Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 429 (2008) 819–834
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Geometric interpolation in symmetrically-normed ideals

Cristian Conde*,1

IAM-CONICET, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina

Received 13 December 2007; accepted 8 April 2008

Submitted by R.A. Brualdi

Abstract

The aim of this work is to apply the complex interpolation method to norms of n-tuples of operators in a symmetrically-normed ideal $\mathscr{J}_\phi \subseteq B(\mathscr{H})$ defined by a ϕ symmetric norming function (s.n.f.). The norms considered define Finsler metrics in a certain manifold of positive operators, and can be regarded as weighted ϕ-norms, the weight being a positive invertible operator.

AMS classification: Primary 46B70; Secondary 05E05, 47B10

Keywords: Complex interpolation method; Finsler norm; Symmetrically-normed ideal

1. Introduction

Let $B(\mathscr{H})$ denote the algebra of bounded operators acting on a complex and separable Hilbert space $\mathscr{H}$ (with norm $\|\cdot\|$), $Gl(\mathscr{H})$ the group of invertible elements of $B(\mathscr{H})$ and $Gl(\mathscr{H})^+$ the set of all positive elements of $Gl(\mathscr{H})$.

In a previous work [5], we studied the effect of the complex interpolation method on the p-Schatten classes (the idea was motivated by [1]). These ideals belong to a larger class of ideals called symmetrically-normed ideals. The aim of this work is to generalize the results [5] to this larger class of ideals.

* Tel.: +54 11 4954 6781; fax: +54 11 4954 6782.
E-mail address: conde@fceia.unr.edu.ar
1 Supported by CONICET.

doi:10.1016/j.laa.2008.04.011

By Calkin's theorem (see, e.g. [9, Chapter 3, Th. 1.1]), if $\mathscr{I}$ is a two-sided ideal then

$$\mathscr{I} \subseteq B_0(\mathscr{H}),$$

where $B_0(\mathscr{H})$ is the ideal of all compact operators in $B(\mathscr{H})$. If $X \in B_0(\mathscr{H})$, we denote by $s(X) = \{s_j(X)\}_{j\in\mathbb{N}}$ the sequence of singular values of X, decreasingly ordered:

$$s_j(X) = \inf\{\|X - S\|: \operatorname{rank}(S) \leqslant j\}.$$

Note that $s_j(X) \searrow 0$. There are other alternatives to describe the sequence $s(X)$. For instance, the $s_j(X)$'s are the eigenvalues of $|X| = (X^*X)^{1/2}$.

If ϕ is a symmetric norming function (the definition will be given below), we consider the ideal of $B(\mathscr{H})$

$$\mathscr{I}_\phi = \{X \in B_0(\mathscr{H}): \|X\|_\phi < \infty\},$$

where

$$\|X\|_\phi = \phi(s(X)).$$

On $\mathscr{I}_\phi$ we define the following norm associated with $a \in Gl(\mathscr{H})^+$:

$$\|X\|_{\phi,a} := \|a^{-1/2}Xa^{-1/2}\|_\phi = \phi(s(a^{-1/2}Xa^{-1/2})).$$

The use of this norm has a geometrical meaning which shall be explained later.

The material is organized as follows. In Section 2, we recall some basic facts about symmetric norming functions and the corresponding symmetrically-normed ideals. Section 3 contains a brief summary of the complex interpolation method. In Section 4, we apply this method and obtain that the curve of interpolation coincides with the curve of weighted norms determined by the positive invertible elements

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}.$$

In Section 5, we present an elementary interpolation argument to obtain Corach–Porta–Recht type inequalities.

Finally, in Section 6 we present the geometrical meaning of the interpolating curve $\gamma_{a,b}$.

2. Symmetrically-normed ideals

We begin by recalling some facts concerning normed ideals (see [9]).

Let $\mathscr{I}$ be a proper two-sided ideal of $B(\mathscr{H})$, it is well known that

$$B_{0,0}(\mathscr{H}) \subseteq \mathscr{I} \subseteq B_0(\mathscr{H}),$$

where $B_{0,0}(\mathscr{H})$ is the ideal of finite rank operators.

$\mathscr{I}$ is a symmetrically-normed ideal if $\mathscr{I}$ is an ideal of $B(\mathscr{H})$ and a Banach space with respect to the norm $\|\cdot\|_{\mathscr{I}}$ satisfying:

(1) $\|XTY\|_{\mathscr{I}} \leqslant \|X\|\|T\|_{\mathscr{I}}\|Y\|$ for $T \in \mathscr{I}$ and $X, Y \in B(\mathscr{H})$,
(2) $\|X\|_{\mathscr{I}} = \|X\|$ if T is the rank one.

In particular, condition (1) implies that the norm $\|\cdot\|_{\mathscr{I}}$ is unitarily invariant

$$\|UXV^*\|_{\mathscr{I}} = \|X\|_{\mathscr{I}}$$

for $X \in \mathscr{I}$ and any pair U, V of unitary operators.

Let c_0 and $c_{0,0}$ be the spaces of sequences of real numbers defined by

$$c_0 = \left\{\xi = \{\xi_i\}: \lim_{i\to\infty} \xi_i = 0\right\}$$

and

$$c_{0,0} = \{\xi = \{\xi_i\} \in c_0: \text{only finitely many } \xi_i\text{'are nonzero}\},$$

respectively.

A function ϕ on $c_{0,0}$ is said to be a symmetric norming function (s.n.f.) if it satisfies:

(1) ϕ is a norm on $c_{0,0}$;
(2) $\phi(\{1, 0, 0, \dots\}) = 1$;
(3) $\phi(\{\xi_j\}) = \phi(\{|\xi_{\pi(j)}|\})$ for any bijection $\pi: \mathbb{N} \to \mathbb{N}$.

Two s.n. functions ϕ and ψ are equivalent if

$$\sup_{\xi\in c_{0,0}} \frac{\phi(\xi)}{\psi(\xi)} < \infty \quad \text{and} \quad \sup_{\xi\in c_{0,0}} \frac{\psi(\xi)}{\phi(\xi)} < \infty.$$

Let us denote by c_ϕ the set of all sequences $\xi \in c_0$ for which

$$\sup_n \phi(\xi^{(n)}) < \infty,$$

where $\xi^{(n)} = \{\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots\}$. We extend the domain of the function ϕ by putting, for each $\xi \in c_\phi$

$$\phi(\xi) = \lim_{n\to\infty} \phi(\xi^{(n)}).$$

If $X \in B_0(\mathscr{H})$ and ϕ is a symmetric norming function, let us denote

$$\|X\|_\phi = \phi(s(X)).$$

There are two symmetrically-normed ideals [9, Th. 4.1] related to ϕ:

$$\mathscr{J}_\phi = \{X \in B_0(\mathscr{H}): \|X\|_\phi < \infty\}$$

and $\mathscr{J}_\phi^{(0)}$ the closure of $B_{0,0}(\mathscr{H})$ with respect to the norm $\|\cdot\|_\phi$. Note that $\mathscr{J}_\phi^{(0)}$ does not coincide with $\mathscr{J}_\phi$ in general. Both spaces coincide if and only if ϕ is a regular function, that is

$$\lim_{n\to\infty} \phi(\xi_{n+1}, \xi_{n+2}, \dots) = 0.$$

Let ϕ be a s.n. function. The function

$$\phi'(\eta) = \sup\left\{\sum_i \eta_i\xi_i: \xi \in c_{0,0}, \|\xi\|_\phi \leqslant 1\right\}$$

makes sense for any $\eta \in c_{0,0}$, and clearly is an s.n. function. The function ϕ' is called the conjugate function of ϕ.

For any s.n. function ϕ one has $(\phi')' = \phi$, and if ϕ is not equivalent to $\phi_1(\xi) = \sum_i |\xi_i|$, one has the following duality:

$$\|X\|_{\phi'} = \sup\{|\mathrm{tr}(XY)| : Y \in \mathscr{J}_\phi^{(0)}, \|Y\|_\phi \leqslant 1\}.$$

For example for $1 \leqslant p < \infty$, the functions

$$\phi_p(\xi) = \left(\sum_i |\xi_i|^p\right)^{1/p}$$

and

$$\phi_\infty(\xi) = \max_i |\xi_i|$$

give rise to

$$\mathscr{I}_\phi^{(0)} = \mathscr{I}_\phi = B_p(\mathscr{H}),$$

where $B_p(\mathscr{H})$ denotes the p-Schatten class.

From now on, we will assume that ϕ' is not equivalent to ϕ_1.

3. The complex interpolation method

We recall the construction of interpolation spaces, usually called the complex interpolation method. We follow the notation used in [2] and we refer to [14,4] for details on the complex interpolation method. From now on, we simply denote by $\mathscr{I}$ the ideal defined by ϕ.

A compatible couple of Banach spaces is a pair $\overline{X} = (X_0, X_1)$ of Banach spaces X_0, X_1 such that both are continuously embedded in some Hausdorff topological vector space $\mathscr{U}$. Observe that for all $a, b \in Gl(\mathscr{H})^+$ the Banach spaces $(\mathscr{I}, \|\cdot\|_{\phi,a})$ and $(\mathscr{I}, \|\cdot\|_{\phi,b})$ are compatible. We will simply write this pair of spaces $\overline{\mathscr{I}}$ when no confusion can arise.

If X_0 and X_1 are compatible, then one can form their sum $X_0 + X_1$ and their intersection $X_0 \cap X_1$. The sum consists of all $x \in \mathscr{U}$ such that one can write $x = y + z$ for some $y \in X_0$ and $z \in X_1$.

Suppose that X_0 and X_1 are compatible Banach spaces. Then $X_0 \cap X_1$ is a Banach space with its norm defined by

$$\|x\|_{X_0 \cap X_1} = \max(\|x\|_{X_0}, \|x\|_{X_1}).$$

Moreover, $X_0 + X_1$ is also a Banach space with the norm

$$\|x\|_{X_0 + X_1} = \inf\{\|y\|_{X_0} + \|z\|_{X_1} : x = y + z,\ y \in X_0, z \in X_1\}.$$

A Banach space X is said to be an intermediate space with respect to $\overline{X}$ if

$$X_0 \cap X_1 \subset X \subset X_0 + X_1$$

and both inclusions are continuous.

Given a compatible pair $\overline{X} = (X_0, X_1)$, one considers the space $\mathscr{F}(\overline{X}) = \mathscr{F}(X_0, X_1)$ of all functions f defined in the strip

$$S = \{z \in \mathbb{C} : 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$$

with values in $X_0 + X_1$, and having the following properties:

(1) $f(z)$ is continuous and bounded in norm of $X_0 + X_1$ on the strip S.
(2) $f(z)$ is analytic relative to the norm of $X_0 + X_1$ on $S^\circ = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$.
(3) $f(j + \mathrm{i}y)$ assumes values in the space X_j $(j = 0, 1)$ and is continuous and bounded in the norm of this space.

One equips the vector space $\mathscr{F}(\overline{X})$ with the norm

$$\|f\|_{\mathscr{F}(\overline{X})} = \max\left\{\sup_{y\in\mathbb{R}}\|f(\mathrm{i}y)\|_{X_0}, \sup_{y\in\mathbb{R}}\|f(1+\mathrm{i}y)\|_{X_1}\right\}.$$

The space $(\mathscr{F}(\overline{X}), \|\ \|_{\mathscr{F}(\overline{X})})$ is a Banach space.

For each $0 < t < 1$ the complex interpolation space, associated with the couple $\overline{X}$, $\overline{X}_{[t]} = (X_0, X_1)_{[t]}$ is the set of all elements $x \in X_0 + X_1$ representable in the form $x = f(t)$ for function $f \in \mathscr{F}(\overline{X})$, equipped with the complex interpolation norm

$$\|x\|_{[t]} = \inf\{\|f\|_{\mathscr{F}(\overline{X})}: f \in \mathscr{F}(\overline{X}), f(t) = x\}.$$

The two main results of the theory are:

Theorem A. *The space $\overline{X}_{[t]}$ is a Banach space and an intermediate space with respect to $\overline{X}$.*

Theorem B. *Let $\overline{X}$ and $\bar{Y}$ two compatible couples. Assume that T is a linear operator from X_j to Y_j bounded by M_j, $j = 0, 1$. Then for $t \in [0, 1]$*

$$\|T\|_{\overline{X}_{[t]}\to\overline{Y}_{[t]}} \leqslant M_0^{1-t}M_1^t.$$

4. Geometric interpolation

In this section, we state the main result of this paper. First, we introduce the notation.

For $n \in \mathbb{N}$, $s \geqslant 1$ and $a \in Gl(\mathscr{H})^+$, let

$$\mathscr{J}^{(n)} = \{(X_0, \ldots, X_{n-1}) : X_i \in \mathscr{J}\}$$

with the norm

$$\|(X_0, \ldots, X_{n-1})\|_{\phi,a;s} = (\|X_0\|_{\phi,a}^s + \cdots + \|X_{n-1}\|_{\phi,a}^s)^{1/s}$$

and $\mathbb{C}^n$ with the norm

$$|(z_0, \ldots, z_{n-1})|_s = (|z_0|^s + \cdots + |z_{n-1}|^s)^{1/s}.$$

We consider the action of $Gl(\mathscr{H})$ on $\mathscr{J}^{(n)}$, defined by

$$l: Gl(\mathscr{H}) \times \mathscr{J}^{(n)} \longrightarrow \mathscr{J}^{(n)}, l_g((X_0, \ldots, X_{n-1})) = (gX_0g^*, \ldots, gX_{n-1}g^*). \tag{4.1}$$

From now on, we denote with $\mathscr{J}^{(n)}_{\phi,a;s}$ the space $\mathscr{J}^{(n)}$ endowed with the norm $\|(\cdot, \ldots, \cdot)\|_{\phi,a;s}$.

Proposition 4.1. *The norm in $\mathscr{J}^{(n)}_{\phi,a;s}$ is invariant for the action of the group of invertible elements. By this we mean that for each $(X_0, \ldots, X_{n-1}) \in \mathscr{J}^{(n)}$, $a \in Gl(\mathscr{H})^+$ and $g \in Gl(\mathscr{H})$, we have*

$$\|(X_0, \ldots, X_{n-1})\|_{\phi,a;s} = \|l_g((X_0, \ldots , X_{n-1}))\|_{\phi,gag^*;s}.$$

Proof. It is sufficient to prove that $s(a^{-1/2}Xa^{-1/2})$ and $s\left((gag^*)^{-\frac{1}{2}}gXg^*(gag^*)^{-\frac{1}{2}}\right)$ coincide. For $j \in \mathbb{N}$, we get

$$\begin{aligned} s_j\left((gag^*)^{-\frac{1}{2}}gXg^*(gag^*)^{-\frac{1}{2}}\right)^2 &= \lambda_j((gag^*)^{-\frac{1}{2}}gX^*g^*(gag^*)^{-\frac{1}{2}}(gag^*)^{-\frac{1}{2}} \\ &\quad \times gXg^*(gag^*)^{-\frac{1}{2}}) \\ &= \lambda_j\left((gag^*)^{-\frac{1}{2}}gX^*g^*(gag^*)^{-1}gXg^*(gag^*)^{-\frac{1}{2}}\right) \\ &= \lambda_j\left((gag^*)^{-\frac{1}{2}}gX^*g^*(g^*)^{-1}a^{-1}g^{-1}gXg^*(gag^*)^{-\frac{1}{2}}\right) \\ &= \lambda_j\left((gag^*)^{-\frac{1}{2}}gX^*a^{-1}Xg^*(gag^*)^{-\frac{1}{2}}\right) \\ &= \lambda_j\left((gag^*)^{-\frac{1}{2}}gX^*a^{-\frac{1}{2}}a^{-\frac{1}{2}}Xg^*(gag^*)^{-\frac{1}{2}}\right) \\ &= \lambda_j\left(a^{-\frac{1}{2}}Xg^*(gag^*)^{-\frac{1}{2}}(gag^*)^{-\frac{1}{2}}gX^*a^{-\frac{1}{2}}\right) \\ &= \lambda_j\left(a^{-\frac{1}{2}}Xg^*(gag^*)^{-1}gX^*a^{-\frac{1}{2}}\right) \\ &= \lambda_j\left(a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}a^{-\frac{1}{2}}X^*a^{-\frac{1}{2}}\right) \\ &= s_j\left(a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\right)^2, \end{aligned}$$

where $\lambda_j(X)$ denotes the jth eigenvalue of X decreasingly ordered. □

Theorem 4.2. *Let* $a, b \in Gl(\mathscr{H})^+$, $1 \leqslant s < \infty$, $n \in \mathbb{N}$ *and* $t \in (0, 1)$. *Then*
$(\mathscr{I}^{(n)}_{\phi,a;s}, \mathscr{I}^{(n)}_{\phi,b;s})_{[t]} = \mathscr{I}^{(n)}_{\phi,\gamma_{a,b}(t);s}$.

Proof. Recall Hadamard's classical three lines theorem [18, p. 33]:

Let $f(z)$ be a Banach space-valued function, bounded and continuous on the strip S, analytic in the interior, satisfying

$$\|f(z)\|_X \leqslant M_0 \text{ if } \mathrm{Re}(z) = 0$$

and

$$\|f(z)\|_X \leqslant M_1 \text{ if } \mathrm{Re}(z) = 1,$$

where $\|\cdot\|_X$ denotes the norm of the Banach space X. Then

$$\|f(z)\|_X \leqslant M_0^{1-\mathrm{Re}(z)} M_1^{\mathrm{Re}(z)}$$

for all $z \in S$.

In order to simplify, we will only consider the case $n = 2$. The proof below works for n-tuples ($n \geqslant 3$) with obvious modifications.

By Proposition 4.1, we have that $\|(X_1, X_2)\|_{[t]}$ is equal to the norm of $a^{-1/2}(X_1, X_2)a^{-1/2}$ interpolated between the norms $\|(\cdot,\cdot)\|_{\phi,1;s}$ and $\|(\cdot,\cdot)\|_{\phi,c;s}$. Consequently, it is sufficient to prove our statement for these two norms.

The proof consists of showing that for all $t \in (0, 1)$, $\|(X_1, X_2)\|_{[t]}$ and $\|(X_1, X_2)\|_{\phi,c^t;s}$ coincide in $\mathscr{I}^{(2)}$.

Let $t \in (0, 1)$ and $(X_1, X_2) \in \mathscr{I}^{(2)}$ such that $\|(X_1, X_2)\|_{\phi,c^t;s} = 1$, and define

$$f(z) = c^{\frac{z}{2}}c^{-\frac{t}{2}}(X_1, X_2)c^{-\frac{t}{2}}c^{\frac{z}{2}} = (f_1(z), f_2(z)).$$

Then for each $z \in S$, $f(z) \in \mathscr{J}^{(2)}$

$$\|f(\mathrm{i}y)\|_{\phi,1;s} = \|c^{\frac{\mathrm{i}y}{2}} c^{-\frac{t}{2}}(X_1, X_2)c^{-\frac{t}{2}} c^{\frac{\mathrm{i}y}{2}}\|_{\phi,1;s} = \left(\sum_{k=1}^{2} \|c^{\frac{\mathrm{i}y}{2}} c^{-\frac{t}{2}} X_k c^{-\frac{t}{2}} c^{\frac{\mathrm{i}y}{2}}\|_{\phi,1}^s\right)^{1/s} \leqslant 1$$

and

$$\|f(1+\mathrm{i}y)\|_{\phi,c;s} = \left(\sum_{k=1}^{2} \|c^{\frac{1}{2}} c^{\frac{\mathrm{i}y}{2}} c^{-\frac{t}{2}} X_k c^{-\frac{t}{2}} c^{\frac{\mathrm{i}y}{2}} c^{\frac{1}{2}}\|_{\phi,c}^s\right)^{1/s} \leqslant 1.$$

Since $f(t) = (X_1, X_2)$ and $f = (f_1, f_2) \in \mathscr{F}(\mathscr{J}^{(2)})$ we have $\|(X_1, X_2)\|_{[t]} \leqslant 1$. Thus we have shown that

$$\|(X_1, X_2)\|_{[t]} \leqslant \|(X_1, X_2)\|_{\phi,c^t;s}.$$

To prove the converse inequality, let $f = (f_1, f_2) \in \mathscr{F}(\mathscr{J}^{(2)})$; $f(t) = (X_1, X_2)$ and $k = 1, 2$, we consider $Y_k \in \mathscr{J}_{\phi'}^{(0)}$ with $\|Y_k\|_{\phi'} \leqslant 1$. Let

$$g_k(z) = c^{-\frac{z}{2}} Y_k c^{-\frac{z}{2}}.$$

Consider the function $h: S \to (\mathbb{C}^2, |(\cdot,\cdot)|_s)$,

$$h(z) = (\mathrm{tr}(f_1(z)g_1(z)), \mathrm{tr}(f_2(z)g_2(z))).$$

Since $f(z)$ is analytic in S° and bounded in S as a $\mathscr{J}^{(2)}$-valued function, then h is analytic in S° and bounded in S, and

$$h(t) = \left(\mathrm{tr}\left(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1\right), \mathrm{tr}\left(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2\right)\right).$$

By Hadamard's three lines theorem, applied to h and the Banach space $(\mathbb{C}^2, |(\cdot,\cdot)|_s)$, we have

$$|h(t)|_s \leqslant \max\left\{\sup_{y\in\mathbb{R}} |h(\mathrm{i}y)|_s, \sup_{y\in\mathbb{R}} |h(1+\mathrm{i}y)|_s\right\}.$$

For $j = 0, 1$

$$\begin{aligned}\sup_{y\in\mathbb{R}} |h(j+\mathrm{i}y)|_s &= \sup_{y\in\mathbb{R}} \left(\sum_{k=1}^{2} |\mathrm{tr}(f_k(j+\mathrm{i}y)g_k(j+\mathrm{i}y))|^s\right)^{1/s} \\ &= \sup_{y\in\mathbb{R}} \left(\sum_{k=1}^{2} |\mathrm{tr}(c^{-j/2} f_k(j+\mathrm{i}y)c^{-j/2} g_k(\mathrm{i}y))|^s\right)^{1/s} \\ &\leqslant \sup_{y\in\mathbb{R}} \left(\sum_{k=1}^{2} \|f_k(j+\mathrm{i}y)\|_{\phi,c^j}^s\right)^{1/s} \leqslant \|f\|_{\mathscr{F}(\mathscr{J}^{(2)})},\end{aligned}$$

then

$$\begin{aligned}\|X_1\|_{\phi,c^t}^s + \|X_2\|_{\phi,c^t}^s &= \sup_{\|Y_2\|_{\phi'}\leqslant 1, \|Y_1\|_{\phi'}\leqslant 1} \left\{|\mathrm{tr}\left(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1\right)|^s + |\mathrm{tr}(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2)|^s\right\} \\ &\leqslant \sup_{\|Y_2\|_{\phi'}\leqslant 1, \|Y_1\|_{\phi'}\leqslant 1} |h(t)|_s^s \leqslant \|f\|_{\mathscr{F}(\mathscr{J}^{(2)})}^s.\end{aligned}$$

Since the previous inequality is valid for each $f \in \mathscr{F}(\mathscr{J}^{(2)})$ with $f(t) = (X_1, X_2)$, we have

$$\|(X_1, X_2)\|_{\phi,c^t;s} \leqslant \|(X_1, X_2)\|_{[t]}. \qquad \square$$

In the special case $n = s = 1$ we obtain

Corollary 4.3. *Given $a, b \in Gl(\mathscr{H})^+$ we have for all $t \in [0, 1]$*

$$(\mathscr{J}_{\phi,a}, \mathscr{J}_{\phi,b})_{[t]} = \mathscr{J}_{\phi,\gamma_{a,b}(t)}.$$

Remark 4.4. Note that when a and b commute the curve is given by $\gamma_{a,b}(t) = a^{1-t}b^t$. The previous corollary tells us that the interpolating space, $\mathscr{J}_{\phi,\gamma_{a,b}(t)}$ can be regarded as a weighted $\mathscr{J}_\phi$ space with weight $a^{1-t}b^t$ (see [2, Th. 5.5.3]).

By Theorem B, we obtain the following result of interpolation:

Corollary 4.5. *Let $a, b, c, d \in Gl(\mathscr{H})^+$, $s \geqslant 1$, $n \in \mathbb{N}$ and T a linear operator such that the norm of T is at most M_0(between the spaces $\mathscr{J}^{(n)}_{\phi,a;s}$ and $\mathscr{J}^{(n)}_{\phi,b;s}$) and the norm of T is at most M_1 (between the spaces $\mathscr{J}^{(n)}_{\phi,c;s}$ and $\mathscr{J}^{(n)}_{\phi,d;s}$). Then, for all $t \in [0, 1]$ we have*

$$\|T(x)\|_{\phi,\gamma_{b,d}(t);s} \leqslant (M_0)^{1-t} M_1^t \|x\|_{\phi,\gamma_{a,c}(t);s}.$$

5. On the Corach–Porta–Recht inequality

In their work on the geometry of the space of self-adjoint invertible elements of a C^*-algebra, Corach et al. proved in [6] that if S is invertible and self-adjoint in $B(\mathscr{H})$, then for all $X \in B(\mathscr{H})$

$$\|SXS^{-1} + S^{-1}XS\| \geqslant 2\|X\|.$$

In [12], Kittaneh proved a more general version of the CPR inequality: for any norm ideal $(\mathscr{I}, \|\cdot\|_{\mathscr{I}})$ of $B(\mathscr{H})$ and for all $X \in \mathscr{I}$ we have

$$2\|X\|_{\mathscr{I}} \leqslant \|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}}. \tag{5.1}$$

In [19], Seddik obtained the following inequality for any norm ideal $\mathscr{I}$ of $B(\mathscr{H})$.

Theorem 5.1. *For all $X \in \mathscr{I}$*

$$\|SXS^{-1} - S^{-1}XS\|_{\mathscr{I}} \leqslant (\|S\|\|S^{-1}\| - 1)\|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}}. \tag{5.2}$$

Recently, Larotonda in [16] obtained the following inequality for any norm ideal.

Theorem 5.2 [16, Corollary 28]. *For all $X \in \mathscr{I}$*

$$\|SXS^{-1} - S^{-1}XS\|_{\mathscr{I}} \leqslant \|L_T - R_T\|_{B(\mathscr{I})}\|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}}, \tag{5.3}$$

where $T = \log|S|$ and L_T, R_T are the left and right multiplication representations of T in $B(\mathscr{I})$, $L_T(U) = TU$ and $R_T(U) = UT$.

Here $\|P\|_{B(\mathscr{I})}$ denotes the norm of the linear operator $P: \mathscr{I} \to \mathscr{I}$, that is

$$\|P\|_{B(\mathscr{I})} = \sup\{\|P(x)\|_{\mathscr{I}}: \|x\|_{\mathscr{I}} = 1\}.$$

The bound in (5.3) is related to the theory of generalized derivations. If $A, B \in B(\mathscr{H})$ let

$$\delta_{A,B}: X \to \delta_{A,B}(X) := AX - XB = L_A(X) - R_B(X).$$

The theory of generalized derivations has been extensively studied in the literature, see for example [7]. In [20], Stampfli proved the following equality :

$$\|\delta_{A,B}\| = \inf\{\|A-\lambda\| + \|B-\lambda\| : \lambda \in \mathbb{C}\}. \tag{5.4}$$

If $(\mathscr{I}, \|\cdot\|_{\mathscr{I}})$ is a norm ideal in $B(\mathscr{H})$ and $X \in \mathscr{I}$, then for all $\lambda \in \mathbb{C}$

$$\|\delta_{A,B}(X)\|_{\mathscr{I}} = \|(A-\lambda)X + X(B-\lambda)\|_{\mathscr{I}} \leqslant (\|A-\lambda\| + \|B-\lambda\|)\|X\|_{\mathscr{I}}. \tag{5.5}$$

It follows from (5.4) that

$$\|\delta_{A,B}\|_{B(\mathscr{I})} \leqslant \|\delta_{A,B}\|.$$

From these facts we get

$$\begin{aligned}\|SXS^{-1} - S^{-1}XS\|_{\mathscr{I}} &\leqslant \|L_T - R_T\|_{B(\mathscr{I})}\|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}}\\ &= \|\delta_{T,T}\|_{B(\mathscr{I})}\|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}}\\ &\leqslant \|\delta_{T,T}\|\|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}}.\end{aligned} \tag{5.6}$$

From (5.6) and (5.2) we obtain that

$$\|SXS^{-1} - S^{-1}XS\|_{\mathscr{I}} \leqslant \min\{\|\delta_{T,T}\|, \|S\|\|S^{-1}\| - 1\}\|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}}.$$

Note that the bound in Theorem 5.2 is a refinement of (5.2). We start by recalling the next.

Corollary 5.3 [20, Corollary 1]. *Let T be a normal operator. Then*

$$\|\delta_{T,T}\| = \sup\{\|TX - XT\| : T \in B(\mathscr{H}) \text{ and } \|T\| = 1\} = 2r(T) = \lambda_{\max}(T) - \lambda_{\min}(T),$$

where $r(T)$ is the radius of the spectrum of T.

First, we shall assume that S is positive. Then

$$\|\delta_{T,T}\| = \lambda_{\max}(T) - \lambda_{\min}(T) = \log(\lambda_{\max}(S)) - \log(\lambda_{\min}(S)) = \log\left(\frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}\right),$$

therefore

$$\|S\| = \lambda_{\max}(S) \quad \text{and} \quad \|S^{-1}\| = \frac{1}{\lambda_{\min}(S)}.$$

So

$$\|\delta_{T,T}\| = \log\left(\frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}\right) < \frac{\lambda_{\max}(S)}{\lambda_{\min}(S)} - 1 = \|S\|\|S^{-1}\| - 1.$$

Here we use the fact that $\log(t) < t - 1$, for all $t > 1$.
In the general case (i.e. S invertible and self-adjoint) we have

$$\|\delta_{T,T}\| = \log\left(\frac{\lambda_{\max}(|S|)}{\lambda_{\min}(|S|)}\right) < \||S|\|\,\||S|^{-1}\| - 1 = \|S\|\,\|S^{-1}\| - 1.$$

Now, we are ready to state the next

Theorem 5.4. *Let $\mathscr{I}$ be a norm ideal, then for all $X \in \mathscr{I}$*

$$\|SXS^{-1} - S^{-1}XS\|_{\mathscr{I}} \leqslant \|\delta_{T,T}\| \|SXS^{-1} + S^{-1}XS\|_{\mathscr{I}} \tag{5.7}$$

with $T = \log|S|$.

Note that the inequality holds for any norm ideal $\mathscr{I}$ where the explicit bound $\|\delta_{T,T}\|$ depends only on the operator S and the norm in $B(\mathscr{H})$ and not on the given unitarily invariant norm.

Now, we shall apply Corollary 4.5 to the inequality obtained above. Consider

$$T_{\mathscr{J},S} : \mathscr{J} \longrightarrow \mathscr{J} \quad T_{\mathscr{J},S}(X) = SXS^{-1} - S^{-1}XS$$

and

$$R_{\mathscr{J},S} : \mathscr{J} \longrightarrow \mathscr{J} \quad R_{\mathscr{J},S}(X) = SXS^{-1} + S^{-1}XS.$$

From Corollary 4.5 and (5.7), we obtain

Corollary 5.5. *For $a, b \in Gl(\mathscr{H})^{+}$, $X \in \mathscr{J}$ and $t \in [0, 1]$, we have*

$$\|SXS^{-1} - S^{-1}XS\|_{\phi} \leqslant 2\mu\|\delta_{T,T}\|\|a\|^{1-t}\|b\|^{t}\|X\|_{\phi,\gamma_{a,b}(t)}, \tag{5.8}$$

where $\mu = \|S\|\,\|S^{-1}\|$.

Proof. We will denote by $\gamma(t) = \gamma_{a,b}(t)$, when no confusion can arise. By (5.7) the norm of $T_{\mathscr{J},S}$ is at most $2\mu\|\delta_{T,T}\|\|a\|$ when

$$T_{\mathscr{J},S} : (\mathscr{J}, \|\cdot\|_{\phi,a}) \longrightarrow (\mathscr{J}, \|\cdot\|_{\phi})$$

and the norm of $T_{\mathscr{J},S}$ is at most $2\mu\|\delta_{T,T}\|\|b\|$ when

$$T_{\mathscr{J},S} : (\mathscr{J}, \|\cdot\|_{\phi,b}) \longrightarrow (\mathscr{J}, \|\cdot\|_{\phi}).$$

Therefore, using the complex interpolation, we obtain the following diagram of interpolation for $t \in [0, 1]$:

$$\begin{tikzcd}
(\mathcal{J}, \|.\|_{\phi,a}) \arrow[dr, "T"] & \\
(\mathcal{J}, \|.\|_{\phi,\gamma(t)}) \arrow[r, "T_t"] & (\mathcal{J}, \|.\|_{\phi}) \\
(\mathcal{J}, \|.\|_{\phi,b}) \arrow[ur, "T"] &
\end{tikzcd}$$

By Corollary 4.5

$$\begin{aligned}
\|T_{\mathscr{J},S}(X)\|_{\phi} &\leqslant (2\mu\|\delta_{T,T}\|\|b\|)^{t}(2\mu\|\delta_{T,T}\|\|a\|)^{1-t}\|X\|_{\phi,\gamma(t)} \\
&= 2\mu\|\delta_{T,T}\|\|a\|^{1-t}\|b\|^{t}\|X\|_{\phi,\gamma(t)}. \qquad \square
\end{aligned}$$

With a slight change in the previous proof, we get the inequality

$$\|SXS^{-1} + S^{-1}XS\|_{\phi} \leqslant 2\mu\|a\|^{1-t}\|b\|^{t}\|X\|_{\phi,\gamma_{a,b}(t)}. \tag{5.9}$$

We need to consider the operator $R_{\mathscr{J},S}$ and the fact that the norm of this operator is at most $2\mu\|a\|$ when

$$R_{\mathscr{J},S}\colon (\mathscr{J}, \|\cdot\|_{\phi,a}) \longrightarrow (\mathscr{J}, \|\cdot\|_{\phi}).$$

Corollary 5.6. *For $X \in \mathscr{J}$ and $t \in [0, 1]$, we have*

$$\|SXS^{-1} - S^{-1}XS\|_{\phi} \leqslant 2\mu\|\delta_{T,T}\| \inf\{\|a\|^{1-t}\|b\|^{t}\|X\|_{\phi,\gamma_{a,b}(t)}\colon t \in [0, 1], a, b \in Gl(\mathscr{H})^{+}\}$$

and

$$\begin{aligned} 2\|X\|_{\phi} &\leqslant \|SXS^{-1} + S^{-1}XS\|_{\phi} \\ &\leqslant 2\mu \inf\{\|a\|^{1-t}\|b\|^{t}\|X\|_{\phi,\gamma_{a,b}(t)}\colon t \in [0, 1], a, b \in Gl(\mathscr{H})^{+}\}, \end{aligned}$$

where $\mu = \|S\|\|S^{-1}\|$.

Remark 5.7. In [5], we obtained Clarkson's type inequality from 4.5 for the p-Schatten ideals with $1 \leqslant p < \infty$, i.e. with $\phi = \phi_p$.

6. The geometry of $\Delta^{1}_{\mathscr{J}}$

In this section, we give a geometric context to what has been previously presented. More precisely we prove that the curves $\gamma_{a,b}$ are minimal curves of a Finsler geometry for a manifold of positive and invertible operators.

6.1. Topological and differentiable structure of $\Delta^{1}_{\mathscr{J}}$

Given a s.n. ideal $\mathscr{J}_{\phi}$, which we denote from now on $\mathscr{J}$, we consider:

$$\widetilde{\mathscr{J}} = \{\lambda + X \in B(\mathscr{H})\colon \lambda \in \mathbb{C}, X \in \mathscr{J}\}.$$

There is a natural norm for this subspace

$$\|\lambda + X\|_{\tilde{\phi}} = |\lambda| + \|X\|_{\phi}.$$

Lemma 6.1. *Let $\lambda + X, \mu + Y \in \widetilde{\mathscr{J}}$. Then*

(1) $\|\lambda + X\| \leqslant \|\lambda + X\|_{\tilde{\phi}}$,
(2) $\|(\lambda + X)(\mu + Y)\|_{\tilde{\phi}} \leqslant \|\lambda + X\|_{\tilde{\phi}}\|\mu + Y\|_{\tilde{\phi}}$.

In particular, $(\widetilde{\mathscr{J}}, +, \cdot)$ is a Banach algebra.

Proof. One has the usual estimates

(1) $\|\lambda + X\| \leqslant |\lambda| + \|X\| \leqslant |\lambda| + \|X\|_{\phi} = \|\lambda + X\|_{\tilde{\phi}}$;

(2) $$\begin{aligned} \|(\lambda + X)(\mu + Y)\|_{\tilde{\phi}} &= \|\lambda\mu + \lambda Y + \mu X + XY\|_{\tilde{\phi}} = |\lambda\mu| + \|\lambda Y + \mu X + XY\|_{\phi} \\ &\leqslant |\lambda||\mu| + |\lambda|\|Y\|_{\phi} + |\mu|\|X\|_{\phi} + \|XY\|_{\phi} \\ &\leqslant |\lambda||\mu| + |\lambda|\|Y\|_{\phi} + |\mu|\|X\|_{\phi} + \|X\|_{\phi}\|Y\|_{\phi} \\ &= (|\lambda| + \|X\|_{\phi})(|\mu| + \|Y\|_{\phi}). \qquad \square \end{aligned}$$

The self-adjoint part of $\widetilde{\mathscr{J}}$ is

$$\widetilde{\mathscr{J}}^{\mathrm{sa}} = \{\lambda + X \in \widetilde{\mathscr{J}} : (\lambda + X)^* = \lambda + X\}.$$

Remark 6.2. (1) $(\widetilde{\mathscr{J}}, \|\cdot\|_{\tilde{\phi}})$ is the minimal unitization of $(\mathscr{J}, \|\cdot\|_{\phi})$.

(2) Note that the multiples of the identity $\lambda 1$ and the operators $X \in \mathscr{J}$ are linearly independent. Therefore

$$\lambda + X \in \widetilde{\mathscr{J}}^{\mathrm{sa}} \text{ if and only if } \lambda \in \mathbb{R}, \quad X^* = X.$$

Formally

$$\widetilde{\mathscr{J}} = \mathbb{C} \oplus \mathscr{J}, \quad \widetilde{\mathscr{J}}^{\mathrm{sa}} = \mathbb{R} \oplus \mathscr{J}^{\mathrm{sa}},$$

where $\mathscr{J}^{\mathrm{sa}}$ denotes the set of self-adjoint operators in $\mathscr{J}$.

Inside $\widetilde{\mathscr{J}}^{\mathrm{sa}}$, we consider

$$\Delta_\phi = \{\lambda + X \in \widetilde{\mathscr{J}} : \lambda + X > 0\} \subset Gl(\mathscr{H})^+$$

and

$$\Delta_\phi^1 = \{1 + X \in \widetilde{\mathscr{J}} : 1 + X > 0\}.$$

Apparently Δ_ϕ is an open subset of $\widetilde{\mathscr{J}}^{\mathrm{sa}}$, and therefore, a differentiable (analytic) submanifold.

The next step is to prove that Δ_ϕ^1 is a submanifold of Δ_ϕ. For this purpose, we consider

$$\theta : \Delta_\phi \to \mathbb{R}, \quad \theta(\lambda + X) = \lambda.$$

Lemma 6.3. *θ is a submersion.*

Proof. It is sufficient to show that $\mathrm{d}\theta_{\lambda+X}$ is surjective and $\ker(\mathrm{d}\theta_{\lambda+X})$ is complemented [15, Th. 2.2].

Since $\widetilde{\mathscr{J}}^{\mathrm{sa}}$ and $\mathbb{R}$ are Banach spaces and θ is a continuous linear map we get that $\mathrm{d}\theta_{\lambda+X} = \theta$.

Apparently, $\mathrm{d}\theta_{\lambda+X}$ is surjective and $\ker(\mathrm{d}\theta_{\lambda+X})$ has codimension 1 and hence is complemented. $\square$

It follows that Δ_ϕ^1 is a submanifold, since $\Delta_\phi^1 = \theta^{-1}(\{1\})$. These facts imply that, for $1 + X \in \Delta_\phi^1$, $(T\Delta_\phi^1)_{1+X}$ identifies with $\mathscr{J}^{\mathrm{sa}}$.

If $\mathscr{I}$ is a Banach algebra and an ideal in the algebra $B(\mathscr{H})$, then we denote by $Gl(\mathscr{H}, \mathscr{I})$ the subset of $Gl(\mathscr{H})$ consisting of those operators of the form $1 + a$ with $a \in \mathscr{I}$, i.e.

$$Gl(\mathscr{H}, \mathscr{I}) = \{1 + a \in Gl(\mathscr{H}) : a \in \mathscr{I}\} = \{b \in Gl(\mathscr{H}) : b - 1 \in \mathscr{I}\}.$$

The standard examples are when $\mathscr{I}$ is the ideal of compact operators $B_0(\mathscr{H})$ – in which case $Gl(\mathscr{H}, B_0(\mathscr{H}))$ is the so-called *Fredholm group* of $\mathscr{H}$ – when $\mathscr{I}$ is the ideal of Hilbert–Schmidt operators and when $\mathscr{I}$ is the ideal of trace class operators $B_1(\mathscr{H})$. The classical work here is [10]. There is a natural action of $Gl(\mathscr{H}, \mathscr{J})$ on Δ_ϕ^1, defined analogously to l in (4.1):

$$l : Gl(\mathscr{H}, \mathscr{J}) \times \Delta_\phi^1 \longrightarrow \Delta_\phi^1, \ l_g(1 + X) = g(1 + X)g^*. \tag{6.1}$$

This action is clearly differentiable and transitive, since if $1 + X, 1 + Y \in \Delta_\phi^1$ then

$$l_r(1 + X) = (1 + Y)$$

for $r = (1 + Y)^{\frac{1}{2}}(1 + X)^{-\frac{1}{2}} \in Gl(\mathscr{H}, \mathscr{J})$.

If $1 + Y \in \Delta_\phi^1$, we define the length of a tangent vector $X \in (T\Delta_\phi^1)_{1+Y}$ by

$$\|X\|_{\phi,1+Y} = \|(1 + Y)^{-\frac{1}{2}} X (1 + Y)^{-\frac{1}{2}}\|_\phi = \|l_{(1+Y)^{-1/2}}(X)\|_\phi.$$

By Proposition 4.1 the Finsler norm is invariant for the action of $Gl(\mathscr{H}, \mathscr{J})$.

6.2. Minimal curves

In this section, we study the existence of minimal curves for the Finsler metric just defined. The expression "minimal" is understood in terms of the length functional (or more generally q-energy functional). We prove that the interpolating curve $\gamma_{a,b}$ joining a with b is the minimun of the q-energy functional for $q \geqslant 1$. We observe that this curve looks formally equal to the geodesic between positive definitive matrices (regarded as a symmetric space, see [17]).

For a piecewise differentiable curve $\alpha\colon [0,1] \to \Delta^1_\phi$, one computes the *length* of the curve α by

$$L_\phi(\alpha) = \int_0^1 \|\dot\alpha(t)\|_{\phi,\alpha(t)}\,\mathrm{d}t.$$

Proposition 6.4. *Given a, b in Δ^1_ϕ, the curve $\gamma_{a,b}$ has length $\|\log(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|_\phi$.*

Proof. Since the group $Gl(H, \mathscr{J})$ acts isometrically and transitively on Δ^1_ϕ, it suffices to prove the theorem for $a = 1$. Then

$$\|\dot\gamma_{1,b}(t)\|_{\phi,\gamma_{1,b}(t)} = \|\log(b)b^t\|_{\phi,b^t} = \|b^{t/2}\log(b)b^{-t/2}\|_\phi = \|\log(b)\|_\phi,$$

because $\log(b)$ and b^t conmute for every $t \in \mathbb{R}$. $\square$

Definition 6.5. Let $a, b \in \Delta^1_\phi$. We denote

$$\Omega_{a,b} = \{\alpha\colon [0,1] \to \Delta^1_\phi\colon \alpha \text{ is a } \mathscr{C}^1 \text{ curve},\ \alpha(0) = a \text{ and } \alpha(1) = b\}.$$

As in classical differential geometry, we consider the geodesic distance between a and b (in the Finsler metric) defined by

$$d_\phi(a,b) = \inf\{L_\phi(\alpha)\colon \alpha \in \Omega_{a,b}\}.$$

The next step consists of showing that $\gamma_{a,b}$ are short curves, i.e. if $\delta \in \Omega_{a,b}$ then

$$L_\phi(\gamma_{a,b}) \leqslant L_\phi(\delta)$$

and hence

$$d_\phi(a,b) = \|\log(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|_\phi.$$

The proof of this fact requires some preliminaries.

We begin with the following inequalities :

Lemma 6.6 [11]. *Let A, B, X be Hilbert space operators with $A, B \geqslant 0$. For any unitarily invariant norm $|||\cdot|||$ we have*

$$|||A^{1/2}XB^{1/2}||| \leqslant \left|\left|\left|\int_0^1 A^tXB^{1-t}\,\mathrm{d}t\right|\right|\right| \leqslant \frac{1}{2}|||AX + XB|||. \tag{6.2}$$

The proof of the next inequality, called by Bhatia (in the context of matrices) the *exponential metric increasing property*, is based on a similar argument used in [3].

Proposition 6.7. *For all $X, Y \in \mathscr{I}^{\text{sa}}$*

$$\|Y\|_\phi \leqslant \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}\|_\phi,$$

where $dexp_X$ denotes the differential of the exponential map at X.

Proof. The proof is based on the inequality (6.2) and the well-known formula below:

$$dexp_X(Y) = \int_0^1 e^{tX} Y e^{(1-t)X}\, dt.$$

Let $X, Y \in \mathscr{I}^{\text{sa}}$. Write $Y = e^{\frac{X}{2}}(e^{-\frac{X}{2}} Y e^{-\frac{X}{2}})e^{\frac{X}{2}}$. Then using the inequalities (6.2) we obtain

$$\|Y\|_\phi \leqslant \left\| \int_0^1 e^{tX}(e^{-\frac{X}{2}} Y e^{-\frac{X}{2}}) e^{(1-t)X}\, dt \right\|_\phi = \left\| e^{-\frac{X}{2}} \int_0^1 e^{tX} Y e^{(1-t)X}\, dt e^{-\frac{X}{2}} \right\|_\phi$$
$$= \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}\|_\phi.$$

This proves the proposition. □

We are now ready to prove the main result in this section.

Theorem 6.8. *Let $a, b \in \Delta_\phi^1$, then $\gamma_{a,b}$ is the shortest curve joining them. So*

$$d_\phi(a, b) = \|\log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_\phi.$$

Proof. Since the group $Gl(H, \mathscr{I})$ acts isometrically and transitively on Δ_ϕ^1, it is sufficient to prove the statement for $a = 1$. Then

$\gamma_{1,b} = b^t = e^{t\log(b)}$ and $L_\phi(\gamma_{1,b}) = \|\log(b)\|_\phi$.

Let $\gamma \in \Omega_{1,b}$; so write $\gamma(t) = e^{\alpha(t)}$ we get

$$\|\gamma(t)^{-\frac{1}{2}} \dot{\gamma}(t) \gamma(t)^{-\frac{1}{2}}\|_\phi = \left\| e^{-\frac{\alpha(t)}{2}} \left(e^{\alpha(t)}\right)^{\cdot} e^{-\frac{\alpha(t)}{2}} \right\|_\phi = \left\| e^{-\frac{\alpha(t)}{2}} dexp_{\alpha(t)}(\dot{\alpha}(t)) e^{-\frac{\alpha(t)}{2}} \right\|_\phi$$
$$\geqslant \|\dot{\alpha}(t)\|_\phi.$$

Finally

$$L_\phi(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\phi,\gamma(t)} dt = \int_0^1 \|\gamma(t)^{-\frac{1}{2}} \dot{\gamma}(t) \gamma(t)^{-\frac{1}{2}}\|_\phi dt \geqslant \int_0^1 \|\dot{\alpha}(t)\|_\phi dt$$
$$\geqslant \left\| \int_0^1 \dot{\alpha}(t) dt \right\|_\phi = \|\alpha(t)|_0^1\|_\phi = \|\alpha(1) - \alpha(0)\|_\phi = \|\log(b)\|_\phi. \qquad \square$$

Remark 6.9. The geometrical result described above can be translated to the language of the operator entropy

$$S(a|b) = a^{1/2} \log(a^{-1/2} b a^{-1/2}) a^{1/2}$$

with $a, b \in Gl(\mathscr{H})^+$ defined by Fujii and Kamei [8]. Then

$$d_\phi(a, b) = \|S(a|b)\|_{\phi,a}$$

for $a, b \in \Delta_\phi^1$.

Corollary 6.10. *Let $a, b \in \Delta_\phi^1$ be commuting operators. Then the exponential function maps the line segment determined by $\log(a)$ and $\log(b)$ in $\mathcal{J}$ isometrically to the geodesic $\gamma_{a,b}$. In particular, $\|\log(a) - \log(b)\|_\phi = d_\phi(a, b)$.*

In particular for all real numbers t, s and for all $X \in \mathcal{J}$

$$\|tX - sX\|_\phi = d_\phi(\mathrm{e}^{tX}, \mathrm{e}^{sX}).$$

Thus the exponential map is distance-preserving on all rays through the origin in $\mathcal{J}$.

Proposition 6.11. *Given $a, b \in \Delta_\phi^1$, $g \in Gl(H, \mathcal{J})$ we get*

(1) $d_\phi(a, b) = d_\phi(a^{-1}, b^{-1})$.
(2) *For all $t \in \mathbb{R}$*

$$d_\phi(a, \gamma_{a,b}) = |t|\, d_\phi(a, b).$$

(3) *Invariance under the action by $Gl(H, \mathcal{J})$*

$$d_\phi(a, b) = d_\phi(gag^*, gbg^*).$$

Proof. (1) It is easy to see that $S(a|b) = -a^{\frac{1}{2}} \log\left(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\right) a^{\frac{1}{2}}$, as a consequence from $\log(1/t) = -\log(t)$. Then

$$\begin{aligned} d_\phi(a, b) &= \|\log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_\phi = \|a^{-\frac{1}{2}} S(a|b) a^{-\frac{1}{2}}\|_\phi \\ &= \| - a^{-\frac{1}{2}} a^{\frac{1}{2}} \log\left(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\right) a^{-\frac{1}{2}} a^{\frac{1}{2}}\|_\phi \\ &= \| - \log\left(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\right)\|_\phi = d_\phi(a^{-1}, b^{-1}). \end{aligned}$$

(2) It is obvious that $S(a|\gamma_{a,b}(t)) = tS(a|b)$, then

$$d_\phi(a, \gamma_{a,b}) = \|a^{-\frac{1}{2}} S(a|\gamma_{a,b}(t)) a^{-\frac{1}{2}}\|_\phi = |t|\, \|a^{-\frac{1}{2}} S(a|b) a^{-\frac{1}{2}}\|_\phi = |t| d_\phi(a, b).$$

(3) Note that if $\gamma_{a,b}$ is the geodesic joining a with b, then

$$\|g\dot{\gamma}_{a,b}(t)g^*\|_{\phi, g\gamma_{a,b}(t)g^*} = \|\dot{\gamma}_{a,b}(t)\|_{\phi, \gamma_{a,b}(t)}. \qquad \square$$

Definition 6.12. For $a, b \in \Delta_\phi^1$, we call the midpoint of a and b, and we denote by $m(a, b)$ (following the notation used in [13]) to

$$m(a, b) := \gamma_{a,b}(1/2).$$

By Proposition 6.11 and the last definition we have that:

(1) $m(a, b) = \gamma_{a,b}(1/2) = \gamma_{b,a}(1/2) = m(b, a)$.
(2) $d_\phi(a, m(a, b)) = \frac{1}{2} d_\phi(a, b) = \frac{1}{2} d_\phi(b, a) = d_\phi(b, m(b, a))$.

Definition 6.13. For every $q \in \mathbb{R} - \{0\}$ we define the *q-energy functional*

$$E_q : \Omega_{a,b} \to \mathbb{R}^+, \quad E_q(\alpha) := \int_0^1 \|\dot{\alpha}(t)\|_{\phi, \alpha(t)}^q \mathrm{d}t.$$

Remark 6.14. (1) For $q = 1$ we obtain the *length functional*, and for $q = 2$ we obtain the *energy functional*.

(2) For any curve α such that $\|\dot{\alpha}(t)\|_{\phi,\alpha(t)}$ is constant we have

$$E_q(\alpha) = (L_\phi(\alpha))^q = (E(\alpha))^{\frac{q}{2}}.$$

In Theorem 6.8, we proved that the curve between a and b minimizes the length functional. This fact is valid also for the q-energy functional (associated with $\Omega_{a,b}$) for $q \in (1, \infty)$.

Proposition 6.15. *Let $a, b \in \Delta_\phi^1$ and $q \in [1, \infty)$. Then the q-energy functional achieves its global minimun $d_\phi^q(a, b)$ precisely at $\gamma_{a,b}$.*

Proof. Now, let $\alpha \in \Omega_{a,b}$ and $q \in (1, \infty)$ then by Hölder's inequality

$$(L_\phi(\alpha))^q = \left(\int_0^1 \|\dot{\alpha}(t)\|_{\phi,\alpha(t)}\mathrm{d}t\right)^q \leqslant \int_0^1 \|\dot{\alpha}(t)\|_{\phi,\alpha(t)}^q\mathrm{d}t = E_q(\alpha).$$

On the other hand, $(L_\phi(\gamma_{a,b}))^q = E_q(\gamma_{a,b})$. This implies that

$$E_q(\gamma_{a,b}) = (L_\phi(\gamma_{a,b}))^q \leqslant (L_\phi(\alpha))^q \leqslant E_q(\alpha). \qquad \square$$

References

[1] E. Andruchow, G. Corach, M. Milman, D. Stojanoff, Geodesics and interpolation, Rev. Unión Mat. Argentina, 40 (3 and 4) (1997) 83–91.
[2] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Springer-Verlag, New York, 1976
[3] R. Bhatia, On the exponential metric increasing property, Linear Algebra Appl. 375 (2003) 211–220.
[4] A. Calderón, Intermediate spaces and interpolation, the complex method, Stud. Math. 24 (1964) 113–190.
[5] C. Conde, Geometric interpolation in p-Schatten class, J. Math. Anal. Appl. 340 (2) (2008) 920–931.
[6] G. Corach, H. Porta, L. Recht, An operator inequality, Linear Algebra Appl. 142 (1990) 153–158.
[7] R. Curto, Structural properties of elementary operators, in: M. Mathieu (Ed.), Proc. Int. Workshop, Blaubeurn, Germany, 1991, Elementary Operators and Applications, 1992, pp. 3–52.
[8] J. Fujii, E. Kamei, R. Nakamoto, Relative entropy in noncommutative information theory, Math. Japonica 34 (1989) 341–348.
[9] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, American Mathematical Society, Providence, RI, 1969.
[10] P. de la Harpe, Classical Banach–Lie Algebras and Banach–Lie Groups of Operators in Hilbert Space, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1972.
[11] F. Hiai, H. Kosaki, Comparison of various means of operators, J. Funct. Anal. 163 (1999) 300–323.
[12] F. Kittaneh, On some operator inequalities, Linear Algebra Appl. 208–209 (1994) 19–28.
[13] J. Jost, Nonpositive Curvature: Geometric and Analytic Aspects, Birkhuser Verlag, 1997.
[14] S. Krein, J. Petunin, E. Semenov, Interpolation of Linear Operators, Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1982.
[15] S. Lang, Differential and Riemannian Manifolds, Springer, Berlin, 1995.
[16] G. Larotonda, Norm Inequalities in Operator Ideals, Preprint 2007.
[17] G. Mostow, Some new decomposition theorems for semi-simple groups, Mem. Amer. Math. Soc. 14 (1955) 31–54.
[18] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. II, Academic Press, Orlando, FL, 1975.
[19] A. Seddik, On some operator norm inequalities, Linear Algebra Appl. 389 (2004) 183–187.
[20] J. Stampfli, The norm of a derivation, Pacific J. Math. 33 (3) (1970) 737–747.