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Integral Equations and Operator Theory

Differential Geometry for Nuclear Positive Operators

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To my parents

Abstract. Let H be a Hilbert space, $\dim H = \infty$. The set $\Delta_1 = \{1 + a : a \text{ in the trace class}, 1 + a \text{ positive and invertible}\}$ is a differentiable manifold of operators, and a homogeneous space under the action of the invertible operators g which are themselves nuclear perturbations of the identity (one of the called classical Banach-Lie groups):

$$l_g(1 + a) = g(1 + a)g^*.$$

In this paper we introduce a Finsler metric in Δ_1 , which is invariant under the action. We investigate the metric space thus induced. For instance, we prove that it is complete non-positively curved (in the sense of Busemann). Other geometric properties are derived.

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1. Introduction

The purpose of this paper is to introduce a Finsler structure and expose several results about the geometrical structure of the set Δ_1 , defined by

$$\Delta_1 = \{1 + a \in \mathcal{L}_1 : \ 1 + a > 0\},$$

where $\mathcal{L}_1$ denotes the trace class perturbations of multiples of the identity.

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This study relates to previous work on differential geometry of positive operators (or positive definite matrices). Mainly a series of papers [6], [7] and [8] by Corach, Porta and Recht, where the geometry of the set of positive invertible of a C* algebra was studied. Also this study is related to classical work on the geometry of positive matrices ([18]).
Basically, there are two reasons why we have selected the set Δ_1 . The first is that the trace class operators (which are not invertible) usually appear in Physics ([21]) and other sciences. The second is the duality that exists between the space of positive functional in $B(H)$ and the positive nuclear operators.
Let $Gl(H)$ the general linear group of all invertible bounded operators on a separable and infinite dimensional Hilbert space H and $Gl(H, B_1(H))$ the subgroup of invertible trace class perturbations of the identity, i.e.

$$Gl(H, B_1(H)) = \{1 + a \in Gl(H) : a \in B_1(H)\} = \{g \in Gl(H) : g - 1 \in B_1(H)\}.$$

The subgroup $Gl(H, B_1(H))$ is a Banach-Lie group locally diffeomorphic to $B_1(H)$. The classical reference for this subject and notation is [12].
Consider the homogeneous space $Gl(H, B_1(H))/\mathcal{U}_1$, where $\mathcal{U}_1$ is the subgroup of $Gl(H, B_1(H))$ of unitary operators (this space can be identified with Δ_1). Then there exists a Finsler metric on the tangent bundle of Δ_1 which is given by the 1-norm on $(T\Delta_1)_1$. We show that the Finsler metric induces the following metric on Δ_1

$$d(a, b) = \|log(a^{-\frac{1}{2}} ba^{-\frac{1}{2}})\|_1 \quad a, b \in \Delta_1.$$

The main object of this paper is to determine the properties of the metric space (Δ_1 , d).
The material is organized as follows. Section 2 contains a survey of the topogical structure and differential geometry of Δ_1 , with a description of its structure as a reductive homogeneous space.
In Section 3 we investigate the minimality properties of the geodesics. In Section 4 we prove that Δ_1 shares some properties with Riemannian manifolds of non-positive sectional curvature (though we can not define the sectional curvature in this space). For instance, the metric increasing property (MIP) of the exponential map (Theorem 4.3).
Finally, in Section 5 we prove that Δ_1 has non-positive curvature (in Busemann's sense [14]) with respect to the Finsler metric (or geodesic distance).

2. Some aspects of the geometry of Δ_1 .

2.1. Topological and differentiable structure of Δ_1 .

Let $B(H)$ denote the algebra of bounded operators acting on a complex and separable Hilbert space H.

Throughout, $B_1(H)$ stands for the bilateral ideal of trace class operators of $B(H)$, that is the subset of compact operators with singular values in l_1.
Recall that $B_1(H)$ is a Banach algebra without unit, with the norm

$$\|a\|_1 = tr\,|a| = tr(a^*a)^{\frac{1}{2}} = \sum_{i\in N} \langle |a|\, e_i, e_i\rangle\,,$$

where $\{e_i\}_{i\in N}$ is any given orthonormal basis of H.
We consider a certain subset of Fredholm operators, namely

$$\mathcal{L}_1 = \{\lambda + a \in B(H) : \lambda \in \mathbb{C}, \quad a \in B_1(H)\},$$

the complex linear subalgebra consisting of the trace class perturbations of multiples of the identity. There is a natural (not quadratic) norm for this subspace

$$\|\lambda + X\|_{(1)} = |\lambda| + \|X\|_1.$$

The selfadjoint part of $\mathcal{L}_1$ is

$$\mathcal{L}^1_{\mathbb{R}} = \{\lambda + a \in \mathcal{L}_1 : (\lambda + a)^* = \lambda + a\},$$

Remark 2.1. 1. $(\mathcal{L}^1, \|.\|_{(1)})$ is the unitazion of $(B_1(H), \|.\|_1)$.

2. Note that the multiples of identity $\lambda 1$ and the operators $a \in B_1(H)$ are linearly independent. Therefore

$$\lambda + a \in \mathcal{L}^1_{\mathbb{R}} \text{ if and only if } \lambda \in \mathbb{R}\,,\ a^* = a.$$

   Formally,

$$\mathcal{L}_1 = \mathbb{C} \oplus B_1(H) \qquad\qquad \mathcal{L}^1_{\mathbb{R}} = \mathbb{R} \oplus B_1(H)_h,$$

   where $B_1(H)_h$ denotes the set of selfadjoint trace class operators.
3. One has the usual estimates
   (a) $\|\lambda + a\| \leq \|\lambda + a\|_{(1)}$,
   (b) $\|(\lambda + a)(\mu + b)\|_{(1)} \leq \|\lambda + a\|_{(1)} \|\mu + b\|_{(1)}$.
   for all $\lambda + a, \mu + b \in \mathcal{L}_1$. In particular, $(\mathcal{L}_1, +, .)$ is a Banach algebra.

Inside $\mathcal{L}^1_{\mathbb{R}}$, we consider

$$\Delta = \{\lambda + a \in \mathcal{L}_1 :\ \lambda + a > 0\},$$

and

$$\Delta_1 = \{1 + a \in \mathcal{L}_1 :\ 1 + a > 0\}.$$

Apparently Δ is an open subset of $\mathcal{L}^1_{\mathbb{R}}$, and therefore a differentiable (analytic) submanifold.
The next step is to prove that Δ_1 is a submanifold of Δ. For this purpose, we consider

$$\theta : \Delta \to \mathbb{R}\,,\ \theta(\lambda + a) = \lambda.$$

Lemma 2.1. θ is a submersion.

Proof. It is sufficient to show that $d\theta_{\lambda+a}$ is surjective and $ker(d\theta_{\lambda+a})$ is complemented ([15], Theorem 2.2).
Since $\mathcal{L}^1_{\mathbb{R}}$ and $\mathbb{R}$ are Banach spaces and θ is a continuous linear map we get that $d\theta_{\lambda+a} = \theta$
Apparently, $d\theta_{\lambda+a}$ is suryective and $ker(d\theta_{\lambda+a})$ has codimension 1 and hence is complemented. □

It follows that Δ_1 is a submanifold, since $\Delta_1 = \theta^{-1}(\{1\})$. These facts imply that, for $1+a \in \Delta_1$, $(T\Delta_1)_{1+a}$ identifies with $B_1(H)_h$.
There is a natural action of $Gl(H, B_1(H))$ over Δ_1 , defined by

$$l : Gl(H, B_1(H)) \times \Delta_1 \longrightarrow \Delta_1, l_g(1 + a) = g(1 + a)g^*.$$

This action is clearly differentiable and transitive, since if $1 + a, 1 + b \in \Delta_1$ then

$$l_r(1 + a) = (1 + b),$$

for $r = (1 + b)^{\frac{1}{2}}(1 + a)^{-\frac{1}{2}} \in Gl(H, B_1(H))$. For $1 + a \in \Delta_1$, let

$$\mathcal{I}_{1+a} = \{g \in Gl(H, B_1(H)) : g(1 + a)g^* = 1 + a\},$$

the *isotropy group* of $1 + a$. In particular, for $1 \in \Delta_1$

$$\mathcal{I}_1 = \{g \in Gl(H, B_1(H)) : gg^* = 1\} = U(H) \cap Gl(H, B_1(H)) = \mathcal{U}_1,$$

where $U(H)$ denotes the unitary operators on H.

2.2. Reductive structure.

Let us recall the definition of homogeneous reductive space

Definition 2.1. A homogeneous space G/F is reductive (RHS) if there exists a vector space decomposition $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G, such that $\mathfrak{m}$ is invariant under the action of F

In order to give an RHS structure to Δ_1 (or equivalently to $Gl(H, B_1(H))/\mathcal{U}_1$) under the action of $Gl(H, B_1(H))$ we must find a decomposition

$$\underbrace{(TGl(H, B_1(H)))_1}_{\mathfrak{g}} = \underbrace{(T\mathcal{U}_1)_1}_{\mathfrak{f}} \oplus \mathfrak{m}.$$

Recall that $\mathfrak{g} = (TGl(H, B_1(H)))_1$ and $\mathfrak{f} = (T\mathcal{U}_1)_1$ can be identified with $B_1(H)$ and $iB_1(H)_h$, respectively. Then, we have

$$B_1(H) = iB_1(H)_h \oplus \mathfrak{m}.$$

The most natural choice is $\mathfrak{m} = B_1(H)_h$. Note that $B_1(H)_h$ is $\mathcal{U}_1$-invariant:

$$l_g(B_1(H)_h) = \{gXg^* : X \in B_1(H)_h\} = B_1(H)_h.$$

From the above remarks, we get

Proposition 2.1. *Δ_1 has an RHS structure under the action of $Gl(H, B_1(H))$.*

Now, in order to construct a covariant derivative in Δ_1 , we use its reductive structure. We introduce the transport equation whose solutions give the horizontal lifts to $Gl(H, B_1(H))$ of curves on Δ_1 following the lines of [16].

Definition 2.2. The differential equation

$$\dot{\Gamma} = \frac{1}{2}\dot{\gamma}\gamma^{-1}\Gamma,$$

is called the *transport equation* for γ, and the solution $\Gamma(t)$ with initial condition $\Gamma(0) = 1 \in Gl(H, B_1(H))$ is called the *horizontal lift* of $\gamma(t)$.

The transport equation induces a covariant derivative of a tangent field X along γ, namely

$$\frac{DX}{dt} = \Gamma(t)\frac{d}{dt}((Tl_{\Gamma(t)^{-1}})_{\gamma(t)}X(t))\Gamma(t)^* = \dot{X} - \frac{1}{2}(X\gamma^{-1}\dot{\gamma} + \dot{\gamma}\gamma^{-1}X).$$

From now on, we denote with $a, b, ..$ etc. the elements of Δ_1 .
The curvature tensor for this connection is

$$R(X, Y)Z = -\frac{1}{4}a[[a^{-1}X, a^{-1}Y], a^{-1}Z],$$

for $X, Y, Z \in (T\Delta_1)_a$.
The corresponding exponential at $a \in \Delta_1$ is

$$exp_a : (T\Delta_1)_a \to \Delta_1,\ \ exp_a X = a^{\frac{1}{2}} e^{a^{-\frac{1}{2}} X a^{-\frac{1}{2}}} a^{\frac{1}{2}}.$$

Notice that exp_a is a diffeomorphism and its inverse map is

$$log_a : \Delta_1 \to (T\Delta_1)_a,\ \ log_a b = a^{\frac{1}{2}} log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) a^{\frac{1}{2}}.$$

The covariant derivative can now be used to define a parallel vector field and geodesics on Δ_1 as solutions to ordinary differential equations.
A curve γ is a *geodesic* if $\dot{\gamma}$ is parallel, i.e.

$$\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}. \tag{2.1}$$

The basics properties of the geodesics can be summarized in the following statement.

Proposition 2.2. *Let $a \in \Delta_1$, $X \in (T\Delta_1)_a$ and γ a geodesic. Then*

1. *The curve $g\gamma g^*$ is also a geodesic for all $g \in Gl(H, B_1(H))$,*

2. *The unique geodesic* γ *such that* $\gamma(0) = a$ *and* $\dot\gamma(0) = X$, *is*
$$\gamma(t) = a^{\frac{1}{2}} e^{ta^{-\frac{1}{2}} X a^{-\frac{1}{2}}} a^{\frac{1}{2}} \quad t \in \mathbb{R},$$
3. *Let* $b \in \Delta_1$. *There is one and only one geodesic* $\gamma_{a,b}$ *such that* $\gamma_{a,b}(0) = a$ *and* $\gamma_{a,b}(1) = b$, *namely*
$$\gamma_{a,b}(t) = a^{\frac{1}{2}} (a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^t a^{\frac{1}{2}} \quad t \in \mathbb{R}.$$

Proof. The proof is straightforward. □

Through this paper, we use the following notation
$$a \sharp_t b = a^{\frac{1}{2}} (a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^t a^{\frac{1}{2}} = exp_a(t\, exp_a^{-1}(b)),$$
which is called the t-power mean between a and b (see [19]), and the *relative operator entropy*
$$S(a/b) = a^{\frac{1}{2}} log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) a^{\frac{1}{2}},$$
defined in [11].
Lemma 2 in [10] shows that for $a, b \in \Delta_1$ and $t \in \mathbb{R}$
$$a \sharp_t b = b \sharp_{1-t} a.$$

2.3. Finsler Structure

We define the length of a tangent vector for $X \in (T\Delta_1)_a$ by
$$\|X\|_a = \left\| a^{-\frac{1}{2}} X a^{-\frac{1}{2}} \right\|_1 .$$
where $\|.\|_1$ denotes the norm of $B_1(H)$.

Proposition 2.3. *The metric in* $T\Delta_1$ *is invariant for the action of the group of invertible elements, i.e. for each* $a \in \Delta_1$, $g \in Gl(H, B_1(H))$ *and* $X \in B_1(H)_h$, *we have* $\|X\|_a = \|gXg^*\|_{gag^*}$

Proof. Let $a \in \Delta_1$, $g \in Gl(H, B_1(H))$ and $X \in B_1(H)$, observe that
$$gXg^* = g a^{\frac{1}{2}} a^{-\frac{1}{2}} X a^{-\frac{1}{2}} a^{\frac{1}{2}} g^*.$$
Denote by $z = g a^{\frac{1}{2}}$ then
$$(gag^*)^{-\frac{1}{2}} = (g a^{\frac{1}{2}} a^{\frac{1}{2}} g^*)^{-\frac{1}{2}} = (zz^*)^{-\frac{1}{2}} = |z^*|^{-1} .$$
therefore
$$(gag^*)^{-\frac{1}{2}} gXg^* (gag^*)^{-\frac{1}{2}} = |z^*|^{-1} z a^{-\frac{1}{2}} X a^{-\frac{1}{2}} z^* |z^*|^{-1} .$$
From the polar decomposition applied to $z \in Gl(H)$, $z = |z^*| \rho_z$ with ρ_z unitary, we have
$$(gag^*)^{-\frac{1}{2}} gXg^* (gag^*)^{-\frac{1}{2}} = \rho_z a^{-\frac{1}{2}} X a^{-\frac{1}{2}} \rho_z^*.$$

Now, since $|srs^*| = s\,|r|\,s^*$ for all unitary s, we get

$$\begin{aligned} \|gXg^*\|_{gag^*} &= tr\left|\rho_z a^{-\frac{1}{2}} X a^{-\frac{1}{2}} \rho_z^*\right| = tr(\rho_z \left|a^{-\frac{1}{2}} X a^{-\frac{1}{2}}\right| \rho_z^*) \\ &= tr\left|a^{-\frac{1}{2}} X a^{-\frac{1}{2}}\right| = \left\|a^{-\frac{1}{2}} X a^{-\frac{1}{2}}\right\|_1 = \|X\|_a . \end{aligned}$$

□

3. Minimality of geodesics

In this section we investigate the minimality properties of the geodesics; the expresion "minimal" is understood in terms of the length (or more generally p-energy functional). We prove that the unique geodesic joining two points is the minimun of the p-energy functional for $p \geq 1$.

For a piecewise differentiable curve $\alpha : [0,1] \to \Delta_1$ we now compute the *length* of the curve α by

$$l(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_{\alpha(t)} dt.$$

Note that given a, b in Δ_1 , if $\gamma_{a,b} : [0,1] \longrightarrow \Delta_1$ is the unique geodesic joining them, then

$$l(\gamma_{a,b}) = \|log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_1.$$

Definition 3.1. Let $a, b \in \Delta_1$. We denote

$$\Omega_{a,b} = \{\alpha : [0,1] \to \Delta_1 : \alpha \text{ is a } \mathcal{C}^1 \text{ curve, } \alpha(0) = a \text{ and } \alpha(1) = b\}.$$

The geodesic distance between a and b (in the Finsler metric) is defined by

$$d(a,b) = inf\{l(\alpha) : \alpha \in \Omega_{a,b}\}.$$

If K $\subseteq \Delta_1$, let

$$d(a,K) = inf\{d(a,k) : k \in K\}.$$

The next step consists in showing that geodesics are short curves, i.e. if δ is another curve joining a to b then

$$l(\gamma_{a,b}) \leq l(\delta).$$

and hence

$$d(a,b) = \|log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_1.$$

The proof of this fact requires some preliminaries. We begin with the following inequalities (see [13]):

Let a, b, c be Hilbert space operators with $a, b \geq 0$. For any unitarily invariant norm $|||.|||$ we have

$$|||a^{1/2}cb^{1/2}||| \leq ||| \int_0^1 a^t cb^{1-t} dt||| \leq \frac{1}{2}|||ac + cb|||. \tag{3.1}$$

Theorem 3.1. *For all $X, Y \in B_1(H)_h$*

$$\|Y\|_1 \leq \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}\|_1,$$

where $dexp_X$ denote the derivate, at a point X, of the exponential map.

This inequality was proved by R. Bhatia for matrices ([5]).

Proof. Our proof uses two ingredients. The first is the well-know formula

Claim 3.1. $dexp_X(Y) = \int_0^1 e^{tX} Y e^{(1-t)X} dt.$

We provide here a simple proof of this equality. Since

$$\frac{d}{dt}(e^{tx}e^{(1-t)y}) = e^{tx}(x - y)e^{(1-t)y},$$

we have

$$e^x - e^y = \int_0^1 e^{tx}(x - y)e^{(1-t)y} dt,$$

and hence

$$\lim_{h\to 0} \frac{e^{X+hY} - e^X}{h} = \int_0^1 e^{tX} Y e^{(1-t)X} dt.$$

Let $X, Y \in B_1(H)_h$. Write $Y = e^{\frac{X}{2}} (e^{-\frac{X}{2}} Y e^{-\frac{X}{2}}) e^{\frac{X}{2}}$, and then using the inequalities (3.1) to get from this

$$\begin{aligned} \|Y\|_1 &\leq \| \int_0^1 e^{tX}(e^{-\frac{X}{2}} Y e^{-\frac{X}{2}})e^{(1-t)X} dt\|_1 = \|e^{-\frac{X}{2}} \int_0^1 e^{tX} Y e^{(1-t)X} dt e^{-\frac{X}{2}}\|_1 \\ &= \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}\|_1. \end{aligned}$$

This proves the theorem. □

We are now ready to prove the main result in this section.

Theorem 3.2. *Let $a, b \in \Delta_1$, the geodesic $\gamma_{a,b}$ is the shortest curve joining them. So*

$$d(a, b) = \|log(a^{-\frac{1}{2}} ba^{-\frac{1}{2}})\|_1.$$

Proof. Since the group $Gl_1(H, B_1(H))$ acts isometrically and transitively on Δ_1 , is suffices to prove the theorem for $a = 1$.

Then

$$\gamma_{1,b} = b^t = e^{t\, logb} \qquad \text{and} \qquad l(\gamma_{1,b}) = \|log\ b\|_1.$$

Let $\gamma \in \Omega_{1,b}$; so write $\gamma(t) = e^{\alpha(t)}$ we get

$$\begin{aligned} \|\gamma(t)^{-\frac{1}{2}}\dot{\gamma}(t)\gamma(t)^{-\frac{1}{2}}\|_1 &= \|e^{-\frac{\alpha(t)}{2}}e^{\dot{\alpha}(t)}e^{-\frac{\alpha(t)}{2}}\|_1 = \|e^{-\frac{\alpha(t)}{2}}dexp_{\alpha(t)}(\dot{\alpha}(t))e^{-\frac{\alpha(t)}{2}}\|_1 \\ &\geq \|\dot{\alpha}(t)\|_1. \end{aligned}$$

Finally,

$$\begin{aligned} l(\gamma) &= \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)}dt = \int_0^1 \|\gamma(t)^{-\frac{1}{2}}\dot{\gamma}(t)\gamma(t)^{-\frac{1}{2}}\|_1 dt \geq \int_0^1 \|\dot{\alpha}(t)\|_1 dt \\ &\geq \|\int_0^1 \dot{\alpha}(t)dt\|_1 = \|\alpha(t)|_0^1\|_1 = \|\alpha(1)-\alpha(0)\|_1 = \|log\ b\|_1. \end{aligned}$$

□

Remark 3.1. 1. The geometrical result described above can be translated to the language of the relative entropy

$$d(a,b) = \left\|a^{-\frac{1}{2}}S(a/b)a^{-\frac{1}{2}}\right\|_1 = \|S(a/b)\|_a.$$

2. For each $a \in \Delta_1$ and $\alpha > 0$ the exponential map $exp_a : (T\Delta_1)_a \to \Delta_1$ maps the ball $\{X \in (T\Delta_1)_a : \|X\|_a \leq \alpha\}$ onto the ball $\{x \in \Delta_1 : d(a,x) \leq \alpha\}$, since

$$d(a, exp_a(X)) = d(a, a^{\frac{1}{2}}e^{a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}}a^{\frac{1}{2}}) = \|X\|_a.$$

Corollary 3.1. *If $X, Y \in B_1(H)_h$ commute we have*

$$\|X-Y\|_1 = d(e^X, e^Y).$$

In particular on each line $\mathbb{R}X \subseteq B_1(H)_h$ the exponential map preserves distances.

Definition 3.2. For every $p \in \mathbb{R} - \{0\}$ we define the *p-energy functional*

$$E_p : \Omega_{a,b} \to \mathbb{R}^+, \qquad E_p(\alpha) := \int_0^1 (\|\dot{\alpha}(t)\|_{\alpha(t)})^p dt.$$

Remark 3.2. 1. For $p = 1$ we obtain the *length functional*

$$l(\alpha) := \int_0^1 \|\dot{\alpha}(t)\|_{\alpha(t)}dt,$$

and for $p = 2$ we obtain the *energy functional*

$$E(\alpha) := \int_0^1 (\|\dot{\alpha}(t)\|_{\alpha(t)})^2 dt,$$

2. For any curve α such that $\|\dot{\alpha}(t)\|_{\alpha(t)}$ is constant we have

$$E_p(\alpha) = (l(\alpha))^p = (E(\alpha))^{\frac{p}{2}}.$$

In Theorem 3.2 we have proved that the geodesic between a and b minimizes the length functional. This fact is valid also for the p-energy functional (associated with $\Omega_{a,b}$) for $p \in (1, \infty)$.

Proposition 3.1. *Let $a, b \in \Delta_1$ and $p \in [1, \infty)$. Then the p-energy functional*

$$E_p : \Omega_{a,b} \to \mathbb{R}^+ \qquad E_p(\alpha) := \int_0^1 (\|\dot{\alpha}(t)\|_{\alpha(t)})^p dt,$$

takes on its minumum global $d^p(a, b)$ precisely on $\gamma_{a,b}$.

Proof. Now, let $\alpha \in \Omega_{a,b}$ and $p \in (1, \infty)$ then by Hölder´s inequality

$$(l(\alpha))^p = (\int_0^1 \|\dot{\alpha}(t)\|_{\alpha(t)} dt)^p \leq \int_0^1 (\|\dot{\alpha}(t)\|_{\alpha(t)})^p dt = E_p(\alpha).$$

On the other hand, $(l(\gamma_{a,b}))^p = E_p(\gamma_{a,b})$. This implies

$$E_p(\gamma_{a,b}) = (l(\gamma_{a,b}))^p \leq (l(\alpha))^p \leq E_p(\alpha).$$

□

Proposition 3.2. *Given $a, b \in \Delta_1$, $g \in Gl(H, B_1(H))$ we get*

1. $$d(a, b) = d(a^{-1}, b^{-1}).$$
2. *For all $t \in \mathbb{R}$*
$$d(a, a \sharp_t b) = |t|\, d(a, b).$$
3. *Invariance under the action by $Gl(H, B_1(H))$*
$$d(a, b) = d(gag^*, gbg^*).$$

Proof. 1. It is easy to see that $S(a/b) = -a^{\frac{1}{2}} log(b^{-1}/a^{-1})a^{\frac{1}{2}}$, as a consequence from $log(1/t) = -log(t)$. Then

$$\begin{aligned} d(a, b) &= \|log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_1 = \|a^{-\frac{1}{2}} S(a/b) a^{-\frac{1}{2}}\|_1 \\ &= \| - a^{-\frac{1}{2}} a^{\frac{1}{2}} log(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}) a^{-\frac{1}{2}} a^{\frac{1}{2}}\|_1 \\ &= \| - log(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}})\|_1 = d(a^{-1}, b^{-1}). \end{aligned}$$

2. It is obvious that $S(a, a \sharp_t b) = tS(a/b)$, then
$$d(a, a \sharp_t b) = \left\|a^{-\frac{1}{2}} S(a/a \sharp_t b) a^{-\frac{1}{2}}\right\|_1 = |t| \left\|a^{-\frac{1}{2}} S(a/b) a^{-\frac{1}{2}}\right\|_1 = |t|\, d(a, b).$$
3. Note that if $\gamma_{a,b}$ is the geodesic joining a with b, then
$$\|g\dot{\gamma_{a,b}}(t)g^*\|_{g\gamma_{a,b}(t)g^*} = \|\dot{\gamma_{a,b}}(t)\|_{\gamma_{a,b}(t)}.$$

□

Definition 3.3. For $a, b \in \Delta_1$, we call the midpoint of a and b, and we denote by $m(a, b)$ (following the notation used in [14]) to

$$m(a, b) := a\sharp_{\frac{1}{2}} b.$$

By the Proposition 3.2 and the last definition we have that:

1. $m(a, b) = a\sharp_{\frac{1}{2}} b = b\sharp_{\frac{1}{2}} a = m(b, a)$.
2. $d(a, m(a, b)) = \frac{1}{2} d(a, b) = \frac{1}{2} d(b, a) = d(b, m(b, a))$.

4. Convexity of the geodesic distance

The purpose of this section is to show that the norm of the Jacobi field along to a geodesic γ is a convex function.

Definition 4.1. A vector field J along to a geodesic γ (i.e. $J(t) \in (T\Delta_1)_{\gamma(t)}$ for all t) is a *Jacobi field* if

$$\frac{D^2 J}{dt^2} + R(J, V)V = 0. \tag{4.1}$$

where $V(t) = \dot{\gamma}(t)$ and $R(X, Y)Z$ the curvature tensor.

Theorem 4.1. *If $J(t)$ is a Jacobi field along the geodesic $\gamma(t)$, then $\|J(t)\|_{\gamma(t)}$ is a convex map of $t \in \mathbb{R}$. $\in \mathbb{R}$.*

The method of the following proof is based on a similar argument used in [8].

Proof. Notice that by the invariance of the connection and the metric under the action of $Gl(H, B_1(H))$ we may supose that $\gamma(t) = e^{tX}$ is a geodesic starting at $\gamma(0) = 1$, where $X \in B_1(H)_h$.
Then for the field $K(t) = e^{-\frac{tX}{2}} J(t) e^{-\frac{tX}{2}}$ the differential equation (4.1) changes to

$$4\ddot{K} = KX^2 + X^2K - 2XKX. \tag{4.2}$$

Since the group $Gl(H, B_1(H))$ acts by isometries, we have

$$\|J(t)\|_{\gamma(t)} = \|\gamma(t)^{-\frac{1}{2}} J(t) \gamma(t)^{-\frac{1}{2}}\|_1 = \|K(t)\|_1,$$

thus the proof reduces to show that for any solution $K(t)$ of (4.2), the map $t \to \|K(t)\|_1$ is convex for $t \in \mathbb{R}$.
Fix $u < v \in \mathbb{R}$ and let $t \in [u, v]$. We shall prove that

$$\|K(t)\|_1 \leq \frac{v-t}{v-u}\|K(u)\|_1 + \frac{t-u}{v-u}\|K(v)\|_1.$$

Let $X = \sum_{i\in\mathbb{N}} \lambda_i \langle ., e_i\rangle e_i$ be the spectral decomposition of $X \in B_1(H)_h$ where $\{e_i : i \in \mathbb{N}\}$ is an orthonormal basis of H.
Consider the matrix valued map

$$k(t) = (k_{ij}(t))_{i,j\in\mathbb{N}},$$

where $k_{ij}(t) = \langle K(t)e_i, e_j\rangle$ for all $t \in \mathbb{R}$.
The differential equation (4.2) is equivalent to the equations

$$\ddot{k_{ij}}(t) = \delta_{ij}^2 k_{ij}(t),$$

where $\delta_{ij} = \frac{\lambda_i - \lambda j}{2}$.
A simple verification shows that all solutions of $\ddot{f}(t) = c^2 f(t)$ satisfy

$$f(t) = \phi(u, v, c; t)f(u) + \psi(u, v, c; t)f(v),$$

where

$$\phi(u, v, c; t) = \begin{cases} \frac{\operatorname{Sinh} c(v-t)}{\operatorname{Sinh} c(v-u)} & \text{if c} \neq 0; \\ \frac{(v-t)}{(v-u)}, & \text{if c=0.} \end{cases}$$

$$\psi(u, v, c; t) = \begin{cases} \frac{\operatorname{Sinh} c(t-u)}{\operatorname{Sinh} c(v-u)} & \text{if c} \neq 0; \\ \frac{(t-u)}{(v-u)}, & \text{if c=0.} \end{cases}$$

Then each $k_{ij}(t)$ satisfies

$$k_{ij}(t) = \phi_{ij}(t)k_{ij}(u) + \psi_{ij}(t)k_{ij}(v),$$

where $\phi_{ij}(t) = \phi(u, v, \delta_{ij}; t)$ and $\psi_{ij}(t) = \psi(u, v, \delta_{ij}; t)$. In matrix form

$$k(t) = \Phi(t) \circ k(u) + \Psi(t) \circ k(v),$$

where $\Phi(t) = \{\phi_{ij}(t)\}$, $\Psi(t) = \{\psi_{ij}(t)\}$ and $\circ$ denotes the Schur product of matrices, i.e. $\{a_{ij}\} \circ \{b_{ij}\} = \{a_{ij}b_{ij}\}$. Thus we have that

$$\|k(t)\|_1 \leq \|\Phi(t) \circ k(u)\|_1 + \|\Psi(t) \circ k(v)\|_1. \tag{4.3}$$

We make the following claim:

Claim 4.1. Let $\Psi(t), \Phi(t)$ *and* $k(t)$ as above, then

1. $\|\Phi(t) \circ k(u)\|_1 \leq \frac{v-t}{v-u}\|k(u)\|_1$,
2. $\|\Psi(t) \circ k(v)\|_1 \leq \frac{t-u}{v-u}\|k(v)\|_1$.

Proof. We only prove the first inequality, the second is analogous. Define for each $n \in \mathbb{N}$ and $A = \{a_{ij}\}_{i,j\in\mathbb{N}}$

$$A_n = \begin{cases} a_{ij} & \text{if } 1 \leq i, j \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $n \to \infty$,

$$\Phi(t) \circ k(u)_n \xrightarrow{\|\cdot\|_1} \Phi(t) \circ k(u), \tag{4.4}$$

since

$$\begin{aligned} \|\Phi(t) \circ k(u)_n - \Phi(t) \circ k(u)\|_1 &\leq \max_{i>n} |\phi_{ii}(t)| \sum_{i>n} |k_{ii}(u)| \qquad (4.5) \\ &= \frac{(v-t)}{(v-u)} \sum_{i>n} |\langle K(u)e_i, e_i\rangle| \\ &\leq \sum_{i>n} |\langle K(u)e_i, e_i\rangle| \to 0. \end{aligned}$$

Next we use a theorem by Ando, Horn and Johnson ([3]), according to which if A and P are $n \times n$ matrices, with P positive semidefinite, then

$$\|A \circ P\|_1 \leq (\max_{1 \leq i \leq n} p_{ii}) \, \|A\|_1 \, .$$

Thus

$$\|\Phi(t) \circ k(u)_n\|_1 = \|\Phi(t)_n \circ k(u)_n\|_1 \leq (\max_{1 \leq i \leq n} \phi_{ii}(t)) \|k(u)_n\|_1. \tag{4.6}$$

We conclude from (4.4) and (4.6) that

$$\|\Phi(t) \circ k(u)\|_1 \leq \frac{v-t}{v-u} \|k(u)\|_1.$$

□

Consequently we get

$$\|k(t)\|_1 \leq \frac{v-t}{v-u} \|k(u)\|_1 + \frac{t-u}{v-u} \|k(v)\|_1.$$

□

Remark 4.1. For each $n \in \mathbb{N}$ both matrices, $\Phi(t)_n$ and $\Psi(t)_n$, are positive definite. This follows from Bochner's Theorem applied to $\Phi(u, v, c; t)_n$ and $\Psi(u, v, c; t)_n$ considered as functions of c. In both cases the matrix is of the form $\{F(\lambda_i - \lambda_j)\}_n$ where $F(c)$ is the Fourier transform of a positive function (see [9], formula 1.9.14, page 31).

A consequence of this result follows:

Theorem 4.2. *Let $\gamma(t), \rho(t)$ be geodesics in Δ_1 , then $t \to d(\gamma(t), \rho(t))$ is a convex map in $\mathbb{R}$.*

Proof. Suppose the $\gamma(t)$ and $\sigma(t)$ are defined in $[u, v]$. We consider $h(s, t)$ defined as follows:

1. the map $s \to h(s, u), 0 \leq s \leq 1$ is the geodesic joining $\gamma(u)$ with $\rho(u)$;

2. the map $s \to h(s,v), 0 \le s \le 1$ is the geodesic joining $\gamma(v)$ with $\rho(v)$;
3. for each s, the function $t \to h(s,t), u \le s \le v$ is the geodesic joining $h(s,u)$ with $h(s,v)$.

Let $J(s,t) = \frac{\partial h(s,t)}{\partial s}$. Hence, for each fixed $s, t \to J(s,t)$ is Jacobi field along the geodesic $t \to h(s,t)$. Finally, we define

$$f(t) = \int_0^1 \|J(s,t)\|_{h(s,t)} ds.$$

From Theorem 4.1, $t \to \|J(s,t)\|_{h(s,t)}$ is a convex function for each s. Hence, $t \to f(t)$ ia also convex for $t \in [u,v]$. But $f(u) = \int_0^1 \|J(s,u)\|_{h(s,u)} ds$ is the lenght of $s \to h(s,u)$ and therefore $f(u) = d(\gamma(u), \rho(u))$. Similarly, $f(v) = d(\gamma(v), \rho(v))$. Now, for $u \le t \le v$ $f(t) = \int_0^1 \|J(s,t)\|_{h(s,t)} ds$ is the lenght of the curve $s \to h(s,t)$ which joins $\gamma(t)$ with $\rho(t)$ and then we get $d(\gamma(t), \rho(t)) \le f(t)$. Convexity of $d(\gamma(t), \sigma(t))$ follows and the Theorem is proved. □

Remark 4.2. A particular consequence of the above theorem is that there are no closed nonconstant geodesics in Δ_1. Indeed if $\alpha : [0,1] \to \Delta_1$ is a nonconstant geodesic such that $\alpha(0) = \alpha(1) = a$, then for all $t \in (0,1)$

$$0 < d(a, \alpha(t)) \le t d(a, \alpha(0)) + (1-t) d(a, \alpha(1)) = 0.$$

Definition 4.2. A subset K of Δ_1 is called *convex* if for all $a, b \in K$ the geodesic $\gamma_{a,b}$, joining a and b, is contained in K.

Corollary 4.1. *Let $a, b, c \in \Delta_1$. Then for all $t \in [0,1]$*

$$d(a \sharp_t b, a \sharp_t c) \le t d(b,c). \tag{4.7}$$

In particular,

$$d(b^t, c^t) \le t d(b,c).$$

There is a clear interpretation of the corollary above. In a Riemannian manifold M, the sectional curvature is nonpositve if and only if

$$d(\rho_s(x), \rho_s(y)) \le s d(x,y),$$

for all $x, y \in M$ and all $s \in [0,1]$, where $\rho_s(x) = exp_p(s\ exp_p^{-1}(x))$ and $p \in M$ is fixed (see [4]). This expression reduces, in our (non Riemannian) case to

$$d(p \sharp_s x, p \sharp_s y) \le s d(x,y),$$

which is (4.7).

Corollary 4.2. *Let $a \in \Delta_1$, a fixed. Then*

$$f(\gamma(t)) \leq (1-t)f(\gamma(0)) + tf(\gamma(1)),$$

where $f(x) = d(a,x)$ and $\gamma(t)$ is a geodesic. In particular, geodesics spheres are convex sets.

4.1. The Metric Increasing Property of the Exponential Map

In this section we provide a proof of the metric increasing property (MIP) of the exponential map (Theorem 4.3) which is based on the exposition in Corach, Porta and Recht [8]. We begin with a lemma of approximation.

Lemma 4.1. Let $\gamma(t)$ be a curve in Δ_1 , then $log(\gamma(t))$ can be approximated uniformly by polynomials for $t \in [t_0, t_1]$.

Proof. Throughout the proof $Hol(G)$ and S^2 denote the set of all complex analytic functions defined in G, with G an open set of complex plane and the Riemann sphere, respectively. Let $\sigma(t)$ be the spectrum of $\gamma(t)$, $\sigma(\gamma) = \bigcup_{t\in[t_0,t_1]} \sigma(t)$ be the spectrum of γ in the algebra $C([0,1], \mathcal{L}_1)$ and $G \subseteq \mathbb{C} - \{z : Im(z) \leq 0\}$ an open neighbourhood of $\sigma(\gamma)$.
Since $\sigma(\gamma)$ is compact, $S^2 - \sigma(\gamma)$ is connected and $log(z) \in Hol(G)$ then there is a sequence P_n of polynomials such that $P_n(z) \to log(z)$ uniformly on $\sigma(\gamma)$ ([20], Theorem 13.7).
Since $P_n(z)$ are all analytic on G, $\sigma(\gamma) \subseteq G$, and $P_n(z) \to log(z)$ uniformly on compact subsets of G, then $\|P_n(\gamma(t)) - log(\gamma(t))\|_1 \to 0$ as $n \to \infty$. $\square$

The Finsler structure of Δ_1 is not Riemannian. However Δ_1 shares some property with Riemannian manifolds of non-positive sectional curvature. For instance, the following

Theorem 4.3. *The exponential map in Δ_1 increases distances, i.e. for all $a \in \Delta_1$, $X, Y \in B_1(H)_h$ we have*

$$d(exp_a(X), exp_a(Y)) \geq \|X - Y\|_a . \tag{4.8}$$

Proof. Let $\gamma_1(t) = e^{tX}$,$\gamma_2(t) = e^{tY}$ and $f(t) = d(\gamma_1(t), \gamma_2(t))$. By Theorem 4.2, f is convex, with $f(0) = 0$. Hence $\frac{f(t)}{t} \leq f(1)$ for all $t \in (0,1]$. Note that

$$\frac{f(t)}{t} = \frac{1}{t} \left\| log(e^{tX/2}e^{-tY}e^{tX/2}) \right\|_1 = tr \left| \frac{1}{t}\, log(e^{tX/2}e^{-tY}e^{tX/2}) \right| .$$

Taking limits we have

$$\lim_{t\to 0^+} \frac{f(t)}{t} \leq f(1).$$

Observe next that by the previous lemma, $log\ x$ can be approximated on any interval $[x_0, x_1]$ with $0 < x_0 < x_1$ uniformly by polinomials $P_n(x)$. In particular

$$\lim_{n\to\infty} P_n(x) = log\ x \qquad \text{and} \qquad \lim_{n\to\infty} \dot{P}_n(x) = \frac{1}{x}.$$

(in morm $\|.\|_1$).
Then

$$\lim_{t\to 0^+} \left| \frac{1}{t}\ log(e^{tX/2} e^{-tY} e^{tX/2}) \right| = |X - Y|\,.$$

From this inequality and convexity we conclude that

$$f(t) \geq t\,\|X - Y\|_1\,.$$

This implies that
$d(exp_a(tX), exp_a(tY)) \geq t\,\|X - Y\|_a$ for all $a \in \Delta_1$, and all $X, Y \in B_1(H)_h$. $\square$

Remark 4.3. For $a = 1$, from the theorem above we get

$$\|X - Y\|_1 \leq \left\| log(e^{-\frac{X}{2}} e^{Y} e^{-\frac{X}{2}}) \right\|_1,$$

for $X, Y \in B_1(H)_h$, which can also be written

$$\|logx - logy\|_1 \leq \left\| log(x^{-\frac{1}{2}} y x^{-\frac{1}{2}}) \right\|_1, \tag{4.9}$$

with $x, y \in \Delta_1$.

Proposition 4.1. *Δ_1 is a complete metric space with the geodesic distance.*

Proof. Consider a Cauchy sequence $X_n \subset \Delta_1$. By (4.9) $Y_n = log(X_n)$ is a Cauchy sequence in $B_1(H)_h$. Then there exists an operator $Y \in B_1(H)_h$ such that $Y_n \xrightarrow{\|\cdot\|_1} Y$. Hence

$$d(X_n, e^{Y}) = \left\| log(e^{\frac{Y}{2}} e^{-Y_n} e^{\frac{Y}{2}}) \right\|_1 \to 0,$$

when $n \to \infty$. $\square$

5. Non-positive Curvature

5.1. Metrics spaces of non-positive curvature

It would be very interesting to understand the relations between the geodesic distance and general metric spaces with non-positive curvature.

In this section, we will briefly review some basic facts about these spaces. About fifty years ago Alexandrov showed that the notions of upper and lower curvature bounds make sense for a more general class of metric spaces than Riemannian manifolds, namely, for *geodesic spaces*. One of the first papers on non-positively curved spaces was written by Busemann in 1948. For more details on metric spaces with non-positive curvature we refer to [14].

We now introduce the notion of midpoint maps and Busemann's notion of non-positive curvature in a metric space (X, d).

Definition 5.1. Let (X, d) be a metric space. A symmetric map $M : X \times X \longrightarrow X$ is called a midpoint map for (X, d) if for all $x, y \in X$

$$d(M(x, y), x) = \frac{1}{2}d(x, y) = d(M(x, y), y).$$

Definition 5.2. A complete metric space (X, d) is called a *geodesic length space*, or simply a *geodesic space*, if for any two points $x, y \in X$, there exists a shortest geodesic joining them, i.e. a continuous curve such that $\gamma : [0, 1] \to X$ with $\gamma(0) = x, \gamma(1) = y$, and

$$d(x, y) = l_d(\gamma).$$

Here, $l_d(\gamma)$ denotes the *length* of γ (respect to the metric d) and it is defined as

$$l_d(\gamma) := sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\}.$$

A curve $\gamma : [0, 1] \to X$ is called a *geodesic* if there exists $\epsilon > 0$ such that

$$l_d(\gamma\,\big|_{[t,t']}) = d(\gamma(t), \gamma(t')) \quad whenever\, |t - t'| < \epsilon.$$

Finally, a geodesic $\gamma : [0, 1] \to X$ is called a *shortest geodesic* if

$$l_d(\gamma) = d(\gamma(0), \gamma(1)).$$

In particular, for the metric space (Δ_1, d) the geodesics $\gamma_{a,b}$ (in the sense of the equation (2.1)) are also shortest geodesic, since

$$
\begin{aligned}
l_d(\gamma_{a,b}) &= sup\{\sum_{i=1}^{n} d(\gamma_{a,b}(t_{i-1}), \gamma_{a,b}(t_i)) : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\} \\
&= sup\{\sum_{i=1}^{n} l(\gamma_{a,b}\big|_{[t_{i-1},t_i]}) : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\} \\
&= sup\{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\dot{\gamma}_{a,b}(t)\|_{\gamma_{a,b}(t)}\, dt : 0 = t_0 < t_1 < ... < t_n = 1, n \in \mathbb{N}\} \\
&= l(\gamma_{a,b}).
\end{aligned}
$$

this equality implies that $\gamma_{a,b}$ is also a geodesic in the metric sense

$$
l_d(\gamma_{a,b}\big|_{[t,t']}) = l(\gamma_{a,b}\big|_{[t,t']}) = d(\gamma_{a,b}(t), \gamma_{a,b}(t')).
$$

Then

$$
l_d(\gamma_{a,b}) = l(\gamma_{a,b}) = d(\gamma_{a,b}(0), \gamma_{a,b}(1)).
$$

By the above argument, we have the following statement

Proposition 5.1. *The metric space (Δ_1, d) is a geodesic space and $m(\ .\ ,\ .\ )$ is a midpoint point corresponding to the shortest geodesic $\gamma_{a,b}$ for all $a, b \in \Delta_1$.*

Definition 5.3. Let (X, d) be a metric space and $m : X \times X \longrightarrow X$ be a midpoint map for (X, d). Then (X, d) is said to be a *m-global Busemann non-positive curvature space* ($m-global\ BNPC$) if for all $x, y, z \in X$

$$
d(m(x, y), m(x, z)) \leq \frac{1}{2} d(y, z). \tag{5.1}
$$

Remark 5.1. The m-global BNPC condition is equivalent to m is a convex midpoint map, i.e. for all $x_1, x_2, y_1, y_2 \in X$

$$
d(m(x_1, y_1), m(x_2, y_2)) \leq \frac{1}{2}(d(x_1, x_2) + d(y_1, y_2)).
$$

Theorem 5.1. *The space (Δ_1, d) is a m-global Busemann NPC space.*

Here, we introduce the notion of convex hull of a subset K.

Definition 5.4. Let (X, d) be a geodesic length space. The *convex hull C(K)* of a subset K of X is the smallest convex subset of X containing K.

In general, the convex hull of a set K as defined above need not exist, because the intersection of convex subsets of X need not be convex. Nevertheless, for (Δ_1, d) there is a constructive approach to compute $C(K)$ due to its Busemann NPC structure.

Proposition 5.2. *For any $K \subseteq \Delta_1$, the convex hull $C(K)$ exists and can be obtained as follows*

$$C(K) = \bigcup_{n=0}^{\infty} K_n.$$

where $K_0 = K$ and $K_n := \bigcup\{a \sharp_t b : a, b \in K_{n-1}\}$.

Proof. Lemma 3.3.1 ([14]). □

5.2. An alternative definition of sectional curvature

In this section, we shall see that it is possible to give an alternative definition of sectional curvature in Δ_1 . For this, we remember that in [17] Milnor recalls that the sectional curvature, $s_a(X,Y)$, can be obtained by the following limit

$$s_a(X,Y) = 6 \lim_{r\to 0^+} \frac{r\,\|X-Y\|_a - d(exp_a(rX), exp_a(rY))}{r^2 d(exp_a(rX), exp_a(rY))}.$$

where X, Y are tangent vectors at a point a. We will see that this limit makes sense in our context.
Suppose that $r > 0$ is close to 0 such that $e^{-rX/2}e^{rY}e^{-rX/2}$ lies within the radius of convergence of the series $log(u)$. Them

$$log(e^{-rX/2}e^{rY}e^{-rX/2}) = r(Y-X) + r^3\kappa(X,Y) + o(r^3),$$

where

$$\kappa(X,Y) = \frac{1}{6}YXY + \frac{1}{12}XYX - \frac{1}{12}(XY^2 + Y^2X) - \frac{1}{24}(X^2Y + YX^2).$$

Before stating the existence of the limit above we need the following definiton and lemmas.

Definition 5.5. Let V a vector space and f be a function from V to $\mathbb{R} \cup \{+\infty\}$. We shall say that $Df(x_0)(v)$ is the *right derivate of f at x_0 in the direction v* if the limit

$$Df(x_0)(v) = \lim_{t\to 0^+} \frac{f(x_0+tv) - f(x_0)}{t}.$$

exists. In this case, we denote by $v \to Df(x_0)(v)$ the right derivate of f at x_0.

Remark 5.2. ([1], Proposition 4.1) Let V a vector space and f be a nontrivial convex function from V to $\mathbb{R} \cup \{+\infty\}$. Suppose $x_0 \in Dom(f)$ and $v \in V$. Then the limit $Df(x_0)(v)$ exists in $\overline{\mathbb{R}}$ and satisfies

$$f(x_0) - f(x_0 - v) \leq Df(x_0)(v) \leq f(x_0 + v) - f(x_0).$$

For $a \in \Delta_1$, we denote by

$$P_a : (T\Delta_1)_a \to \mathbb{R}^+, \qquad P_a(X) = \|X\|_a$$

Lemma 5.1. For $a \in \Delta_1$, P_a is a convex function. Moreover, P_a is right differentiable on $B_1(H)_h$ and satisfies

$$\|X\|_a - \|X-Y\|_a \leq DP_a(X)(Y) \leq \|X+Y\|_a - \|X\|_a .$$

Proof. By the remark 5.2 it suffices to prove that P_a is convex. Clearly this is obvious for the usual properties of a norm, since for all $\lambda \in (0,1)$

$$P_a(\lambda X + (1-\lambda)Y) \leq \lambda P_a(X) + (1-\lambda)P_a(Y).$$

□

Theorem 5.2. *Let $a \in \Delta_1$ and $X, Y \in B_1(H)_h$. The limit*

$$s_a(X,Y) = \lim_{r\to 0^+} \frac{r\|X-Y\|_a - d(exp_a(rX), exp_a(rY))}{r^2 d(exp_a(rX), exp_a(rY))}$$

exists and verifies

$$1 - \frac{\|Y-X+\kappa(X,Y)\|_1}{\|X-Y\|_1} \leq s_a(X,Y) \leq 0.$$

Proof. Since the metric on $B_1(H)_h$ and the geodesic distance are invariant by the action of $Gl(H, B_1(H))$, it suffices to consider the case $a = 1$. Note that

$$\lim_{r\to 0^+} \frac{1}{r} d(e^{rX}, e^{rY}) = \lim_{r\to 0^+} \left\|Y - X + r^2\kappa(X,Y) + o(r^2)\right\|_1 = \|Y-X\|_1 .$$

Them it is enough to show the existence of the following limit

$$\lim_{r\to 0^+} \frac{1}{r^3}(r\|X-Y\|_1 - \left\|r(Y-X) + r^3\kappa(X,Y) + o(r^3)\right\|_1),$$

which is equivalent to the existence of the limit

$$\lim_{r\to 0^+} \frac{1}{r^2}(\|X-Y\|_1 - \left\|(Y-X) + r^2\kappa(X,Y)\right\|_1).$$

But this exists and is equal to $-DP_1(Y-X)(\kappa(X,Y))$ and therefore

$$s_1(X,Y) = \frac{-DP_1(Y-X)(\kappa(X,Y))}{\|Y-X\|_1}.$$

By the MIP property this limit is non positive. On the other hand,

$$-DP_1(Y-X)(\kappa(X,Y)) \geq \|Y-X\|_1 - \|Y-X+\kappa(X,Y)\|_1$$

and therefore $s_1(X,Y) \geq 1 - \frac{\|Y-X+\kappa(X,Y)\|_1}{\|X-Y\|_1}$. □

References

[1] J. Aubin, *Optima and Equilibria - An introduction to nonlinear analysis.* 2nd Edition, Springer Verlag, 1998.

[2] E. Andruchow and L. Recht, *Sectional curvature and commutation of pairs of seladjoint operators.* Journal of Operator Theory(to appear).

[3] T, Ando; R. Horn and Ch. Johnson, *The Singular Values of a Hadamard product: a basic inequality.* Linear Multinear Algebra **21** (1987), 345–365.

[4] W. Ballmann; M. Gromov and V. Schroeder, *Manifolds of non positive curvature.* Birkhäuser Verlag, 1985.

[5] R. Bhatia, *On the exponential metric increasing property.* Linear Algebra and its applications **375** (2003), 211–220.

[6] G. Corach; H. Porta and L. Recht, *A geometric interpretation of Segal´s inequality.* Proc. Amer. Math. Soc. **115** (1992), 229–231.

[7] G. Corach; H. Porta and L. Recht, *The geometry of spaces of selfadjoint invertible elements of a C^*-algebra.* Integral Equations and Operator Theory **16** (1993), 333–359.

[8] G. Corach; H. Porta and L. Recht, *Convexity of the geodesic distance on spaces of positive operators.* Illinois Journal of Mathematics **38** N$^{\text{o}}$ 1 (1994), 87–94.

[9] Erdelyi, A. et al.: *Tables of integral transforms.* (The Bateman manuscript), Volumen 1, Mc Graw Hill, 1954.

[10] M. Fujii; R. Furuta and R. Nakamoto, *Norm inequalities in the Corach-Porta-Recht theory and operator means.* Illinois Journal of Mathematics **40** N$^{\text{o}}$ 4 (1996), 1–8.

[11] J. Fujii and E. Kamei, *Relative operator entropy in nonconmutative information theory.* Mathematica Japonica **34** (1989), 341–348.

[12] P. de la Harpe, *Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space.* Lecture Notes in Mathematics 285, Springer-Verlag, 1972.

[13] F. Hiai and H. Kosaki, *Comparison of various Means of operators.* Journal of Functional Analysis **163** (1999), 300–323.

[14] J. Jost, *Nonpositive Curvature: Geometric and Analytic Aspects,* Birkhäuser Verlag, 1997.

[15] S. Lang, *Differential and Riemannian Manifolds*, Springer, 1995.

[16] L. Mata Lorenzo and L. Recht, *Infinite Dimensional Homogeneous Reductive Spaces.* Acta Científica Venezolana **43** N^{o} 2 (1992), 76–90.

[17] J. Milnor, *Morse Theory* Annals of Mathematical Studies 54, Princeton University Press, 1963.

[18] G. Mostow, *Some new decomposition theorems for semi-simple groups*, Mem. Amer. Math. Soc. **14** (1955), 31-54.

[19] W. Pusz and S. Woronowicz, *Functional Calculus for sesquilinear forms and the purification map.* Rep. Math. Phys. **8** (1975), 159–170.

[20] W. Rudin, *Real and Complex Analysis,* Tata Mc Graw Hill, 1974.

[21] A. Uhlmann, *Density operators as an arena for differential geometry.* Rep. Math. Phys. **36** (1993), 253–263.

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