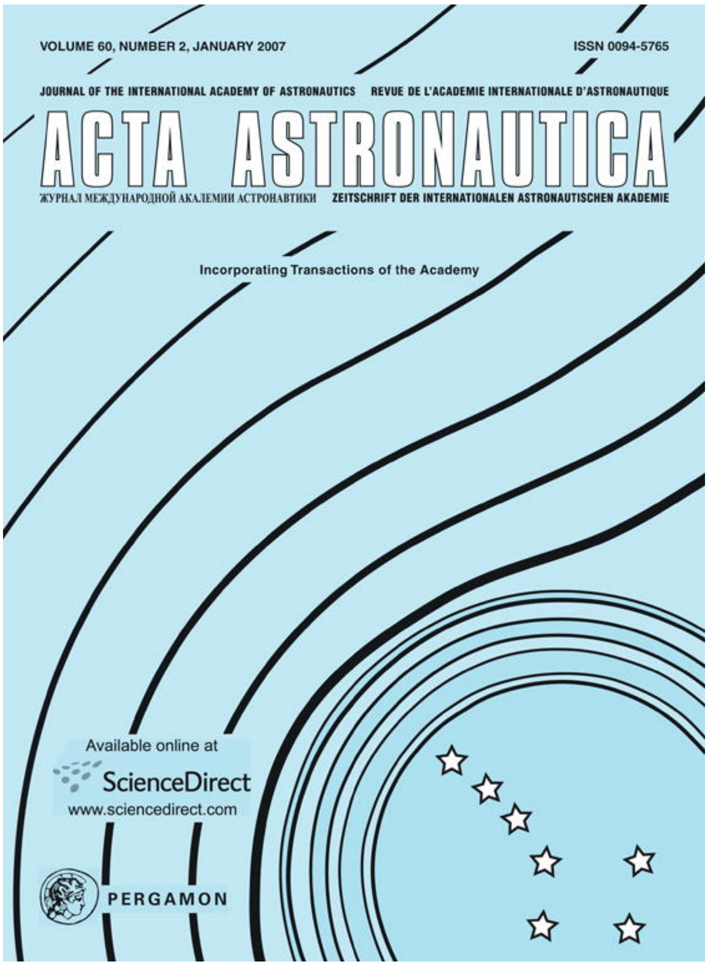
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Synthetic data for validation of navigation systems

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Abstract

Pre-flight design assessment and validation of navigation systems require emulating data acquired by a range of different hypothetical configurations of onboard sensors with different quality standards and/or sampling time specifications. A proper validation procedure requires the emulated data to be available at arbitrarily specified sampling times and for different hypothetical test-trajectories. The problem thus formulated entails obtaining, for the different case studies, closed solutions of the nonlinear kinematics equations as explicit functions of time. The method proposed is based on constructing a subset of the solutions of the kinematics equations contained in a vector spline functional space whose order may be arbitrarily specified. Particular consideration is devoted to the solutions of the attitude quaternion. The method is illustrated with experimental data obtained during a sub-orbital flight of the VS30 vehicle of the Brazilian Space Agency.

Keywords: Navigation; INS data synthesis; B-splines; Trajectory generation

1. Introduction

Design and validation of navigation systems require pre-flight cost-benefit ratio evaluation of competing configurations involving the measurement system, the complexity and performance of the involved navigation algorithms and required onboard computing resources. The process calls for emulating the data delivered by the measurement system aboard a vehicle moving along a hypothetical test-trajectory. Since measurements' sampling times are critical for the overall navigation system's performance, it is highly desirable to be able to arbitrarily specify the sampling times of the emulated sensors. Only a closed solution of the nonlinear kinematical equations will enable this possibility.

Throughout this paper, any set of "sufficiently differentiable" vector functions of time, $P(t) : \mathbb{R} \to \mathbb{R}^3$, $V(t) : \mathbb{R} \to \mathbb{R}^3$, $a(t) : \mathbb{R} \to \mathbb{R}^3$, $q(t) : \mathbb{R} \to \mathbb{R}^4$ and $\omega(t) : \mathbb{R} \to \mathbb{R}^3$ which, if taken, respectively, as the position, velocity, linear acceleration, quaternion attitude and angular rate in a given coordinate system, satisfy the kinematics equations are called *consistent kinematical magnitudes* (or briefly, *consistent magnitudes*). When taken altogether, this set will be called a *consistent trajectory*. Given any consistent trajectory of explicit functions of time, the kinematical magnitudes and the specific force experienced by the vehicle may be evaluated at arbitrary sampling times (via usage of a position dependent gravitational model [1]). These samples are processed by the sensors' models (inertial or external aiding sensors for integrated navigation configurations) to produce the simulated sampled data entering the navigation algorithm. Consistent trajectories

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are essential for the evaluation of the numeric performance of navigation algorithms. Indeed, *inconsistent* data would induce artificial errors on the navigation algorithm thus distorting its performance assessment.

The purpose of this paper is to present a method for synthesizing consistent trajectories as explicit functions of time. It consists in constructing a subset of solutions of the kinematics equations contained in a vector splines functional space with arbitrarily specified order. The navigation system validation procedure that ensues does not require a physical model of the vehicle. Actually, what challenges the navigation system's effectiveness is the dynamic evolution of the kinematics variables along a specific trajectory but not the vehicle that it travels through. In order to exploit prior information concerning typical or pre-designed trajectories, the method starts off from "*coarse initial data*" including the initial conditions (position, velocity and attitude) and samples of position, velocity, attitude and angular rate describing, more or less roughly, a vehicle's trajectory. These data are usually available as (a) samples acquired during a previous flight experiment of a particular vehicle; (b) a nominal or theoretical test trajectory (e.g. issued from an optimal guidance problem); (c) data provided by a simulated vehicle.

Section 5 illustrates the proposed method using measured data obtained during the sub-orbital flight of a VS30 Brazilian Space Agency rocket.

2. Piecewise polynomial functions and B-splines

This paragraph introduces B-splines theory basic notions, required later, as applied to function approximation.

Definition 1 (*De Boor [2]*). Given the strictly growing sequence $\{t\}_T \triangleq (t_i)_{i=1,\dots,T}$ in $\mathbb{R}$ and the sequence of polynomials $\{P\}_{T-1} \triangleq (P_i)_{i=1,\dots,T-1}$ of order (degree + 1) k, a *piecewise polynomial function* (*ppf*), *of order* k *over the sequence* $\{t\}_T$ is a function $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$f(x) \triangleq \begin{cases} P_1(x), & x < t_1, \\ P_i(x), & t_i \leqslant x < t_{i+1} \quad \forall i = 1, \dots, T-1, \\ P_{T+1}(x), & x \geqslant t_T. \end{cases} \tag{1}$$

In practice, the domain of definition of f is limited to the interval $[t_1, t_T]$. In the sequel, $P_{k,\{t\}}$ denotes the set of all the *ppf* of order k defined over the sequence $\{t\}_T$. As shown in Ref. [2], $P_{k,\{t\}}$ is a vector space of dimension $k \times (T-1)$.

Definition 2 (*De Boor [2]*). Given a "sufficiently differentiable" function $g : \mathbb{R} \to \mathbb{R}$ and a non-decreasing sequence $\{\xi\}_{n+k} \triangleq (\xi_i)_{i=1,\dots,n+k}$ in $\mathbb{R}$ the family (θ_i) *of divided differences* associated with g is defined as

for $i = 1, \dots, n, \quad \theta_0(\{\xi\}, i, g) \triangleq g(\xi_i) \quad \forall \xi_i;$

for $i = 1, \dots, n-1$,

$$\theta_1(\{\xi\}, i, g) \triangleq \begin{cases} \frac{\theta_0(\{\xi\}, i+1, g) - \theta_0(\{\xi\}, i, g)}{\xi_{i+1} - \xi_i}, & \xi_i \neq \xi_{i+1}, \\ g'(\xi_i)/1!, & \xi_i = \xi_{i+1}; \end{cases}$$

$\vdots$

for $i = 1, \dots, n-k$,

$$\theta_k(\{\xi\}, i, g) \triangleq \begin{cases} \frac{\theta_{k-1}(\{\xi\}, i+1, g) - \theta_{k-1}(\{\xi\}, i, g)}{\xi_{i+k} - \xi_i}, & \xi_i \neq \xi_{i+k}, \\ g^{(k)}(\xi_i)/k!, & \xi_i = \xi_{i+k}. \end{cases} \tag{2}$$

Definition 3 (*De Boor [2]*). (B-splines functions over a sequence). Given the non-decreasing sequence $\{\xi\}_{n+k}$ the *ith normalized B-spline function of order k* is defined as

$$B_{i,k,\{\xi\}}(x) \triangleq (\xi_{i+k} - \xi_i)\theta_k(\{\xi\}, i, g_x) \quad \forall x \in \mathbb{R} \text{ for } i = 1 \dots, n, \tag{3}$$

where

$$g_x(y) \triangleq \begin{cases} (y-x)^{k-1}, & y \geqslant x, \\ 0, & y < x. \end{cases}$$

From the above definitions, the following *property of the B-spline functions* ensues [2]:

$$B_i(x) = 0 \quad \forall x \notin [\xi_i, \xi_{i+k}]; \text{ only } B_{i-k+1}, B_{i-k+2}, \dots, B_i \text{ are non-zero for } x \in [\xi_i, \xi_{i+k}]. \tag{4}$$

As shown in Ref. [2], the jth derivative $D^j f \ (j < k)$ of $f \in P_{k,\{t\}}$, obtained by differentiating j-times each of the elements of the sequence $\{P\}_{T-1}$, is also a ppf.

Notation. Given a sequence of non-negative integers $\{v\}_T \triangleq (v_i)_{i=1,\dots,T}$ with $v_i \leqslant k$, the set of polynomials of order k over the sequence $\{t\}_T$ with v_i the first discontinuous derivative at t_i is denoted as

$$P_{k,\{t\},\{v\}} \triangleq \{f \in P_{k,\{t\}} : \text{jump}_{t_i} D^j f = 0; \ j = 1, \dots, v_i - 1\}, \tag{5}$$

where $\text{jump}_\alpha f \triangleq f(\alpha^+) - f(\alpha^-)$. As shown in Ref. [2], $P_{k,\{t\},\{v\}}$ is a vector subspace of $P_{k,\{t\}}$ with $n \triangleq \dim(P_{k,\{t\},\{v\}}) = k \times (T-1) - \sum_{i=2}^{T-1} v_i$. The next theorem gives a basis for the subspace $P_{k,\{t\},\{v\}}$.

Theorem 1 (*Schoemberg and Curry [3]*). *Given* (a) *a strictly growing sequence* $\{t\}_T$ *in* $\mathbb{R}$; (b) *a sequence of non-negative integers* $\{v\}_T$ *with* $v_1 = v_T = 0$ *and* $v_i < k\ \forall i$, (c) *the non-decreasing sequence* $\{\xi\}_{n+k}$ *in* $\mathbb{R}$, *satisfying*

(i) $$\xi_1 = \xi_2 = \cdots = \xi_k = t_1, \tag{6}$$

(ii) $$\xi_{1+\sum_{r=1}^{j-1}(k-v_r)} = \cdots = \xi_{\sum_{r=1}^{j}(k-v_r)} = t_j, \quad j = 2, \ldots, T-1, \tag{7}$$

(iii) $$t_T = \xi_{n+1} = \cdots = \xi_{n+k}. \tag{8}$$

The sequence of B-spline functions of order k for $n = k \times (T-1) - \sum_{i=2}^{T-1} v_i$, *defined over the sequence* $\{\xi\}_{n+k}$ *according to* Eqs. (2) *and* (3), *namely* $(B_{i,k,\{\xi\}})_{i=1,\ldots,n}$, *is a basis of the subspace* $P_{k,\{t\},\{v\}}$ *over the interval* $[\xi_k, \xi_{n+1}] \supseteq [t_1, t_T]$, *so that, for some* $\{\alpha\}_n$ *real, any ppf* $h \in P_{k,\{t\},\{v\}}$ *may be expressed as*

$$h = \sum_{i=1}^{n} \alpha_i B_{i,k,\{\xi\}}(x). \tag{9}$$

Notice that condition (ii) implies that t_i is replicated $k - v_i$ times within the sequence $\{\xi\}_{n+k}$.

2.1. Approximation of m-vector functions with m-vector splines of order k

Given the order k of an m-vector pp-expansion and the following assumptions on the *initial data*:

(i) A set of values of a vector function $\mathbf{g} : \mathbb{R} \to \mathbb{R}^m$ and of its first μ derivatives evaluated, respectively, over non-decreasing sequences in $\mathbb{R}$: $\{\tau\}_N, \{\tau^1\}_{N^1}, \ldots, \{\tau^\mu\}_{N^\mu}$.

(ii) m strictly growing sequences $\{t^l\}_{T_l}$, for $l = 1, \ldots, m$ (each characterizing a set P_{k,t^l} defined as in (1)) delimiting T_{l-1} adjacent intervals $[t_i^l, t_{i+1}^l]$ in $\mathbb{R}$, satisfying (a) $t_1^l \leqslant \min\{\tau_1, \tau_1^1, \ldots, \tau_1^\mu\}$; (b) the $\{t^l\}_{T_l}$ include those time instants at which any derivative (of order $< k$) of the l-component is discontinuous; (c) at least k distinct $\tau_j \in \{\tau\}_N$ are included in each interval $[t_i^l, t_{i+1}^l]$.

(iii) m non-negative integer sequences $\{v^l\}_T$, with $v_i^l < k$; $v_1^l = v_T^l = 0$, indicating the lowest discontinuous derivative order of the l-component ($l = 1, \ldots, m$) of the approximating pp vector function at instant t_i^l.

Based on sequences $\{t^l\}_{T_l}, \{v^l\}_T$, for $l = 1$ to m, introduce the l-indexed sequences $\{\xi^l\}_{n(l)+k}$ chosen, according to Eqs. (6)–(8) of Theorem 1 with $n(l) = k \times (T_l - 1) - \sum_{i=2}^{T-1} v_i$ and denote with $B^l_{i,k,\{\xi^l\}}$ the corresponding basis generating $P_{k,\{t^l\},\{v^l\}}$.

Problem 1. Determine the m-vector ppf $\hat{\mathbf{f}} : \mathbb{R} \to \mathbb{R}^m$ with components $\hat{f}^l \in P_{k,\{t^l\},\{v^l\}}$ for $l = 1, \ldots, m$, minimizing the distance:

$$\mathrm{dist}(\mathbf{f}, \mathbf{g}) \triangleq \left(\sum_{i=1}^{N} |\mathbf{f}(\tau_i) - \mathbf{g}(\tau_i)|^2 w_i + \sum_{j=1}^{N_1} |\mathbf{f}'(\tau_j^1) - \mathbf{g}'(\tau_j^1)|^2 w_j^1 + \cdots + \sum_{k=1}^{N_\mu} |\mathbf{f}^{(\mu)}(\tau_k^\mu) - \mathbf{g}^{(\mu)}(\tau_k^\mu)|^2 w_k^\mu \right)^{1/2}. \tag{10}$$

Remarks. $|\cdot|$ denotes the standard Euclidian norm of an m-vector. The weights $w_i > 0$, $w_j^s \geqslant 0$, $s = 1, \ldots, \mu$, reflect the relative importance assigned to the errors of the vector function and its derivatives. As usual, for measured "initial data", the weights correspond to the inverse of error variances. The order k of the approximating polynomials is chosen according to the "smoothness" sought for the approximating pp vector function (a high k may induce undesired oscillations in the approximating solution). If for any i and l, $v_i^l > 1$ the first derivative of the l-component of $\hat{\mathbf{f}}$ will be continuous at t_i. Non-decreasing sequences $\{\tau^\mu\}_{N^\mu}$ allow usage of simultaneous measurements—possibly acquired by different sensors. Including more than k different τ_j within $[t_i^l, t_{i+1}^l]$ has a filtering effect on the original data over that interval. An efficient data compression could be achieved through a careful selection of those intervals. Even though very much involved notationwise, coordinate dependent values of the order k, the sequences $\{\tau^j\}_N$ and the weights w_j^s may also be considered.

Solution of Problem 1. We first define the scalar functions: $l = 1, \ldots, m$; $f^l, h^l \in P_{k,\{t^l\},\{v^l\}}$: $\langle , \rangle_l : P_{k.\{t^l\},\{v^l\}} \times P_{k.\{t^l\},\{v^l\}} \to \mathbb{R}$:

$$\langle f^l, h^l \rangle_l \triangleq \sum_{i=1}^{N} f^l(\tau_i) h^l(\tau_i) w_i + \sum_{j=1}^{N_1} f^{l^{(1)}}(\tau_j^1) h^{l^{(1)}}(\tau_j^1) w_j^1 + \cdots + \sum_{k=1}^{N_\mu} f^{l^{(\mu)}}(\tau_k^\mu) h^{l^{(\mu)}}(\tau_k^\mu) w_k^\mu. \tag{11}$$

As a direct consequence of Schoenberg and Withney's Theorem [2] and assumption (ii-c), it may be shown that (11) is an inner product of the $n(l)$-dimensional space $P_{k.\{t^l\},\{v^l\}}$. Moreover, with (11), the objective (10) is rewritten as

$$\mathrm{dist}(\mathbf{f},\mathbf{g})=\left(\sum_l \langle f^l-g^l, f^l-g^l\rangle_l\right)^{1/2} \triangleq \|\mathbf{f}-\mathbf{g}\| \tag{12}$$

and Problem 1 restated as the following optimization problem for the given "initial data":

$$\hat{\mathbf{f}}=[\hat{f}^1 \ \cdots \ \hat{f}^m]^{\mathrm{T}}=\underset{\hat{f}^l\in P_{k,t^l,v^l}}{\mathrm{argmin}}\ \{\|\mathbf{f}-\mathbf{g}\|\}. \tag{13}$$

Following Theorem 1, the solution to Problem 1 is an m-vector ppf $\hat{\mathbf{f}}$ with components $\hat{f}^l=\sum_{i=1}^{n(l)}\alpha_i^l B^l_{i,k,\{\xi^l\}}$, where, for each $l=1,\ldots,m$, the α_i^l, $i=1,\ldots,n(l)$, are the unique solutions of the following *normal equation* formulated in terms of the product (11):

$$\langle B^l_{j,k,\{\xi^l\}}, g^l\rangle_l=\sum_{i=1}^{n(l)}\langle B^l_{j,k,\{\xi^l\}}, B^l_{i,k,\{\xi^l\}}\rangle_l\alpha_i^l,$$
$$j=1,\ldots,n(l),\quad l=1,\ldots,m. \tag{14}$$

Proof hint. Notice that since Eq. (12) involves a sum of positive and independent terms, the minimization of each term assures the minimization of Eq. (12). On the other hand, from standard Hilbert space arguments, the function $\hat{f}^l\in P_{k.\{t^l\},\{v^l\}}$ that best approximates g^l satisfies $\langle \hat{f}^l-g^l, B^l_{i,k,\{\xi^l\}}\rangle_l=0$ (for $l=1,\ldots,m$) and condition (14) follows using $\hat{f}^l=\sum_{i=1}^{n(l)}\alpha_i^l B^l_{i,k,\{\xi^l\}}$. Moreover, for each $l=1,\ldots,m$, the matrices $[m_{ij}]^l=[\langle B^l_{j,k,\{\xi\}^l}, B^l_{j,k,\{\xi\}^l}\rangle_l]$ are positive definite since by definition the set $\{B^l_{j,k,\{\xi\}^l}\}_{n(l)}$ is a basis of the $\langle,\rangle_l$-inner product space $P_{k.\{t^l\},\{v^l\}}$. This guarantees the existence and uniqueness of the solution of Eq. (14).

Remark. From the Property P1 above (4), one has that, for $|i-j|\geqslant k$, $\langle B_i, B_j\rangle_l=m_{i,j}=0$, therefore, $M=[m_{ij}]$ is a "striped" matrix. This fact with the positiveness of M allows an efficient numeric solution of system (14) (e.g. Cholesky' factorization [4].)

The results presented thus far are now applied to the synthesis of explicit functions of time consistent trajectories.

3. Synthesis of consistent position, velocity, acceleration and specific force

The synthesis procedure consists in determining a twice-differentiable vector ppf for the position in inertial coordinates. Velocity and acceleration are then obtained by successive differentiation. The specific force is subsequently calculated by subtracting the gravitational force—evaluated using a position dependent gravitational model—from the acceleration vector.

Problem PV. For a given order k and the following *initial data*:

- $\mathbb{R}^3$– samples of position, velocity and acceleration in inertial frame: $\{\tilde{\mathbf{P}}\}_N$, $\{\tilde{\mathbf{V}}\}_{N_1}\{\tilde{\mathbf{A}}\}_{N_2}$ specified or measured (possibly noise corrupted), respectively, at the times: $\{\tau\}_N$, $\{\tau'\}_{N_1}$ and $\{\tau^2\}_{N_2}$;
- $\mathbb{R}^3$-sequences: $\{\boldsymbol{\sigma}^2_{\mathbf{p}}\}_N$, $\{\boldsymbol{\sigma}^2_{\mathbf{v}}\}_{N_2}$ and $\{\boldsymbol{\sigma}^2_a\}_{N_3}$ of position velocity and acceleration measurements' uncertainties (measurement's error variances);
- user defined non-decreasing sequences of times $\{t^l\}_{T^l}$ for $l=1,2,3$ (may be coincident) satisfying (ii) of Problem 1;
- non-negative integer sequences $\{v^l\}_T$, for $l=1,2,3$, satisfying (iii) of Problem 1 (e.g. $2\leqslant v_i^l\leqslant k\ \forall i$ implies continuous velocity),

find the vector ppf $\hat{\mathbf{P}}(t)\triangleq[\hat{P}^1\ \hat{P}^2\ \hat{P}^3]^{\mathrm{T}}\in\mathbb{R}^3$ minimizing the distance

$$\mathrm{dist}(\hat{\mathbf{P}},\tilde{\mathbf{P}})\triangleq\left(\sum_{i=1}^{N}|\hat{\mathbf{P}}(\tau_i)-\tilde{\mathbf{P}}_i|^2 w_i^p+\sum_{j=1}^{N_1}|\hat{\mathbf{P}}'(\tau_j^1)-\tilde{\mathbf{V}}_j|^2 w_j^v+\sum_{r=1}^{N_2}|\hat{\mathbf{P}}^{(2)}(\tau_r^2)-\tilde{\mathbf{A}}_r|^2 w_r^a\right)^{1/2},$$
$$w_i^p>0,\ w_j^v\geqslant 0,\ w_r^a\geqslant 0. \tag{15}$$

Solution of Problem PV. Under the conditions required by Theorem 1 and Problem 1, the solution $\hat{\mathbf{P}}(t)$ to problem (15) is

$$\hat{P}^l(t)=\sum_{i=1}^{n(l)}\alpha_i^l B^l_{i,k,\{\xi^l\}},\quad l=1,2,3, \tag{16}$$

where $n(l)=k(T-1)-\sum_{i=2}^{T-1}v_i^l$ and $\boldsymbol{\alpha}^l\triangleq[\alpha_1^l\ \alpha_2^l\ \ldots\ \alpha_n^l]^{\mathrm{T}}\in\mathbb{R}^n$ corresponds to the unique solutions of the system of equations

$$\langle B^l_{j,k,\{\xi^l\}}, \tilde{P}^l\rangle=\sum_{i=1}^{n(l)}\langle B^l_{j,k,\{\xi^l\}}, B^l_{i,k,\{\xi^l\}}\rangle\alpha_i^l,$$
$$j=1,\ldots,n,\quad l=1,2,3. \tag{17}$$

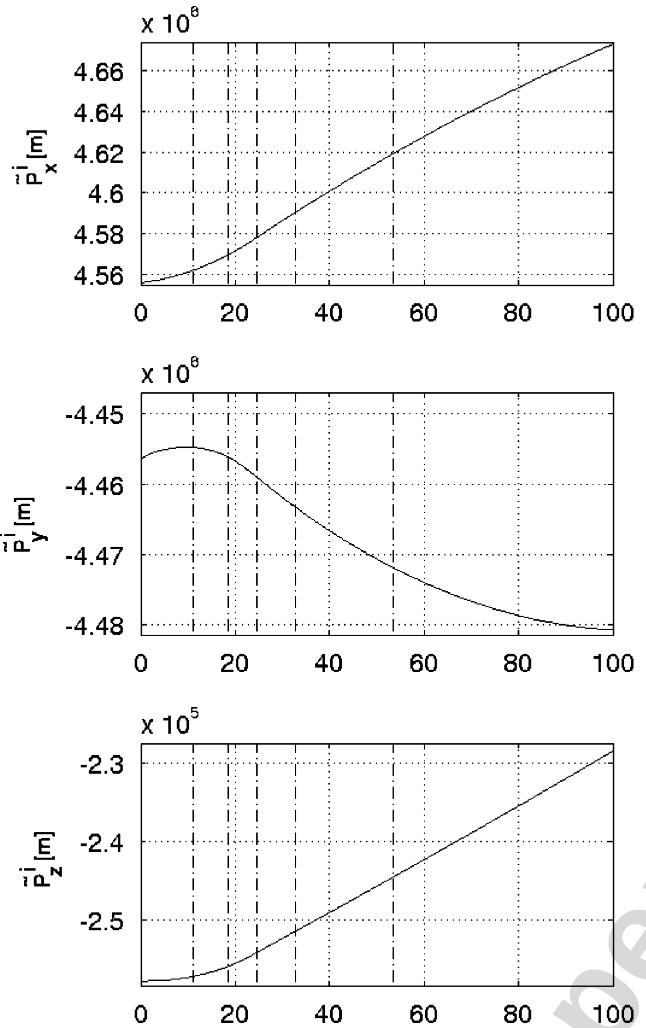

Fig. 1. Position data in inertial frame of VS30 vehicle and the selected partition for the pp functions.

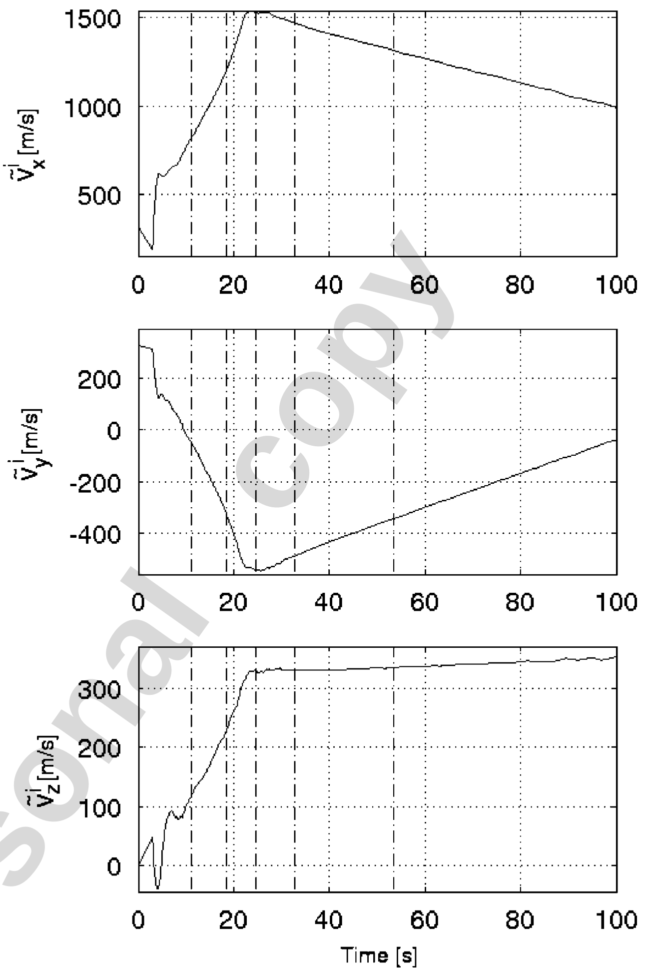

Fig. 2. Velocity data in inertial frame of the VS30 vehicle and the selected partition for the pp functions.

The B-spline functions $B^l_{j,k,\{\xi^l\}}$ defined over the sequences $\{\xi^l\}$ are defined as in Theorem 1 (Eqs. (6)–(8)).

Remarks. Weight in Eq. (15) are chosen as $w^s_i = 1/(\sigma^s_i)^2$; $s = p, v, a$, with $(\sigma^s_i)^2$; $s = p, v, a$, respectively, the specified uncertainties or measurement error variances of the position, velocity and acceleration for $l = 1, 2, 3$. The velocity and acceleration ppf are obtained by successive derivations of the ppf $\hat{P}^l(t)$, solution of Eq. (16). The velocity's continuity is assured by the condition: $1 < v_i \leqslant k\ \forall i$.

4. Synthesis of consistent attitude and angular rate vectors

Problem ωq**.** Generate a *consistent* time explicit pair of functions $(\mathbf{q^i_{ib}}(t), \boldsymbol{\omega}^{\mathbf{b}}_{\mathbf{ib}}(t))$ of attitude quaternion and angular rate, with the vehicle's initial quaternion and angular rate samples measured (possibly noise corrupted) or pre-specified at arbitrary discrete times, as initial data. The procedure is subdivided into three sub-problems, solved successively.

Problem ω**.** For a given order k and the following *initial data*:

- $\mathbb{R}^3$-samples of body angular rate: $\tilde{\boldsymbol{\omega}} = (\tilde{\boldsymbol{\omega}}_i)_{i=1,\ldots,N}$ (possibly noise corrupted measurements) in inertial frame, evaluated on the non-decreasing time sequence: $(\tau_i)_{i=1,\ldots,N}$;
- the $\mathbb{R}^3$-sequence $(\boldsymbol{\sigma}^{\mathbf{2}}_{\boldsymbol{\omega}_{\mathbf{i}}})_{i=1,\ldots,N}$ of uncertainties (error measurements' variances) of the available angular rate data;
- user defined non-decreasing sequences of times $\{t^l\}_{T^l}$ for $l = 1, 2, 3$ (may be coincident) satisfying (ii) of Problem 1;

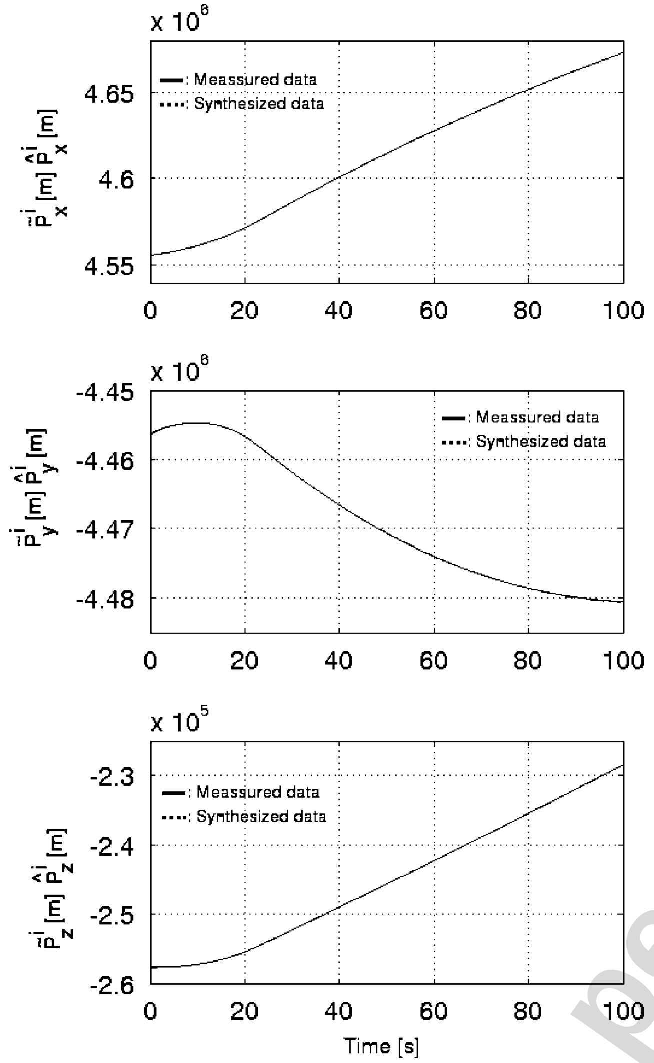

Fig. 3. Measured position and synthesized position in inertial frame of the VS30 vehicle.

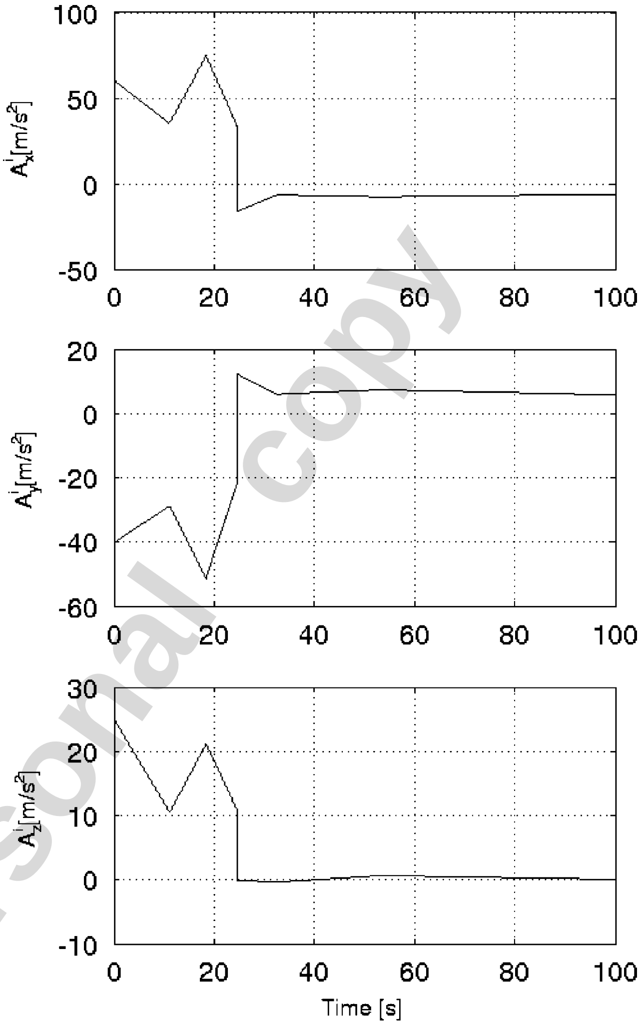

Fig. 4. Synthesized acceleration in inertial frame.

- non-negative integer sequences $\{v^l\}_T$, for $l=1,2,3$, satisfying (iii) of Problem 1. In general, $v_i^l > 0$ except when an abrupt change in $\boldsymbol{\omega}(t)$ is explicitly assumed at time t_i in which case $v_i^l = 0$ is adopted,

find those elements $\hat{\boldsymbol{\omega}}(t) \triangleq [\hat{\omega}^1 \; \hat{\omega}^2 \; \hat{\omega}^3]^{\mathrm{T}} \in \mathbb{R}^3$ minimizing the distance

$$\mathrm{dist}(\hat{\boldsymbol{\omega}}, \tilde{\boldsymbol{\omega}}) \triangleq \left(\sum_{i=1}^{N} |\hat{\boldsymbol{\omega}}(\tau_i) - \tilde{\boldsymbol{\omega}}_i|^2 w_i^{\omega} \right)^{1/2}, \quad w_i^{\omega} > 0. \quad (18)$$

Solution of Problem ω. Under the conditions established by Theorem 1 and Problem 1, the solution $\hat{\boldsymbol{\omega}}(t)$ to Eq. (18) is

$$\hat{\omega}^l(t) = \sum_{i=1}^{n(l)} \alpha_i^l B_{i,k,\{\xi^l\}}^l \quad (l=1,2,3), \quad (19)$$

with $n(l) = k(T-1) - \sum_{i=2}^{T-1} v_i^l$ and $\boldsymbol{\alpha}^l \triangleq [\alpha_1^l \; \alpha_2^l \; \ldots \; \alpha_n^l]^{\mathrm{T}} \in \mathbb{R}^n$ corresponds to the unique solutions of the system of equations

$$\langle B_{j,k,\{\xi^l\}}^l, \tilde{\omega}^l \rangle = \sum_{i=1}^{n(l)} \langle B_{j,k,\{\xi^l\}}^l, B_{i,k,\{\xi^l\}}^l \rangle \alpha_i^l, \quad j=1,\ldots,n, \quad l=1,2,3. \quad (20)$$

The B-spline functions $B_{j,k,\{\xi^l\}}^l$ defined over the sequence $(\xi_i^l)_{i=1,\ldots,n+k}$ are defined as in Theorem 1 (Eqs. (6)–(8)).

Problem $q1$. Given the initial quaternion $\mathbf{q}_b^i(0)$ and the ppf $\hat{\boldsymbol{\omega}}(t)$ (Eq. (19)), calculate the sequence of quaternions $(\tilde{\mathbf{q}}_{\mathbf{j}})_{j=1,\ldots,M} \in \mathbb{H}$ (the unitary sphere in $\mathbb{R}^4$) by numerically integrating the kinematic (21) over a strictly growing sequence of times $\{\tilde{\tau}\}_N \triangleq (\tilde{\tau}_j)_{i=1,\ldots,N}$.

$$\dot{\tilde{\mathbf{q}}} = \frac{1}{2} \tilde{\mathbf{q}} \circ \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{q}}(0) = \mathbf{q}_{\mathbf{b}}^{\mathbf{i}}(0). \quad (21)$$

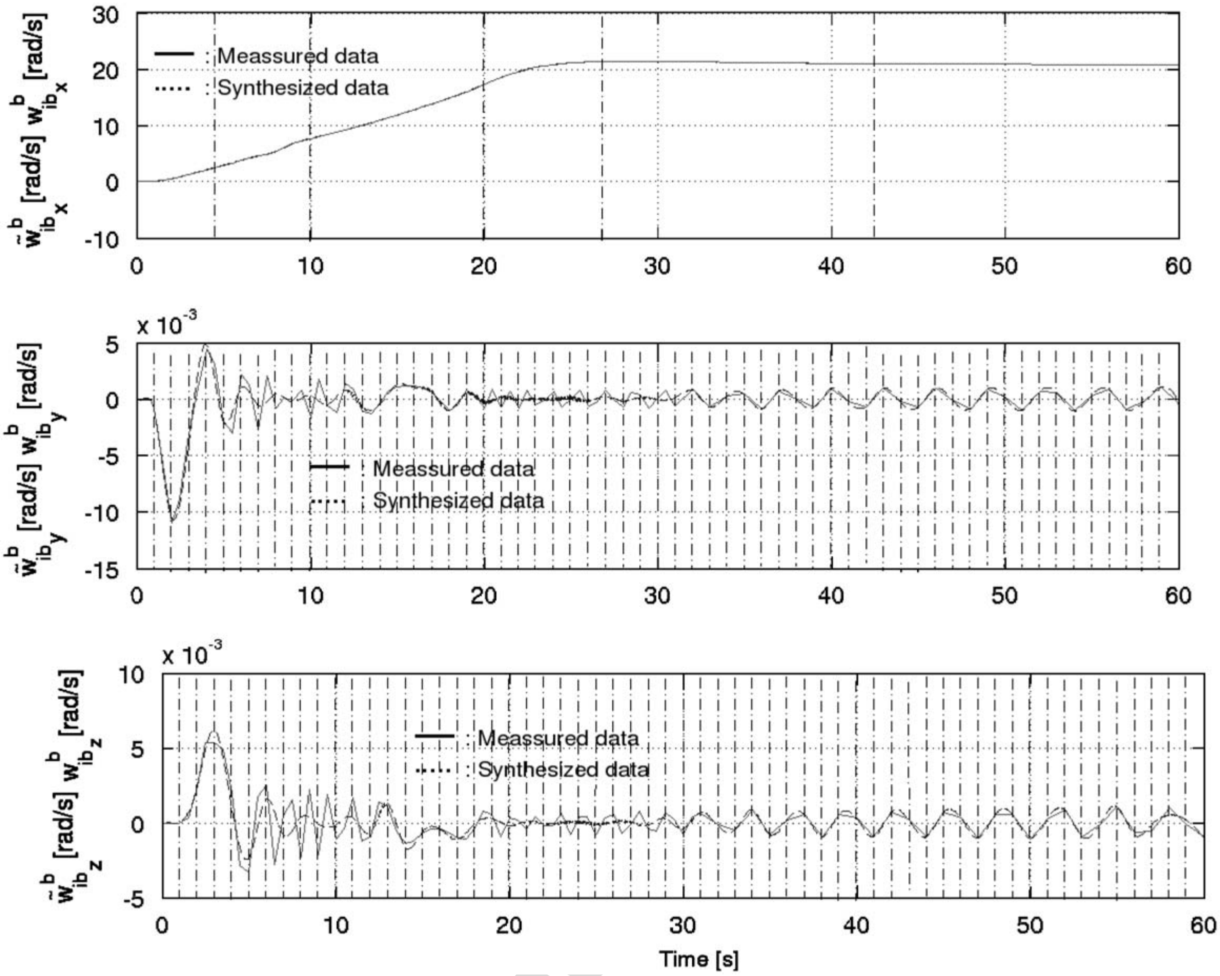

Fig. 5. Inertial angular velocity data and synthesized angular velocity in body frame.

In practice, this sequence coincides with the integration intervals of a variable step numerical integration algorithm, thus assuring a sampling sequence in accordance with the local dynamic of the kinematical equations. The tolerance parameter of the integration method is used to trade among precision and data compression.

Problem $q2$**.** Interpolate the pairs $((\tilde{\mathbf{q}}_j);(\tilde{\tau}_j))_{j=1,\dots,M}$, resulting from Problem $q1$, with a differentiable explicit function of time $\mathbf{q}(t):\mathbb{R}\to\mathbb{H}$ whereby, a consistent couple $(\mathbf{q}(t),\boldsymbol{\omega}(t))$ of explicit functions of time are readily obtained by introducing $\mathbf{q}(t)$ and its derivative $\dot{\mathbf{q}}(t)$ into the kinematic (21) and solving for $\boldsymbol{\omega}(t)$.

Solution of Problem $q2$**.** A cubic spline (see for instance [5]) is used to interpolate the couples $((\tilde{q}^l_j),(\tilde{\tau}^l_j))_{j=1,\dots,M}$ for $l=1,2,3,4$ (see for instance [1]). The result: $\bar{\mathbf{q}}(\mathbf{t})\triangleq[\bar{q}^1(t)\ \bar{q}^2(t)\ \bar{q}^3(t)\ \bar{q}^4(t)]^{\mathrm{T}}$ is a ppf at least three times differentiable passing over the points $(\tilde{\mathbf{q}})_{i=1,\dots,M}\in\mathbb{H}$ at instants $(\tilde{\tau}_j)_{j=1,\dots,M}$.

We now define the projection along the radius vector in $\mathbb{R}^4: P:\mathbb{R}^4\to\mathbb{H}$ as

$$P(\mathbf{q})\triangleq\begin{cases}\mathbf{q}/\|\mathbf{q}\| & \mathbf{q}\in\mathbb{R}^4 \text{ if } \mathbf{q}\neq 0,\\ 0 & \text{if } \mathbf{q}=0,\end{cases}$$

together with the composition of functions $\mathbf{q}(t)\triangleq P\circ\bar{\mathbf{q}}(t)$. Clearly, since $\bar{\mathbf{q}}(\tau_i)=\tilde{\mathbf{q}}_{\mathbf{i}}\in\mathbb{H}$ and P is a projection over $\mathbb{H}$, $\mathbf{q}(\tau_i)=P(\bar{\mathbf{q}}(\boldsymbol{\tau}_{\mathbf{i}}))=\tilde{\mathbf{q}}_{\mathbf{i}}$, thus, $\mathbf{q}(t)$ is an interpolating function over $\mathbb{H}$ of the data $(\tilde{\mathbf{q}})_{i=1,\dots,M}$. Moreover, except for the practically irrelevant case $\bar{\mathbf{q}}=0$, $\dot{\mathbf{q}}(t)$ may be explicitly evaluated at any time t, with

$$\dot{\mathbf{q}}(t)=\frac{1}{\|\bar{\mathbf{q}}(t)\|}(I-\bar{\mathbf{q}}(t)\bar{\mathbf{q}}(t)^{\mathrm{T}})\dot{\bar{\mathbf{q}}}(t)\quad\text{for } \mathbf{q}\neq 0 \tag{22}$$

obtained differentiating with respect to time the expression $\mathbf{q}=P(\bar{\mathbf{q}})=\bar{\mathbf{q}}/\|\bar{\mathbf{q}}\|$. This solves Problem $q2$.

Solution of Problem ωq**.** With the quaternion $\mathbf{q}^{\mathbf{i}}_{\mathbf{b}}(t)=\mathbf{q}(t)$ together with its derivative $\dot{\mathbf{q}}(t)$ (22), both solutions of Problem $q2$, an explicit expression of the consistent angular rate $\boldsymbol{\omega}_{\mathbf{ib}}^{\mathbf{b}}(t) = \boldsymbol{\omega}(t)$ is readily obtained for any time t by solving Eq. (21) so as to obtain

$$\begin{bmatrix} \boldsymbol{\omega}_{\mathbf{ib}}^{\mathbf{b}}(t) \\ 0 \end{bmatrix} = 2\mathbf{q}(t)^* \circ \dot{\mathbf{q}}(t) \quad \forall t. \tag{23}$$

5. Experimental results

Figs. 1 and 2 show, respectively, position and velocity data (acquired by radar) in an inertial frame of a VS30 sound rocket trajectory. The sequence $\{\tau\}_N$ corresponds to a sampling period of 0.1 s. The vertical lines indicate the $\{t\}_T$ sequence chosen so as to better reflect abrupt changes in the acceleration occurring at the inflection points of the curve.

In Fig. 3, the experimental data are compared with the almost coincident spline function $\mathbf{P}^i(t)$ of order $k = 4$, solution of problem PV. Since no reliable velocity and acceleration data were available, $w_j^v = w_r^a = 0$ was set in Eq. (15). The synthesized vehicle's inertial acceleration $\mathbf{a}^i(t)$ $(=\ddot{\mathbf{P}}^i(t))$ is shown in Fig. 4.

While the thrust is active, during approximately the first 20 s of flight, four canted fins in the back spin up the vehicle along its longitudinal x-axis from zero to approximately 3 rps. This motion allows for a distribution of environmental pitching moments leading to residual attitude rate in the y–z body axis. Fig. 5 displays the initial data $[\{\tilde{\boldsymbol{\omega}}\}_N, \{\tau\}_N]$ of the angular rate vector in body coordinates, measured in flight, and the function $\boldsymbol{\omega}_{\mathrm{ib}}^b(t)$, solution of problem ωq (Eq. (23)). The user defined sequence $\{t\}_T$, indicated with vertical lines in the figure, is chosen so as to fulfill Shannon's condition for the fundamental frequency of oscillations visible mainly on the y–z body axis. Notice that higher frequencies are being filtered out. Should these higher frequencies need to be included in the final synthesized (consistent) $\boldsymbol{\omega}_{\mathrm{ib}}^b(t)$ solution, shorter time intervals within the $\{t\}_T$ have to be chosen. In order to assure sharp corners in $\boldsymbol{\omega}_{\mathrm{ib}}^b(t)$, occurring at times $t_1 = 61$ s, $t_2 = 66$ s due to the de-spinning maneuver, these two time instants are included in the sequence with $v_1 = v_2 = 1$. The synthesized specific force in body axis $\mathbf{f}^b(\mathbf{t})$, shown in Fig. 6, is determined through

$$\mathbf{f}^b(t) = \mathbf{q}_i^b(t) \circ (\mathbf{a}^i(t) - \mathbf{g}^i(\mathbf{P}^i(t))), \tag{24}$$

where the attitude quaternion function $\mathbf{q}_{\mathbf{b}}^{\mathbf{i}}(t) = P(\bar{\mathbf{q}}(t))$, solution of the Problem ωq, and a gravitation position dependent model $\mathbf{g}^i(\mathbf{P}^i)$ have been used.

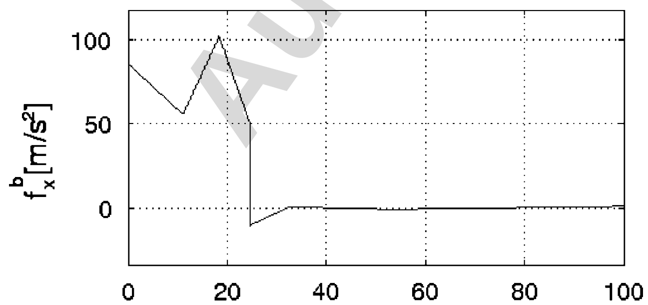

Fig. 6. Specific force in the axial direction.

6. Conclusions

Pre-flight design assessment and validation of navigation systems require emulating data possibly acquired by a range of potential configurations of onboard sensors with diverse quality standards, arbitrary sampling times and for a variety of hypothetical test-trajectories. The problem, thus formulated, entails obtaining for the different case studies, closed solutions of the nonlinear kinematics differential equations. Consistent sampled data derived from these trajectories are particularly relevant for the validation of strap-down navigation systems. Indeed, inconsistent data induce artificial errors into the navigation algorithm thus impairing the whole system's validation process. Coarse initial data, indicative of desired passage points, sampled velocities and attitude quaternion of a hypothetical vehicle's trajectory are the inputs to the method. Via embedding the solutions of the kinematics equations into a B-spline functional space with arbitrary order, the method provides time explicit functions consistent trajectories. As valuable byproducts, the method can filter noisy sampled measurements, allows the user to decide the degree of compression of ensuing data and makes it possible to represent discontinuities on higher derivatives of the kinematics variables such as those arising during launchers' take off, fast de-spinning maneuvers or abrupt deceleration during atmospheric reentry.

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