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BEST LOCAL APPROXIMATION IN ORLICZ SPACES

H. Cuenya, F. Levis, M. Marano, and C. Ridolfi

[1]*Department of Mathematics, Universidad Nacional de Rio Cuarto (UNRC), Río Cuarto, Argentina*
[2]*Department of Mathematics, University of Jaén, Jaén, Spain*
[3]*Department of Mathematics, Universidad Nacional de San Luis (UNSL), San Luis, Argentina*

□ *We get results in Orlicz spaces L^ϕ about best local approximation on non-balanced neighborhoods when ϕ satisfies a certain asymptotic condition. This fact generalizes known previous results in L^p spaces.*

Keywords Best local approximation; Orlicz spaces; Non-balanced neighborhoods.

AMS Subject Classification Primary 41A10; Secondary 41A30.

1. INTRODUCTION

Let $\emptyset \neq X \subset \mathbb{R}$ be an open and bounded set. We denote by $\mathcal{M} = \mathcal{M}(X)$ the equivalence class of all real Lebesgue measurable functions on X. Let Φ be the set of convex functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\phi(x) > 0$ for $x > 0$ and $\phi(0) = 0$.

For each $\phi \in \Phi$, define

$$L^\phi = L^\phi(X) = \left\{ f \in \mathcal{M} : \int_X \phi(\alpha|f(x)|)\,dx < \infty, \text{ for some } \alpha > 0 \right\}.$$

The space L^ϕ is called an Orlicz space determined by ϕ. This space is endowed with the Luxemburg norm defined by

$$\|f\|_\phi = \inf\left\{ \lambda > 0 : \int_X \phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\},$$

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Address correspondence to C. Ridolfi, Department of Mathematics, Universidad Nacional de San Luis (UNSL), San Luis 5700, Argentina; E-mail: ridolfi@unsl.edu.ar

and so it becomes a Banach space. Sometimes we write $\|\cdot\|_{L^\phi(W)}$ instead of $\|f\chi_W\|_\phi$, where χ_W denotes the characteristic function on $W \subset X$.

We assume that $\phi \in \Phi$ satisfies the Δ_2-condition, that is, there exists a constant $\gamma > 0$ such that $\phi(2x) \le \gamma\phi(x)$ for all $x \ge 0$. In this case, $\int_X \phi\left(\frac{|f(x)|}{\|f\|_\phi}\right)dx = 1$. A detailed treatment about these subjects may be found in [4].

Given $x_1 < \cdots < x_n$ in X, $n \ge 2$, for $\delta > 0$ small enough we define a net of pairwise disjoint sets $V_k = V_k(\delta) := x_k + \varepsilon_k(\delta)A_k \subset X$, $1 \le k \le n$, where $\varepsilon_k = \varepsilon_k(\delta) \searrow 0$ as $\delta \to 0$, and each interval A_k, independent of δ, has Lebesgue measure 1. Given a function $f \in L^\phi$ and a subspace $S \subset L^\phi$ we denote by g_δ an element in S that satisfies

$$\|f - g_\delta\|_{L^\phi(V)} \le \|f - h\|_{L^\phi(V)},$$

for all $h \in S$, where $V = \bigcup_{k=1}^n V_k$. Such a function g_δ is called a *best* $\|\cdot\|_\phi$-*approximation of* f *from* S *over* V. It is well known that g_δ exists when S has finite dimension. If the net $\{g_\delta\}_{\delta>0}$ has a limit in S as $\delta \to 0$, this limit is called a *best local* $\|\cdot\|_\phi$-*approximation of* f *from* S *on* $\{x_1, \ldots, x_n\}$.

Now we make an assumption on the n-tuple $\langle\varepsilon_k\rangle := (\varepsilon_1, \ldots, \varepsilon_n)$, which allows us to compare the following expressions, as functions of δ,

$$v_k(\alpha) := \|\chi_{V_k}\|_\phi \varepsilon_k^\alpha = \frac{\varepsilon_k^\alpha}{\phi^{-1}\left(\frac{1}{\varepsilon_k}\right)}, \quad \text{where } \alpha \text{ is a nonnegative integer.}$$

More precisely, for each pair of nonnegative integers α and β, and any pair j, k, $1 \le j, k \le n$, we suppose

$$v_k(\alpha) = O(v_j(\beta)), \quad \text{or } v_j(\beta) = o(v_k(\alpha)) \quad \text{as } \delta \to 0. \tag{1}$$

Let $\langle i_k\rangle$ be an ordered n-tuple of nonnegative integers. We say that $v_j(i_j)$ is *maximal* if $v_k(i_k) = O(v_j(i_j))$ for all k, $1 \le k \le n$. We denote it by

$$v_j(i_j) = \max\{v_k(i_k)\}.$$

An n-tuple $\langle i_k\rangle$ of nonnegative integers is said to be $\|\cdot\|_\phi$-balanced if for each $i_j > 0$,

$$\frac{1}{v_j(i_j - 1)}\max\{v_k(i_k)\} = o(1).$$

An integer $N \ge 0$ is called $\|\cdot\|_\phi$-*balanced* if there exists a $\|\cdot\|_\phi$-balanced n-tuple $\langle i_k\rangle$, with $N = \sum_{k=1}^n i_k$. It easy to see that such n-tuple is unique.

As seen in [2], for each $N \ge 0$ there exists the smallest $\|\cdot\|_\phi$-balanced integer greater than or equal to N, and the greatest $\|\cdot\|_\phi$-balanced integer smaller than or equal to N, which we denote by $\overline{N}$ and $\underline{N}$, respectively. We write $\sum_{k=1}^{n} \overline{i}_k = \overline{N}$ and $\sum_{k=1}^{n} \underline{i}_k = \underline{N}$, where $\langle \overline{i}_k \rangle$ and $\langle \underline{i}_k \rangle$ are $\| \cdot \|_\phi$-balanced n-tuples. If N is not a $\| \cdot \|_\phi$-balanced integer, then $\underline{N} < N < \overline{N}$ and there are no $\| \cdot \|_\phi$-balanced integers between $\underline{N}$ and $\overline{N}$.

Generally, if $\langle i_k \rangle$ is a $\| \cdot \|_\phi$-balanced n-tuple, let K be the set of indexes k with the property that $v_k(i_k) = \max\{v_l(i_l)\}$. As a consequence of the algorithm established in [2] for computing the $\| \cdot \|_\phi$-balanced integers, we deduce that the smallest $\| \cdot \|_\phi$-balanced integer greater than $\sum_{k=1}^{n} i_k$ is $\sum_{k=1}^{n} i'_k$, where $\langle i'_k \rangle$ is a $\| \cdot \|_\phi$-balanced n-tuple, $i'_k = i_k + 1$ for $k \in K$ and $i'_k = i_k$ for $k \notin K$. Therefore, the cardinal of K is $\sum_{k=1}^{n} i'_k - \sum_{k=1}^{n} i_k$.

We denote by $PC^t(X)$ the class of functions with $t-1$ continuous derivatives and bounded, piecewise continuous tth derivative on X. Consider $f \in PC^t(X)$, and a subspace $S_N \subset PC^t(X)$ of dimension N with the property that if $\langle i_k \rangle$ is an ordered n-tuple of nonnegative integers with $i_k \leq t$ and $\sum_{k=1}^{n} i_k = N$, and $\{a_{j,k}\}$ is an arbitrary set of real numbers, then there exists a unique $g \in S_N$ such that $g^{(j)}(x_k) = a_{j,k}$, $k = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, i_k - 1$.

In this work, we study the existence and characterization of best local approximations of f from S_N when the dimension N of the approximating space S_N is not a $\| \cdot \|_\phi$-balanced integer (Theorem 3.1). This result has been proved in [1] (Theorem 3) for L^p spaces, $1 \leq p \leq \infty$. In Orlicz spaces the existence of best local approximation can be seen in [2] in the case where N is a $\| \cdot \|_\phi$-balanced integer. More precisely, if $N = \sum_{k=1}^{n} i_k$ and $\langle i_k \rangle$ is a $\| \cdot \|_\phi$-balanced n-tuple, then the best local aproximation of $f \in PC^t(X)$ is the function $g \in S_N$ that satisfies $g^{(j)}(x_k) = f^{(j)}(x_k)$ for $k = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, i_k - 1$. In the non-balanced case $\lim_{\delta \to 0} g_\delta$ may not exist, as we see in the example of Section 4, and to get existence more conditions on ϕ are required.

2. PRELIMINARY RESULTS

In this section, we prove some auxiliary results, which are necessary for proving the main theorem. Let N be a non-$\| \cdot \|_\phi$-balanced integer. Then $\underline{N} = \sum_{k=1}^{n} \underline{i}_k < \sum_{k=1}^{n} \overline{i}_k = \overline{N}$, where $\langle \underline{i}_k \rangle$ and $\langle \overline{i}_k \rangle$ are $\| \cdot \|_\phi$-balanced n-tuples,

$$K := \{k, 1 \leq k \leq n : v_k(\underline{i}_k) = \max\{v_l(\underline{i}_l)\}\} = \{k, 1 \leq k \leq n : \overline{i}_k = \underline{i}_k + 1\}, \tag{2}$$

and $\overline{i}_k = \underline{i}_k$ for $k \notin K$.

Set $E = \max\{v_l(\underline{i}_l)\}$.

Lemma 2.1. *If N is not a $\| \cdot \|_\phi$-balanced integer, then* $\max\{v_k(\overline{i}_k)\} = o(E)$.

Proof. Since $\underline{N} < \overline{N}$, there exists a subindex j, $1 \le j \le n$, such that $\underline{i}_j = \overline{i}_j - 1$. As $\overline{N}$ is $\|\cdot\|_\phi$-balanced,

$$\max\{v_k(\overline{i}_k)\} = o(v_j(\overline{i}_j - 1)) = o(v_j(\underline{i}_j)) = o(E). \qquad \square$$

Lemma 2.2. *For $k = 1, 2, \ldots, n$,*

(a) $\|(x - x_k)^{\overline{i}_k}\|_{L^\phi(V_k)} = o(E)$, *and*
(b) $\|(x - x_k)^{\underline{i}_k}\|_{L^\phi(V_k)} = O(v_k(\underline{i}_k)) = O(E)$.

Proof. We have

$$\begin{aligned}\|(x - x_k)^{\overline{i}_k}\|_{L^\phi(V_k)} &= \inf\left\{\lambda > 0 : \int_{V_k} \phi\left(\frac{|(x - x_k)^{\overline{i}_k}|}{\lambda}\right) dx \le 1\right\}\\ &= \inf\left\{\lambda > 0 : \int_{A_k} \epsilon_k \phi\left(\frac{|\epsilon_k^{\overline{i}_k} y^{\overline{i}_k}|}{\lambda}\right) dy \le 1\right\}\\ &\le M v_k(\overline{i}_k),\end{aligned}$$

where $M = \max_{1\le k\le n} |y|^{\overline{i}_k}$. Then Lemma 2.1 implies

$$\|(x - x_k)^{\overline{i}_k}\|_{L^\phi(V_k)} = O(v_k(\overline{i}_k)) = O(\max\{v_k(\overline{i}_k)\}) = o(E),$$

which proves (a). The proof of (b) is similar. $\square$

Henceforth, $\{g_\delta\}$ stands for a net of best $\|\cdot\|_\phi$-approximations of f from S_N on V, and

$$c_{j,k} = c_{j,k}(\delta) := (f - g_\delta)^{(j)}(x_k)\frac{v_k(j)}{E}, \qquad (3)$$

for $k = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, \underline{i}_k - 1$.

The following result is proved in [2] (Theorem 4.1).

Lemma 2.3. *If $\{g_\delta\}_{\delta>0}$ is a net of best $\|\cdot\|_\phi$-approximations of f from S_N on V, then $\{g_\delta\}_{\delta>0}$ is uniformly bounded on X.*

Lemma 2.4. *If $\{g_\delta\}_{\delta>0}$ is a net of best $\|\cdot\|_\phi$-approximations of f from S_N on V, then*

$$\frac{|(f - g_\delta)^{(j)}(x_k)\varepsilon_k^j|}{\phi^{-1}(\frac{1}{\varepsilon_k})} = O(E), \quad k = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, \underline{i}_k - 1.$$

Proof. For each k with $\underline{i}_k > 0$, expanding $f - g_\delta$ using Taylor polynomials we obtain

$$(f - g_\delta)(x) = \sum_{j=0}^{\underline{i}_k - 1} \frac{(f - g_\delta)^{(j)}(x_k)}{j!}(x - x_k)^j + T_{\delta,k}(x), \tag{4}$$

where $T_{\delta,k}(x) = O((x - x_k)^{\underline{i}_k})$. Using Lemma 2.3 and the equivalence of the norms in S_N we can show that $T_{\delta,k}(x) = O((x - x_k)^{\underline{i}_k})$ uniformly in δ. By Lemma 2.2(b),

$$\| T_{\delta,k} \|_{L^\phi(V_k)} = O(E). \tag{5}$$

Let h be a fixed function in S_N interpolating the derivatives $f^{(j)}(x_k)$, $k = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, \underline{i}_k - 1$. Thus, $(f - h)(x) = O((x - x_k)^{\underline{i}_k})$ for $k = 1, 2, \ldots, n$. We have

$$\|f - g_\delta\|_{L^\phi(V_k)} \le \|f - g_\delta\|_{L^\phi(V)} \le \|f - h\|_{L^\phi(V)} \le \sum_{k=1}^{n} \|f - h\|_{L^\phi(V_k)} = O(E),$$

where the equality follows from Lemma 2.2(b). Then, from (4) and (5),

$$\left\| \sum_{j=0}^{\underline{i}_k - 1} \frac{(f - g_\delta)^{(j)}(x_k)}{j!}(x - x_k)^j \right\|_{L^\phi(V_k)} \le \|f - g_\delta\|_{L^\phi(V_k)} + \| T_{\delta,k} \|_{L^\phi(V_k)} = O(E). \tag{6}$$

Let $H_{\delta,k}(x) := \sum_{j=0}^{\underline{i}_k - 1} \frac{(f - g_\delta)^{(j)}(x_k)}{j!}(x - x_k)^j$. Assume $\|H_{\delta,k}\|_{L^\phi(V_k)} > 0$ for each δ and k. Thus,

$$O(E) = \|H_{\delta,k}\|_{L^\phi(V_k)} =: \lambda_0.$$

Since ϕ satisfies the Δ_2-condition,

$$1 = \int_{V_k} \phi\left(\frac{|H_{\delta,k}(x)|}{\lambda_0} \right) dx = \int_{A_k} \epsilon_k \phi\left(\frac{|H_{\delta,k}(x_k + \epsilon_k y)|}{\lambda_0} \right) dy.$$

Suppose

$$\max_{y \in A_k} \frac{|H_{\delta,k}(x_k + \epsilon_k y)|}{\lambda_0} = C_{\delta,k} \phi^{-1}\left(\frac{1}{\epsilon_k} \right) = \frac{|H_{\delta,k}(x_k + \epsilon_k y_0)|}{\lambda_0}, \tag{7}$$

where $C_{\delta,k}$ is a constant and y_0 depends on δ (and k). Let (a_0, b_0) be the connected component of

$$\left\{y \in A_k : \frac{|H_{\delta,k}(x_k + \epsilon_k y)|}{\lambda_0} > \frac{C_{\delta,k}}{2}\phi^{-1}\left(\frac{1}{\epsilon_k}\right)\right\}$$

which contains y_0. Note that a_0 and b_0 depend on δ.

We assert that $\liminf_{\delta\to 0}(b_0 - a_0) > 0$. Otherwise, with $y_1 = a_0$ or $y_1 = b_0$ we get $\limsup_{\delta\to 0} \frac{|H'_{\delta,k}(x_k+\epsilon_k y_1)|}{|H_{\delta,k}(x_k+\epsilon_k y_0)|} = \infty$, which contradicts the equivalence of the two norms $\|P\|_\infty$ and $\|P\|_\infty + \|P'\|_\infty$ in the space of $(\underline{i}_k - 1)$th degree polynomials P on A_k.

If $C_{\delta,k}/2 \geq 1$ then, by the definition of (a_0, b_0),

$$1 = \int_{A_k} \epsilon_k \phi\left(\frac{|H_{\delta,k}(x_k + \epsilon_k y)|}{\lambda_0}\right) dy > \int_{a_0}^{b_0} \epsilon_k \phi\left(\frac{C_{\delta,k}}{2}\phi^{-1}\left(\frac{1}{\epsilon_k}\right)\right) dy \geq \frac{C_{\delta,k}}{2}(b_0 - a_0),$$

where in the last inequality we are using that $\phi(vx) \geq v\phi(x)$ for $v \geq 1$, since ϕ is a convex function, $\phi(0) = 0$. We deduce that $C_{\delta,k}$ is bounded as a function of δ.

Therefore, by (7), $\max_{y\in A_k}\left|\sum_{j=0}^{\underline{i}_k-1} \frac{(f-g_\delta)^{(j)}(x_k)\epsilon_k^j y^j}{\lambda_0\phi^{-1}(1/\epsilon_k)}\right|$ is bounded. Applying again the equivalence of norms and recalling that $\lambda_0 = O(E)$, we conclude the lemma. □

As a consequence of Lemma 2.4 we have

$$c_{j,k} = O(1), \quad k = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, \underline{i}_k - 1. \tag{8}$$

Since $g_\delta^{(i)}(x_s) = f^{(i)}(x_s) - c_{i,s}\frac{E}{v_s(i)}$ and $\frac{E}{v_s(i)} = o(1)$ for $i = 0, 1, \ldots, \underline{i}_s - 1$, from (8) we obtain

$$g_\delta^{(i)}(x_s) = f^{(i)}(x_s) + o(1), \quad s = 1, 2, \ldots, n, \quad i = 0, 1, \ldots, \underline{i}_s - 1. \tag{9}$$

Consider the following basis for S_N, say $\{u_{j,k}\} \cup \{w_r\}$, with $k = 1, 2, \ldots, n$, $j = 0, 1, \ldots, \underline{i}_k - 1$, and $r = 1, 2, \ldots, N - \underline{N}$, which satisfies

$$u_{j,k}^{(j')}(x_{k'}) = \delta_{(j,k),(j',k')} \quad \text{and} \quad w_r^{(j')}(x_{k'}) = 0,$$
$$k' = 1, 2, \ldots, n \text{ and } j' = 0, 1, \ldots, \underline{i}_{k'} - 1,$$

where $\delta_{(j,k),(j',k')}$ is the Kronecker delta. Observe that if $g \in S_N$, then

$$g(x) = \sum_{k=1}^{n}\sum_{j=0}^{\underline{i}_k-1} a_{j,k}u_{j,k}(x) + \sum_{r=1}^{N-\underline{N}} b_r w_r(x),$$

where $g^{(j)}(x_k) = a_{j,k}$, $k = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, \underline{i}_k - 1$.

Lemma 2.5. *There holds*

$$\Gamma := \left\| \sum_{k\in K} \left(\sum_{s=1}^{n} \sum_{i=0}^{\underline{i}_s-1} \frac{c_{i,s}}{v_s(i)} u_{i,s}^{(\underline{i}_k)}(x_k) \right) \frac{(x-x_k)^{\underline{i}_k}}{\underline{i}_k!} \chi_{V_k}(x) \right\|_{L^{\phi}(V)} = o(1),$$

where K is defined in (2).

Proof. By (9), we have

$$M_\delta := \max_{i,s,k} |(f-g_\delta)^{(i)}(x_s) u_{i,s}^{(\underline{i}_k)}(x_k)| \to 0 \quad \text{as } \delta \to 0.$$

It follows from Lemma 2.2 (b) that

$$\begin{aligned} \Gamma &= \left\| \sum_{k\in K} \left(\sum_{s=1}^{n} \sum_{i=0}^{\underline{i}_s-1} \frac{(f-g_\delta)^{(i)}(x_s)}{E} u_{i,s}^{(\underline{i}_k)}(x_k) \right) \frac{(x-x_k)^{\underline{i}_k}}{\underline{i}_k!} \chi_{V_k}(x) \right\|_{L^{\phi}(V)} \\ &\le \frac{M_\delta \underline{N}}{E} \sum_{k\in K} \left\| \frac{(x-x_k)^{\underline{i}_k}}{\underline{i}_k!} \right\|_{L^{\phi}(V_k)} = \frac{M_\delta}{E} O(E) = o(1). \end{aligned}$$

□

Lemma 2.6. *Suppose that $g_\delta = \sum_{k=1}^{n} \sum_{j=0}^{\underline{i}_k-1} a_{j,k}(\delta) u_{j,k} + \sum_{r=1}^{N-\underline{N}} b_r(\delta) w_r$. If*

$$\Omega_k(\delta) := f^{(\underline{i}_k)}(x_k) - \sum_{s=1}^{n} \sum_{i=0}^{\underline{i}_s-1} f^{(i)}(x_s) u_{i,s}^{(\underline{i}_k)}(x_k) - \sum_{r=1}^{N-\underline{N}} b_r(\delta) w_r^{(\underline{i}_k)}(x_k), \tag{10}$$

then

$$\left\| \sum_{k\in K} \Omega_k(\delta) \frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)\underline{i}_k!} o_k(1) \chi_{V_k}(x) \right\|_{L^{\phi}(V)} = o(1).$$

Proof. Due to Lemma 2.3, the net g_δ is uniformly bounded. Then $b_r(\delta) = O(1)$ and, as a consequence,

$$M_\delta := \sup_k |\Omega_k(\delta) o_k(1)| \to 0,$$

as $\delta \to 0$. Using Lemma 2.2(b), we conclude that

$$\begin{aligned} \left\| \sum_{k\in K} \Omega_k(\delta) \frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)\underline{i}_k!} o_k(1) \chi_{V_k}(x) \right\|_{L^{\phi}(V)} &\le M_\delta \sum_{k\in K} \frac{\|(x-x_k)^{\underline{i}_k}\|_{L^{\phi}(V_k)}}{v_k(\underline{i}_k)\underline{i}_k!} \\ &= M_\delta \sum_{k\in K} \frac{O(v_k(\underline{i}_k))}{v_k(\underline{i}_k)} = o(1). \end{aligned}$$

□

Lemma 2.7. *Let $\phi \in \Phi$ satisfying the Δ_2-condition. If for each $x \geq 0$ there exists $\lim_{\alpha\to\infty} \frac{\phi(\alpha x)}{\phi(\alpha)} =: \psi(x)$, then $\psi(x) = x^p$ for some $p \geq 1$.*

Proof. As ϕ is Δ_2, ψ assumes only finite values. By definition, $\psi(0) = 0$ and $\psi(1) = 1$. Observe that

1) *ψ is strictly increasing.*
Indeed, if $0 < x < y$, then $\phi(\alpha x) \leq \alpha x \phi'(\alpha x)$ and $\phi(\alpha y) - \phi(\alpha x) \geq \alpha(y - x)\phi'(\alpha x)$. Thus, $\frac{\phi(\alpha y)}{\phi(\alpha x)} \geq 1 + \frac{y-x}{x}$ and $\psi(y) \geq \frac{y}{x}\psi(x)$.
2) *ψ is multiplicative.*
This follows because for $x, y > 0$, $\psi(x) = \lim_{\alpha\to\infty} \frac{\phi(\alpha x)}{\phi(\alpha)} = \lim_{\alpha\to\infty} \frac{\phi(y\alpha x)}{\phi(y\alpha)} \frac{\phi(\alpha)}{\phi(\alpha)} = \frac{\psi(xy)}{\psi(y)}$. So

$$\psi(xy) = \psi(x)\psi(y).$$

3) *ψ is continuous.*
To prove this observe that by 1), ψ is continuous at almost any point. Now, let $x_0 \neq 0$ be a continuity point of ψ, that is,

$$\lim_{x\to x_0} \psi(x) = \psi(x_0).$$

If $x_1 > 0$ and $z := \frac{x x_0}{x_1}$, then 2) implies

$$\lim_{x\to x_1} \psi(x) = \lim_{x\to x_1} \psi\left(z\frac{x_1}{x_0}\right) = \lim_{z\to x_0} \psi(z)\psi\left(\frac{x_1}{x_0}\right) = \psi(x_0)\psi\left(\frac{x_1}{x_0}\right) = \psi(x_1),$$

which proves the continuity of ψ.

Under these conditions, it is well known that there exists a constant $p \geq 0$ such that $\psi(x) = x^p$ for $x > 0$. Clearly, 1) implies $p > 0$. As ϕ is convex, $\phi(\alpha x) \leq x\phi(\alpha)$ for $0 \leq x \leq 1$. From the definition of ψ it follows that $x^p \leq x$ for $0 \leq x \leq 1$, whence $p \geq 1$. □

Lemma 2.8. *For $\delta > 0$ and small enough let β and α be two functions of δ such that $\lim_{\delta\to 0} \beta(\delta) = \beta_0 > 0$ and $\lim_{\delta\to 0} \alpha(\delta) = \infty$. Then, under the same conditions of Lemma 2.7, we have $\lim_{\delta\to 0} \frac{\phi(\alpha(\delta)\beta(\delta))}{\phi(\alpha(\delta))} = \beta_0^p$.*

Proof. Given $0 < \epsilon < \beta_0$, there exists $\delta_0 > 0$ such that $\beta_0 - \epsilon \leq \beta(\delta) \leq \beta_0 + \epsilon$ for $\delta < \delta_0$. Thus, we have

$$\phi(\alpha(\delta)(\beta_0 - \epsilon)) \leq \phi(\alpha(\delta)\beta(\delta)) \leq \phi(\alpha(\delta)(\beta_0 + \epsilon)), \quad \delta < \delta_0,$$

and, therefore,

$$(\beta_0 - \epsilon)^p \le \liminf_{\delta\to 0} \frac{\phi(\alpha(\delta)\beta(\delta))}{\phi(\alpha(\delta))} \le \limsup_{\delta\to 0} \frac{\phi(\alpha(\delta)\beta(\delta))}{\phi(\alpha(\delta))} \le (\beta_0 + \epsilon)^p.$$

Since ϵ is arbitrary, we get $\lim_{\delta\to 0} \frac{\phi(\alpha(\delta)\beta(\delta))}{\phi(\alpha(\delta))} = \beta_0^p$. □

Lemma 2.9. *For every $k \in K$ set $P_{k,\delta}(y) := \sum_{j=0}^{i_k-1} \frac{c_{j,k}(\delta)}{j!} y^j + c_k^*(\delta) y^{i_k}$, $k \in K$, such that $\lim_{\delta\to 0} c_{j,k}(\delta) = d_{j,k}$ and $\lim_{\delta\to 0} c_k^*(\delta) = m_k$. If $\lim_{\alpha\to\infty} \frac{\phi(\alpha x)}{\phi(\alpha)} =: \psi(x)$ exists for $x \ge 0$ and $\lim_{\delta\to 0} \alpha_k(\delta) = \infty$ for each $k \in K$, then*

$$\lim_{\delta\to 0}\left[\inf\left\{\lambda > 0 : \sum_{k\in K}\int_{A_k} \frac{\phi\left(\alpha_k(\delta)\frac{|P_{k,\delta}(y)|}{\lambda}\right)}{\phi(\alpha_k(\delta))} dy \le 1\right\}\right]$$
$$= \inf\left\{\lambda > 0 : \sum_{k\in K}\int_{A_k} \psi\left(\frac{|P_k(y)|}{\lambda}\right) dy \le 1\right\},$$

where $P_k(y) = \sum_{j=0}^{i_k-1} \frac{d_{j,k}}{j!} y^j + m_k y^{i_k}$.

Proof. Note that $c_{j,k}(\delta)$ is defined in (3). By Lemma 2.7, $\psi(x) = x^p$ for some $p \ge 1$. Since $P_{k,\delta} \to P_k$ as $\delta \to 0$ uniformly in X for each $k \in K$, from Lemma 2.8 and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{\delta\to 0} \sum_{k\in K}\int_{A_k} \frac{\phi\left(\alpha_k(\delta)\frac{|P_{k,\delta}(y)|}{\lambda}\right)}{\phi(\alpha_k(\delta))} dy = \sum_{k\in K}\int_{A_k} \psi\left(\frac{|P_k(y)|}{\lambda}\right) dy. \tag{11}$$

Let $A := \inf\left\{\lambda > 0 : \sum_{k\in K}\int_{A_k} \psi\left(\frac{|P_k(y)|}{\lambda}\right) dy \le 1\right\}$ and for each $\delta > 0$,

$$B(\delta) := \left\{\lambda > 0 : \sum_{k\in K}\int_{A_k} \frac{\phi\left(\alpha_k(\delta)\frac{|P_{k,\delta}(y)|}{\lambda}\right)}{\phi(\alpha_k(\delta))} dy \le 1\right\} \quad \text{and}$$
$$C(\delta) := \inf\{\lambda > 0 : \lambda \in B(\delta)\}.$$

Suppose $A > 0$. Given $0 < \epsilon < A$, we have

$$\sum_{k\in K}\int_{A_k} \psi\left(\frac{|P_k(y)|}{A-\epsilon}\right) dy > 1.$$

By (11), there exists $\delta_0 > 0$ such that

$$\sum_{k\in K}\int_{A_k} \frac{\phi\left(\alpha_k(\delta)\frac{|P_{k,\delta}(y)|}{A-\epsilon}\right)}{\phi(\alpha_k(\delta))} dy > 1, \quad \delta < \delta_0.$$

Therefore, $A - \epsilon \notin B(\delta)$ for $\delta < \delta_0$. Then

$$A - \epsilon < C(\delta), \quad \delta < \delta_0. \tag{12}$$

On the other hand, $\sum_{k\in K}\int_{A_k}\psi\Big(\frac{|P_k(y)|}{A+\epsilon}\Big)dy < 1$, and (11) implies again that there exists δ_1 such that

$$\sum_{k\in K}\int_{A_k}\frac{\phi\Big(\alpha_k(\delta)\frac{|P_{k,\delta}(y)|}{A+\epsilon}\Big)}{\phi(\alpha_k(\delta))}dy < 1, \quad \delta < \delta_1.$$

Thus,

$$C(\delta) \le A + \epsilon, \quad \delta < \delta_1,$$

and this fact together with (12) prove the lemma for the case $A > 0$.

Assume now $A = 0$. Given $\lambda > 0$, (11) implies that there exists $\delta(\lambda) > 0$ such that $\lambda \in B(\delta)$, $\delta < \delta(\lambda)$. It immediately follows that $C(\delta) \le \lambda$, $\delta < \delta(\lambda)$. So $\limsup_{\delta\to 0} C(\delta) \le \lambda$. As λ is arbitrary we conclude the lemma. □

3. THE MAIN RESULT

We now show under certain conditions the existence of the best local $\|\cdot\|_\phi$-approximation of a given function from S_N. It generalizes the result proved in [1] for L^p spaces.

Theorem 3.1. *Let $\phi \in \Phi$ and satisfying the Δ_2-condition. Assume that there exists $\lim_{\alpha\to\infty}\frac{\phi(\alpha x)}{\phi(\alpha)}$ for all $x \ge 0$, and therefore this limit is x^p for some $p \ge 1$. Let N be a non-$\|\cdot\|_\phi$-balanced integer and $t = \max_{1\le k\le n}\{\bar{i}_k\}$. For each $k \in K$ suppose*

$$\lim_{\delta\to 0}\frac{v_k(\underline{i}_k)}{E} = e_k > 0. \tag{13}$$

If $f \in PC^t(X)$ and $S_N \subset PC^t(X)$ then, for $\delta \to 0$, the limit of any convergent subsequence of $\{g_\delta\}$, a net of best $\|\cdot\|_\phi$-approximations of f from S_N, is a solution of the following minimization problem in $\mathbb{R}^{\overline{N}-\underline{N}}$:

$$\begin{cases} \min\limits_{h\in S_N}\|\langle e_k J_{A_k}(\underline{i}_k, p)(f-h)^{(\underline{i}_k)}(x_k)/\underline{i}_k!\rangle_{k\in K}\|_{l_p}, \\ \text{with the constraints } (f-h)^{(j)}(x_k) = 0, \quad k = 1,2,\ldots,n \quad \text{and} \\ \quad j = 0,1,\ldots,\underline{i}_k - 1, \end{cases} \tag{14}$$

where, for $k \in K$, $J_{A_k}(\underline{i}_k, p)$ is the minimum L_p norm over A_k of an $\underline{i}_k$th degree polynomial with unit leading coefficient. In particular, if (14) has a unique *solution g, then $g = \lim_{\delta\to 0} g_\delta$ and therefore this is a best local $\|\cdot\|_\phi$-approximation of f from S_N on $\{x_1, \ldots, x_n\}$.*

Proof. The fact that $\lim_{\alpha\to\infty} \frac{\phi(\alpha x)}{\phi(\alpha)} = x^p$, $1 \le p < \infty$, is a consequence of Lemma 2.7. Let $h \in S_N$. Approximating $f - h$ by the Taylor polynomial around x_k, we get

$$(f-h)(x) = \sum_{j=0}^{\bar{i}_k - 1} \frac{(f-h)^{(j)}(x_k)}{j!}(x-x_k)^j + R_k(x), \quad x \in V_k(\delta),$$

where $R_k(x) = O((x-x_k)^{\bar{i}_k})$. We have

$$\begin{aligned}\left\|\frac{f-g_\delta}{E}\right\|_{L^\phi(V)} &= \left\| \sum_{k=1}^{n} \left(\sum_{j=0}^{\bar{i}_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!E}(x-x_k)^j + \frac{R_{\delta,k}(x)}{E}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)} \\ &\ge \left\| \sum_{k=1}^{n} \left(\sum_{j=0}^{\bar{i}_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!E}(x-x_k)^j \right)\chi_{V_k}(x)\right\|_{L^\phi(V)} \\ &\quad - \left\| \sum_{k=1}^{n} \left(\frac{R_{\delta,k}(x)}{E}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)}.\end{aligned}$$

By Lemma 2.3 the net $\{g_\delta\}$ is uniformly bounded in δ, on X. The equivalence of norms in S_N shows that $R_{\delta,k}(x) = O((x-x_k)^{\bar{i}_k})$ uniformly in δ. Thus, by Lemma 2.2 (a) we have

$$\begin{aligned}\left\| \sum_{k=1}^{n} \left(\frac{R_{\delta,k}(x)}{E}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)} &\le \sum_{k=1}^{n} \frac{\|R_{\delta,k}\|_{L^\phi(V_k)}}{E} \\ &\le C \sum_{k=1}^{n} \left\| \frac{(x-x_k)^{\bar{i}_k}}{E}\right\|_{L^\phi(V_k)} = o(1),\end{aligned}$$

where C is a constant. Hence, as $\bar{i}_k - 1 = \underline{i}_k$ for $k \in K$,

$$\begin{aligned}\left\|\frac{f-g_\delta}{E}\right\|_{L^\phi(V)} &\ge \left\| \sum_{k=1}^{n} \left(\sum_{j=0}^{\bar{i}_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!E}(x-x_k)^j \right)\chi_{V_k}(x)\right\|_{L^\phi(V)} + o(1) \\ &\ge \left\| \sum_{k\in K} \left(\sum_{j=0}^{\underline{i}_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!E}(x-x_k)^j \right.\right. \\ &\qquad \left.\left. + \frac{(f-g_\delta)^{(\underline{i}_k)}(x_k)(x-x_k)^{\underline{i}_k}}{\underline{i}_k!E}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)} + o(1)\end{aligned}$$

$$= \left\| \sum_{k\in K} \left(\sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j!v_k(j)}(x-x_k)^j \right.\right.$$
$$\left.\left. + \frac{(f-g_\delta)^{(\underline{i}_k)}(x_k)(x-x_k)^{\underline{i}_k}}{\underline{i}_k!E} \right)\chi_{V_k}(x) \right\|_{L^\phi(V)} + o(1), \tag{15}$$

where $c_{j,k}$ is defined in (3). On the other hand, as in Lemma 2.6, we write

$$g_\delta = \sum_{k=1}^{n}\sum_{j=0}^{\underline{i}_k-1} a_{j,k}(\delta)u_{j,k} + \sum_{r=1}^{N-\underline{N}} b_r(\delta)w_r.$$

Since $a_{j,k}(\delta) = g_\delta^{(j)}(x_k) = f^{(j)}(x_k) - c_{j,k}(\delta)\frac{E}{v_k(j)}$ for $k = 1,2,\ldots,n$ and $j = 0,1,\ldots,\underline{i}_k - 1$, we obtain

$$(f-g_\delta)^{(\underline{i}_k)}(x_k) = \Omega_k(\delta) + \sum_{s=1}^{n}\sum_{i=0}^{\underline{i}_s-1} c_{i,s}(\delta)\frac{E}{v_s(i)}u_{i,s}^{(\underline{i}_k)}(x_k),$$

where $\Omega_k(\delta)$ is defined in (10). Therefore,

$$\left\|\frac{f-g_\delta}{E}\right\|_{L^\phi(V)} \geq \left\| \sum_{k\in K}\left(\sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j!v_k(j)}(x-x_k)^j + \Omega_k(\delta)\frac{(x-x_k)^{\underline{i}_k}}{\underline{i}_k!E}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)}$$
$$- \left\| \sum_{k\in K}\left(\sum_{s=1}^{n}\sum_{i=0}^{\underline{i}_s-1} \frac{c_{i,s}}{v_s(i)}u_{i,s}^{(\underline{i}_k)}(x_k)\right)\frac{(x-x_k)^{\underline{i}_k}}{\underline{i}_k!}\chi_{V_k}(x)\right\|_{L^\phi(V)} + o(1).$$

Let us denote

$$c_k^*(\delta) = \Omega_k(\delta)\frac{e_k}{\underline{i}_k!}.$$

By (13), $\frac{v_k(\underline{i}_k)}{Ee_k} = 1 + o_k(1)$ for $k \in K$. Thus, Lemmas 2.5 and 2.6 imply

$$\left\|\frac{f-g_\delta}{E}\right\|_{L^\phi(V)} \geq \left\| \sum_{k\in K}\left(\sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j!v_k(j)}(x-x_k)^j \right.\right.$$
$$\left.\left. + c_k^*(\delta)\frac{(x-x_k)^{\underline{i}_k}}{Ee_k}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)} + o(1)$$

$$= \left\| \sum_{k\in K} \left(\sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j! v_k(j)} (x-x_k)^j \right.\right.$$

$$\left.\left. + c_k^*(\delta) \frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)} \Big(1+o_k(1)\Big)\right)\chi_{V_k}(x) \right\|_{L^\phi(V)} + o(1)$$

$$\geq \left\| \sum_{k\in K} \left(\sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j! v_k(j)} (x-x_k)^j + c_k^*(\delta) \frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)} \right)\chi_{V_k}(x) \right\|_{L^\phi(V)}$$

$$- \left\| \sum_{k\in K} c_k^*(\delta) \frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k) e_k} o_k(1) \chi_{V_k}(x) \right\|_{L^\phi(V)} + o(1)$$

$$= \left\| \sum_{k\in K} \left(\sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j! v_k(j)} (x-x_k)^j \right.\right.$$

$$\left.\left. + c_k^*(\delta) \frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)} \right)\chi_{V_k}(x) \right\|_{L^\phi(V)} + o(1)$$

$$= \inf \left\{ \lambda > 0 : \sum_{k\in K} \int_{A_k} \right.$$

$$\left. \epsilon_k \phi\left(\phi^{-1}\left(\frac{1}{\epsilon_k}\right) \frac{\left| \sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j!} y^j + c_k^*(\delta) y^{\underline{i}_k} \right|}{\lambda} \right) \, dy \leq 1 \right\} + o(1).$$

Writing $\alpha_k = \phi^{-1}\left(\frac{1}{\epsilon_k}\right)$, we have

$$\left\| \frac{f-g_\delta}{E} \right\|_{L^\phi(V)}$$

$$\geq \inf \left\{ \lambda > 0 : \sum_{k\in K} \int_{A_k} \frac{1}{\phi(\alpha_k)} \phi\left(\alpha_k \frac{\left| \sum_{j=0}^{\underline{i}_k-1} \frac{c_{j,k}(\delta)}{j!} y^j + c_k^*(\delta) y^{\underline{i}_k} \right|}{\lambda} \right) \, dy \leq 1 \right\} + o(1).$$

Since $c_{j,k}(\delta)$ and $b_r(\delta)$ are bounded, $k = 1, 2, \ldots, n$, $j = 0, 1, \ldots, \underline{i}_k - 1$ and $r = 1, 2, \ldots, N - \underline{N}$ (see (8)), there is a subsequence δ_a such that $b_r(\delta_a) \to b_r$ and $c_{j,k}(\delta_a) \to d_{j,k}$ as $\delta_a \to 0$. Denote $m_k = \lim_{\delta_a \to 0} c_k^*(\delta_a)$. Then by Lemma 2.9,

$$\liminf_{\delta_a \to 0} \left\| \frac{f - g_{\delta_a}}{E} \right\|_{L^\phi(V)} \geq \left(\sum_{k\in K} \left\| \sum_{j=0}^{\underline{i}_k-1} \frac{d_{j,k}}{j!} y^j + m_k y^{\underline{i}_k} \right\|_{L^p(A_k)}^p \right)^{1/p}. \tag{16}$$

On the other hand, given two sets of real numbers (independent of δ), say $\{\tilde{b}_r\}_{r=1}^{N-\underline{N}}$ and $\{\tilde{c}_{j,k}\}_{j=0}^{\underline{i}_k-1}$, $k \in K$, consider the following net of functions $\{h_\delta\} \subset S_N$,

$$h_\delta = \sum_{s \notin K} \sum_{i=0}^{\underline{i}_s - 1} f^{(i)}(x_s) u_{i,s} + \sum_{s\in K} \sum_{i=0}^{\underline{i}_s - 1} \left(f^{(i)}(x_s) - \tilde{c}_{i,s} \frac{E}{v_s(i)} \right) u_{i,s} + \sum_{r=1}^{N-\underline{N}} \tilde{b}_r w_r.$$

Since g_δ is a best $\|\cdot\|_\phi$-approximation and the net $\{h_\delta\}$ is uniformly bounded in δ, on X, we may use the above analysis to deduce as in (15) that

$$\begin{aligned} \left\| \frac{f - g_\delta}{E} \right\|_{L^\phi(V)} &\le \left\| \frac{f - h_\delta}{E} \right\|_{L^\phi(V)} \\ &\le \left\| \sum_{k=1}^{n} \left(\sum_{j=0}^{\bar{i}_k - 1} \frac{(f - h_\delta)^{(j)}(x_k)}{j! E} (x - x_k)^j \right) \chi_{V_k}(x) \right\|_{L^\phi(V)} + o(1). \end{aligned}$$

If we note that $(f - h_\delta)^{(j)}(x_k) = 0$ for $k \notin K$ and $(f - h_\delta)^{(j)}(x_k) = \tilde{c}_{j,k} \frac{E}{v_k(j)} = o(1)$ for $k \in K$, $j = 0, 1, \ldots, \underline{i}_k - 1$, we obtain

$$\begin{aligned} &\left\| \frac{f - g_\delta}{E} \right\|_{L^\phi(V)} \\ &\le \left\| \sum_{k\in K} \left(\sum_{j=0}^{\underline{i}_k - 1} \frac{\tilde{c}_{j,k}}{j! v_k(j)} (x - x_k)^j + \frac{(f - h_\delta)^{(\underline{i}_k)}(x_k)(x - x_k)^{\underline{i}_k}}{\underline{i}_k! E} \right) \chi_{V_k}(x) \right\|_{L^\phi(V)} + o(1), \end{aligned}$$

where we are using that $\bar{i}_k - 1 = \underline{i}_k$ for $k \in K$. For these indexes k we denote

$$\tilde{c}_k^* = \left(f^{(\underline{i}_k)}(x_k) - \sum_{s=1}^{n} \sum_{i=0}^{\underline{i}_s - 1} f^{(i)}(x_s) u_{i,s}^{(\underline{i}_k)}(x_k) - \sum_{r=1}^{N-\underline{N}} \tilde{b}_r w_r^{(\underline{i}_k)}(x_k) \right) \frac{e_k}{\underline{i}_k!}. \tag{17}$$

Since, for $k \in K$,

$$\begin{aligned} \frac{(f - h_\delta)^{(\underline{i}_k)}(x_k)}{\underline{i}_k! E} &= \frac{1}{\underline{i}_k! E} \Bigg(f^{(\underline{i}_k)}(x_k) - \sum_{\substack{0 \le i \le \underline{i}_s - 1 \\ 1 \le s \le n}} f^{(i)}(x_s) u_{i,s}^{(\underline{i}_k)}(x_k) \\ &\qquad - \sum_{r=1}^{N-\underline{N}} \tilde{b}_r w_r^{(\underline{i}_k)}(x_k) + \sum_{s\in K} \sum_{i=0}^{\underline{i}_s - 1} \tilde{c}_{i,s} \frac{E}{v_s(i)} u_{i,s}^{(\underline{i}_k)}(x_k) \Bigg) \\ &= \frac{\tilde{c}_k^*}{v_k(\underline{i}_k)} \frac{v_k(\underline{i}_k)}{E e_k} + \sum_{s\in K} \sum_{i=0}^{\underline{i}_s - 1} \frac{\tilde{c}_{i,s}}{\underline{i}_k! v_s(i)} u_{i,s}^{(\underline{i}_k)}(x_k) \\ &= \frac{\tilde{c}_k^*}{v_k(\underline{i}_k)} (1 + o_k(1)) + \sum_{s\in K} \sum_{i=0}^{\underline{i}_s - 1} \frac{\tilde{c}_{i,s}}{\underline{i}_k! v_s(i)} u_{i,s}^{(\underline{i}_k)}(x_k), \end{aligned}$$

we get

$$\left\|\frac{f-g_\delta}{E}\right\|_{L^\phi(V)} \le \left\|\sum_{k\in K}\left(\sum_{j=0}^{\underline{i}_k-1}\frac{\tilde{c}_{j,k}}{j!v_k(j)}(x-x_k)^j+\tilde{c}_k^*\frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)}$$
$$+\left\|\sum_{k\in K}\tilde{c}_k^*\frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)}o_k(1)\chi_{V_k}(x)\right\|_{L^\phi(V)}$$
$$+\left\|\sum_{k\in K}\left(\left(\sum_{s\in K}\sum_{i=0}^{\underline{i}_s-1}\frac{\tilde{c}_{i,s}}{v_s(i)}u_{i,s}^{(\underline{i}_k)}(x_k)\right)\frac{(x-x_k)^{\underline{i}_k}}{\underline{i}_k!}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)}$$
$$+\,o(1).$$

Now, an analogous analysis to that in Lemmas 2.5 and 2.6 implies that

$$\left\|\frac{f-g_\delta}{E}\right\|_{L^\phi(V)} \le \left\|\sum_{k\in K}\left(\sum_{j=0}^{\underline{i}_k-1}\frac{\tilde{c}_{j,k}}{j!v_k(j)}(x-x_k)^j+\tilde{c}_k^*\frac{(x-x_k)^{\underline{i}_k}}{v_k(\underline{i}_k)}\right)\chi_{V_k}(x)\right\|_{L^\phi(V)}+o(1)$$
$$=\inf\left\{\lambda>0:\sum_{k\in K}\int_{A_k}\epsilon_k\phi\left(\phi^{-1}\left(\frac{1}{\epsilon_k}\right)\frac{\left|\sum_{j=0}^{\underline{i}_k-1}\frac{\tilde{c}_{j,k}}{j!}y^j+\tilde{c}_k^*y^{\underline{i}_k}\right|}{\lambda}\right)\,dy\le 1\right\}$$
$$+\,o(1)$$
$$=\inf\left\{\lambda>0:\sum_{k\in K}\int_{A_k}\frac{1}{\phi(\alpha_k)}\phi\left(\alpha_k\frac{\left|\sum_{j=0}^{\underline{i}_k-1}\frac{\tilde{c}_{j,k}}{j!}y^j+\tilde{c}_k^*y^{\underline{i}_k}\right|}{\lambda}\right)\,dy\le 1\right\}$$
$$+\,o(1),$$

where, recall, $\alpha_k=\phi^{-1}\left(\frac{1}{\epsilon_k}\right)$. By Lemma 2.9, we get

$$\limsup_{\delta\to 0}\left\|\frac{f-g_\delta}{E}\right\|_{L^\phi(V)}\le\left(\sum_{k\in K}\left\|\sum_{j=0}^{\underline{i}_k-1}\frac{\tilde{c}_{j,k}}{j!}y^j+\tilde{c}_k^*y^{\underline{i}_k}\right\|_{L^p(A_k)}^p\right)^{1/p},\tag{18}$$

for any set $\{\tilde{c}_{j,k}\}$ and $\{\tilde{b}_r\}$. Thus, from (16) and (18) we obtain

$$\left(\sum_{k\in K}\left\|\sum_{j=0}^{\underline{i}_k-1}\frac{d_{j,k}}{j!}y^j+m_ky^{\underline{i}_k}\right\|_{L^p(A_k)}^p\right)^{1/p}\le\left(\sum_{k\in K}\left\|\sum_{j=0}^{\underline{i}_k-1}\frac{\tilde{c}_{j,k}}{j!}y^j+\tilde{c}_k^*y^{\underline{i}_k}\right\|_{L^p(A_k)}^p\right)^{1/p},$$

for any set $\{\tilde{c}_{j,k}\}$ and $\{\tilde{b}_r\}$. Therefore, the arbitrarily of $\{\tilde{c}_{j,k}\}$ implies

$$\left(\sum_{k\in K}|m_k|^pJ_{A_k}^p(\underline{i}_k,p)\right)^{1/p}\le\left(\sum_{k\in K}|\tilde{c}_k^*|^pJ_{A_k}^p(\underline{i}_k,p)\right)^{1/p}$$

for any set $\{\tilde{b}_r\}$. Hence, since $m_k = \tilde{c}_k^*$ taking $\tilde{b}_r = b_r$, we get

$$\left(\sum_{k\in K} |m_k|^p J_{A_k}^p(\underline{i}_k, p)\right)^{1/p} = \min_{\{\tilde{b}_r\}} \left(\sum_{k\in K} |\tilde{c}_k^*|^p J_{A_k}^p(\underline{i}_k, p)\right)^{1/p}. \tag{19}$$

Then (9) implies that g_{δ_a} tends to

$$g = \sum_{s=1}^{n} \sum_{i=0}^{\underline{i}_s - 1} f^{(i)}(x_s) u_{i,s} + \sum_{r=1}^{N-\underline{N}} b_r w_r$$

as $\delta_a \to 0$, and using (10),

$$m_k = \lim_{\delta_a \to 0} \frac{e_k}{\underline{i}_k} \Omega_k(\delta_a) = e_k \frac{(f-g)^{\underline{i}_k}(x_k)}{\underline{i}_k!}.$$

Finally, taking into account (17) and (19) we conclude that g is a solution of (14). In particular, if (14) has a unique solution, then this is $g = \lim_{\delta\to 0} g_\delta$. □

The following example shows that $\lim_{\delta\to 0} g_\delta$ may not exist if ϕ does not satisfy the assumption that $\lim_{\alpha\to\infty} \frac{\phi(\alpha x)}{\phi(\alpha)}$ exists for all $x \geq 0$.

4. EXAMPLE

Let $x_1 = 0$, $x_2 = 1$, $A_1 = A_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\epsilon_1 = 2\delta$, $\epsilon_2 = \delta$, $0 < \delta < \frac{1}{3}$, and let S_1 be the subspace formed by the constant functions in L^ϕ. Define

$$\phi(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2x - 1 & \text{if } x \in [1,2] \\ 2\ 3^\eta x - 3^{2\eta} & \text{if } x \in [2\ 3^{\eta-1}, 2\ 3^\eta], \eta \in \mathbb{N}, \end{cases} \tag{20}$$

and $f(x) = 0$ if $x \in [-\frac{1}{3}, \frac{1}{3}]$, $f(x) = 1$ if $x \in [\frac{5}{6}, \frac{7}{6}]$.

Observe the following conditions:

- *The function ϕ satisfies the Δ_2-condition.*
  Clearly, $\phi(2x) = 2x \leq 27\phi(x)$ if $x \in \left[0, \frac{1}{2}\right]$ and $\phi(2x) = 4x - 1 \leq 27\phi(x)$ if $x \in \left[\frac{1}{2}, 1\right]$. On the other hand, if $x \in [3^{\eta-1}, 3^\eta]$, $\eta \in \mathbb{N}$, $\phi(2x) \leq \phi(2\ 3^\eta) = 3^{2\eta+1} = 27\phi(3^{\eta-1}) \leq 27\phi(x)$. So the condition is proved.
- $\lim_{\alpha\to\infty} \frac{\phi(\alpha x)}{\phi(\alpha)}$ *does not exist for $x = \frac{1}{2}$.*

Let $\alpha_\eta = 2\ 3^\eta$ and $\alpha'_\eta = 3^\eta$, $\eta \in \mathbb{N}$. From (20) it follows that $\frac{\phi(\alpha_\eta x)}{\phi(\alpha_\eta)} = \frac{\phi(3^\eta)}{\phi(2\,3^\eta)} = \frac{3^{2\eta}}{3^{2\eta+1}} = \frac{1}{3}$ and $\frac{\phi(\alpha'_\eta x)}{\phi(\alpha'_\eta)} = \frac{\phi\left(\frac{3^\eta}{2}\right)}{\phi(3^\eta)} = \frac{2\,3^{2\eta-2}}{3^{2\eta}} = \frac{2}{9}$, $\eta \in \mathbb{N}$. This proves the assertion.

- *The* 2*-tuple* $(\epsilon_1, \epsilon_2) = (2\delta, \delta)$ *satisfies* (1). We know that

$$\frac{v_1(\alpha)}{v_2(\beta)} = 2^\alpha \delta^{\alpha-\beta} \frac{\phi^{-1}\left(\frac{1}{\delta}\right)}{\phi^{-1}\left(\frac{1}{2\delta}\right)}. \tag{21}$$

Given $\delta > 0$, $\frac{1}{\delta} \in [3^{2\eta-1}, 3^{2\eta+1}]$ for some $\eta \in \mathbb{N}$, it follows that $\frac{1}{2\delta} \in [3^{2\eta-1}, 3^{2\eta+1}]$ if $\frac{1}{\delta} > 2\,3^{2\eta-1}$ and $\frac{1}{2\delta} \in [3^{2\eta-2}, 3^{2\eta-1}]$ if $\frac{1}{\delta} < 2\,3^{2\eta-1}$.
As

$$\phi^{-1}(x) = \frac{x}{2\ 3^\eta} + \frac{3^\eta}{2}, \quad x \in [3^{2\eta-1}, 3^{2\eta+1}],$$

in the first case we obtain

$$\frac{\phi^{-1}\left(\frac{1}{\delta}\right)}{\phi^{-1}\left(\frac{1}{2\delta}\right)} = \frac{\frac{1}{2\ 3^\eta} + \frac{3^\eta\delta}{2}}{\frac{1}{4\ 3^\eta} + \frac{3^\eta\delta}{2}} \to 1 \quad \text{as } \eta \to \infty, \tag{22}$$

and in the second case we get

$$\frac{\phi^{-1}\left(\frac{1}{\delta}\right)}{\phi^{-1}\left(\frac{1}{2\delta}\right)} = \frac{\frac{1}{2\ 3^\eta} + \frac{3^\eta\delta}{2}}{\frac{1}{4\ 3^{\eta-1}} + \frac{3^{\eta-1}\delta}{2}} \to 3 \quad \text{as } \eta \to \infty. \tag{23}$$

Therefore, (21) implies (1).

- $N = 1$ *is not a* $\|\cdot\|_\phi$*-balanced integer.*
From (21), (22) and (23), there exist $M_1, M_2 > 0$ such that $M_1 \le \frac{v_2(0)}{v_1(0)} \le M_2$ as $\delta \to 0$. Thus, since $(0, 0)$ is a $\|\cdot\|_\phi$-balanced 2-tuple, as a consequence of the Algorithm in [2] we deduce that the smallest $\|\cdot\|_\phi$-balanced integer greater than 0 is 2. Therefore, $N = 1$ is not a $\|\cdot\|_\phi$-balanced integer.
- *The limit in* (13) *does not exist.*
Let $\delta = \frac{1}{3^\eta}$, $\eta \in \mathbb{N}$. Since

$$\phi^{-1}(3^\eta) = \begin{cases} 3^{\frac{\eta}{2}} & \text{if } \eta \text{ is even} \\ 2\ 3^{\frac{\eta-1}{2}} & \text{if } \eta \text{ is odd} \end{cases} \quad \text{and}$$

$$\phi^{-1}\left(\frac{3^\eta}{2}\right) = \begin{cases} \frac{1}{4} 3^{\frac{\eta}{2}+1} & \text{if } \eta \text{ is even} \\ \frac{5}{4} 3^{\frac{\eta-1}{2}} & \text{if } \eta \text{ is odd} \end{cases}, \text{ then}$$

$$\frac{\phi^{-1}\left(\frac{3^\eta}{2}\right)}{\phi^{-1}(3^\eta)} = \begin{cases} \dfrac{3}{4} & \text{if } \eta \text{ is even} \\ \dfrac{5}{8} & \text{if } \eta \text{ is odd} \end{cases}. \tag{24}$$

Clearly, $(\underline{i}_1, \underline{i}_2) = (0, 0)$. Thus, the equality $\frac{v_2(\underline{i}_2)}{\max\{v_k(\underline{i}_k)\}} = \frac{\phi^{-1}\left(\frac{3^\eta}{2}\right)}{\phi^{-1}(3^\eta)}$ and (24) complete the proof.

Now let us denote by g_η a best $\|\cdot\|_\phi$-approximation of f from S_1 over

$$V = V_1(3^{-\eta}) \cup V_2(3^{-\eta}),$$

and let $u_\eta = \|f - g_\eta\|_{L^\phi(V)}$.

Recall that que g_η is a best $\|\cdot\|_\phi$-approximation of f from S_1 over V if and only if $\frac{g_\eta}{u_\eta}$ is a best ϕ-approximation of $\frac{f}{u_\eta}$ from $\frac{S_1}{u_\eta}$ over V (see [3]), or equivalently, for every constant function $c \neq 0$, $\gamma_\phi^+\left(\frac{f-g_\eta}{u_\eta}, \frac{c}{u_\eta}\right) \geq 0$, where

$$\gamma_\phi^+(g, h) = \lim_{t \to 0^+} \int_V \frac{\phi(|g + th|) - \phi(|g|)}{t}, \quad g, h \in L^\phi(V).$$

This fact is an immediate consequence of a modified version of Theorem 1.6 in [5], for convex functionals. We assert the following:

- *If η is odd, then $g_\eta = \frac{2}{7}$ is a best $\|\cdot\|_\phi$-approximation of f, and $u_\eta = \frac{1}{7} 3^{\frac{3-\eta}{2}}$.*
  Assume $\eta = 2k - 1$, $k > 1$. Then $\frac{g_\eta}{u_\eta} = 2\ 3^{k-2} \in \left[2\ 3^{k-2}, 2\ 3^{k-1}\right]$ and $\frac{1-g_\eta}{u_\eta} = 5\ 3^{k-2} \in \left(2\ 3^{k-2}, 2\ 3^{k-1}\right)$. Since

$$\phi\left(\frac{g_\eta}{u_\eta}\right)\frac{2}{3^\eta} + \phi\left(\frac{1-g_\eta}{u_\eta}\right)\frac{1}{3^\eta} = 3^{2k-3}\frac{2}{3^{2k-1}} + 7\ 3^{2k-3}\frac{1}{3^{2k-1}} = 1,$$

  we have $u_\eta = \left\|f - \frac{2}{7}\right\|_{L^\phi(V)} = \frac{1}{7} 3^{\frac{3-\eta}{2}}$.
  On the other hand, for small $t > 0$ there holds

$$\int_V \frac{\phi\left(\left|\frac{f-g_\eta}{u_\eta} + t\frac{c}{u_\eta}\right|\right) - \phi\left(\left|\frac{f-g_\eta}{u_\eta}\right|\right)}{t} = \frac{\phi\left(2\ 3^{k-2} - t\frac{c}{u_\eta}\right) - \phi\left(2\ 3^{k-2}\right)}{t}\frac{2}{3^{2k-1}} + \frac{\phi\left(5\ 3^{k-2} + t\frac{c}{u_\eta}\right) - \phi\left(5\ 3^{k-2}\right)}{t}\frac{1}{3^{2k-1}}.$$

Then

$$\gamma^+_\phi\Big(\frac{f-g_\eta}{u_\eta},\frac{c}{u_\eta}\Big)=\begin{cases}\dfrac{c}{u_\eta}\Big(-\phi'_-(2\ 3^{k-2})\dfrac{2}{3^{2k-1}}+\phi'(5\ 3^{k-2})\dfrac{1}{3^{2k-1}}\Big) & \text{if } c>0\\ \dfrac{c}{u_\eta}\Big(-\phi'_+(2\ 3^{k-2})\dfrac{2}{3^{2k-1}}+\phi'(5\ 3^{k-2})\dfrac{1}{3^{2k-1}}\Big) & \text{if } c<0\end{cases},$$

or equivalently, $\gamma^+_\phi\Big(\frac{f-g_\eta}{u_\eta},\frac{c}{u_\eta}\Big)$ is equal to $\frac{14}{27}c$ if $c>0$ and to $-\frac{14}{9}c$ if $c<0$. Hence $g_\eta=\frac{2}{7}$ is a best $\|\cdot\|_\phi$-approximation of f from S_1 over V.

- *If η is even, then $g_\eta=\frac{1}{2}$ is a best $\|\cdot\|_\phi$-approximation of f, and $u_\eta=\frac{1}{4}3^{\frac{2-\eta}{2}}$.*

Assume $\eta=2k$, $k\geq 1$. Then $\frac{g_\eta}{u_\eta}=\frac{1-g_\eta}{u_\eta}=2\ 3^{k-1}\in\Big[2\ 3^{k-2},2\ 3^{k-1}\Big]$. Since

$$\phi\Big(\frac{g_\eta}{u_\eta}\Big)\frac{2}{3^\eta}+\phi\Big(\frac{1-g_\eta}{u_\eta}\Big)\frac{1}{3^\eta}=3^{2k-1}\frac{3}{3^{2k}}=1,$$

we get $u_\eta=\frac{1}{4}3^{\frac{2-\eta}{2}}$.

Now, for small $t>0$ we have

$$\int_V\frac{\phi\Big(\Big|\frac{f-g_\eta}{u_\eta}+t\frac{c}{u_\eta}\Big|\Big)-\phi\Big(\Big|\frac{f-g_\eta}{u_\eta}\Big|\Big)}{t}=\frac{\phi\Big(2\ 3^{k-1}-t\frac{c}{u_\eta}\Big)-\phi\Big(2\ 3^{k-1}\Big)}{t}\frac{2}{3^{2k}}+\frac{\phi\Big(2\ 3^{k-1}+t\frac{c}{u_\eta}\Big)-\phi\Big(2\ 3^{k-1}\Big)}{t}\frac{1}{3^{2k}}.$$

Then

$$\gamma^+_\phi\Big(\frac{f-g_\eta}{u_\eta},\frac{c}{u_\eta}\Big)=\begin{cases}\dfrac{c}{u_\eta}\Big(-\phi'_-(2\ 3^{k-1})\dfrac{2}{3^{2k}}+\phi'_+(2\ 3^{k-1})\dfrac{1}{3^{2k}}\Big) & \text{if } c>0\\ \dfrac{c}{u_\eta}\Big(-\phi'_+(2\ 3^{k-1})\dfrac{2}{3^{2k}}+\phi'_-(2\ 3^{k-1})\dfrac{1}{3^{2k}}\Big) & \text{if } c<0\end{cases},$$

or equivalently, $\gamma^+_\phi\big(\frac{f-g_\eta}{u_\eta},\frac{c}{u_\eta}\big)$ is equal to $\frac{8}{9}c$ if $c>0$ and to $-\frac{40}{9}c$ if $c<0$. Therefore $g_\eta=\frac{1}{2}$ is a best $\|\cdot\|_\phi$-approximation of f from S_1 over V.

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