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Normal operators and inequalities in norm ideals

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In this work we characterize normal invertible operators via inequalities with unitarily invariant norm of elementary operators.

1. Introduction

Let $(B(H), \|.\|)$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space H with the usual norm. We denote by $Gl(H)$ the group of invertible elements of $B(H)$ and by $U(H)$ the unitary operators.

In [8], Nakamoto proved that a bounded linear operator X on H is normal if and only if $\|XY - YX\|_2 = \|X^*Y - YX^*\|_2$ for every $Y \in B_2(H)$ (Hilbert–Schmidt class).

In [2], Corach et al. proved that if S is invertible and selfadjoint in $B(H)$, then for all $X \in B(H)$

$$\|SXS^{-1} + S^{-1}XS\| \geqslant 2\|X\|, \tag{1}$$

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this inequality is called in the literature as CPR inequality. For sake of simplicity we denote by $\Phi_S(X) = SXS^{-1} + S^{-1}XS$. In [9], Seddik obtained the following characterization: an invertible S is a non zero complex multiple of some selfadjoint operator if and only if $\|\Phi_S(X)\| \geqslant 2\|X\|$ for all $X \in B(H)$, i.e. the author characterizes the operators S for which the CPR inequality holds.

In [5], Kittaneh proved that if R, S, X are operators in $B(H)$ such that R and S are invertible and $\mathcal{I}$ is a norm ideal then

$$\|R^*XS^{-1} + R^{-1}XS^*\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}}. \tag{2}$$

The author proves this inequality as a consequence of the arithmetic–geometric mean inequality (for different proofs and several applications of this inequality the reader is referred to [1,4,7]) that states

$$\|AA^*X + XBB^*\|_{\mathcal{I}} \geqslant 2\|A^*XB\|_{\mathcal{I}}, \tag{3}$$

for every $A, B, X \in B(H)$.

If $S = R$ is an invertible selfadjoint operator in $B(H)$ the inequality (2) implies (1) in any normed ideal $\mathcal{I}$, more exactly: for all $X \in \mathcal{I}$ we have

$$2\|X\|_{\mathcal{I}} \leqslant \|\Phi_S(X)\|_{\mathcal{I}}. \tag{4}$$

In [6], the authors to ask whether the same characterization obtained by Seddik is true for other unitarily invariant norm. They proved that S is necessarily a normal operator if $2\|X\|_{\mathcal{I}} \leqslant \|\Phi_S(X)\|_{\mathcal{I}}$ for all $X \in B(H)$, with rank one (see Corollary 2.2).

The objective of this work, motivated by Theorem 2.1 and Corollary 2.2 in [6], is to obtain a characterization of normal invertible operators in $B(H)$ (or some subclass of them) using unitarily invariant norms.

2. Preliminaries

We recall that $\mathcal{I}$ is a norm ideal of $B(H)$ if $\mathcal{I}$ is a two-sided ideal of $B(H)$ and a Banach space with respect to the norm $\|.\|_{\mathcal{I}}$ satisfying:

1. $\|XTY\|_{\mathcal{I}} \leqslant \|X\|\|T\|_{\mathcal{I}}\|Y\|$ for $T \in \mathcal{I}$ and $X, Y \in B(H)$,
2. $\|X\|_{\mathcal{I}} = \|X\|$ if X is the rank one.

In particular, condition 1. implies that the norm $\|.\|_{\mathcal{I}}$ is unitarily invariant, that is $\|UXV^*\|_{\mathcal{I}} = \|X\|_{\mathcal{I}}$ for $X \in \mathcal{I}$ and any $U, V \in U(H)$. If $\mathcal{I}$ is a proper two-sided ideal of $B(H)$, it is well-known that $B_{0,0}(H) \subseteq \mathcal{I} \subseteq B_0(H)$, where $B_{0,0}(H)$ is the ideal of finite rank operators and $B_0(H)$ the set of compact operators.

The most known examples of norm ideals of $B(H)$ are the called p-Schatten class with $p \geqslant 1$ defined by $B_p(H) = \{X \in B_0(H) : \{s_j(X)\} \in l^p\}$, where $\{s_j(X)\}$ denotes the sequence of singular values of X, rearranged such that $s_1(X) \geqslant s_2(X) \geqslant \cdots$ with multiplicies counted, with norm given by $\|X\|_p = (\sum s_j(X)^p)^{1/p}$. When $p = \infty$, the norm $\|.\|_\infty$ coincides with the usual norm $\|X\| = s_1(X)$. For a complete account of the theory of unitarily invariant norms the reader is referred to [3].

For sake of completness, we recall the following statement of Magajna et al. in [6] that we will use in the following section.

Theorem 2.1. *Let $A, B \in B(H)$ be positive invertible operators. Then the inequality*

$$\|AXA^{-1}\| + \|B^{-1}XB\| \geqslant 2\|X\|, \tag{5}$$

holds for all operators $X \in B(H)$ of rank 1 if and only if $B = f(A)$, where $f : \sigma(A) \to \sigma(B)$ is a strictly increasing positive continuous function satisfying

$$\frac{f(s)}{t} \leqslant \frac{f(t) - f(s)}{t - s} \leqslant \frac{f(t)}{s}, \tag{6}$$

for all $s < t$ in the spectrum of A. In particular, for two unitarily equivalent operators A and B, (5) implies that $A = B$. Each function f satisfying (6) is differentiable at any interior point of $\sigma(A)$ and if $\sigma(A)$ is an interval, then f is of the form $f(t) = ct$ for some constant c.

Note that if $X \in \mathcal{I}$ is the rank 1, by condition 2. we get

$$\|AXA^{-1}\|_{\mathcal{I}} + \|B^{-1}XB\|_{\mathcal{I}} = \|AXA^{-1}\| + \|B^{-1}XB\|,$$

and if futhermore $A = B = S \in Gl(H)$ satisfies the inequality (4), we get

$$\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} \geqslant \|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}},$$

then by Corollary 2.2 in [6], S is necessarily a normal operator.

Now, we ask which are all invertible operators S that hold the inequality

$$\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}}$$

for every $X \in \mathcal{I}$.

3. Main results

Theorem 3.1. *Let $S \in Gl(H)$ and $\mathcal{I}$ a norm ideal. Then the following conditions are equivalent*:

1. *S is normal,*
2. *$\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} = \|S^*XS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS^*\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$,*
3. *$\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} \geqslant \|S^*XS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS^*\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$,*
4. *$\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$,*
5. *$\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$, with rank 1.*

Proof. 1. $\Rightarrow$ 2. By hypothesis, the polar decomposition of S is $S = PU = UP$ with P positive and $U \in U(H)$, then

$$\begin{aligned}\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} &= \|UPXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XPU\|_{\mathcal{I}} = \|PXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XP\|_{\mathcal{I}} \\ &= \|U^*PXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XPU^*\|_{\mathcal{I}} = \|S^*XS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS^*\|_{\mathcal{I}}.\end{aligned}$$

2. $\Rightarrow$ 3. The implication is trivial.

3. $\Rightarrow$ 4. Let $X \in \mathcal{I}$, then

$$\begin{aligned}\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} &\geqslant \|S^*XS^{-1} + S^{-1}XS^*\|_{\mathcal{I}} = \|PU^*XP^{-1}U^* + P^{-1}U^*XPU^*\|_{\mathcal{I}} \\ &= \|PU^*XP^{-1} + P^{-1}U^*XP\|_{\mathcal{I}} = \|\Phi_P(U^*X)\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}}\end{aligned}$$

in the last inequality we use (4).

4. $\Rightarrow$ 5. The implication is trivial.

5. $\Rightarrow$ 1. We consider $X \in \mathcal{I}$ with rank 1, $A = (S^*S)^{1/2}$ and $B = (SS^*)^{1/2}$, then

$$\|AXA^{-1}\|_{\mathcal{I}} + \|B^{-1}XB\|_{\mathcal{I}} = \|UAXA^{-1}U^*\|_{\mathcal{I}} + \|U^*B^{-1}XBU\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}},$$

where $S = UA$ and $S^* = U^*B$. Since A and B are unitarily equivalent, we have by Theorem 2.1 in [6] that $A = B$, hence S is normal. $\square$

Remark 3.2. The previous theorem is a generalization of Proposition 5 in [10].

Specializing the previous theorem to the Hilbert–Schmidt class and using the Nakamoto's characterization of normal operators we obtain the following statement.

Corollary 3.3. *If $S \in Gl(H)$ the following conditions are equivalent*:

1. *S is normal,*
2. *$\|SXS^{-1}\|_2 + \|S^{-1}XS\|_2 = \|S^*XS^{-1}\|_2 + \|S^{-1}XS^*\|_2$ for every $X \in B_2(H)$,*
3. *$\|SXS^{-1}\|_2 + \|S^{-1}XS\|_2 \geqslant 2\|X\|_2$ for every $X \in B_2(H)$,*

4. $\|SXS^{-1}\|_2 + \|S^{-1}XS\|_2 \geqslant 2\|X\|_2$ *for every* $X \in B_2(H)$, *with rank 1*,
5. $\|SX - XS\|_2 = \|S^*X - XS^*\|_2$ *for every* $X \in B_2(H)$.

Now, we shall characterize another subclass of normal operators. The following characterization of the nonzero complex multiple of selfadjoint operators is easily deduced (see [6], Theorem 2.5). More precisely,

Proposition 3.4. *Let* $S \in Gl(H)$ *and* $\mathcal{I}$ *a norm ideal. Then the following conditions are equivalent*

1. γS *is selfadjoint for some* $\gamma \in \mathbb{C} - \{0\}$,
2. $\inf_{t>0} \|tSXS^{-1} + \frac{1}{t}S^{-1}XS\|_{\mathcal{I}} = \inf_{r>0} \|rS^*XS^{-1} + \frac{1}{r}S^{-1}XS^*\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$,
3. $\inf_{t>0} \|tSXS^{-1} + \frac{1}{t}S^{-1}XS\|_{\mathcal{I}} \geqslant \inf_{r>0} \|rS^*XS^{-1} + \frac{1}{r}S^{-1}XS^*\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$,
4. $\inf_{t>0} \|tSXS^{-1} + \frac{1}{t}S^{-1}XS\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$, *with rank 1*.

Proof. The implications 1. $\Rightarrow$ 2. and 2. $\Rightarrow$ 3. are trivial.

3. $\Rightarrow$ 4. From the polar decomposition of $S = UP = QU$ with $P, Q > 0$ and $U \in U(H)$, we get for every $X \in \mathcal{I}$ with rank one

$$\begin{aligned}\inf_{t>0} \|tSXS^{-1} + \frac{1}{t}S^{-1}XS\|_{\mathcal{I}} &\geqslant \inf_{r>0} \|rS^*XS^{-1} + \frac{1}{r}S^{-1}XS^*\|_{\mathcal{I}} \\ &= \inf_{r>0} \|rPU^*XP^{-1}U^* + \frac{1}{r}P^{-1}U^*XPU^*\|_{\mathcal{I}} \\ &= \inf_{r>0} \|rP(U^*X)P^{-1} + \frac{1}{r}P^{-1}(U^*X)P\|_{\mathcal{I}} \geqslant 2\|X\|_{\mathcal{I}}\end{aligned}$$

in the last inequality we use the inequality (2).

4. $\Rightarrow$ 1. The implication follows of [6]. □

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