gamma^v(t)\right)\right] \\ &= \left[\gamma(t), A^{\bullet}\left(\gamma^{(l)}(t), X_{\eta(t)}(\gamma(t))\right)\right] = [\gamma(t), \eta(t)] = \bar{\eta}(t). \end{aligned}$$

It rests to show that $\delta\gamma(t) \in C_V\left(\left[\gamma^{(l)}(t)\right]\right)$. From calculations above it follows that

$$\alpha_{A^{\bullet}}^{\left[\gamma^{(l)}(t)\right]}([\delta\gamma(t)]) = \delta x(t) \oplus \bar{\eta}(t) \in \mathfrak{C}^{\bullet}_V\left(\left[\gamma^{(l)}(t)\right]\right).$$

In particular,

$$\alpha_{A^{\bullet}}^{\left[\gamma^{(l)}(t)\right]}([\delta\gamma(t)]) \in \alpha_{A^{\bullet}}^{\left[\gamma^{(l)}(t)\right]} \circ p\left(C_V\left(\gamma^{(l)}(t)\right)\right).$$

But $\alpha_{A^{\bullet}}^{[\zeta]} \circ p$ is injective for all ζ. Then

$$\delta\gamma(t) \in C_V\left(\gamma^{(l)}(t)\right),$$

and the proof is over. □

4.4. **The case of trivial bundles revisited.** Let us come back to the notation of the Section 3. Consider a HOCS (L, C_K, C_V) with configuration space $Q = \mathcal{X} \times G$, and suppose that the triple is G-invariant under the left action (32). In particular, Q defines a trivial principal bundle with structure group G and base $\mathcal{X}$.

4.4.1. *l-Connections and isomorphisms α_A.* An l-connection $A : T^{(l)}Q \times_Q TQ \to \mathfrak{g}$, in the notation of the mentioned Section, satisfies

$$A\left(g\,\zeta; x, g\,h, \dot{x}, g\,\dot{h}\right) = Ad_g\left(A\left(\zeta; x, h, \dot{x}, \dot{h}\right)\right)$$

and

$$A\left(\zeta; x, e, 0, h^{-1}\,\dot{h}\right) = h^{-1}\,\dot{h},$$

being $g\,\zeta = \rho_g^{(l)}(\zeta)$. Note that ζ belongs to $T^{(l)}_{(x,h)}Q$ in the first equation and to $T^{(l)}_{(x,e)}Q$ in the second one. Note also that the classes of $T^{(l)}Q/G$ are in bijection with the elements of $T^{(l)}_{(x,e)}Q$. Then, from above equations and linearity of A,

$$\begin{aligned} A\left(\zeta; x, h, \dot{x}, \dot{h}\right) &= A(\zeta; x, h, \dot{x}, 0) + A\left(\zeta; x, h, 0, \dot{h}\right) \\ &= Ad_h\left(A\left(h^{-1}\zeta; x, e, \dot{x}, 0\right)\right) \\ &\quad + Ad_h\left(A\left(h^{-1}\zeta; x, e, 0, h^{-1}\dot{h}\right)\right) \\ &= Ad_h\left(A\left(h^{-1}\zeta; x, e, \dot{x}, 0\right)\right) + Ad_h\left(h^{-1}\dot{h}\right) \\ &= Ad_h\left(\mathcal{A}([\zeta])\,\dot{x}\right) + \dot{h}\,h^{-1}, \end{aligned}$$

where

$$\mathcal{A} : T^{(l)}Q\big/G \to T^*\mathcal{X} \otimes \mathfrak{g} \tag{77}$$

is given by

$$\mathcal{A}[\zeta]\,\dot{x} = A(\zeta; x, e, \dot{x}, 0), \tag{78}$$

with $\zeta \in T^{(l)}_{(x,e)}Q$.

Using the identification between $\widetilde{\mathfrak{g}}$ and $\mathcal{X} \times \mathfrak{g}$ mentioned in the Section 3.1.2, α_A can be seen as a map

$$\alpha_A : T^{(l)}Q\big/G \times_{\mathcal{X}} TQ/G \to T\mathcal{X} \times \mathfrak{g}.$$

Moreover, it is easy to show that [compare to Eq. (35)]

$$\alpha_A^{[\zeta]} \circ p\left(x, h, \dot{x}, \dot{h}\right) = (x, \dot{x}) \oplus \left(x, \mathcal{A}[\zeta]\,\dot{x} + h^{-1}\dot{h}\right). \tag{79}$$

4.4.2. *The map φ.* Consider the higher order connection $A^\bullet$ related to (L, C_K, C_V). Given $\zeta \in T^{(l)}_{(x,h)}Q$, we can write the map $\varphi^{[\zeta]} : T_x\mathcal{X} \to \widetilde{\mathfrak{g}}_x$, defined as

$$\varphi^{[\zeta]}(x, \dot{x}) = P_{\widetilde{\mathfrak{g}}} \circ \alpha_A\left[\left(\alpha_{A^\bullet}^{[\zeta]}\right)^{-1}\left(I_{T\mathcal{X}}(x, \dot{x})\right)\right],$$

completely in terms of $\mathcal{A}^\bullet$ when the principal connection A, appearing in formula above, is taken as the trivial one. In fact, following the same steps as in Section 3.1.3 [this time using Eq. (79) instead of (35)], it can be proved that

$$\varphi^{[\zeta]}(x, \dot{x}) = \left(x, -\mathcal{A}^\bullet[\zeta]\,\dot{x}\right), \tag{80}$$

where we are identifying $\widetilde{\mathfrak{g}}$ and $\mathcal{X} \times \mathfrak{g}$ as in last section.

4.4.3. *The reduced equations.* For simplicity, let us suppose also that $T\mathcal{X}$ is a trivial bundle. According to the calculations of the Section 3, yhe Eqs. (74) and (75) translate to

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial l}{\partial x} + (\varphi^{\mathfrak{c}})^{*}\left(-\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathsf{ad}_{\xi}^{*}\frac{\partial l}{\partial \xi}\right), \delta x^{\bullet}\right\rangle = 0 \tag{81}$$

and

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathsf{ad}_{\xi}^{*}\frac{\partial l}{\partial \xi}, \eta^{\bullet}\right\rangle = 0 \tag{82}$$

respectively [see Eqs. see (46) and (48)], where

$$\mathfrak{c} = \left(x^{(l)}; \oplus_{i=1}^{l}\frac{d^{i-1}}{dt^{i-1}}\xi\right).$$

We are writing the unknown as a curve $(x(t), \dot{x}(t), \xi(t))$ on $T\mathcal{X} \times \mathfrak{g}$. Using the Equation (80), the horizontal equations translates to

$$\left\langle -\frac{d}{dt}\frac{\partial l}{\partial \dot{x}} + \frac{\partial l}{\partial x}, \delta x^{\bullet}\right\rangle = \left\langle -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathsf{ad}_{\xi}^{*}\frac{\partial l}{\partial \xi}, \mathcal{A}^{\bullet}(\mathfrak{c})\,\delta x^{\bullet}\right\rangle$$

The kinematical constraints can be written

$$\left(x^{(k)}; \oplus_{i=1}^{k}\frac{d^{i-1}}{dt^{i-1}}\xi\right) \in \mathfrak{C}_K.$$

Again, if we consider the right action (33), we must change the sign of $\mathsf{ad}_{\xi}^{*}\frac{\partial l}{\partial \xi}$.

4.5. **The ball on the plane.** In this section we shall calculate the higher order connection and write down the horizontal and vertical higher order LDP equations for a concrete constrained system.

Let us consider a ball of mass m and moment of inertia I moving along a horizontal plane. This time we do not necessarily consider the rolling constraints. Recall that the configuration space of the (unconstrained) system is $Q = \mathbb{R}^2 \times SO(3)$. Consider again the right action ρ of $G = SO(3)$ on Q given by (52). Recall that ρ defines on Q the structure of a trivial principal bundle. Using such an action, we can identify TQ and

$$\mathbb{R}^2 \times SO(3) \times \mathbb{R}^2 \times \mathfrak{so}(3). \tag{83}$$

Note that under this identification, the action ρ_* on TQ translate into the right multiplication of the second factor, i.e. the action

$$((\mathbf{a}, R, \dot{\mathbf{a}}, \xi), M) \mapsto ((\mathbf{a}, R\,M, \dot{\mathbf{a}}, \xi)). \tag{84}$$

This in turn gives rise to the identification

$$TQ/G = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathfrak{so}(3). \tag{85}$$

In the following, we shall also use (53) to identify $\mathfrak{so}(3)$ and $\mathbb{R}^3$.

Now, we shall impose on the system kinematic constraints $C_K \subset T^{(k)}Q$ with $k \geq 0$, that we shall not specify, and variational constraints $C_V \subset TQ \times_Q TQ$ given below. Under the identification (83), C_V can be seen as a subset of

$$\left(\mathbb{R}^2 \times SO(3)\right) \times \left(\mathbb{R}^2 \times \mathfrak{so}(3)\right) \times \left(\mathbb{R}^2 \times \mathfrak{so}(3)\right). \tag{86}$$

Suppose that C_V is given by the points $(\mathbf{a}, R, \dot{\mathbf{a}}, \xi, \mathbf{x}, \mathbf{y})$ such that

$$\langle f_1(\mathbf{a}, R, \dot{\mathbf{a}}, \xi), \mathbf{x}\rangle + \langle f_2(\mathbf{a}, R, \dot{\mathbf{a}}, \xi), \mathbf{y}\rangle = 0,$$

being

$$f_1 : TQ \to \mathbb{R}^2 \quad \text{and} \quad f_2 : TQ \to \mathfrak{so}(3).$$

In other words,

$$C_V(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathfrak{so}(3) : \langle f_1, \mathbf{x} \rangle + \langle f_2, \mathbf{y} \rangle = 0 \right\}.$$

The symbol $\langle \cdot, \cdot \rangle$ indicates the euclidean inner product in $\mathbb{R}^2$ and $\mathbb{R}^3 \simeq \mathfrak{so}(3)$.

Remark 14. Examples of constraints $C_K \subset T^{(2)}Q$ can be found in [9, 10, 21, 22, 23, 24, 31], where dissipative systems and applications to control of servomechanisms are described.

Remark 15. The constraints C_V given above can be interpreted as the annihilator of a space of constraint forces F_V generated by the (co)vector $(f_1(v), f_2(v))$ at the point $v \in TQ$.

We shall assume that the above defined triple (L, C_K, C_V) is a G-invariant HOCS, with $G = SO(3)$. We already know that L is G-invariant, so we are assuming that

$$\rho_M^{(k)}(C_K) = C_K, \quad \forall M \in SO(3),$$

and, for all $v \in TQ$,

$$(\rho_M)_*(C_V(v)) = C_V((\rho_M)_*(v)), \quad \forall M \in SO(3),$$

which is equivalent to ask that [recall (84)]

$$f_i((\mathbf{a}, R\,M, \dot{\mathbf{a}}, \xi)) = f_i((\mathbf{a}, R, \dot{\mathbf{a}}, \xi)), \quad \forall M \in SO(3), \quad i = 1, 2.$$

Note that f_1 and f_2 do not depend on the group variable. So we can assume that they are functions

$$f_1 : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$$

and

$$f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathfrak{so}(3) \to \mathfrak{so}(3).$$

In other words, f_1 and f_2 defines functions on TQ/G [see (85)].

In order to write down the reduced equations of motion for (L, C_K, C_V), we need to calculate the higher order connection of the triple, which in the present case is a 1-connection

$$A^\bullet : TQ \times_Q TQ \to \mathfrak{so}(3),$$

and the related map [see (77)]

$$\mathcal{A}^\bullet : TQ/G \to T^*\mathbb{R}^2 \otimes \mathfrak{so}(3). \tag{87}$$

Using (85), $\mathcal{A}^\bullet$ can be seen as a map

$$\mathcal{A}^\bullet : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathfrak{so}(3) \to T^*\mathbb{R}^2 \otimes \mathfrak{so}(3).$$

We must follow the steps of Section 4.3.2.

1. Let us fix on Q the Riemannian metric given by the euclidean inner product on the fibers of TQ (which we have identified with $\mathbb{R}^2 \times \mathbb{R}^3$).

2. Since the vertical subspaces are

$$\begin{aligned} \mathcal{V}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) &= \mathcal{V}_{(\mathbf{a},R)} = \{(\mathbf{a}, R, \mathbf{x}, \mathbf{y}) : \mathbf{x} = 0\} \\ &= \{0\} \times \mathfrak{so}(3), \end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{S}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) &= C_V(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) \cap \mathcal{V}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) \\ &= \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} = 0, \quad \langle f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi), \mathbf{y}\rangle = 0\} \\ &= \{0\} \times \langle f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)\rangle^{\perp}.\end{aligned}$$

From now on, suppose that f_2 never vanishes. Then,

$$\mathcal{T}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) = \left\{(\mathbf{x}, \mathbf{y}) : \ \mathbf{y} = -\frac{\langle f_1(\mathbf{a}, \dot{\mathbf{a}}, \xi), \mathbf{x}\rangle}{\|f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)\|^2} f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)\right\}$$

and

$$\begin{aligned}\mathcal{U}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) &= \{(0, \lambda f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)) : \lambda \in \mathbb{R}\} \\ &= \{0\} \times \langle f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)\rangle.\end{aligned}$$

3. Also, using that $f_2 \neq 0$, we have that $C_V(\mathbf{a}, R, \dot{\mathbf{a}}, \xi)$ is given by the points $(\mathbf{x}, \mathbf{y})$ such that

$$\mathbf{y} = -\frac{\langle f_1(\mathbf{a}, \dot{\mathbf{a}}, \xi), \mathbf{x}\rangle}{\|f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)\|^2} f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi) + \widehat{\mathbf{y}},$$

with

$$\langle f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi), \widehat{\mathbf{y}}\rangle = 0,$$

that is to say

$$\widehat{\mathbf{y}} \in \langle f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)\rangle^{\perp}.$$

For simplicity, let us take $\|f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)\| = 1$. Then

$$C_V(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) = \left\{(\mathbf{x}, - \ \langle f_1, \mathbf{x}\rangle \ f_2 + \widehat{\mathbf{y}}) : \ \widehat{\mathbf{y}} \in \langle f\rangle^{\perp}\right\}. \tag{88}$$

As a consequence,

$$C_V(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) + \mathcal{V}_{(\mathbf{a},R)} = T_{(\mathbf{a},R)}Q,$$

and

$$\mathcal{R}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) = 0.$$

4. Summing up, the horizontal spaces defining $A^{\bullet}$ are

$$\begin{aligned}\mathcal{H}^{\bullet}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) &= \mathcal{T}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi) \\ &= \left\{(\mathbf{x}, - \ \langle f_1(\mathbf{a}, \dot{\mathbf{a}}, \xi), \mathbf{x}\rangle \ f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi)) : \ \mathbf{x} \in \mathbb{R}^2\right\}.\end{aligned}$$

It is easy to show that the spaces $\mathcal{H}^{\bullet}(\mathbf{a}, R, \dot{\mathbf{a}}, \xi)$ define effectively a 1-connection. Using the Proposition 3,

$$A^{\bullet} : TQ \times_Q TQ \to \mathfrak{so}(3)$$

is given by [see (86)]

$$A^{\bullet}((\mathbf{a}, R, \dot{\mathbf{a}}, \xi, \mathbf{x}, \mathbf{y})) = \langle f_1(\mathbf{a}, \dot{\mathbf{a}}, \xi), \mathbf{x}\rangle \ f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi) + \mathbf{y},$$

and from the Eqs. (77), (78) and (87), it follows that

$$\mathcal{A}^{\bullet}(\mathbf{a}, \dot{\mathbf{a}}, \xi) \ \mathbf{x} = \langle f_1(\mathbf{a}, \dot{\mathbf{a}}, \xi), \mathbf{x}\rangle \ f_2(\mathbf{a}, \dot{\mathbf{a}}, \xi).$$

The maps

$$\alpha_{A^\bullet}^{(\mathbf{a},\dot{\mathbf{a}},\xi)}: TQ/G \to T\mathbb{R}^2 \times \mathfrak{so}(3),$$

according to (79), (85) and the Eqs. above, are given by

$$\alpha_{A^\bullet}^{(\mathbf{a},\dot{\mathbf{a}},\xi)}(\mathbf{a},\mathbf{x},\mathbf{y}) = (\mathbf{x}, \langle f_1(\mathbf{a},\dot{\mathbf{a}},\xi), \mathbf{x}\rangle\ f_2(\mathbf{a},\dot{\mathbf{a}},\xi) + \mathbf{y}).$$

Applying $\alpha_{A^\bullet}^{(\mathbf{a},\dot{\mathbf{a}},\xi)} \circ p$ to each $C_V(\mathbf{a}, R, \dot{\mathbf{a}}, \xi)$ [see (88)], we can construct subspaces

$$\mathfrak{C}_V^{\mathrm{hor}}(\mathbf{a},\dot{\mathbf{a}},\xi) \quad \text{and} \quad \mathfrak{C}_V^{\mathrm{ver}}(\mathbf{a},\dot{\mathbf{a}},\xi),$$

which in this case are (see Proposition 4)

$$\mathfrak{C}_V^{\mathrm{hor}}(\mathbf{a},\dot{\mathbf{a}},\xi) = T_{\mathbf{a}}\mathbb{R}^2 \times \{0\}$$

and

$$\mathfrak{C}_V^{\mathrm{ver}}(\mathbf{a},\dot{\mathbf{a}},\xi) = \{0\} \times \langle f_2(\mathbf{a},\dot{\mathbf{a}},\xi)\rangle^{\perp}.$$

We have now all the elements to write the reduced equations. From 3.3, we know that the reduced Lagrangian corresponding to the trivial connection is

$$l(\mathbf{a},\dot{\mathbf{a}},\xi) = \frac{1}{2} I\ \langle \xi, \xi\rangle + \frac{1}{2} m\, \dot{\mathbf{a}}^2,$$

and that

$$-\frac{d}{dt}\frac{\partial l}{\partial \dot{\mathbf{a}}} + \frac{\partial l}{\partial \mathbf{a}} = -m\,\ddot{\mathbf{a}}$$

and

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathsf{ad}_\xi^* \frac{\partial l}{\partial \xi} = I\,\dot{\xi}.$$

So, from (81) and (82) (changing the sign of $\mathsf{ad}_\xi^* \frac{\partial l}{\partial \xi}$), the higher order LDP equations for the system are

$$\langle -m\,\ddot{\mathbf{a}}, \delta x^\bullet\rangle + \left\langle I\,\dot{\xi}, \mathcal{A}^\bullet(\mathbf{a},\dot{\mathbf{a}},\xi)\ \delta x^\bullet \right\rangle = 0,$$

or equivalently,

$$\langle -m\,\ddot{\mathbf{a}}, \delta x^\bullet\rangle + \left\langle I\,\dot{\xi}, \langle f_1(\mathbf{a},\dot{\mathbf{a}},\xi), \delta x^\bullet\rangle\ f_2(\mathbf{a},\dot{\mathbf{a}},\xi) \right\rangle = 0, \tag{89}$$

and

$$\left\langle I\,\dot{\xi}, \eta^\bullet \right\rangle = 0, \tag{90}$$

for all $\delta x^\bullet \in \mathbb{R}^2$ and $\eta^\bullet \in \langle f_2(\mathbf{a},\dot{\mathbf{a}},\xi)\rangle^{\perp}$. Eq. (89) says that

$$-m\,\ddot{\mathbf{a}} + \left\langle I\,\dot{\xi}, f_2(\mathbf{a},\dot{\mathbf{a}},\xi) \right\rangle\ f_1(\mathbf{a},\dot{\mathbf{a}},\xi) = 0$$

and Eq. (90) that

$$I\,\dot{\xi} = \lambda\, f_2(\mathbf{a},\dot{\mathbf{a}},\xi)$$

for some number λ. Combining both equations we have

$$m\,\ddot{\mathbf{a}} = \lambda\ f_1(\mathbf{a},\dot{\mathbf{a}},\xi) \quad \text{and} \quad I\,\dot{\xi} = \lambda\, f_2(\mathbf{a},\dot{\mathbf{a}},\xi).$$

Then, the functions f_1 and f_2 represent directions of constraint forces (see Remark 15), and λ a Lagrange multiplier which gives the strength of such forces. The latter can be determined by using the kinematic constraints $\mathfrak{C}_K^\bullet$.

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REFERENCES

[1] R. Abraham and J. E. Marsden, "Foundation of Mechanics," Benjaming Cummings, New York, 1985.

[2] P. Balseiro and J. Solomin, *On generalized non-holonomic systems*, Letters of Mathematical Physics, **84** (2008), 15–30.

[3] A. Bloch, P. S. Krishnaprasad, J. E. Marsden and R. M. Murray, *Nonholonomic mechanical systems with symmetry*, Arch. Rat. Mech. Anal., **136** (1996), 21–99.

[4] W. M. Boothby, "An Introduction to Differentiable Manifolds and Riemannian Geometry," Second edition, Pure and Applied Mathematics, **120**, Academic Press, Orlando, FL, 1986.

[5] F. Bullo and A. Lewis, "Geometric Control of Mechanical Systems. Modeling, Analysis, and Design for Simple Mechanical Control Systems," Texts in Applied Mathematics, **49**, Springer-Verlag, New York, 2005.

[6] F. Cantrijn, M. de León, J. C. Marrero and D. Martín de Diego, *Reduction of constrained systems with symmetries*, J. Math. Phys., **40** (1999), 795–820.

[7] H. Cendra, S. Ferraro and S. Grillo, *Lagrangian reduction of generalized nonholonomic systems*, Journal of Geometry and Physics, **58** (2008), 1271–1290.

[8] H. Cendra and S. Grillo, *Generalized nonholonomic mechanics, servomechanisms and related brackets*, J. Math. Phys., **47** (2006), 022902, 29 pp.

[9] H. Cendra and S. Grillo, *Lagrangian systems with higher order constraints*, J. Math. Phys., **48** (2007), 052904, 35 pp.

[10] H. Cendra, A. Ibort, M. de León and D. de Diego, *A generalization of Chetaev's principle for a class of higher order nonholonomic constraints*, J. Math. Phys., **45** (2004), 2785–2801.

[11] H. Cendra, E. A. Lacomba and W. A. Reartes, *The Lagrange D'Alembert-Poincaré equations for the symmetric rolling sphere*, Actas del VI Congreso Antonio Monteiro, (2002), 19–32.

[12] H. Cendra, J. E. Marsden, S. Pekarsky and T. S. Ratiu, *Variational principles for Lie-Poisson and Hamilton-Poincaré equations*, Moscow Mathematical Journal, **3** (2003), 833–867, 1197–1198.

[13] H. Cendra, J. E. Marsden and T. S. Ratiu, *Lagrangian reduction by stages*, Memoirs of the American Mathematical Society, **152** (2001), x+108 pp.

[14] H. Cendra, J. E. Marsden and T. S. Ratiu, *Geometric mechanics, Lagrangian reduction, and non-holonomic systems*, in "Mathematics Unlimited–2001 and Beyond" (eds. B. Enguist and W. Schmid), Springer, Berlin, (2001), 221–273.

[15] N. G. Chetaev, *On Gauss principle*, Izv. Fiz-Mat. Obsc. Kazan Univ., **7** (1934), 68–71.

[16] M. Crampin, W. Sarlet and F. Cantrijn, *Higher-order differential equations and higher-order Lagrangian mechanics*, Math. Proc. Camb. Phil. Soc., **99** (1986), 565–587.

[17] M. de León and P. R. Rodrigues, "Generalized Classical Mechanics and Field Theory. A Geometrical Approach of Lagrangian and Hamiltonian Formalisms Involving Higher Order Derivatives," North-Holland Mathematics Studies, **112**, Notes on Pure Mathematics, **102**, North-Holland Publishing Co., Amsterdam, 1985.

[18] J. Fernandez, C. Tori and M. Zuccalli, *Lagrangian reduction of nonholonomic discrete mechanical systems*, J. Geom. Mech., **2** (2010), 69–111.

[19] C. Godbillon, "Géométrie Différentielle et Mécanique Analytique," Hermann, Paris, 1969.

[20] J. H. Greidanus, "Besturing en Stabiliteit van het Neuswielonderstel," Rapport V 1038, Nationaal Luchtvaartlaboratorium, Amsterdam, 1942.

[21] S. Grillo, "Sistemas Noholónomos Generalizados," (Spanish), Ph.D thesis, Universidad Nacional del Sur, 2007.

[22] S. Grillo, *Higher order constrained Hamiltonian systems*, J. Math. Phys., **50** (2009), 082901, 34 pp.

[23] S. Grillo, F. Maciel and D. Pérez, *Closed-loop and constrained mechanical systems*, International Journal of Geometric Methods in Modern Physics, **7** (2010), 1–30.

[24] S. Grillo, J. Marsden and S. Nair, *Lyapunov constraints and global asymptotic stabilization*, Journal of Geometric Mechanics, **3** (2011), 145–196.

[25] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry," New York, John Wiley & Son, 1963.

[26] O. Krupková, *Higher-order mechanical systems with constraints*, J. Math. Phys., **41** (2000), 5304–5324.

[27] C.-M. Marle, *Kinematic and geometric constraints, servomechanism and control of mechanical systems*, Rend. Sem. Mat. Univ. Pol. Torino, **54** (1996), 353–364; *Various approaches*

to conservative and nonconservative non-holonomic systems, Rep. Math. Phys., **42** (1998), 211–229, MR1656282.

[28] J. E. Marsden and T. S. Ratiu, "Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems," Texts in Applied Mathematics, **17**, Springer-Verlag, New York, 1994.

[29] J. E. Marsden and T. S. Ratiu, "Manifolds, Tensor Analysis and Applications," New York, Springer-Verlag, 2001.

[30] D. Pérez, "Sistemas Dinámicos no Holónomos Generalizados y su Aplicación a la Teoría de Control Automático Mediante Vínculos Cinemáticos," (Spanish), Proyecto Integrador, Carrera de Ingeniería Mecánica del Instituto Balseiro, 2006.

[31] D. Pérez, "Sistemas Mecánicos con Vínculos de Orden Superior: Aplicaciones a la Teoría de Control," (Spanish), Maestría en Cs. Físicas del Instituto Balseiro, Orientación Física Aplicada, 2007.

[32] Y. Pironneau, *Sur les liaisons non holonomes non linéaires déplacement virtuels à travail nul, conditions de Chetaev*, in "Proceedings of the IUTAM–ISIMMM Symposium on Modern Developments in Analytical Mechanics," Vol. II (Torino, 1982) (eds. S. Benenti, M. Francaviglia and A. Lichnerowicz), Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur., **117** (1983), 671–686.

[33] J. W. S. Rayleigh, "The Theory of Sound," Dover Publications, New York, 1945.

[34] Do Shan, *Equations of motion of systems with second-order nonlinear non-holonomic constraints*, Prikl. Mat. Mekh., **37** (1973), 349–354.

[35] W. M. Tulczyjew and P. Urbanski, *A slow and careful Legendre transformation for singular Lagrangians*, Acta Physica Polonica B, **30** (1999), 2909–2978.

[36] V. Vâlcovici, *Une extension des liaisons non holonomes et des principes variationnels*, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl., **102** (1958), 39 pp.

[37] E. T. Whittaker, "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies," Cambridge University Press, Cambriedge, 1937.

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