CLARKSON-MCCARTHY INTERPOLATED INEQUALITIES IN FINSLER NORMS

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Abstract. We apply the complex interpolation method to prove that, given two spaces $B^{(n)}_{p_0,a;s_0}, B^{(n)}_{p_1,b;s_1}$ of n-tuples of operators in the p-Schatten class of a Hilbert space H, endowed with weighted norms associated to positive and invertible operators a and b of $B(H)$ then, the curve of interpolation $(B^{(n)}_{p_0,a;s_0}, B^{(n)}_{p_1,b;s_1})_{[t]}$ of the pair is given by the space of n-tuples of operators in the p_t-Schatten class of H, with the weighted norm associated to the positive invertible element $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$.

1. Introduction

In [6], J. Clarkson introduced the concept of uniformly convexity in Banach spaces and obtain that spaces L_p (or l_p) are uniformly convex for $p > 1$ throughout the following inequalities

$$2(\|f\|_p^p + \|g\|_p^p) \le \|f-g\|_p^p + \|f+g\|_p^p \le 2^{p-1}(\|f\|_p^p + \|g\|_p^p), \tag{1.1}$$

Let $(B(H), \|.\|)$ denote the algebra of bounded operators acting on a complex and separable Hilbert space H, $Gl(H)$ the group of invertible elements of $B(H)$ and $Gl(H)^+$ the set of all positive elements of $Gl(H)$.

If $X \in B(H)$ is compact we denote by $\{s_j(X)\}$ the sequence of singular values of X (decreasingly ordered). For $0 < p < \infty$, let

$$\|X\|_p = (\sum s_j(X)^p)^{1/p},$$

and the linear space

$$B_p(H) = \{X \in B(H) : \|X\|_p < \infty\},$$

For $1 \le p < \infty$, this space is called the $p-Schatten\, class$ of $B(H)$ (to simplify notation we use B_p) and by convention $\|X\| = \|X\|_\infty = s_1(X)$. A reference for this subject is [9].

C. McCarthy proved in [14], among several other results, the following inequalities for p-Schatten norms of Hilbert space operators:

$$2(\|A\|_p^p + \|B\|_p^p) \le \|A-B\|_p^p + \|A+B\|_p^p \le 2^{p-1}(\|A\|_p^p + \|B\|_p^p), \tag{1.2}$$

for $2 \le p < \infty$, and

$$2^{p-1}(\|A\|_p^p + \|B\|_p^p) \le \|A-B\|_p^p + \|A+B\|_p^p \le 2(\|A\|_p^p + \|B\|_p^p), \tag{1.3}$$

for $1 \le p \le 2$.

These are non-commutative versions of Clarkson's inequalities. These estimates have been found to be very powerful tools in operator theory (in particular they imply the uniform convexity of B_p for $1 < p < \infty$) and in mathematical physics (see [16]).

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M. Klaus has remarked that there is a simple proof of the Clarkson-McCarthy inequalities which results from mimicking the proof that Boas [4] gave of the Clarkson original inequalities via the complex interpolation method.

In a previous work [7], motivated by [1], we studied the effect of the complex interpolation method on $B_p^{(n)}$ (this set will be defined below) for $p, s \geq 1$ and $n \in \mathbb{N}$ with a Finsler norm associated with $a \in Gl(H)^+$:

$$\|X\|_{p,a;s} := \|a^{-1/2} X a^{-1/2}\|_p^s.$$

From now on, for sake of simplicity, we denote with lower case letters the elements of $Gl(H)^+$.

As a by-product, we obtained Clarkson's type inequalities using the Klaus idea with the linear operator $T_n : B_p^{(n)} \longrightarrow B_p^{(n)}$ given by

$$T_n(\bar{X}) = (T_n(X_1, ..., X_n) = (\sum_{j=1}^{n} X_j, \sum_{j=1}^{n} \theta_j^1 X_j, ..., \sum_{j=1}^{n} \theta_j^{n-1} X_j),$$

where $\theta_1, ..., \theta_n$ are the n roots of unity.

Recently, Kissin in [12], motivated by [3], obtain analogues of Clarkson-McCarthy inequalities for n-tuples of operators from Schatten ideals. In this work the author consider H^n the orthogonal sum of n copies of the Hilbert space H and for each operator $R \in B(H^n)$ can be represented as an $n \times n$ block-matrix operator $R = (R_{jk})$ with $R_{jk} \in B(H)$ and the linear operator $T_R : B_p^{(n)} \to B_p^{(n)}$ defined by $T_R(\overline{A}) = R\overline{A}$. Finally we remark that the works [3] and [11] are generalizations of [10].

In these notes we obtain inequalities for the linear operator T_R in the Finsler norm $\|.\|_{p,a;s}$ as by-product of the complex interpolation method and Kissin's inequalities.

2. Geometric Interpolation

We follow the notation used in [2] and we refer to [13] and [5] for details on the complex interpolation method. For completeness, we recall the classical Calderón-Lions theorem.

Theorem 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ two compatible couples. Assume that T is a linear operator from $\mathcal{X}_j$ to $\mathcal{Y}_j$ bounded by M_j, $j = 0, 1$. Then for $t \in [0, 1]$*

$$\|T\|_{\mathcal{X}_{[t]} \to \mathcal{Y}_{[t]}} \leq M_0^{1-t} M_1^t.$$

Here and subsequently, let $1 \leq p < \infty, n \in \mathbb{N}, s \geq 1$, $a \in Gl(H)^+$ and

$$B_p^{(n)} = \{\overline{A} = (A_1, \dots, A_n)^t : A_i \in B_p\},$$

(where with t we denote the transpose of the n-tuple) endowed with the norm

$$\|\overline{A}\|_{p,a;s} = (\|A_1\|_{p,a}^s + ... + \|A_n\|_{p,a}^s)^{1/s},$$

and $\mathbb{C}^n$ endowed with the norm

$$|(a_0, ..., a_{n-1})|_s = (|a_0|^s + ... + |a_{n-1}|^s)^{1/s}.$$

From now on, we denote with $B_{p,a;s}^{(n)}$ the space $B_p^{(n)}$ endowed with the norm $\|(., \dots, .)\|_{p,a;s}$.

From the Calderón-Lions interpolation theory we get that for $p_0, p_1, s_0, s_1 \in [1, \infty)$

$$(B_{p_0,1;s_0}^{(n)}, B_{p_1,1;s_1}^{(n)})_{[t]} = B_{p_t,1;s_t}^{(n)}, \tag{2.1}$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{s_t} = \frac{1-t}{s_0} + \frac{t}{s_1}$$

Note that for $p = 2$, (1.2) and (1.3) both reduce to the *parallelogram law*

$$2(\|A\|_2^2 + \|B\|_2^2) = \|A - B\|_2^2 + \|A + B\|_2^2\,,$$

while that for the cases $p = 1, \infty$ these inequalities follows from the triangle inequality for B_1 and $B(H)$ respectively. Then the inequalities (1.2) and (1.3) can be proved (for $n = 2$ via Th. 2.1) by interpolation between the previous elementary cases with the linear operator $T_2 : B^{(2)}_{p,1;p} \longrightarrow B^{(2)}_{p,1;p}$, $T_2(\overline{A}) = (A_1 + A_2, A_1 - A_2)^t$ as observed Klaus.

In this section, we generalize (2.1) for the Finsler norms $\|(.,..,.)\|_{p,a;s}$. In [7], we have obtained this extension for the particular case in that $p_0 = p_1 = p$ and $s_0 = s_1 = s$. For sake of completeness, we recall this result

Theorem 2.2. *([7], Th. 3.1.) Let $a, b \in Gl(H)^+, 1 \le p, s < \infty$, $n \in \mathbb{N}$ and $t \in (0, 1)$. Then*

$$(B^{(n)}_{p,a;s}, B^{(n)}_{p,b;s})_{[t]} = B^{(n)}_{p,\gamma_{a,b}(t);s},$$

where $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$.

Remark 2.1. Note that when a and b commute the curve is given by $\gamma_{a,b}(t) = a^{1-t}b^t$. The previous corollary tells us that the interpolating space, $B_{p,\gamma_{a,b}(t);s}$ can be regarded as a weighted p-Schatten space with weight $a^{1-t}b^t$ (see [2], Th. 5.5.3).

We observe that the curve $\gamma_{a,b}$ looks formally equal to the geodesic (or shortest curve) between positive definitive matrices ([15]), positive invertible elements of a C*-algebra ([8]) and positive invertible operators that are perturbations of the p-Schatten class by multiples of the identity ([7]).

There is a natural action of $Gl(H)$ on $B^{(n)}_p$, defined by

$$l : Gl(H) \times B^{(n)}_p \longrightarrow B^{(n)}_p,\ l_g(\overline{A}) = (gA_1g^*, \dots , gA_ng^*)^t. \tag{2.2}$$

Proposition 2.3. *([7], Prop. 3.1.) The norm in $B^{(n)}_{p,a;s}$ is invariant for the action of the group of invertible elements. By this we mean that for each $\overline{A} \in B^{(n)}_p$, $a \in Gl(H)^+$ and $g \in Gl(H)$, we have*

$$\|\overline{A}\|_{p,a;s} = \|l_g(\overline{A})\|_{p,gag^*;s}\,.$$

Now, we state the main result of this paper, the general case $1 \le p_0, p_1, s_0, s_1 < \infty$.

Theorem 2.4. *Let $a, b \in Gl(H)^+, 1 \le p_0, p_1, s_0, s_1 < \infty$, $n \in \mathbb{N}$ and $t \in (0, 1)$. Then*

$$(B^{(n)}_{p_0,a;s_0}, B^{(n)}_{p_1,b;s_1})_{[t]} = B^{(n)}_{p_t,\gamma_{a,b}(t);s_t},$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{s_t} = \frac{1-t}{s_0} + \frac{t}{s_1}.$$

Proof. In order to simplify, we will only consider the case $n = 2$ and we omit the transpose. The proof below works for n-tuples ($n \ge 3$) with obvious modifications.

By the previous proposition, $\|(X_1, X_2)\|_{[t]}$ is equal to the norm of $a^{-1/2}(X_1, X_2)a^{-1/2}$ interpolated between the norms $\|(., .)\|_{p_0,1;s_0}$ and $\|(., .)\|_{p_1,c;s_1}$. Consequently it is sufficient to prove our statement for these two norms.

Let $t \in (0, 1)$ and $(X_1, X_2) \in B^{(2)}_{p_t}$ such that $\|(X_1, X_2)\|_{p_t,c^t;s_t} = 1$, and define

$$g(z) = (U_1|c^{\frac{z}{2}}c^{-\frac{t}{2}}X_1c^{\frac{-t}{2}}c^{\frac{z}{2}}|^{\lambda(z)}, U_2|c^{\frac{z}{2}}c^{-\frac{t}{2}}X_2c^{-\frac{t}{2}}c^{\frac{z}{2}}|^{\lambda(z)}) = (g_1(z), g_2(z)),$$

where $\lambda(z) = p_t(\frac{1-z}{p_0} + \frac{z}{p_1})s_t(\frac{1-z}{s_0} + \frac{z}{s_1})$ and $X_i = U_i|X_i|$ is the polar decomposition of X_i for $i = 1, 2$.

Then for each $z \in S$, $g(z) \in B^{(2)}_{p_0} + B^{(2)}_{p_1}$ and

$$\begin{aligned}\|g(iy)\|^{s_0}_{p_0,1;s_0} &= (\sum_{k=1}^{2} \|U_k|c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_k c^{-\frac{t}{2}} c^{\frac{iy}{2}}|^{\lambda(iy)}\|^{s_0}_{p_0}) \\ &\leq (\sum_{k=1}^{2} \|c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_k c^{-\frac{t}{2}} c^{\frac{iy}{2}}\|^{p_t}_{p_t}) \\ &\leq (\sum_{k=1}^{2} \|X_k\|^{p_t}_{p_t,c^t}) = 1\end{aligned}$$

and

$$\|g(1+iy)\|^{s_1}_{p_1,c;s_1} \leq (\sum_{k=1}^{2} \|X_k\|^{p_t}_{p_t,c^t}) = 1.$$

Since $g(t) = (X_1, X_2)$ and $g = (g_1, g_2) \in \mathcal{F}(B^{(2)}_{p_0,1;s_0}, B^{(2)}_{p_1,c;s_1})$ we have $\|(X_1, X_2)\|_{[t]} \leq 1$. Thus we have shown that

$$\|(X_1, X_2)\|_{[t]} \leq \|(X_1, X_2)\|_{p_t,c^t;s_t}.$$

To prove the converse inequality, let $f = (f_1, f_2) \in \mathcal{F}(B^{(2)}_{p_0,1;s_0}, B^{(2)}_{p_1,c;s_1})$; $f(t) = (X_1, X_2)$ and $Y_1, Y_2 \in B_{0,0}(H)$ (the set of finite-rank operators) with $\|Y_k\|_{q_t} \leq 1$, where q_t is the conjugate exponent for $1 < p_t < \infty$ (or a compact operator and $q = \infty$ if $p = 1$). For $k = 1, 2$, let

$$g_k(z) = c^{-\frac{z}{2}} Y_k c^{-\frac{z}{2}}.$$

Consider the function $h : S \to (\mathbb{C}^2, |(.,.)|_{s_t})$,

$$h(z) = (tr(f_1(z)g_1(z)), tr(f_2(z)g_2(z))).$$

Since $f(z)$ is analytic in $\overset{\circ}{S}$ and bounded in S, then h is analytic in $\overset{\circ}{S}$ and bounded in S, and

$$h(t) = (tr(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1), tr(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2)) = (h_1(t), h_2(t)).$$

By Hadamard's three line theorem, applied to h and the Banach space $(\mathbb{C}^2, |(.,.)|_{s_t})$, we have

$$|h(t)|_{s_t} \leq max\{\sup_{y\in\mathbb{R}} |h(iy)|_{s_t}, \sup_{y\in\mathbb{R}} |h(1+iy)|_{s_t}\}.$$

For $j = 0, 1$,

$$\begin{aligned}\sup_{y\in\mathbb{R}} |h(j+iy)|_{s_t} &= \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} |tr(f_k(j+iy)g_k(j+iy))|^{s_t})^{1/s_t} \\ &= \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} |tr(c^{-j/2} f_k(j+iy) c^{-j/2} g_k(iy))|^{s_t})^{1/s_t} \\ &\leq \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} \|f_k(j+iy)\|^{s_t}_{p,c^j})^{1/s_t} \leq \|f\|_{\mathcal{F}(B^{(2)}_{p_0,1;s_0}, B^{(2)}_{p_1,c;s_1})},\end{aligned}$$

then

$$\begin{aligned}\|X_1\|^{s_t}_{p_t,c^t}+\|X_2\|^{s_t}_{p_t,c^t} &= \sup_{\substack{\|Y_1\|_{q_t}\le 1, Y_1\in B_{00}(H)\\ \|Y_2\|_{q_t}\le 1, Y_2\in B_{00}(H)}} \{|h_1(t)|^{s_t}+|h_2(t)|^{s_t}\}\\ &= \sup_{\substack{\|Y_1\|_{q_t}\le 1, Y_1\in B_{00}(H)\\ \|Y_2\|_{q_t}\le 1, Y_2\in B_{00}(H)}} |h(t)|^{s_t}_{s_t} \le \|f\|^{s_t}_{\mathcal{F}(B^{(2)}_{p_0,1;s_0},B^{(2)}_{p_1,c;s_1})}.\end{aligned}$$

Since the previous inequality is valid for each $f\in\mathcal{F}(B^{(2)}_{p_0,1;s_0},B^{(2)}_{p_1,c;s_1})$ with $f(t)=(X_1,X_2)$, we have

$$\|(X_1,X_2)\|_{p_t,c^t;s_t}\le\|(X_1,X_2)\|_{[t]}.$$

□

In the special case that $p_0=p_1=p$ and $s_0=s_1=s$ we obtain the Theorem 2.2.

3. Clarkson-Kissin's type inequalities

Bhatia and Kittaneh [3] proved that if $2\le p<\infty$, then

$$n^{\frac{2}{p}}\sum_{j=1}^n\|A_j\|_p^2\le\sum_{j=1}^n\|B_j\|_p^2\le n^{2-\frac{2}{p}}\sum_{j=1}^n\|A_j\|_p^2. \tag{3.1}$$

$$n\sum_{j=1}^n\|A_j\|_p^p\le\sum_{j=1}^n\|B_j\|_p^p\le n^{p-1}\sum_{j=1}^n\|A_j\|_p^p. \tag{3.2}$$

(for $0<p\le 2$, these two inequalities are reversed) where $B_j=\sum_{k=1}^n\theta_k^jA_k$ with $\theta_1,...,\theta_n$ the n roots of unity.

If we interpolate these inequalities we obtain that

$$n^{\frac{1}{p}}(\sum_{j=1}^n\|A_j\|_p^{s_t})^{\frac{1}{s_t}}\le(\sum_{j=1}^n\|B_j\|_p^{s_t})^{\frac{1}{s_t}}\le n^{(1-\frac{1}{p})}(\sum_{j=1}^n\|A_j\|_p^{s_t})^{\frac{1}{s_t}}, \tag{3.3}$$

where

$$\frac{1}{s_t}=\frac{1-t}{2}+\frac{t}{p}.$$

Dividing by n^{s_t}, we obtain

$$n^{\frac{1}{p}}(\frac{1}{n}\sum_{j=1}^n\|A_j\|_p^{s_t})^{\frac{1}{s_t}}\le(\frac{1}{n}\sum_{j=1}^n\|B_j\|_p^{s_t})^{\frac{1}{s_t}}\le n^{(1-\frac{1}{p})}(\frac{1}{n}\sum_{j=1}^n\|A_j\|_p^{s_t})^{\frac{1}{s_t}}. \tag{3.4}$$

This inequality can be rephrased as follows, if $\mu\in[2,p]$ then

$$n^{\frac{1}{p}}(\frac{1}{n}\sum_{j=1}^n\|A_j\|_p^{\mu})^{\frac{1}{\mu}}\le(\frac{1}{n}\sum_{j=1}^n\|B_j\|_p^{\mu})^{\frac{1}{\mu}}\le n^{(1-\frac{1}{p})}(\frac{1}{n}\sum_{j=1}^n\|A_j\|_p^{\mu})^{\frac{1}{\mu}}. \tag{3.5}$$

In each of the following statements $R\in Gl(H^n)$ and we denote by T_R the linear operator

$$T_R:B_p^{(n)}\longrightarrow B_p^{(n)}\qquad T_R(\overline{A})=R\overline{A}=(B_1,...,B_n)^t,$$

with $B_j=\sum_{k=1}^nR_{jk}A_k$ and $\alpha=\|R^{-1}\|,\beta=\|R\|$ (we use the same symbol to denote the norm in $B(H)$ and $B(H^n)$).

We observe that if the norm of T_R is at most M when

$$T_R:(B_p^{(n)},\|(.,...,.)\|_{p,1,s})\to(B_p^{(n)},\|(.,...,.)\|_{p,1,r}),$$

then if we consider the operator T_R between the spaces

$$T_R : (B_p^{(n)}, \|(., ..., .)\|_{p,a,s}) \to (B_p^{(n)}, \|(., ..., .)\|_{p,b,r}),$$

its norm is at most $F(a,b)M$ with

$$F(a,b) = \begin{cases} \min\{\|b^{-1}\|\|a\|, \|a^{1/2}b^{-1}a^{1/2}\|\|a^{-1}\|\|a\|\} & \text{if } a \neq b, \\ \|a^{-1}\|\|a\| & \text{if } a = b. \end{cases}$$

Remark 3.1. If $a^{-1/2} \in Gl(H)$ commutes with $R \in B(H^n)$, by this we mean that $a^{-1/2}$ commutes with R_{jk} for all $1 \leq j, k \leq n$, then F reduced to

$$F(a,b) = \begin{cases} \min\{\|b^{-1}\|\|a\|, \|a^{1/2}b^{-1}a^{1/2}\|\} = \|a^{1/2}b^{-1}a^{1/2}\| & \text{if } a \neq b, \\ 1 & \text{if } a = b. \end{cases}$$

In [12], Kissin proved the following Clarkson's type inequalities for the n-tuples $\overline{A} \in B_p^{(n)}$. If $2 \leq p < \infty$ and $\lambda, \mu \in [2,p]$, or if $0 < p \leq 2$ and $\lambda, \mu \in [p,2]$, then

$$n^{-f(p)}\alpha^{-1}(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{\mu})^{\frac{1}{\mu}} \leq (\frac{1}{n}\sum_{j=1}^{n}\|B_j\|_p^{\lambda})^{\frac{1}{\lambda}} \leq n^{f(p)}\beta(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{\mu})^{\frac{1}{\mu}}, \quad (*)$$

where $f(p) = |\frac{1}{p} - \frac{1}{2}|$.

Remark 3.2. This result extends the results of Bhatia and Kittaneh proved for $\mu = \lambda = 2$ or p and $R = (R_{jk})$ where

$$R_{jk} = e^{(i\frac{2\pi(j-1)(k-1)}{n})}1.$$

We use the inequalities $(*)$ and the interpolation method to obtain the following inequalities.

Theorem 3.1. *Let $a, b \in Gl(H)^+, \overline{A} \in B_p^{(n)}, 1 \leq p < \infty$ and $t \in [0,1]$, then*

$$\tilde{k}(\sum_{j=1}^{n}\|A_j\|_{p,a}^{\mu})^{\frac{1}{\mu}} \leq (\sum_{j=1}^{n}\|B_j\|_{p,\gamma_{a,b}(t)}^{\lambda})^{\frac{1}{\lambda}} \leq \tilde{K}(\sum_{j=1}^{n}\|A_j\|_{p,a}^{\mu})^{\frac{1}{\mu}} \tag{3.6}$$

where

$$\tilde{k} = \tilde{k}(p,a,b,t) = F(a,a)^{t-1}F(b,a)^{-t}n^{\frac{1}{\lambda}-\frac{1}{\mu}-|\frac{1}{p}-\frac{1}{2}|}\alpha^{-1},$$

and

$$\tilde{K} = \tilde{K}(p,a,b,t) = F(a,a)^{1-t}F(a,b)^{t}n^{\frac{1}{\lambda}-\frac{1}{\mu}+|\frac{1}{p}-\frac{1}{2}|}\beta,$$

if $2 \leq p$ *and* $\lambda, \mu \in [2,p]$ *or if* $1 \leq p \leq 2$ *and* $\lambda, \mu \in [p,2]$.

Proof. We will denote by $\gamma(t) = \gamma_{a,b}(t)$, when no confusion can arise.

Consider the space $B_p^{(n)}$ with the norm:

$$\|\overline{A}\|_{p,a;s} = (\|A_1\|_{p,a}^{s} + ... + \|A_n\|_{p,a}^{s})^{1/s},$$

where $a \in Gl(H)^+$.

By $(*)$, the norm of T_R is at most $F(a,a)n^{\frac{1}{\lambda}-\frac{1}{\mu}+|\frac{1}{p}-\frac{1}{2}|}\beta$ when

$$T_R : (B_p^{(n)}, \|(\ ., ..., .\ )\|_{p,a;\mu}) \longrightarrow (B_p^{(n)}, \|(\ ., ..., .\ )\|_{p,a;\lambda}),$$

and the norm of T_R is at most $F(a,b)n^{\frac{1}{\lambda}-\frac{1}{\mu}+|\frac{1}{p}-\frac{1}{2}|}\beta$ when

$$T_R : (B_p^{(n)}, \|(., ..., .)\|_{p,a;\mu}) \longrightarrow (B_p^{(n)}, \|(., ..., .)\|_{p,b;\lambda}).$$

Therefore, using the complex interpolation, we obtain the following diagram of interpolation for $t \in [0,1]$

$$\begin{array}{ccc} & & (B_p^{(n)}, \|(.,...,.)\|_{p,a;\lambda}) \\ & \nearrow^{T_R} & \\ (B_p^{(n)}, \|(.,...,.)\|_{p,a;\mu}) & \xrightarrow{T_R} & (B_p^{(n)}, \|(.,...,.)\|_{p,\gamma(t);\lambda}) \\ & \searrow_{T_R} & \\ & & (B_p^{(n)}, \|(.,...,.)\|_{p,b;\lambda}). \end{array}$$

By Theorem 2.1, T_R satisfies

$$\|T_R(\overline{A})\|_{p,\gamma(t);\lambda} \leq F(a,a)^{1-t}F(a,b)^t n^{\frac{1}{\lambda}-\frac{1}{\mu}+|\frac{1}{p}-\frac{1}{2}|}\beta\|\overline{A}\|_{p,a;\mu}. \tag{3.7}$$

Now applying the Complex method to

$$\begin{array}{ccc} (B_p^{(n)}, \|(.,...,.)\|_{p,a;\lambda}) & & \\ & \searrow^{T_{R^{-1}}} & \\ (B_p^{(n)}, \|(.,...,.)\|_{p,\gamma(t);\lambda}) & \xrightarrow{T_{R^{-1}}} & (B_p^{(n)}, \|(.,...,.)\|_{p,a;\mu}) \\ & \nearrow_{T_{R^{-1}}} & \\ (B_p^{(n)}, \|(.,...,.)\|_{p,b;\lambda}) & & \end{array}$$

one obtains

$$\|T_{R^{-1}}(\overline{A})\|_{p,a;\mu} \leq F(a,a)^{1-t}F(b,a)^t n^{\frac{1}{\mu}-\frac{1}{\lambda}+|\frac{1}{p}-\frac{1}{2}|}\alpha\|\overline{A}\|_{p,\gamma(t);\lambda}. \tag{3.8}$$

Replacing in (3.8) $\overline{A}$ by $R\overline{A}$ we obtain

$$\|\overline{A}\|_{p,a;\mu} \leq F(a,a)^{1-t}F(b,a)^t n^{\frac{1}{\mu}-\frac{1}{\lambda}+|\frac{1}{p}-\frac{1}{2}|}\alpha\|R\overline{A}\|_{p,\gamma(t);\lambda}, \tag{3.9}$$

or equivalently

$$F(a,a)^{t-1}F(b,a)^{-t} n^{\frac{1}{\lambda}-\frac{1}{\mu}-|\frac{1}{p}-\frac{1}{2}|}\alpha^{-1}\|\overline{A}\|_{p,a;\mu} \leq \|T_R(\overline{A})\|_{p,\gamma(t);\lambda}. \tag{3.10}$$

Finally, the inequalities (3.7) and (3.10) complete the proof. □

We remark that the previous statement is a generalization of Th. 4.1 in [7] where $T_n = T_R$ with $R = (e^{(i\frac{2\pi(j-1)(k-1)}{n})}1)_{1\leq j,k\leq n}$ and $a^{-1/2}$ commutes with R for all $a \in Gl(H)^+$.

On the other hand, it is well known that if $x_1, ..., x_n$ are non-negative numbers, $s \in \mathbb{R}$ and we denote $\mathcal{M}_s(\overline{x}) = (\frac{1}{n}\sum_{i=1}^n x_i^s)^{1/s}$ then for $0 < s < s'$, $\mathcal{M}_s(\overline{x}) \leq \mathcal{M}_{s'}(\overline{x})$.

If we denote $\|\overline{B}\| = (\|B_1\|_p, ..., \|B_n\|_p)$ and we consider $1 < p \leq 2$, then it holds for $t \in [0,1]$ and $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$ that

$$\mathcal{M}_{s_t}(\|\overline{B}\|) \leq \mathcal{M}_q(\|\overline{B}\|) \leq r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{-1}{q}}(\sum_{j=1}^n \|A_j\|_p^p)^{\frac{1}{p}},$$

or equivalently

$$(\sum_{j=1}^{n}\|B_j\|_p^{s_t})^{\frac{1}{s_t}} \leq r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}(\sum_{j=1}^{n}\|A_j\|_p^{p})^{\frac{1}{p}}. \tag{3.11}$$

Analougsly, for $2 \leq p < \infty$ we get

$$(\sum_{j=1}^{n}\|A_j\|_p^{p})^{\frac{1}{p}} \leq \rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}n^{\frac{1}{q}-\frac{1}{s_t}}(\sum_{j=1}^{n}\|B_j\|_p^{s_t})^{\frac{1}{s_t}}; \tag{3.12}$$

where $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$.

Now we can use the interpolation method with the inequalities (3.11) and $(*)$ (or (3.12) and $(*)$).

If we consider the following diagram of interpolation with $1 < p \leq 2$ and $t \in [0, 1]$,

$$(B_p^{(n)}, \|(.,...,.)\|_{p,1;p}) \xrightarrow{T_R} (B_p^{(n)}, \|(.,...,.)\|_{p,1;p})$$

$$(B_p^{(n)}, \|(.,...,.)\|_{p,1;p}) \xrightarrow{T_R} (B_p^{(n)}, \|(.,...,.)\|_{p,1;s_t})$$

$$(B_p^{(n)}, \|(.,...,.)\|_{p,1;p}) \xrightarrow{T_R} (B_p^{(n)}, \|(.,...,.)\|_{p,1;q}).$$

By Theorem 2.1 and $(*)$, T_R satisfies

$$\|T_R(\overline{A})\|_{p,1;s_t} \leq (n^{f(p)}\beta)^{1-t}(r^{\frac{2}{p}-1}\beta^{\frac{2}{q}})^t\|\overline{A}\|_{p,1;p}. \tag{3.13}$$

Finally, from the inequalities (3.11) and (3.13) we obtain

$$(\sum_{j=1}^{n}\|B_j\|_p^{s_t})^{\frac{1}{s_t}} \leq \min\{r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)}r^{(\frac{2}{p}-1)t}\}(\sum_{j=1}^{n}\|A_j\|_p^{p})^{\frac{1}{p}}.$$

We can summarize the previous facts in the following statement.

Theorem 3.2. *Let $\overline{A} \in B_p^{(n)}$ and $B = R\overline{A}$, where $R = (R_{jk})$ is invertible. Let $r = \max\|R_{jk}\|$, $\rho = \max\|(R^{-1})_{jk}\|$ and q the conjugate exponent of p. Then, for $t \in [0, 1]$ we get*

$$(\sum_{j=1}^{n}\|A_j\|_p^{p})^{\frac{1}{p}} \leq \min\{\rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}n^{\frac{1}{q}-\frac{1}{s_t}}, n^{f(p)t}\alpha^{t+(1-t)\frac{2}{p}}\rho^{(1-\frac{2}{p})(1-t)}\}(\sum_{j=1}^{n}\|B_j\|_p^{s_t})^{\frac{1}{s_t}};$$

if $2 \leq p$ and $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$ or

$$(\sum_{j=1}^{n}\|B_j\|_p^{s_t})^{\frac{1}{s_t}} \leq \min\{r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}n^{\frac{1}{s_t}-\frac{1}{q}}, n^{f(p)(1-t)}\beta^{1+t(\frac{2}{q}-1)}r^{(\frac{2}{p}-1)t}\}(\sum_{j=1}^{n}\|A_j\|_p^{p})^{\frac{1}{p}},$$

if $1 < p \leq 2$ and $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$.

Finally using the Finsler norm $\|(.,...,.)\|_{p,a;s}$, Calderón's method and the previous inequalities we obtain

Corollary 3.3. *Let $a, b \in Gl(H)^+$, $\overline{A} \in B_p^{(n)}$ and $B = R\overline{A}$, where $R = (R_{jk})$ is invertible. Let $r = \max \|R_{jk}\|$, $\rho = \max \|(R^{-1})_{jk}\|$ and q the conjugate exponent of p. Then, for $t, u \in [0, 1]$ we get*

$$(\sum_{j=1}^{n} \|A_j\|_{p,a}^{p})^{\frac{1}{p}} \leq F(a, a)^{1-u} F(b, a)^u M_1 (\sum_{j=1}^{n} \|B_j\|_{p,\gamma_{a,b}(u)}^{s_t})^{\frac{1}{s\,t}}; \tag{3.14}$$

if $2 \leq p$, $\frac{1}{s_t} = \frac{1-t}{q} + \frac{t}{p}$ *and*

$$M_1 = M_1(R, p, t) = \min\{\rho^{1-\frac{2}{p}} \alpha^{\frac{2}{p}} n^{\frac{1}{q} - \frac{1}{s_t}}, n^{f(p)t} \alpha^{t+(1-t)\frac{2}{p}} \rho^{(1-\frac{2}{p})(1-t)}\}$$

or

$$(\sum_{j=1}^{n} \|B_j\|_{p,\gamma_{a,b}(u)}^{s_t})^{\frac{1}{s_t}} \leq F(a, a)^{1-u} F(a, b)^u M_2 (\sum_{j=1}^{n} \|A_j\|_{p}^{p})^{\frac{1}{p}}, \tag{3.15}$$

if $1 < p \leq 2$, $\frac{1}{s_t} = \frac{1-t}{p} + \frac{t}{q}$ *and*

$$M_2 = M_2(R, p, t) = \min\{r^{\frac{2}{p}-1} \beta^{\frac{2}{q}} n^{\frac{1}{s_t} - \frac{1}{q}}, n^{f(p)(1-t)} \beta^{1+t(\frac{2}{q}-1)} r^{(\frac{2}{p}-1)t}\}.$$

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