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The Essential Norm of Operators in the Toeplitz Algebra on $A^p(\mathbb{B}_n)$

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ABSTRACT. Let A^p be the Bergman space on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ for $1 < p < \infty$, and $\mathfrak{T}_p$ be the corresponding Toeplitz algebra. We show that every $S \in \mathfrak{T}_p$ can be approximated by operators that are specially suited for the study of local behavior. This is used to obtain several estimates for the essential norm of $S \in \mathfrak{T}_p$, an estimate for the essential spectral radius of $S \in \mathfrak{T}_2$, and a localization result for its essential spectrum. Finally, we characterize compactness in terms of the Berezin transform for operators in $\mathfrak{T}_p$.

CONTENTS

1. INTRODUCTION AND PRELIMINARIES

For $0 < p \leq \infty$ consider the space $L^p = L^p(\mathbb{B}_n, dv)$, where $\mathbb{B}_n$ is the open unit ball in $\mathbb{C}^n$ and dv is the normalized volume measure on $\mathbb{B}_n$. The Bergman space A^p consists of the analytic functions in L^p (as usual, we write H^∞ if $p = \infty$). When $1 < p < \infty$, the Bergman projection P defines a bounded operator from L^p onto A^p. If $a \in L^\infty$ let $M_a : L^p \to L^p$ be the operator of multiplication by a and $P_a = PM_a$. Then $\|P_a\| \leq C_p \|a\|_\infty$, where C_p is the norm of P acting on L^p. The Toeplitz operator $T_a : A^p \to A^p$ is the restriction of P_a to the space A^p. If E_1 and E_2 are Banach spaces, we write $\mathfrak{L}(E_1, E_2)$ for the space of all bounded operators from E_1 into E_2, or just $\mathfrak{L}(E_1)$ if $E_1 = E_2$. The Toeplitz algebra on A^p is

$$\mathfrak{T}_p = \text{the closed subalgebra of } \mathfrak{L}(A^p) \text{ generated by } \{T_a : a \in L^\infty\}.$$

This paper has three purposes. The first purpose is to approximate in norm an operator $S \in \mathfrak{T}_p$ by a strongly convergent series of operators formed by 'truncations' of S. We call this series a segmented operator. Each truncation of S is associated with a compact set $K \subset \mathbb{B}_n$, so that its value at a given $f \in A^p$ is controlled by the behavior of f in a quantitatively determined hyperbolic neighborhood of K. This means that a segmented operator splits into a sum of operators that in some sense can be localized. This useful approximation-localization scheme will be applied to obtain several estimates of the essential norm for $S \in \mathfrak{T}_p$ (denoted $\|S\|_e$). This is the second purpose of the paper. The most involved estimate of $\|S\|_e$ is given in terms of a family of associated operators $\{S_x\}_{x \in E}$, where E is the complement of $\mathbb{B}_n$ inside a special compactification of $\mathbb{B}_n$. In the particular case $p = 2$, the estimate will turn out to give the exact number $\|S\|_e$. Furthermore, if $p = 2$, the family $\{S_x\}_{x \in E}$ will be used to estimate the essential spectral radius of S and to localize its essential spectrum. This localization takes a distinctively simple form when $S \in \mathfrak{T}_2$ is essentially normal.

The Berezin transform is a bounded linear map $B : \mathfrak{L}(A^p) \to L^\infty$, where $1 < p < \infty$. Since the Berezin transform is one-to-one, every bounded operator S on A^p is determined by $B(S)$. Despite this fact, the information on S that we can collect by only looking at $B(S)$ rarely is in the surface. To further complicate matters, the range of B is not closed, and therefore the inverse map $B^{-1} : B(\mathfrak{L}(A^p)) \to \mathfrak{L}(A^p)$ is not bounded. In the positive direction, there is a growing body of research to establish relations between some properties of S and $B(S)$. This view has been particularly successful when dealing with the compactness of operators related to function theory. If $S \in \mathfrak{L}(A^p)$ is compact, then $B(S)(z) \to 0$ when $|z| \to 1$, while several authors have shown examples where the reciprocal implication does not hold (see [2] and [11]).

On the other hand, when $p = 2$, Coburn [4] showed that the compact operators form the commutator ideal of $\mathfrak{T}_2(C(\bar{\mathbb{B}}_n))$, the closed algebra generated by Toeplitz operators with continuous symbol on the closed ball $\bar{\mathbb{B}}_n$, and Engliš [8] proved that every compact operator is the norm limit of Toeplitz operators with bounded symbol. Any of these results implies that the compact operators are contained in $\mathfrak{T}_2$. We will see that this also holds for $1 < p < \infty$. Therefore, we have the following necessary conditions for $S \in \mathfrak{L}(A^p)$ to be compact

$$S \in \mathfrak{T}_p \quad \text{and} \quad \lim_{|z|\to 1} B(S)(z) = 0. \tag{1.1}$$

The above mentioned counterexamples show that there is no redundance in these conditions, since there are plenty of non-compact operators $S \in \mathfrak{L}(A^2)$ satisfying the second condition. These facts triggered extensive studies showing that for different subclasses $\mathfrak{S} \subset \mathfrak{T}_2$, the implication

$$\lim_{|z|\to 1} B(S)(z) = 0 \;\Rightarrow\; S \text{ is compact} \tag{1.2}$$

holds for $S \in \mathfrak{S}$ (see [2], [9, 10], [12], [14], [16], [18], [20], [22], and [24]). The survey paper of Stroethoff [19] is a good source to get a taste of some of the above results. Clearly, the final goal of these studies is to find a reasonable answer to the question: what operators S satisfy (1.2)?

One of the most general results obtained so far was given by Axler and Zheng [2] for the disk and later generalized by Enlgiš [9, 10] to irreducible bounded symmetric domains in $\mathbb{C}^n$. They proved that if S is a several variables polynomial of Toeplitz operators T_a ($a \in L^\infty$) acting on A^2, then S satisfies (1.2) (the precise statement in [9, 10] is more complicated, since it deals with weighted Bergman spaces of more general domains). This means that (1.2) holds for a dense subclass $\mathfrak{S} \subset \mathfrak{T}_2$, and it suggests that the answer to the question when $p = 2$ should be $\mathfrak{T}_2$.

The third purpose of this paper is to prove that (1.2) holds on the ball $\mathbb{B}_n$ for every $S \in \mathfrak{T}_p$, where $1 < p < \infty$. This is achieved by exploiting the interaction between $B(S)$ and the family $\{S_x\}_{x\in E}$ together with the corresponding characterization of $\|S\|_e$ in terms of this family. This means that the conditions in (1.1) characterize compactness, which gives a complete answer to the question. These results are new even for $n = 1$ and $p = 2$.

2. Operators Associated to Carleson Measures

We fix the dimension n and write $\mathbb{B} = \mathbb{B}_n$. Accordingly, it should be assumed that the multiplicative constants in the paper depend on n, even when this is not always explicitly stated. If $z, w \in \mathbb{B}$, we write $\langle z, w\rangle$ for the inner product in $\mathbb{C}^n$ and $|z|$ for the norm; P_z will be the orthogonal projection onto the complex line $\mathbb{C}z$, and $Q_z = I - P_z$ its complementary projection. The function

$$\varphi_z(\omega) = \frac{z - P_z(\omega) - (1 - |z|^2)^{1/2} Q_z(\omega)}{1 - \langle \omega, z\rangle}$$

is the (unique) automorphism of $\mathbb{B}$ that satisfies $\varphi_z \circ \varphi_z = id$ and $\varphi_z(0) = z$. The pseudo-hyperbolic and hyperbolic metrics on $\mathbb{B}$ are defined, respectively, by

$$\rho(z,\omega) = |\varphi_z(\omega)| \quad \text{and} \quad \beta(z,\omega) = \frac{1}{2}\log\frac{1+\rho(z,\omega)}{1-\rho(z,\omega)}.$$

Thus, $\rho = (e^{2\beta}-1)/(e^{2\beta}+1) = \tanh\beta$. These metrics are invariant under actions of $\mathrm{Aut}(\mathbb{B})$. For $r > 0$ write

$$D(z,r) \overset{\text{def}}{=} \{\omega \in \mathbb{B} : \beta(\omega,z) \leq r\}.$$

Therefore, $D(z,r) = \{\omega \in \mathbb{B} : \rho(\omega,z) \leq s\}$, where $s = \tanh r$. We shall make extensive use of the classical equality

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2}$$

(see [17, Chapter 2]). We will also write $\langle , \rangle$ for the usual integral pairing between functions. If $1 < p < \infty$, the Bergman projection $P : L^p \to A^p$ is defined as $(Pf)(z) = \langle f, K_z\rangle$, where

$$K_z(w) = \frac{1}{(1-\langle w,z\rangle)^{n+1}}, \quad w \in \mathbb{B},$$

is the reproducing kernel for $z \in \mathbb{B}$. If $1/p + 1/q = 1$, there is a constant $c_p > 0$ such that the functions

$$k_z^{(p)}(w) = \frac{(1-|z|^2)^{(n+1)/q}}{(1-\langle w,z\rangle)^{n+1}}, \quad w \in \mathbb{B},$$

satisfy $c_p^{-1} \leq \|k_z^{(p)}\|_p \leq c_p$ for all $z \in \mathbb{B}$. That is, $k_z^{(p)}$ plays the same role for a general p that the normalized reproducing kernel $k_z^{(2)} = K_z/\|K_z\|_2$ plays for $p = 2$. The Berezin transform of $S \in \mathfrak{L}(A^p)$ is the function

$$B(S)(z) = (1-|z|^2)^{n+1}\langle SK_z, K_z\rangle = \langle Sk_z^{(p)}, k_z^{(q)}\rangle, \quad (z \in \mathbb{B}).$$

It is clear that $B(S) \in L^\infty$ and $\|B(S)\|_\infty \leq C_p\|S\|$, where $C_p > 0$ only depends on p.

Unless stated otherwise, by a measure we mean a positive, finite, regular, Borel measure. If $p \geq 1$, a measure ν on $\mathbb{B}$ is called a Carleson measure (for A^p) if there is $C > 0$ such that

$$\int_{\mathbb{B}} |f|^p \, d\nu \leq C \int_{\mathbb{B}} |f|^p \, dv$$

for every $f \in A^p$. When this holds, the inclusion of A^p into $L^p(d\nu)$ will be denoted ι_p. If ν is a measure, the operator

$$T_\nu f(z) = \int_{\mathbb{B}} \frac{f(w)}{(1-\langle z,w\rangle)^{n+1}}\,\mathrm{d}\nu(w),$$

defines an analytic function for every $f \in H^\infty$. So, T_ν is densely defined on A^p and it is well-known that for $1 < p < \infty$, T_ν is bounded if and only if ν is a Carleson measure for A^p. As it turned out, this condition does not depend on p.

The next four lemmas are well-known or easily deduced from well-known results, so proofs are kept to a minimum.

Lemma 2.1. *Let $1 < p < \infty$, ν be a measure on $\mathbb{B}$ and $r > 0$. The following quantities are equivalent (with constants depending on n, r and p).*

(1) $$\|\nu\|_* \overset{\text{def}}{=} \sup_{z\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|z|^2)^{n+1}}{|1-\langle w,z\rangle|^{2(n+1)}}\,\mathrm{d}\nu(w),$$

(2) $$\|\iota_p\|^p = \inf\left\{C > 0 : \int |f|^p\,\mathrm{d}\nu \le C\int |f|^p\,\mathrm{dv} \text{ for } f \in A^p\right\},$$

(3) $$\sup_{z\in\mathbb{B}} \frac{\nu(D(z,r))}{\mathrm{v}(D(z,r))},$$

(4) $$\|T_\nu\|_{\mathfrak{L}(A^p)}.$$

Proof. The equivalence between (1), (2) and (3) is in the proof of Theorem 2.25 in [26]. If (4) holds, then $\|\nu\|_* = \|B(T_\nu)\|_\infty \le C_p\|T_\nu\|$, so (1) holds. Finally, if (1) holds and $f, g \in H^\infty$, Fubini's theorem and Hölder's inequality yield

$$\begin{aligned}|\langle T_\nu f, g\rangle| &= \left|\int_{\mathbb{B}} f\bar{g}\,\mathrm{d}\nu\right| \le \|f\|_{L^p(d\nu)}\,\|g\|_{L^q(d\nu)} \\ &\le \|\iota_p\|\,\|\iota_q\|\,\|f\|_{A^p}\,\|g\|_{A^q} \le C_p\|\nu\|_*\,\|f\|_{A^p}\,\|g\|_{A^q},\end{aligned}$$

where the last inequality follows from the equivalence between (1) and (2). The isomorphism $(A^p)^* \simeq A^q$ then gives (4). □

A measure ν satisfying any of the above conditions will be simply called a Carleson measure.

Lemma 2.2. *Let $1 < p < \infty$, $q = p/(p-1)$, $F \subset \mathbb{B}$ be a compact set and ν be a Carleson measure. Then there exists a constant α_p such that*

$$\|T_{\chi_F\nu} f\|_{A^p} \le \alpha_p\|\iota_q\|\;\|\chi_F f\|_{L^p(d\nu)}$$

for every $f \in A^p$.

Proof. Since F is compact and ν is a finite measure, it is clear that $T_{\chi_F \nu} f$ is a bounded analytic function for any $f \in A^p$. As in the proof of the previous lemma, if $g \in A^q$,

$$|\langle T_{\chi_F \nu} f, g \rangle| \leq \|\chi_F f\|_{L^p(d\nu)} \|g\|_{L^q(d\nu)} \leq \|\chi_F f\|_{L^p(d\nu)} \|\iota_q\| \, \|g\|_{A^q}. \qquad \square$$

The following covering was initially constructed by Coifman and Rochberg in connection with a family of atomic decompositions of $A^p(\Omega)$, for bounded symmetric domains $\Omega \in \mathbb{C}^n$ [5]. The proof depends on simple volume arguments, and a version suited for our purpose can be found in [26, Lemma 2.28].

Lemma 2.3. *Given $\varrho > 0$, there is a family of Borel sets $D_m \subset \mathbb{B}$ and points $w_m \in D_m$ such that*
(a) $D(w_m, \varrho/4) \subset D_m \subset D(w_m, \varrho)$ *for all* $m \geq 1$,
(b) $D_m \cap D_k = \varnothing$ *if* $m \neq k$,
(c) $\bigcup_{m \geq 1} D_m = \mathbb{B}$.

The next result is in [17, Proposition 1.4.10].

Lemma 2.4. *For $z \in \mathbb{B}$, s real and $t > -1$, let*

$$F_{s,t}(z) = \int_{\mathbb{B}} \frac{(1 - |\omega|^2)^t}{|1 - \langle z, \omega \rangle|^s} \, dv(\omega).$$

Then $F_{s,t}$ is bounded if $s < n + 1 + t$ and grows as $(1 - |z|^2)^{n+1+t-s}$ when $|z| \to 1$ if $s > n + 1 + t$.

Lemma 2.5. *Let $1 < p < \infty$, ν be a Carleson measure, F_j, $K_j \subset \mathbb{B}$ be Borel sets such that $\{F_j\}$ are pairwise disjoint and $\beta(F_j, K_j) > \sigma \geq 1$ for every j. If $0 < \gamma < \min\left\{1/((n+1)p),\ 1 - 1/p\right\}$, then*

$$\int_{\mathbb{B}} \sum_j [\chi_{F_j}(z) \chi_{K_j}(\omega)] \frac{(1 - |\omega|^2)^{-1/p}}{|1 - \langle z, \omega \rangle|^{n+1}} \, d\nu(\omega) \leq G \|\nu\|_* (1 - \delta^{2n})^\gamma (1 - |z|^2)^{-1/p}, \tag{2.1}$$

where $\delta = \tanh(\sigma/2)$ and $G > 0$ only depends on n, p and γ.

Proof. Since for $z \in F_j$ and $\omega \in K_j$, $\beta(\omega, z) > \sigma$, then $K_j \subset \mathbb{B} \setminus D(z, \sigma)$ and

$$\sum_j \chi_{F_j}(z) \chi_{K_j}(\omega) \leq \sum_j \chi_{F_j}(z) \chi_{\mathbb{B} \setminus D(z,\sigma)}(\omega).$$

Hence, the integral in (2.1) is bounded by

$$J = \sum_j \chi_{F_j}(z) \int_{\mathbb{B}} \chi_{\mathbb{B} \setminus D(z,\sigma)}(\omega) \frac{(1 - |\omega|^2)^{-1/p}}{|1 - \langle z, \omega \rangle|^{n+1}} \, d\nu(\omega). \tag{2.2}$$

Let $w_m \in D_m \subset \mathbb{B}$ be as in Lemma 2.3 with $\varrho = \frac{1}{10}$. When $w \in D_m$, (a) says that $\beta(w, w_m) \le \frac{1}{10}$. Hence, $(1 - |w|^2)$ and $(1 - |w_m|^2)$ are equivalent, and $|1 - \langle z, w\rangle|$ is equivalent to $|1 - \langle z, w_m\rangle|$ independently of $z \in \mathbb{B}$. This implies that there exists $C_1 > 0$ depending only on n and p such that

$$C_1^{-1} \frac{(1-|\omega|^2)^{-1/p}}{|1-\langle z,\omega\rangle|^{n+1}} \le \frac{(1-|\omega_m|^2)^{-1/p}}{|1-\langle z,\omega_m\rangle|^{n+1}} \le C_1 \frac{(1-|\omega|^2)^{-1/p}}{|1-\langle z,\omega\rangle|^{n+1}} \tag{2.3}$$

for every $w \in D_m$ and $z \in \mathbb{B}$. Also, since ν is a Carleson measure and we have fixed $\varrho = \frac{1}{10}$, Lemma 2.1 and (a) of Lemma 2.3 say that there exists an absolute constant $C_2 > 0$ (depending only on n) such that

$$\nu(D_m) \le C_2 \|\nu\|_* \mathrm{v}(D_m). \tag{2.4}$$

It will be convenient to write

$$\phi(w,z) = \frac{(1-|\omega|^2)^{-1/p}}{|1-\langle z,\omega\rangle|^{n+1}} \quad \text{and} \quad D(z,\sigma)^c = \mathbb{B} \setminus D(z,\sigma).$$

Thus $J = \sum_j \chi_{F_j}(z) J_z$, where

$$\begin{aligned}
J_z &:= \int_{\mathbb{B}} \chi_{D(z,\sigma)^c}(\omega)\phi(w,z)\,\mathrm{d}\nu(\omega) \\
&= \sum_{n\ge 1} \int_{D_m} \chi_{D(z,\sigma)^c}(\omega)\phi(w,z)\,\mathrm{d}\nu(\omega) \\
&\le \sum_{D_m \cap D(z,\sigma)^c \ne \emptyset} \int_{D_m} \phi(w,z)\,\mathrm{d}\nu(\omega) \\
&\le C_1 \sum_{D_m \cap D(z,\sigma)^c \ne \emptyset} \int_{D_m} \phi(w_m,z)\,\mathrm{d}\nu(\omega) && \text{by (2.3)} \\
&\le C_1 C_2 \|\nu\|_* \sum_{D_m \cap D(z,\sigma)^c \ne \emptyset} \int_{D_m} \phi(w_m,z)\,\mathrm{dv}(\omega) && \text{by (2.4)} \\
&\le C_1^2 C_2 \|\nu\|_* \sum_{D_m \cap D(z,\sigma)^c \ne \emptyset} \int_{D_m} \phi(w,z)\,\mathrm{dv}(\omega) && \text{by (2.3).}
\end{aligned}$$

If $D_m \cap D(z,\sigma)^c \ne \emptyset$ and $w \in D_m$, then $\beta(w, D(z,\sigma)^c) \le \operatorname{diam}_\beta D_m \le 2\varrho = \frac{1}{5}$, and since

$$\beta(D(z,\sigma/2), D(z,\sigma)^c) = \frac{\sigma}{2} \ge \frac{1}{2},$$

we get

$$D_m \cap D(z,\sigma/2) = \emptyset \quad \text{whenever } D_m \cap D(z,\sigma)^c \ne \emptyset.$$

Therefore

$$J_z \le C_1^2 C_2 \|\nu\|_* \sum_{m\ge 1} \int_{D_m} \chi_{D(z,\sigma/2)^c}(w)\phi(w,z)\,\mathrm{dv}(\omega)$$

$$= C_1^2 C_2 \|\nu\|_* \int_{\mathbb{B}} \chi_{D(z,\sigma/2)^c}(w)\phi(w,z)\,\mathrm{dv}(\omega).$$

Going back to (2.2), we obtain

$$\begin{aligned} J &= \sum_j \chi_{F_j}(z) J_z \\ &\le C_1^2 C_2 \|\nu\|_* \sum_j \chi_{F_j}(z) \int_{\mathbb{B}} \chi_{D(z,\sigma/2)^c}(w)\phi(w,z)\,\mathrm{dv}(\omega). \end{aligned} \tag{2.5}$$

The last sum in (2.5) is

$$\begin{aligned} &\sum_j \chi_{F_j}(z) \int_{\mathbb{B}} \chi_{D(z,\sigma/2)^c}(\omega) \frac{(1-|\omega|^2)^{-1/p}}{|1-\langle z,\omega\rangle|^{n+1}}\,\mathrm{dv}(\omega) \\ &\quad = \sum_j \chi_{F_j}(z) \int_{|v|>\delta} \frac{(1-|\varphi_z(v)|^2)^{-1/p}}{|1-\langle z,v\rangle|^{n+1}}\,\mathrm{dv}(v) \\ &\quad \le \int_{|v|>\delta} \frac{(1-|v|^2)^{-1/p}}{|1-\langle z,v\rangle|^{n+1-2/p}} (1-|z|^2)^{-1/p}\,\mathrm{dv}(v), \end{aligned} \tag{2.6}$$

where the equality comes from the change of variables $v = \varphi_z(\omega)$ and the observation that $\varphi_z(D(z,\sigma/2)^c) = D(0,\sigma/2)^c = \{v \in \mathbb{B} : |v| > \delta = \tanh(\sigma/2)\}$, and the inequality because the sets F_j are pairwise disjoint. Pick a number $a = a(n,p)$ satisfying simultaneously the conditions

$$1 < a < p \quad \text{and} \quad a(n+1-1/p) < n+1.$$

If $a^{-1} + b^{-1} = 1$, Hölder's inequality gives

$$\begin{aligned} &\int_{|v|>\delta} \frac{(1-|v|^2)^{-1/p}}{|1-\langle z,v\rangle|^{n+1-2/p}}\,\mathrm{dv}(v) \\ &\quad \le \left(\int_{\mathbb{B}} \frac{(1-|v|^2)^{-a/p}}{|1-\langle z,v\rangle|^{a(n+1-2/p)}}\,\mathrm{dv}(v) \right)^{1/a} \mathrm{v}(\{|v|>\delta\})^{1/b}. \end{aligned}$$

Since $a(n+1-2/p) = a(n+1-1/p) - a/p < n+1-a/p$, Lemma 2.4 says that the last expression is bounded by $C_3\mathrm{v}(\{|v|>\delta\})^{1/b} = C_3(1-\delta^{2n})^{1/b}$, where C_3 depends only on n, p and a. Inserting this inequality in (2.6) and the resulting inequality in (2.5), we get

$$J \le C_1^2 C_2 C_3 \|\nu\|_* (1-\delta^{2n})^{1/b}(1-|z|^2)^{-1/p}.$$

Write $G = C_1^2 C_2 C_3$ and observe that since $b^{-1} = 1 - a^{-1}$, the restrictions on a translate in terms of b as $0 < b^{-1} < \min\{1/((n+1)p), 1-1/p\}$. The lemma follows from the last inequality and the paragraph preceding (2.2). ◻

We are going to need one of many known versions of Schur's test. There is a proof for $p = 2$ in [15, p. 282] that can be easily adapted to $1 < p < \infty$. A proof containing the result that we need can be found in [7, Proposition 5.12].

Lemma 2.6. *Let $(X, d\mu)$ and $(X, d\nu)$ be measure spaces, $R(x,y)$ be a non-negative $d\mu \times d\nu$-measurable function on $X \times X$, $1 < p < \infty$ and $q = p/(p-1)$. If h is a positive function on X that is measurable with respect to both $d\mu$ and $d\nu$, and C_q, C_p are positive numbers such that*

$$\int_X R(x,y)h(y)^q \,\mathrm{d}\nu(y) \le C_q h(x)^q, \quad d\mu(x)\text{-almost everywhere,}$$
$$\int_X R(x,y)h(x)^p \,\mathrm{d}\mu(x) \le C_p h(y)^p, \quad d\nu(y)\text{-almost everywhere;}$$

then $Sf(x) = \int_X R(x,y)f(y)\,\mathrm{d}\nu(y)$ defines a bounded operator $S : L^p(X, d\nu) \to L^p(X, d\mu)$ with $\|T\| \le C_q^{1/q} C_p^{1/p}$.

If ν is a Carleson measure and $1 < p < \infty$, for $f \in L^p(d\nu)$ define

$$P_\nu f(z) = \int_{\mathbb{B}} \frac{f(w)}{(1-\langle z,w\rangle)^{n+1}}\,\mathrm{d}\nu(w).$$

The argument in the proof of Lemma 2.1 shows that P_ν is a bounded operator from $L^p(d\nu)$ into A^p. Observe also that $T_\nu = P_\nu \circ \iota_p$. If $a \in L^\infty(d\nu)$, we write M_a for the operator of multiplication by a.

Lemma 2.7. *Suppose that $1 < p < \infty$, ν is a Carleson measure, F_j, $K_j \subset \mathbb{B}$ are Borel sets, and $a_j \in L^\infty(\mathrm{d}\nu)$, $b_j \in L^\infty(d\nu)$ are functions of norm ≤ 1 for all $j \ge 1$. If*

(i) $\beta(F_j, K_j) \ge \sigma \ge 1$,
(ii) $\operatorname{supp} a_j \subset F_j$ *and* $\operatorname{supp} b_j \subset K_j$,
(iii) *every $z \in \mathbb{B}$ belongs to at most N (a positive integer) of the sets F_j,*

then $\sum_{j\ge1} M_{a_j} P_\nu M_{b_j} \in \mathfrak{L}(A^p, L^p(\mathrm{d}\nu))$, and there is a function $\beta_p(\sigma) \to 0$ when $\sigma \to \infty$ such that

$$\Big\|\sum_{j\ge1} M_{a_j} P_\nu M_{b_j}\Big\|_{\mathfrak{L}(A^p, L^p(\mathrm{d}\nu))} \le N\beta_p(\sigma)\|\nu\|_* \tag{2.7}$$

and for every $f \in A^p$ *of norm* ≤ 1,

$$\sum_{j\geq 1} ||M_{a_j} P_\nu M_{b_j} f||^p_{L^p(d\nu)} \leq N\beta_p^p(\sigma)||\nu||_*^p. \tag{2.8}$$

Proof. Write $\delta = \tanh(\sigma/2)$. Since ν is a Carleson measure, Lemma 2.1 says that the norm of the inclusion $\iota_p : A^p \subset L^p(d\nu)$ is bounded by $C_p\|\nu\|_*^{1/p}$, for some constant $C_p > 0$. So, the lemma will follow if we prove that there is a function $k_p(\delta) \to 0$ when $\delta \to 1$ such that

$$\Big|\Big| \sum_{j\geq 1} M_{a_j} P_\nu M_{b_j} \Big|\Big|_{\mathfrak{L}(L^p(d\nu), L^p(d\nu))} \leq N k_p(\delta)||\nu||_*^{(p-1)/p} \tag{2.9}$$

and for every $f \in L^p(d\nu)$ of norm ≤ 1,

$$\sum_{j\geq 1} ||M_{a_j} P_\nu M_{b_j} f||^p_{L^p(d\nu)} \leq N k_p^p(\delta)||\nu||_*^{p-1}. \tag{2.10}$$

First let us assume that $N = 1$, meaning that the family $\{F_j\}$ is pairwise disjoint. Write

$$\Phi(z,\omega) = \sum_{j\geq 1} \chi_{F_j}(z)\chi_{K_j}(\omega)\frac{1}{|1-\langle z,\omega\rangle|^{n+1}}.$$

Let $f \in L^p(d\nu)$. Since $\|a_j\|_\infty, \|b_j\|_\infty \leq 1$ for all j, (ii) yields

$$\begin{aligned}\Big|\Big(\sum_{j\geq 1} M_{a_j} P_\nu M_{b_j} f\Big)(z)\Big| &= \Big|\sum_{j\geq 1} a_j(z)\int_{\mathbb{B}} b_j(\omega) f(\omega)\frac{d\nu(\omega)}{(1-\langle z,\omega\rangle)^{n+1}}\Big| \\ &\leq \int_{\mathbb{B}} \Phi(z,\omega)|f(\omega)|\, d\nu(\omega).\end{aligned}$$

Taking $h(z) = (1-|z|^2)^{-1/pq}$, where $p^{-1}+q^{-1} = 1$, and $\gamma > 0$ as in Lemma 2.5, the lemma asserts that there is a constant $G > 0$ such that

$$\int_{\mathbb{B}} \Phi(z,\omega)h(\omega)^q\, d\nu(\omega) \leq \|\nu\|_* G(1-\delta^{2n})^\gamma h(z)^q.$$

On the other hand, Lemma 2.4 implies that there is some $C > 0$ such that

$$\int_{\mathbb{B}} \Phi(z,\omega)h(z)^p\, d\nu(z) \leq Ch(\omega)^p.$$

By Lemma 2.6 the integral operator with kernel $\Phi(z,\omega)$ is bounded from $L^p(\mathbb{B}, d\nu)$ into $L^p(\mathbb{B}, d\nu)$ and its norm is bounded by $\|\nu\|_*^{1/q}(1-\delta^{2n})^{\gamma/q}G^{1/q}C^{1/p}$. Thus, writing $k_p(\delta) = (1-\delta^{2n})^{\gamma/q}G^{1/q}C^{1/p}$, we obtain (2.9) for $N = 1$. Since in this case,

$$\sum_{j\geq 1} \|M_{a_j}P_\nu M_{b_j}f\|^p_{L^p(d\text{v})} = \Big\|\sum_{j\geq 1}(M_{a_j}P_\nu M_{b_j}f)\Big\|^p_{L^p(d\text{v})},$$

it also proves (2.10).

Now assume that $N > 1$. For $z \in \mathbb{B}$ let $\Lambda(z) = \{j : z \in F_j\}$, ordered in the natural way. Then F_j admits the disjoint decomposition $F_j = A_j^1 \cup \cdots \cup A_j^N$, where $A_j^i = \{z \in F_j : j \text{ is the } i^{th} \text{ element of } \Lambda(z)\}$. It is clear that for each value of $1 \leq i \leq N$, the family $\{A_j^i : j \geq 1\}$ is pairwise disjoint. Thus,

$$\begin{aligned}
&\sum_{j\geq 1} \|M_{a_j}P_\nu M_{b_j}f\|^p_{L^p(d\text{v})} \\
&\quad= \sum_{j\geq 1}\left(\|M_{(a_j\chi_{A_j^1})}P_\nu M_{b_j}f\|^p_{L^p(d\text{v})} + \cdots + \|M_{(a_j\chi_{A_j^N})}P_\nu M_{b_j}f\|^p_{L^p(d\text{v})}\right) \\
&\quad= \sum_{i=1}^{N}\sum_{j\geq 1}\|(M_{(a_j\chi_{A_j^i})}P_\nu M_{b_j}f)\|^p_{L^p(d\text{v})} \leq N k_p^p(\delta)\|\nu\|_*^p,
\end{aligned}$$

where the last inequality follows from the previous case $N = 1$. So, (2.10) holds. To prove (2.9) observe that just as in the above formula, $\sum_{j\geq 1} M_{a_j}P_\nu M_{b_j}$ can be written as a sum of N operators that satisfy the hypotheses of the previous case. ◻

3. A Covering of the Ball

Lemma 3.1. *There is a positive integer N (depending only on the dimension n) such that for any $\sigma > 0$ there is a covering of $\mathbb{B}$ by Borel sets B_j satisfying*

(1) *$B_j \cap B_k = \varnothing$ if $j \neq k$,*
(2) *every point of $\mathbb{B}$ belongs to at most N of the sets $\Omega_\sigma(B_j) = \{z : \beta(z, B_j) \leq \sigma\}$,*
(3) *there is a constant $C(\sigma) > 0$ such that* $\operatorname{diam}_\beta B_j \leq C(\sigma)$ *for every j.*

Proof. First observe that (2) says that every closed hyperbolic ball of radius σ cannot meet more than N sets B_j. Therefore, it is enough to replace (2) by

(2′) *every set of hyperbolic diameter 2σ cannot meet more than N sets B_j.*

Also, we only need to construct a numerable covering $\{B_j'\}$ satisfying (2′) and (3), since the family $B_k = B_k' \setminus \bigcup_{j=1}^{k-1} B_j$ will satisfy the lemma. For $E \subset \mathbb{B}$ write

$$\tilde{E} = \{e^{it}z : z \in E,\ 0 \leq t < 2\pi\}.$$

Given $\sigma > 0$, let $M \geq 2$ be an integer to be chosen later, depending only on σ (and n). Let

$$\Gamma^1 = \{z = (z_1, \ldots, z_n) \in \mathbb{B} : |z|^2 \geq 1 - M^{-6},\ z_1 \in \mathbb{R},\ z_1 \geq 1/(2\sqrt{n})\}.$$

Then $\Gamma^1 \subset (I \times \{0\}) \times I^{2n-2} = I^{2n-1}$, where $I = [-1,1]$. For any integer $k \geq 3$, let $Q_{k,j}$ be the standard decomposition of I^{2n-1} into closed cubes of side length $2/M^{k-1}$, and denote

$$A_{k,j} = Q_{k,j} \cap \{z \in \Gamma^1 : M^{-2k-2} \leq 1 - |z|^2 \leq M^{-2k}\},$$

where we disregard all the indexes for which this intersection is empty. Now pick an arbitrary point $z_{k,j} \in A_{k,j}$ and for all integers $0 \leq \ell < M^{2k-5}$ let

$$A_{k,j,\ell} = \left\{ e^{it} w : w \in \tilde{A}_{k,j},\ \langle w, z_{k,j} \rangle \geq 0,\ \frac{2\pi\ell}{M^{2k-5}} \leq t \leq \frac{2\pi(\ell+1)}{M^{2k-5}} \right\}.$$

Thus $A_{k,j,\ell} \subset \tilde{A}_{k,j}$ for every ℓ, and if $z \in \tilde{A}_{k,j}$, then $(\bar{z}_1/|z_1|)z \in A_{k,j}$. Since $k \geq 3$, it is clear that the sets $A_{k,j,\ell}$ form a covering of $\tilde{\Gamma}^1$. We shall show that if $M = M(\sigma)$ is big enough, this covering of $\tilde{\Gamma}^1$ satisfies properties (2′) and (3) of the lemma. If $S_k = \{z : |z|^2 = 1 - M^{-2k}\}$, an elementary calculation shows that

$$\begin{aligned} \frac{1}{1 - \rho^2(S_k, S_{k+1})} &= \frac{1}{1 - \rho^2((1 - M^{-2k})^{1/2}, (1 - M^{-2k-2})^{1/2})} \\ &= M^2 \left(\frac{1}{4} + h_k(M) \right), \end{aligned}$$

where the pseudohyperbolic metric in the second member is taken on the disk, and $h_k(M)$ are functions that tend to 0 uniformly on k when $M \to \infty$. Hence, by choosing M large enough, we can assure that $4\sigma < \beta(S_k, S_{k+1})$. This inequality guarantees that every set of hyperbolic diameter 2σ meets no more than 2 strips $M^{-2k-2} \leq 1 - |z|^2 \leq M^{-2k}$. So, fix $k \geq 3$.

Sublemma 3.2. *If* $1 - M^{-2k} \leq |z|^2, |w|^2 \leq 1 - M^{-2k-2}$, $|z_1|, |w_1| \geq 1/(2\sqrt{n})$, *and we denote* $\delta = \left| (\overline{z_1}/|z_1|)z - (\overline{w_1}/|w_1|)w \right|$, *then*

$$\frac{M^{2k}\delta^2}{18n} \leq \frac{1 - |\langle z, w \rangle|}{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}} \leq \frac{M^{2k+2}\delta^2}{2} + M^2. \tag{3.1}$$

Proof. If $\tilde{d} = \inf_t |z - e^{it}w|$, then

$$\begin{aligned} \tilde{d}^2 &= |z|^2 + |w|^2 - 2|\langle z, w \rangle| \\ &= (|z|^2 - 1) + (|w|^2 - 1) + 2(1 - |\langle z, w \rangle|). \end{aligned}$$

Hence, $\tilde{d}^2/2 + M^{-2k-2} \leq 1 - |\langle z, w \rangle| \leq \tilde{d}^2/2 + M^{-2k}$, and since

$$M^{2k} \leq [(1 - |z|^2)(1 - |w|^2)]^{-1/2} \leq M^{2k+2}, \tag{3.2}$$

we get

$$\text{(3.3)}\qquad \begin{aligned}\frac{M^{2k}\tilde{d}^2}{2}+M^{-2} &\le \frac{1-|\langle z,w\rangle|}{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}\\ &\le \frac{M^{2k+2}\tilde{d}^2}{2}+M^2.\end{aligned}$$

On the other hand, for any $t\in[0,2\pi)$,

$$\begin{aligned}\delta &= \left|\frac{\overline{z_1}}{|z_1|}z-\frac{\overline{w_1}}{|w_1|}w\right|\\ &\le \left|\frac{\overline{z_1}}{|z_1|}z-e^{it}\frac{\overline{w_1}}{|w_1|}w\right|+\left|e^{it}\frac{\overline{w_1}}{|w_1|}w-\frac{\overline{w_1}}{|w_1|}w\right|.\end{aligned}$$

If we pick $t\in[0,2\pi)$ such that the first summand above is $\tilde{d}$, then

$$\text{(3.4)}\qquad \delta\le\tilde{d}+|w|\,|e^{it}-1|\le\tilde{d}+|e^{it}-1|.$$

By hypothesis we can assume that $1/(2\sqrt{n})\le|z_1|\le|w_1|$, which leads to

$$\begin{aligned}\frac{1}{2\sqrt{n}}|1-e^{it}| &\le |z_1|\,|1-e^{it}|\\ &= \Big|\,|z_1|-|z_1|e^{it}\,\Big| \le \Big|\,|z_1|-|w_1|e^{it}\,\Big|\\ &\le \tilde{d},\end{aligned}$$

where the last inequality holds by our choice of t, and the previous one from a simple drawing. Thus, on (3.4) we get $\delta\le\tilde{d}+2\sqrt{n}\tilde{d}\le3\sqrt{n}\tilde{d}$, and since obviously $\tilde{d}\le\delta$,

$$\frac{\delta^2}{9n}\le\tilde{d}^2\le\delta^2.$$

The sublemma follows by inserting these inequalities in (3.3). ◻

We recall that we have fixed $k\ge3$. An immediate volume argument shows that every cube $Q_{k,j}$ meets no more than $3^{2n}-1$ of the other cubes. So, the same holds for the sets $\tilde{A}_{k,j}$. In addition, if $z\in\tilde{A}_{k,j_1}$, $w\in\tilde{A}_{k,j_2}$, and $Q_{k,j_1}\cap Q_{k,j_2}=\varnothing$, then

$$\left|\frac{\bar{z}_1}{|z_1|}z-\frac{\bar{w}_1}{|w_1|}w\right|\ge\frac{2}{M^{k-1}},$$

which together with the first inequality in (3.1) yields

$$\frac{1}{(1-\rho(z,w)^2)^{1/2}}\ge\frac{1-|\langle z,w\rangle|}{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}\ge\frac{2}{9n}M^2\to\infty$$

when $M \to \infty$. Hence, we can choose M depending on σ big enough so that $\beta(\tilde{A}_{k,j_1}, \tilde{A}_{k,j_2}) > 4\sigma$. Together with the previous comments, this implies that for any fixed value of k, every set of hyperbolic diameter 2σ meets no more than 3^{2n} of the sets $\tilde{A}_{k,j}$. On the other hand, if z, $w \in \tilde{A}_{k,j}$, then

$$\left| \frac{\bar{z}_1}{|z_1|} z - \frac{\bar{w}_1}{|w_1|} w \right| \le \operatorname{diam} Q_{k,j} = \frac{2\sqrt{2n-1}}{M^{k-1}},$$

and the second inequality in (3.1) gives

$$\frac{1 - |\langle z, w \rangle|}{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}} \le 4nM^4. \tag{3.5}$$

Observe that the restriction $k \ge 3$, (3.2) and (3.5) imply that if $w \in \tilde{A}_{k,j}$, then $\langle w, z_{k,j} \rangle \ne 0$ as soon as $M^2 > 4n$. So, assuming this restriction on M, in the definition of $A_{k,j,\ell}$ we could have taken $\langle w, z_{k,j} \rangle > 0$ instead of $\langle w, z_{k,j} \rangle \ge 0$. This guarantees that no point of $\tilde{A}_{k,j}$ is in more than 2 of the sets $A_{k,j,\ell}$.

Finally, we fix the values of k and j, and see what happens inside the set $\tilde{A}_{k,j}$. Since every $A_{k,j,\ell}$ is a rotation of $A_{k,j,0}$, they all have the same hyperbolic diameter. If $w \in A_{k,j,0}$, then $\langle w, z_{k,j} \rangle = e^{it} |\langle w, z_{k,j} \rangle|$, with $0 \le t \le 2\pi M^{-2k+5}$, so

$$\begin{aligned}
\left| 1 - \langle w, z_{k,j} \rangle \right| &= \left| 1 - e^{it} |\langle w, z_{k,j} \rangle| \right| \\
&\le |1 - e^{it}| + 1 - |\langle w, z_{k,j} \rangle| \\
&\le t + 1 - |\langle w, z_{k,j} \rangle|,
\end{aligned}$$

which, together with (3.2) and (3.5), implies

$$\begin{aligned}
\frac{1}{(1 - \rho(w, z_{k,j})^2)^{1/2}} &= \frac{|1 - \langle w, z_{k,j} \rangle|}{(1 - |z_{k,j}|^2)^{1/2}(1 - |w|^2)^{1/2}} \\
&\le 2\pi M^7 + 4nM^4.
\end{aligned}$$

Therefore, the hyperbolic diameter of $A_{k,j,\ell}$ is bounded by a constant that only depends on M. In symbols,

$$\operatorname{diam}_\beta A_{k,j,\ell} \le C_1(M) \quad \text{for all } k,\ j \text{ and } \ell. \tag{3.6}$$

Since k and j are fixed, each $A_{k,j,\ell}$ meets two other of these sets, and we shall see next that disjoint sets are hyperbolically far away (depending on M). So, suppose that $u \in A_{k,j,\ell_1}$, $v \in A_{k,j,\ell_2}$, and $A_{k,j,\ell_1} \cap A_{k,j,\ell_2} = \varnothing$. This means that

$$\frac{\langle u, z_{k,j}\rangle}{|\langle u, z_{k,j}\rangle|} = e^{it_1} \quad \text{and} \quad \frac{\langle v, z_{k,j}\rangle}{|\langle v, z_{k,j}\rangle|} = e^{it_2},$$
$$\text{with } \frac{2\pi}{M^{2k-5}} \le |t_1 - t_2| \le 2\pi - \frac{2\pi}{M^{2k-5}}.$$

We recall that for $z \in \mathbb{B}$, P_z and Q_z denote the projection onto $\mathbb{C}z$ and its orthogonal complement, respectively. Since $|\langle u, z_{k,j}\rangle|^2 = |z_{k,j}|^2\,|P_{z_{k,j}}(u)|^2$, (3.5) and (3.2) yield

$$\begin{aligned}|z_{k,j}|^2\,|Q_{z_{k,j}}(u)|^2 &= |z_{k,j}|^2\,|u|^2 - |z_{k,j}|^2\,|P_{z_{k,j}}(u)|^2\\ &\le 1 - \big|\langle u, z_{k,j}\rangle\big|^2 \le 8nM^{4-2k},\end{aligned}$$

and since the same holds for $Q_{z_{k,j}}(v)$,

$$|z_{k,j}|^2\;|\langle Q_{z_{k,j}}(u), Q_{z_{k,j}}(v)\rangle| \le 8nM^{4-2k}.$$

Together with the equality $|z_{k,j}|^2\langle P_{z_{k,j}}(u), P_{z_{k,j}}(v)\rangle = \langle u, z_{k,j}\rangle\overline{\langle v, z_{k,j}\rangle}$, this gives

$$\begin{aligned}&|z_{k,j}|^2\;|1 - \langle u, v\rangle|\\ &\quad= |z_{k,j}|^2\;|1 - \langle P_{z_{k,j}}(u), P_{z_{k,j}}(v)\rangle - \langle Q_{z_{k,j}}(u), Q_{z_{k,j}}(v)\rangle|\\ &\quad\ge |1 - \langle u, z_{k,j}\rangle\overline{\langle v, z_{k,j}\rangle}| - (1 - |z_{k,j}|^2) - |z_{k,j}|^2\;|\langle Q_{z_{k,j}}(u), Q_{z_{k,j}}(v)\rangle|\\ &\quad\ge |1 - \langle u, z_{k,j}\rangle\overline{\langle v, z_{k,j}\rangle}| - (M^{-2k} + 8nM^{4-2k}).\end{aligned}$$

If $0 < \alpha \le \pi$, the elementary inequality

$$|1 - e^{ix}| = |1 - e^{-ix}| \ge \frac{\alpha}{2\pi} \quad \text{when } x \in [\alpha,\ 2\pi - \alpha]$$

applied to $\alpha = 2\pi/M^{2k-5}$ and $x = |t_1 - t_2|$ gives $|1 - e^{i(t_1-t_2)}| \ge M^{5-2k}$. Hence,

$$\begin{aligned}&|1 - \langle u, z_{k,j}\rangle\overline{\langle v, z_{k,j}\rangle}|\\ &\quad= \big|\,|1 - e^{i(t_1-t_2)}|\langle u, z_{k,j}\rangle\langle v, z_{k,j}\rangle|\,\big|\\ &\quad\ge |1 - e^{i(t_1-t_2)}| - (1 - |\langle u, z_{k,j}\rangle|) - |\langle u, z_{k,j}\rangle|(1 - |\langle v, z_{k,j}\rangle|)\\ &\quad\ge M^{5-2k} - 8nM^{4-2k},\end{aligned}$$

where the last inequality follows from (3.2) and (3.5). The last two chains of inequalities and (3.2) say that

$$\frac{1}{(1-\rho(u,v)^2)^{1/2}} \overset{\text{by (3.2)}}{\geq} M^{2k}|z_{k,j}|^2\,|1-\langle u,v\rangle| \geq M^5 - (16nM^4+1),$$

which tends to infinity as $M \to \infty$. That is, we can choose $M = M(\sigma)$ big enough so that $\beta(u,v) > 4\sigma$ whenever $u \in A_{k,j,\ell_1}$, $v \in A_{k,j,\ell_2}$, and these sets do not meet. Thus, a set of hyperbolic diameter 2σ in $\tilde{A}_{k,j}$ can only intersect 2 of the sets $A_{k,j,\ell}$.

Summing up, any set of hyperbolic diameter 2σ meets at most 2 of the strips $\{M^{-2k} \leq 1-|z|^2 \leq M^{-2k-2}\}$. For any fixed k, it meets at most 3^{2n} sets $\tilde{A}_{k,j}$, and for any fixed pair k, j, it meets at most two sets $A_{k,j,\ell}$. Henceforth, any such set meets at most $2\cdot 3^{2n}\cdot 2$ of the sets $A_{k,j,\ell}$, an absolute constant if we take the dimension as such. That is, we have constructed a covering of $\tilde{\Gamma}^1$ that satisfies conditions (2′) and (3) of the lemma. By permuting the coordinates we obtain similar coverings $\{A^i_{k,j,\ell}\}_{k,j,\ell}$ of

$$\tilde{\Gamma}^i = \left\{z \in \mathbb{B} : |z|^2 \geq 1-M^{-6},\ |z_i| \geq \frac{1}{2\sqrt{n}}\right\} \quad (i = 1,\ldots,n).$$

In addition, since $M \geq 2$, we have $1-M^{-6} > \frac{1}{4}$, which clearly implies that

$$\left\{z \in \mathbb{B} : |z|^2 \geq 1-M^{-6}\right\} = \bigcup_{i=1}^{n} \tilde{\Gamma}^i.$$

So, $\{A^i_{k,j,\ell}\}$ together with the closed Euclidean ball U, centered at the origin and of radius $(1-M^{-6})^{1/2}$, form a covering of $\mathbb{B}$ that satisfies conditions (2′) and (3), where N is bounded by $2\cdot 3^{2n}\cdot 2\cdot n+1$, and such that all its elements have hyperbolic diameter bounded by the maximum between the constant $C_1(M)$ of (3.6) and $\operatorname{diam}_\beta U$, both depending on M, which in turn depends on σ. ◻

Remark 3.3. In the particular case of the disk, the above lemma can be simplified notoriously. The construction is clearer in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. If $M > 1$ is an integer, consider the rectangles

$$V_{j,m} = \left[\frac{j}{M^m}, \frac{j+1}{M^m}\right] \times \left[\frac{1}{M^{m+2}}, \frac{1}{M^{m+1}}\right],$$

where j and m run over all the integers. These sets form an essentially disjoint decomposition of $\mathbb{C}_+$, and since they can be transformed into each other by a real translation followed by a dilation, they have the same hyperbolic size. All the upper horizontal sides of the rectangles are conformally equivalent and their hyperbolic diameter tends to infinity as $M \to \infty$, and the same holds for all the lower horizontal sides and for all the vertical sides. A moment of reflection shows that if $\sigma > 0$, we can take $M = M(\sigma)$ big enough so that any hyperbolic ball of radius σ in $\mathbb{C}_+$ meets no more than 4 of the above rectangles.

Let $\sigma > 0$ and k be a non-negative integer. Let $\{B_j\}$ be a covering of the ball satisfying the conditions of Lemma 3.1 for $(k+1)\sigma$ instead of σ. For $0 \le i \le k$ and $j \ge 1$ write

$$F_{0,j} = B_j, \quad \text{and} \quad F_{i+1,j} = \{z : \beta(z, F_{i,j}) \le \sigma\}. \tag{3.7}$$

The next result is now immediate.

Corollary 3.4. *Let $\sigma > 0$ and k be a non-negative integer. For each $0 \le i \le k+1$ the family $\mathcal{F}^i = \{F_{i,j} : j \ge 1\}$ forms a covering of $\mathbb{B}$ such that*

(a) $F_{0,j_1} \cap F_{0,j_2} = \emptyset$ *if* $j_1 \ne j_2$,
(b) $F_{0,j} \subset F_{1,j} \subset \cdots \subset F_{k+1,j}$ *for all* j,
(c) $\beta(F_{i,j}, F^c_{i+1,j}) \ge \sigma$ *for all* $0 \le i \le k$ *and* $j \ge 1$,
(d) *every point of* $\mathbb{B}$ *belongs to no more than* N *elements of* $\mathcal{F}^i$,
(e) $\operatorname{diam}_\beta F_{i,j} \le C(k,\sigma)$ *for all* i, j, *where* $C(k,\sigma)$ *depends only on* k *and* σ.

The constants N and $C(k,\sigma) = C((k+1)\sigma)$ are given, respectively, by items (2) *and* (3) *of Lemma 3.1.*

4. Approximation by Segmented Operators

Lemma 4.1. *Let $1 < p < \infty$, $\sigma \ge 1$, functions $a_1, \ldots, a_k \in L^\infty$ of norm ≤ 1 and ν be a Carleson measure. Consider the coverings of $\mathbb{B}$ given by* (3.7) *for these values of k and σ. Then there is a positive constant $C_0 = C_0(p,k,n)$ such that*

$$\Big\| T_{a_1} \cdots T_{a_k} T_\nu - \sum_j M_{\chi_{F_{0,j}}} T_{a_1} \cdots T_{a_k} T_{(\chi_{F_{k+1,j}} \nu)} \Big\|_{\mathfrak{L}(A^p, L^p)} \le C_0 \beta_p(\sigma) \|T_\nu\|_{\mathfrak{L}(A^p)}, \tag{4.1}$$

where $\beta_p(\sigma) \to 0$ as $\sigma \to \infty$.

Proof. Step 1. We shall show that there is a constant $C_1 = C_1(p,k,n)$ such that that

$$\left\|T_{a_1}\cdots T_{a_k}T_\nu-\sum_j M_{\chi_{F_{0,j}}}T_{(\chi_{F_{1,j}}a_1)}\cdots T_{(\chi_{F_{k,j}}a_k)}T_{(\chi_{F_{k+1,j}}\nu)}\right\|_{\mathfrak{L}(A^p,L^p)}\leq C_1\beta_p(\sigma)\|T_\nu\|_{\mathfrak{L}(A^p)}. \tag{4.2}$$

For $0\leq m\leq k+1$ define the operators $S_m\in\mathfrak{L}(A^p,L^p)$ as

$$S_m=\sum_j M_{\chi_{F_{0,j}}}T_{(\chi_{F_{1,j}}a_1)}\cdots T_{(\chi_{F_{m,j}}a_m)}T_{a_{m+1}}\cdots T_{a_k}T_\nu.$$

It is clear that

$$S_0=\sum_j(M_{\chi_{F_{0,j}}}T_{a_1}\cdots T_{a_k}T_\nu)=T_{a_1}\cdots T_{a_k}T_\nu,$$

where the series converges in the strong operator topology. If $0\leq m\leq k-1$,

$$\begin{aligned}S_m-S_{m+1}&=\sum_j\left\{M_{\chi_{F_{0,j}}}\Big(\prod_{i=1}^m T_{(\chi_{F_{i,j}}a_i)}\Big)\Big[T_{a_{m+1}}-T_{(\chi_{F_{m+1,j}}a_{m+1})}\Big]\Big(\prod_{i=m+2}^k T_{a_i}\Big)T_\nu\right\}\\&=\sum_j\left\{M_{\chi_{F_{0,j}}}\Big(\prod_{i=1}^m T_{(\chi_{F_{i,j}}a_i)}\Big)T_{(\chi_{F^c_{m+1,j}}a_{m+1})}\Big(\prod_{i=m+2}^k T_{a_i}\Big)T_\nu\right\},\end{aligned}$$

where any of the products above should be understood as the identity when the lower index is bigger than the upper index. For notational reasons we take a_0 as the constant function 1 in the next expression when $m=0$. Hence, if $f\in A^p$ has norm 1, using that the sets $F_{0,j}$ are pairwise disjoint and Lemma 2.7 applied to the measure $d\nu$, we obtain

$$\begin{aligned}\|(S_m-S_{m+1})f\|_p^p&\leq(C_p^p)^m\sum_j\left\|\Big[M_{(\chi_{F_{m,j}}a_m)}PM_{(\chi_{F^c_{m+1,j}}a_{m+1})}\Big]\Big(\prod_{i=m+2}^k T_{a_i}\Big)T_\nu f\right\|_p^p\\&\leq(C_p^p)^m N\beta_p^p(\sigma)\left\|\Big(\prod_{i=m+2}^k T_{a_i}\Big)T_\nu f\right\|_p^p\quad\text{by (2.8)}\\&\leq(C_p^p)^m(C_p^p)^{k-m-1}N\beta_p^p(\sigma)\|T_\nu\|^p\\&=(C_p^p)^{k-1}N\beta_p^p(\sigma)\|T_\nu\|^p\end{aligned}$$

for $0\leq m\leq k-1$, where N is given by Corollary 3.4 and depends only on the dimension n, $\beta_p(\sigma)$ is given by Lemma 2.7, and $C_p=\|P\|_{\mathfrak{L}(L^p)}$. Similarly, since

$$S_k - S_{k+1} = \sum_j M_{\chi_{F_{0,j}}} T_{(\chi_{F_{1,j}} a_1)} \cdots T_{(\chi_{F_{k,j}} a_k)} T_{(\chi_{F^c_{k+1,j}} \nu)},$$

Lemma 2.7 applied to $d\nu$ gives

$$\begin{aligned} ||(S_k - S_{k+1})f||_p^p &\le (C_p^p)^k \sum_j ||M_{(\chi_{F_{k,j}} a_m)} P_\nu M_{(\chi_{F^c_{k+1,j}})} f||_p^p \\ &\le (C_p^p)^k N \beta_p^p(\sigma) ||\nu||_*^p. \quad \text{by (2.8)} \end{aligned}$$

Since Lemma 2.1 says that $\|\nu\|_*$ is equivalent to $\|T_\nu\|_{\mathfrak{L}(A^p)}$, there is a constant $c = c(p,k,n)$ such that

$$\|S_m - S_{m+1}\| \le c(p,k,n)\beta_p(\sigma)\|T_\nu\|, \quad \text{for all } 0 \le m \le k.$$

Consequently

$$\|S_0 - S_{k+1}\| \le \sum_{m=0}^{k} \|S_m - S_{m+1}\| \le (k+1)c(p,k,n)\beta_p(\sigma)\|T_\nu\|,$$

which proves (4.2).

Step 2. We show now that there is a constant $C_2 = C_2(p,k,n)$ such that

$$\begin{aligned} \Big\| \sum_j M_{\chi_{F_{0,j}}} T_{a_1} \cdots T_{a_k} T_{(\chi_{F_{k+1,j}} \nu)} \\ - \sum_j M_{\chi_{F_{0,j}}} T_{(\chi_{F_{1,j}} a_1)} \cdots T_{(\chi_{F_{k,j}} a_k)} T_{(\chi_{F_{k+1,j}} \nu)} \Big\|_{\mathfrak{L}(A^p, L^p)} \\ \le C_2 \beta_p(\sigma) \|T_\nu\|_{\mathfrak{L}(A^p)}. \end{aligned} \tag{4.3}$$

For $0 \le m \le k$, define

$$S_m = \sum_j M_{\chi_{F_{0,j}}} T_{(\chi_{F_{1,j}} a_1)} \cdots T_{(\chi_{F_{m,j}} a_m)} T_{a_{m+1}} \cdots T_{a_k} T_{(\chi_{F_{k+1,j}} \nu)}.$$

Therefore, if $0 \le m \le k-1$,

$$\begin{aligned} S_m - S_{m+1} = \sum_j \Big\{ M_{\chi_{F_{0,j}}} \Big(\prod_{i=1}^{m} T_{(\chi_{F_{i,j}} a_i)} \Big) \Big[T_{a_{m+1}} - T_{(\chi_{F_{m+1,j}} a_{m+1})} \Big] \\ \times \Big(\prod_{i=m+2}^{k} T_{a_i} \Big) T_{(\chi_{F_{k+1,j}} \nu)} \Big\} \quad = \end{aligned}$$

$$= \sum_j \Big\{ M_{\chi_{F_{0,j}}} \Big(\prod_{i=1}^{m} T_{(\chi_{F_{i,j}} a_i)} \Big) T_{(\chi_{F^c_{m+1,j}} a_{m+1})} \Big(\prod_{i=m+2}^{k} T_{a_i} \Big) T_{(\chi_{F_{k+1,j}} \nu)} \Big\},$$

where as before, any of the products above is the identity when the lower index is bigger than the upper index. Hence, if $f \in A^p$ has norm 1,

$$\begin{aligned}
(4.4)\quad & \|(S_m - S_{m+1})f\|_p^p \\
&\le (C_p^p)^m \sum_j \Big\| [M_{(\chi_{F_{m,j}} a_m)} P M_{(\chi_{F^c_{m+1,j}} a_{m+1})}] \Big(\prod_{i=m+2}^{k} T_{a_i} \Big) T_{(\chi_{F_{k+1,j}} \nu)} f \Big\|_p^p \\
&\le (C_p^p)^m \sum_j \Big\{ \|M_{(\chi_{F_{m,j}} a_m)} P M_{(\chi_{F^c_{m+1,j}} a_{m+1})}\|^p_{\mathfrak{L}(A^p, L^p)} \\
&\qquad\qquad \times \Big\| \prod_{i=m+2}^{k} T_{a_i} \Big\|^p \|T_{(\chi_{F_{k+1,j}} \nu)} f\|_p^p \Big\} \\
&\le (C_p^p)^m \sum_j \beta_p^p(\sigma) (C_p^p)^{k-m-1} \|T_{(\chi_{F_{k+1,j}} \nu)} f\|_p^p \\
&\le (C_p^p)^{k-1} \beta_p^p(\sigma) \sum_j \|T_{(\chi_{F_{k+1,j}} \nu)} f\|_p^p,
\end{aligned}$$

where the third inequality holds because $\| \prod_{i=m+2}^{k} T_{a_i} \|_p \le C_p^{k-m-1}$, and (2.7) applied to the measure $d\nu$ implies that

$$\|M_{(\chi_{F_{m,j}} a_m)} P M_{(\chi_{F^c_{m+1,j}} a_{m+1})}\|_{\mathfrak{L}(A^p, L^p)} \le \beta_p(\sigma)$$

for all $j \ge 1$. By Lemma 2.2 there is a constant α_p depending only on p such that $\|T_{(\chi_{F_{k+1,j}} \nu)} f\|_p \le \alpha_p \|\iota_q\| \, \|\chi_{F_{k+1,j}} f\|_{L^p(d\nu)}$, and since every point of $\mathbb{B}$ is in no more than N of the sets $F_{k+1,j}$, we get

$$\begin{aligned}
(4.5)\quad \sum_j \|T_{(\chi_{F_{k+1,j}} \nu)} f\|_p^p &\le \alpha_p^p \|\iota_q\|^p \sum_j \|\chi_{F_{k+1,j}} \nu f\|^p_{L^p(d\nu)} \\
&\le \alpha_p^p \|\iota_q\|^p \, N \|f\|^p_{L^p(d\nu)} \\
&\le \alpha_p^p N \|\iota_q\|^p \, \|\iota_p\|^p \, \|f\|^p_{A^p}.
\end{aligned}$$

Since Lemma 2.1 says that $\|\iota_s\|$ is equivalent to $\|\nu\|_*^{1/s}$ for $s = p, q$, we see that $(\|\iota_q\| \, \|\iota_p\|)^p$ is equivalent to $(\|\nu\|_*^{1/q} \|\nu\|_*^{1/p})^p = \|\nu\|_*^p$, which by the same lemma, is equivalent to $\|T_\nu\|^p_{\mathfrak{L}(A^p)}$. Inserting this equivalence in (4.5) and going back from there to (4.4), we obtain that there is a constant $c(p, k, n)$ such that

$$\|(S_m - S_{m+1})\|_p \le c(p, k, n) \beta_p(\sigma) \|T_\nu\|$$

for all $0 \le m \le k-1$. Consequently,

$$\|S_0 - S_k\| \le \sum_{m=0}^{k-1} \|S_m - S_{m+1}\| \le kc(p,k,n)\beta_p(\sigma)\|T_\nu\|,$$

which proves (4.3). The lemma follows from (4.2) and (4.3) with $C_0 = C_1 + C_2$. ∎

If ν is a complex-valued measure whose total variation $|\nu|$ is a Carleson measure, decompose ν into its real and imaginary parts and then use the Jordan decomposition of each part to obtain $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$, where each ν_j is a positive measure such that $|\nu| \sim \sum_{j=1}^4 |\nu_j|$. Thus, each ν_j is a Carleson measure with $\| \, |\nu| \, \|_* \sim \sum_{j=1}^4 \|\nu_j\|_*$. The comments above and Lemma 2.1 imply that T_ν is a bounded operator on A^p for all $1 < p < \infty$, with norm bounded by a constant that only depends on p and $\| \, |\nu| \, \|_*$.

Lemma 4.2. *Let*

$$S = \sum_{i=1}^m T_{a_1^i} \cdots T_{a_{k_i}^i} T_{\nu_i},$$

where $a_j^i \in L^\infty$, $k_1, \ldots, k_m \le k$, and ν_i are complex-valued measures on $\mathbb{B}$ such that $|\nu_i|$ are Carleson measures. Given $\varepsilon > 0$, there is $\sigma = \sigma(S,\varepsilon) \ge 1$ such that if $\{F_{i,j}\}_{j\ge1}$, $i = 0, \ldots, k+1$, are the sets given by (3.7) for these values of k and σ, then

$$\Big\| S - \sum_j M_{\chi_{F_{0,j}}} \Big(\sum_{i=1}^m T_{a_1^i} \cdots T_{a_{k_i}^i} T_{(\chi_{F_{k+1,j}}\nu_i)} \Big) \Big\|_{\mathfrak{L}(A^p,L^p)} < \varepsilon. \tag{4.6}$$

Proof. Consider first the case where all the measures ν_i are positive (so they are Carleson). We can assume that $k_i = k$ for $i = 1, \ldots, m$ by filling up the 'holes' in each product with products of the identity T_1 if necessary. A straightforward application of Lemma 4.1 tells us that if σ is sufficiently large, then

$$\Big\| T_{a_1^i} \cdots T_{a_k^i} T_{\nu_i} - \sum_j M_{\chi_{F_{0,j}}} T_{a_1^i} \cdots T_{a_k^i} T_{(\chi_{F_{k+1,j}}\nu_i)} \Big\|_{\mathfrak{L}(A^p,L^p)} < \frac{\varepsilon}{m}$$

for $i = 1, \ldots, m$. Summing from $i = 1$ to m yields

$$\Big\| S - \sum_{i=1}^m \Big(\sum_j M_{\chi_{F_{0,j}}} T_{a_1^i} \ldots T_{a_k^i} T_{(\chi_{F_{k+1,j}}\nu_i)} \Big) \Big\|_{\mathfrak{L}(A^p,L^p)} < \varepsilon.$$

Since for every $1 \le i \le m$ the series $S_i = \sum_j M_{\chi_{F_{0,j}}} T_{a_1^i} \ldots T_{a_k^i} T_{(\chi_{F_{k+1,j}}\nu_i)}$ converges in the strong operator topology, the result follows from the above inequality and the linearity of the limit.

In the general case, decompose $\nu_i = \nu_{i,1} - \nu_{i,2} + i(\nu_{i,3} - \nu_{i,4})$, where for $j = 1, \dots, 4$, $\nu_{i,j}$ is a Carleson measure with $\|\nu_{i,j}\|_* \le \|\,|\nu_i|\,\|_* \sim \sum_{\ell=1}^4 \|\nu_{i,\ell}\|_*$. Apply the previous result to $\nu_{i,j}$ for each j and then use again the linearity of the limit in the strong operator topology to get the desired result. ◻

Theorem 4.3. *Let $S \in \mathfrak{T}_p$, ν be a Carleson measure, and $\varepsilon > 0$. Then there are Borel sets $F_j \subset G_j \subset \mathbb{B}$, with $j \ge 1$, such that*

(a) $\mathbb{B} = \bigcup F_j$,
(b) $F_j \cap F_k = \emptyset$ *if* $j \ne k$,
(c) *each point of* $\mathbb{B}$ *is in no more than N sets G_j, where N depends only on n,*
(d) $\mathrm{diam}_\beta\, G_j \le d = d(p, S, \varepsilon)$,

and

$$\Big\|ST_\nu - \sum_j M_{\chi_{F_j}} ST_{\chi_{G_j}\nu}\Big\|_{\mathfrak{L}(A^p, L^p)} < \varepsilon.$$

Proof. Since $S \in \mathfrak{T}_p$, there is

$$S_0 = \sum_{i=1}^m T_{a_1^i} \cdots T_{a_{k_i}^i}$$

such that $\|S - S_0\| < \varepsilon$, where $a_j^i \in L^\infty$, and k_i are positive integers. Let $k = \max\{k_i : 1 \le i \le m\}$. By Lemma 4.2 there are two families of Borel sets, $F_j := F_{0,j}$ and $G_j := F_{k+1,j}$, that satisfy the theorem for S_0. Furthermore, if $f \in A^p$,

$$\begin{aligned}
\Big\|\sum_j M_{\chi_{F_j}}(S - S_0)T_{\chi_{G_j}\nu} f\Big\|_p^p &= \sum_j \|M_{\chi_{F_j}}(S - S_0)T_{\chi_{G_j}\nu} f\|_p^p \\
&\le \varepsilon^p \sum_j \|T_{\chi_{G_j}\nu} f\|_p^p \\
&\le \varepsilon^p \alpha_p^p \|\iota_q\|^p \sum_j \|\chi_{G_j} f\|_{L^p(d\nu)}^p \\
&\le \varepsilon^p \alpha_p^p \|\iota_q\|^p\, N \|f\|_{L^p(d\nu)}^p \\
&\le \varepsilon^p \alpha_p^p\, N \|\iota_q\|^p\, \|\iota_p\|^p\, \|f\|_{A^p}^p \\
&\le \varepsilon^p C_p N \|\nu\|_*^p\, \|f\|_{A^p}^p
\end{aligned}$$

for some constant $C_p > 0$, where the second inequality holds by Lemma 2.2, the third one by item (c), and the last one by Lemma 2.1. ◻

5. Three Characterizations of the Essential Norm

For $\varrho > 0$ let w_m and D_m be as in Lemma 2.3. It is immediate from conditions (a) and (b) of the lemma that $\mu_\varrho = \sum_m \mathrm{v}(D_m)\,\delta_{w_m}$ is a Carleson measure, where δ_w denotes the Dirac measure at w. Therefore T_{μ_ϱ} is bounded on A^p for $1 < p < \infty$.

The next lemma is related to an atomic decomposition of A^p given by Luecking, and it is essentially proved in [13]. Since it is not explicitly stated, we sketch here a proof. For $n = 1$, a detailed proof can be found in [25, Chapter 4].

Lemma 5.1. *$T_{\mu_\varrho} \to I$ on $\mathfrak{L}(A^p)$ when $\varrho \to 0$.*

Proof. If $z \in \mathbb{B}$ and $r > 0$, in [17, p. 30] it is shown that

$$\mathrm{v}(D(z,r)) = s_r^{2n}\left(\frac{1-|z|^2}{1-s_r^2|z|^2}\right)^{n+1}, \tag{5.1}$$

where $s_r = \tanh r$. Assume that $\varrho \le 1$ and write $s = \tanh \varrho$. By (a) of Lemma 2.3, if $z \in \mathbb{B}$ is such that $w_m \in D(z,1)$, then $D_m \subset D(z,2)$. Thus

$$\mu_\varrho(D(z,1)) = \sum_{w_m \in D(z,1)} \mathrm{v}(D_m) \le \mathrm{v}(D(z,2)) \le C\mathrm{v}(D(z,1)),$$

where the last equality follows from (5.1), with $C > 0$ independent of ϱ. The equivalence between (2) and (3) of Lemma 2.1 now says that

$$\sum_m \mathrm{v}(D_m)|g(w_m)|^q \le C_q\|g\|_q^q \tag{5.2}$$

for all $g \in A^q$, where $C_q > 0$ does not depend on ϱ. By [13, Lemma 3.10] applied to our measures $d\mathrm{v}$ and $d\mu_\varrho$, there is a constant $C_p > 0$ independent of ϱ such that

$$\sum_{m\ge 1} \frac{\mathrm{v}(D_m)}{\mathrm{v}(D(w_m,\varrho))}\int_{D(w_m,\varrho)} |f(w) - f(w_m)|^p \,d\mathrm{v}(w) \le C_p s^p \|f\|_p^p$$

for all $f \in A^p$. Since $D(w_m, \varrho/4) \subset D_m \subset D(w_m,\varrho)$, (5.1) leads to $\mathrm{v}(D_m) \sim \mathrm{v}(D(w_m,\varrho))$, with constants not depending on ϱ. Then

$$\sum_{m\ge 1} \int_{D_m} |f(w) - f(w_m)|^p \,d\mathrm{v}(w) \le C'_p s^p \|f\|_p^p. \tag{5.3}$$

If $f, g \in H^\infty$, then

$$\begin{aligned}\langle (I - T_{\mu_\varrho})f, g\rangle &= \int_{\mathbb{B}} f(z)\overline{g(z)}\,\mathrm{dv}(z) - \sum_{m=1}^{\infty} \mathrm{v}(D_m) f(w_m)\langle K_{w_m}, g\rangle \\ &= \sum_{m=1}^{\infty} \int_{D_m} f(z)(\overline{g(z)} - \overline{g(w_m)})\,\mathrm{dv}(z) \\ &\quad + \sum_{m=1}^{\infty} \int_{D_m} (f(z) - f(w_m))\overline{g(w_m)}\,\mathrm{dv}(z).\end{aligned}$$

Applying Hölder's inequality twice (to the integral and the sum) to each one of the above sums, (5.3) and (5.2) show that $|\langle (I - T_{\mu_\varrho})f, g\rangle| \le G_p s \|f\|_p \|g\|_q$, where $G_p > 0$ depends only on p. The density of H^∞ in A^p and A^q, together with the isomorphism $(A^p)^* \simeq A^q$, imply that $\|I - T_{\mu_\varrho}\| \le Cs$ for some constant $C > 0$ depending only on p. Since $s \to 0$ as $\varrho \to 0$, the lemma follows. ◻

By Lemma 5.1, for each $1 < p < \infty$ we can choose $0 < \varrho \le 1$ small enough, so that

$$\|I - T_{\mu_\varrho}\|_{\mathfrak{L}(A^p)} < \frac{1}{4}.$$

This implies that T_{μ_ϱ} is invertible in $\mathfrak{L}(A^p)$, with $\|T_{\mu_\varrho}\|$, $\|T_{\mu_\varrho}^{-1}\| \le \frac{3}{2}$. For the rest of the paper we fix $\varrho = \varrho(p)$ according to these conditions and simply write $\mu = \mu_\varrho$. For $S \in \mathfrak{L}(A^p)$ and $r > 0$, let

$$\alpha_S(r) \overset{\text{def}}{=} \limsup_{|z|\to 1}\ \sup\Big\{\|Sf\| : f \in T_{\chi_{D(z,r)}\mu}(A^p),\ \|f\| \le 1\Big\}.$$

Since $T_{\chi_{D(z,r_1)}\mu}(A^p) \subset T_{\chi_{D(z,r_2)}\mu}(A^p)$ when $r_1 < r_2$, then $\alpha_S(r)$ increases with r, and since $\alpha_S(r) \le \|S\|$ for all r, we have

$$\alpha_S \overset{\text{def}}{=} \lim_{r\to\infty} \alpha_S(r) = \sup_{r>0} \alpha_S(r) \le \|S\|.$$

If E and F are Banach spaces, the essential norm of an operator $R \in \mathfrak{L}(E, F)$ is

$$\|R\|_{\mathrm{e}} \overset{\text{def}}{=} \inf\Big\{\|R - Q\| : Q \in \mathfrak{L}(E, F) \text{ is compact}\Big\}.$$

Theorem 5.2. *Let $1 < p < \infty$ and $S \in \mathfrak{T}_p$. Then $\|S\|_{\mathrm{e}}$ is equivalent to the following quantities* (*with constants depending only on p and n*)

(1) α_S,
(2) $\beta_S = \sup_{d>0} \limsup_{|z|\to 1} \|M_{\chi_{D(z,d)}} S\|_{\mathfrak{L}(A^p, L^p)}$,
(3) $\gamma_S = \lim_{r\to 1} \|M_{\chi_{(rB)^c}} S\|_{\mathfrak{L}(A^p, L^p)}$, *where $(r\mathbb{B})^c = \mathbb{B} \setminus r\mathbb{B}$.*

Beginning of the proof. In order to distinguish between essential norms for operators in $\mathfrak{L}(A^p)$ or $\mathfrak{L}(A^p, L^p)$, we write $\| \ \|_{\mathrm{e}}$ and $\| \ \|_{\mathrm{ex}}$ for the respective essential norm. Any $R \in \mathfrak{L}(A^p)$ can be thought of as belonging to $\mathfrak{L}(A^p, L^p)$, so both quantities apply to R, and since $PR = R$, where P is the Bergman projection, we have

$$\|R\|_{\mathrm{ex}} \le \|R\|_{\mathrm{e}} \le \|P\|_{\mathfrak{L}(L^p)} \|R\|_{\mathrm{ex}}. \tag{5.4}$$

First observe that since $\|T_\mu\|$, $\|T_\mu^{-1}\| \le \frac{3}{2}$, the numbers $\|S\|_{\mathrm{e}}$ and $\|ST_\mu\|_{\mathrm{e}}$ are equivalent. Given $\varepsilon > 0$, there are Borel sets $F_j \subset G_j \subset \mathbb{B}$ as in Theorem 4.3 such that

$$\Big\| ST_\mu - \sum_{j\ge 1} M_{\chi_{F_j}} ST_{\chi_{G_j}\mu} \Big\|_{\mathfrak{L}(A^p, L^p)} < \varepsilon. \tag{5.5}$$

Since $\sum_{j=1}^m M_{\chi_{F_j}} ST_{\chi_{G_j}\mu}$ is compact for any $m \ge 1$, we have

$$\Big\| ST_\mu - \sum_{j\ge m} M_{\chi_{F_j}} ST_{\chi_{G_j}\mu} \Big\|_{\mathrm{ex}} < \varepsilon \tag{5.6}$$

for any $m \ge 1$. Write $S_m = \sum_{j\ge m} M_{\chi_{F_j}} ST_{\chi_{G_j}\mu}$ and let $f \in A^p$ be of norm 1. Since every $z \in \mathbb{B}$ belongs to at most N of the sets G_j, Lemma 2.2 yields

$$\sum_{j\ge m} \|T_{\chi_{G_j}\mu} f\|^p \le \sum_{j\ge 1} C_p^p \|\chi_{G_j} f\|_{L^p(d\mu)}^p \le C_p^p N \|f\|_{L^p(d\mu)}^p = K_p^p,$$

a constant that only depends on p. Therefore

$$\begin{aligned} \|S_m f\|^p &= \sum_{j\ge m} \|M_{\chi_{F_j}} ST_{\chi_{G_j}\mu} f\|^p \\ &= \sum_{j\ge m,\, T_{\chi_{G_j}\mu} f \ne 0} \left(\frac{\|M_{\chi_{F_j}} ST_{\chi_{G_j}\mu} f\|}{\|T_{\chi_{G_j}\mu} f\|} \right)^p \|T_{\chi_{G_j}\mu} f\|^p \\ &\le \sup_{j\ge m} \sup \Big\{ \|M_{\chi_{F_j}} Sg\|^p : g \in T_{\chi_{G_j}\mu}(A^p),\ \|g\| = 1 \Big\} \sum_{j\ge m} \|T_{\chi_{G_j}\mu} f\|^p \\ &\le K_p^p \sup_{j\ge m} \sup \Big\{ \|M_{\chi_{F_j}} Sg\|^p : g \in T_{\chi_{G_j}\mu}(A^p),\ \|g\| = 1 \Big\}. \end{aligned} \tag{5.7}$$

For each j pick $z_j \in G_j$. Since (d) of Theorem 4.3 says that $\operatorname{diam}_\beta G_j \le d$, then $G_j \subset D(z_j, d)$, and consequently $T_{\chi_{G_j}\mu}(A^p) \subset T_{\chi_{D(z_j,d)}\mu}(A^p)$. Also, there is a sequence $0 < \gamma_m < 1$ tending to 1, such that $|z_j| \geq \gamma_m$ when $j \geq m$. So, (5.7) yields

$$\|S_m\|^p \leq K_p^p \sup_{j\geq m} \sup \left\{ \|M_{\chi_{F_j}} S g\|^p : g \in T_{\chi_{D(z_j,d)}\mu}(A^p),\ \|g\| = 1 \right\}$$
$$\leq K_p^p \sup_{|z|\geq \gamma_m} \sup \left\{ \|M_{\chi_{D(z,d)}} S g\|^p : g \in T_{\chi_{D(z,d)}\mu}(A^p),\ \|g\| = 1 \right\}$$
$$\leq K_p^p \sup_{|z|\geq \gamma_m} \sup \left\{ \|S g\|^p : g \in T_{\chi_{D(z,d)}\mu}(A^p),\ \|g\| = 1 \right\}. \tag{5.8}$$

When $m \to \infty$ we have $\gamma_m \to 1$, and consequently

$$\limsup_{m\to\infty} \|S_m\| \leq K_p\, \alpha_S(d).$$

Joining this estimate with (5.6) we get

$$\|ST_\mu\|_{\mathrm{ex}} \leq \limsup_m \|S_m\| + \varepsilon \leq K_p\, \alpha_S(d) + \varepsilon \leq K_p\, \alpha_S + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it can be deleted from the above chain of inequalities. Therefore, (5.4) and the equivalence between $\|ST_\mu\|_{\mathrm{e}}$ and $\|S\|_{\mathrm{e}}$ lead to

$$\|S\|_{\mathrm{e}} \leq G_p \limsup_m \|S_m\| \leq G'_p\, \alpha_S, \tag{5.9}$$

where G_p and G'_p are positive constants depending on p.

It is clear that $\beta_S \leq \gamma_S$. On the other hand, if $0 < r < 1$, there exists a positive integer $m(r) \to \infty$ as $r \to 1$, such that $\bigcup_{j<m(r)} F_j \subset r\mathbb{B}$. By (5.5) then

$$\|M_{\chi_{(rB)^c}} S\|\ \|T_\mu^{-1}\|^{-1} \leq \|M_{\chi_{(rB)^c}} S T_\mu\|$$
$$\leq \left\| M_{\chi_{(rB)^c}} \Big(ST_\mu - \sum_{j\geq 1} M_{\chi_{F_j}} S T_{\chi_{G_j}\mu} \Big) \right\| + \left\| M_{\chi_{(rB)^c}} \sum_{j\geq 1} M_{\chi_{F_j}} S T_{\chi_{G_j}\mu} \right\|$$
$$\leq \varepsilon + \left\| \sum_{j\geq m(r)} M_{\chi_{F_j}} S T_{\chi_{G_j}\mu} \right\| = \varepsilon + \|S_{m(r)}\|.$$

Since $\|T_\mu^{-1}\| \leq \frac{3}{2}$, we get

$$\gamma_S = \limsup_{r\to 1} \|M_{\chi_{(rB)^c}} S\| \leq \tfrac{3}{2}(\varepsilon + \limsup_{r\to 1} \|S_{m(r)}\|) \leq \tfrac{3}{2}(\varepsilon + \limsup_{m\to\infty} \|S_m\|).$$

Since $\varepsilon > 0$ is arbitrary, we can delete it.

Since by (5.8), $\|S_m\| \le K_p \sup_{|z|\ge\gamma_m} \|M_{\chi_D(z,d)}S\|$,

$$\limsup_m \|S_m\| \le K_p \limsup_{|z|\to 1} \|M_{\chi_D(z,d)}S\| \le K_p\beta_S.$$

All this proves the equivalence between β_S, γ_S and $\limsup_{m\to\infty}\|S_m\|$. By (5.9) the theorem will follow if we show that

$$\alpha_S \le C\|S\|_e \tag{5.10}$$

for some constant $C > 0$ depending only on p. The proof of this inequality will be postponed until the proof of Theorem 9.3. □

6. A Uniform Algebra and Its Maximal Ideal Space

Consider the uniform algebra $\mathcal{A}$ of all the bounded functions that are uniformly continuous from the metric space $(\mathbb{B}, \rho)$ into the metric space $(\mathbb{C}, |\ |)$. Clearly, ρ can be replaced by β in the above definition. The maximal ideal space $M_{\mathcal{A}}$ of $\mathcal{A}$ is formed by all the nonzero multiplicative linear maps from $\mathcal{A}$ into $\mathbb{C}$, endowed with the weak star topology. It is a compact Hausdorff space, and the Gelfand transform of $a \in \mathcal{A}$ is the function $\hat{a} \in C(M_{\mathcal{A}})$ defined as $\hat{a}(\varphi) = \varphi(a)$, for $\varphi \in M_{\mathcal{A}}$. Since $\mathcal{A}$ is a commutative C^*-algebra, the Gelfand-Naimark Theorem asserts that the Gelfand transform is an isomorphism (see [6, Theorem 4.29]). That is, we can identify $\mathcal{A}$ with $C(M_{\mathcal{A}})$ via this transform. Evaluations at points of $\mathbb{B}$ are in $M_{\mathcal{A}}$, so $\mathbb{B} \subset M_{\mathcal{A}}$, and the Euclidean topology on $\mathbb{B}$ agrees with the topology induced by $M_{\mathcal{A}}$. Also, the fact that $\mathcal{A}$ is a C^*-algebra easily implies that $\mathbb{B}$ is dense in $M_{\mathcal{A}}$.

In the next lemma, $\bar{E}$ denotes the closure of $E \subset M_{\mathcal{A}}$ in the space $M_{\mathcal{A}}$. By a comment above, when $E \subset r\mathbb{B}$ for some $0 < r < 1$, $\bar{E}$ has the same meaning in both, the $M_{\mathcal{A}}$ and the Euclidean topologies. Also, we shall not write the roof for the Gelfand transform of $a \in \mathcal{A}$.

Lemma 6.1. *Let E, $F \subset \mathbb{B}$. Then $\bar{E} \cap \bar{F} = \emptyset$ if and only if $\rho(E, F) > 0$.*

Proof. If $\bar{E} \cap \bar{F} = \emptyset$, Tietze's theorem says that there is $a \in C(M_{\mathcal{A}}) = \mathcal{A}$ such that $a \equiv 1$ on $\bar{E}$ and $a \equiv 0$ on $\bar{F}$. The uniform ρ-continuity of a on $\mathbb{B}$ implies that $\rho(E, F) > 0$. If $\rho(E, F) > 0$, the function $a(z) = \rho(z, E) \in \mathcal{A}$ and separates $\bar{E}$ from $\bar{F}$, so they are disjoint. □

Lemma 6.2. *Let z, w, $\xi \in \mathbb{B}$. Then there is a constant $G > 0$ depending only on n such that*

$$\rho(\varphi_z(\xi), \varphi_w(\xi)) \le \frac{G}{(1-|\xi|)^2}\rho(z, w).$$

Proof. We are going to need the following elementary inequality for u, $v \in \mathbb{B}$,

$$\rho(u, v) = \frac{|P_u(u-v) + (1-|u|^2)^{1/2}Q_u(u-v)|}{|1-\langle v, u\rangle|} \le \frac{|u-v|}{1-|u|}. \tag{6.1}$$

By Cartan's theorem every automorphisms of $\mathbb{B}$ that fixes the origin has the form $\phi(z) = \mathcal{U}z$, where $\mathcal{U}$ belongs to the complex unitary group $\mathfrak{U}(n) \subset \mathbb{C}^{n\times n}$ (see [17, p. 24]). Hence

$$\varphi_{\varphi_w(z)} \circ \varphi_w \circ \varphi_z = \mathcal{U}$$

for some $\mathcal{U} \in \mathfrak{U}(n)$. Furthermore, in [14, Lemma 2.8] it is shown that

$$\|I + \mathcal{U}\| \le C(n)\rho(z,w). \tag{6.2}$$

We can assume that $z \neq w$. If we write $v = \varphi_w(z)$, then $|v| = \rho(z,w) \neq 0$, and

$$\begin{aligned}\rho(\varphi_z(\xi),\varphi_w(\xi)) = \rho(\varphi_w\circ\varphi_z(\xi),\varphi_w\circ\varphi_w(\xi)) &= \rho(\varphi_{\varphi_w(z)}(\mathcal{U}\xi),\xi)\\ &= \rho(\varphi_v(\mathcal{U}\xi),\xi) \le \rho(\varphi_v(\mathcal{U}\xi),-\mathcal{U}\xi) + \rho(-\mathcal{U}\xi,\xi)\\ &\le \frac{1}{1-|\xi|}(|\varphi_v(\mathcal{U}\xi)+\mathcal{U}\xi| + |\xi+\mathcal{U}\xi|),\end{aligned}$$

where the last inequality comes from (6.1) and $|\mathcal{U}\xi| = |\xi|$. By (6.2) the second summand between brackets is bounded by $C(n)\rho(z,w)$. To estimate the first summand within the brackets, write $\xi' = \mathcal{U}\xi$. Thus

$$\begin{aligned}|\varphi_v(\xi')+\xi'| &= \left|\frac{v - P_v(\xi') - (1-|v|^2)^{1/2}Q_v(\xi')}{1-\langle\xi',v\rangle} + \xi'\right|\\ &= \frac{\left|-\xi'\langle\xi',v\rangle + v + \left(\xi' - \langle\xi',v\rangle\frac{v}{|v|^2}\right)\left[1-(1-|v|^2)^{1/2}\right]\right|}{|1-\langle\xi',v\rangle|}\\ &\le \frac{2|v| + 2[1-(1-|v|^2)^{1/2}]}{(1-|\xi'|)}\\ &\le \frac{4|v|}{(1-|\xi'|)} = \frac{4\rho(z,w)}{(1-|\xi|)}.\end{aligned}$$

□

Let $x \in M_{\mathcal{A}}$ and suppose that (z_α) is a net in $\mathbb{B}$ that tends to x. By compactness, the net (φ_{z_α}) in the product space $M_{\mathcal{A}}^{\mathbb{B}}$ admits a convergent subnet $(\varphi_{z_{\alpha_\beta}})$. This means that there is some function $\varphi : \mathbb{B} \to M_{\mathcal{A}}$ such that $f\circ\varphi_{z_{\alpha_\beta}} \to f\circ\varphi$ pointwise on $\mathbb{B}$ for every $f \in \mathcal{A}$. We show next that the whole net (z_α) tends to φ and that φ does not depend on the net. So, suppose that (ω_γ) is another net in $\mathbb{B}$ converging to x such that φ_{ω_γ} tends to some $\psi \in M_{\mathcal{A}}^{\mathbb{B}}$. If there is $\xi \in \mathbb{B}$ such that $\varphi(\xi) \neq \psi(\xi)$, then there are tails of both nets whose underlying sets

$$E = \left\{\varphi_{z_{\alpha_\beta}}(\xi) : \beta \geq \beta_0\right\} \quad \text{and} \quad F = \left\{\varphi_{\omega_\gamma}(\xi) : \gamma \geq \gamma_0\right\}$$

have disjoint closures in $M_{\mathcal{A}}$. By Lemma 6.1 then $\rho(E,F) > 0$. But Lemma 6.2 says that

$$\begin{aligned}\rho(E,F) &= \inf\left\{\rho(\varphi_{z_{\alpha_\beta}}(\xi), \varphi_{\omega_\gamma}(\xi)) : \beta \geq \beta_0,\ \gamma \geq \gamma_0\right\} \\ &\leq \frac{G}{(1-|\xi|)^2} \inf\left\{\rho(z_{\alpha_\beta}, \omega_\gamma) : \beta \geq \beta_0,\ \gamma \geq \gamma_0\right\} = 0,\end{aligned}$$

where the last equality holds by Lemma 6.1, because both nets (z_{α_β}) and (ω_γ) tend to x. The map φ will be denoted φ_x, and observe that $\varphi_x(0) = \lim \varphi_{z_\alpha}(0) = \lim z_\alpha = x$.

Lemma 6.3. *Let (z_α) be a net in $\mathbb{B}$ converging to $x \in M_{\mathcal{A}}$. Then*

(i) *$a \circ \varphi_x \in \mathcal{A}$ for every $a \in \mathcal{A}$ (hence $\varphi_x : \mathbb{B} \to M_{\mathcal{A}}$ is continuous),*
(ii) *$a \circ \varphi_{z_\alpha} \to a \circ \varphi_x$ uniformly on compact sets of $\mathbb{B}$ for every $a \in \mathcal{A}$.*

Proof. If $a \in \mathcal{A}$, given $\varepsilon > 0$ there is $\delta > 0$ such that if $u, v \in \mathbb{B}$,

$$\rho(u,v) < \delta \ \Rightarrow\ |a(u) - a(v)| < \varepsilon.$$

Since $\rho(\varphi_{z_\alpha}(u), \varphi_{z_\alpha}(v)) = \rho(u,v)$ and $|a(\varphi_x(u)) - a(\varphi_x(v))| = \lim |a(\varphi_{z_\alpha}(u)) - a(\varphi_{z_\alpha}(v))|$, (i) follows. Suppose that (ii) fails. This means that there are $a \in \mathcal{A}$, $0 < r < 1$ and $\varepsilon > 0$ such that

$$\left|(a \circ \varphi_{z_\alpha})(\xi_\alpha) - (a \circ \varphi_x)(\xi_\alpha)\right| > \varepsilon$$

for some points $\xi_\alpha \in r\mathbb{B}$. Taking a suitable subnet we can assume that $\xi_\alpha \to \xi \in \overline{r\mathbb{B}}$. Therefore

$$\begin{aligned}|(a \circ \varphi_{z_\alpha})(\xi_\alpha) - (a \circ \varphi_x)(\xi_\alpha)| &\leq |(a \circ \varphi_{z_\alpha})(\xi_\alpha) - (a \circ \varphi_{z_\alpha})(\xi)| \\ &\quad + |(a \circ \varphi_{z_\alpha})(\xi) - (a \circ \varphi_x)(\xi)| + |(a \circ \varphi_x)(\xi) - (a \circ \varphi_x)(\xi_\alpha)|,\end{aligned}$$

where the first and third summands tend to 0 by the ρ-continuity of a and $a \circ \varphi_x$, respectively, and the second tends to 0 because $a \circ \varphi_{z_\alpha} \to a \circ \varphi_x$ pointwise. This contradicts the previous inequality. □

7. APPROXIMATING TOEPLITZ OPERATORS BY k-BEREZIN TRANSFORMS

Our goal in this section is to show that $\mathfrak{T}_p$ is generated by Toeplitz operators with symbols in $\mathcal{A}$ for every $1 < p < \infty$. Actually, we prove the more general statement that if ν is a complex-valued measure whose total variation is Carleson, then T_ν can be approximated in $\mathfrak{L}(A^p)$-norm by operators of the form T_a, with $a \in \mathcal{A}$. For $n = 1$, $p = 2$, this was proved in [22, Corollary 2.5], and except for some minor simplifications, the proof here is essentially the same. If $z \in \mathbb{B}$, the (complex) Jacobian of the map φ_z is

$$J\varphi_z = (-1)^n \frac{(1-|z|^2)^{(n+1)/2}}{(1-\langle \cdot , z\rangle)^{n+1}} = (-1)^n (1-|z|^2)^{(n+1)/2} K_z.$$

Let ν be a complex-valued, Borel, regular measure on $\mathbb{B}$ of finite total variation. For $z \in \mathbb{B}$ consider the measure $\nu_z = |J\varphi_z|^{-2}(\nu \circ \varphi_z)$, where $(\nu \circ \varphi_z)(E) \stackrel{\text{def}}{=} \nu(\varphi_z(E))$ for every Borel set $E \subset \mathbb{B}$ (i.e., $\nu \circ \varphi_z$ is the pull-back measure). From the identity $(J\varphi_z)(\varphi_z(\xi))(J\varphi_z)(\xi) = 1$ we get

$$\int_{\mathbb{B}} (f \circ \varphi_z)|J\varphi_z|^2 \,\mathrm{d}\nu = \int_{\mathbb{B}} f \,\mathrm{d}\nu_z \tag{7.1}$$

for every bounded continuous function f.

Definition. If $z \in \mathbb{B}$ and $k = 0, 1, \dots,$ the k-Berezin transform of ν is the function

$$B_k(\nu)(z) = \binom{n+k}{n} \int_{\mathbb{B}} |J\varphi_z(w)|^2 (1-|\varphi_z(w)|^2)^k \,\mathrm{d}\nu(w).$$

If $z, w \in \mathbb{B}$, Cartan's theorem implies that $\varphi_w \circ \varphi_z = V \circ \varphi_{\varphi_z(w)}$, where $V \in \mathbb{C}^{n\times n}$ is a unitary matrix, leading to $|(J\varphi_w)\circ\varphi_z|\,|J\varphi_z| = |J\varphi_{\varphi_z(w)}|$. It follows immediately from these equalities and (7.1) that $B_k(\nu)(\varphi_z(w)) = B_k(\nu_z)(w)$ for all $k \geq 0$. In particular, if ν is a Carleson measure,

$$\|\nu\|_* = \|B_0(\nu)\|_\infty = \|B_0(\nu_z)\|_\infty = \|\nu_z\|_*. \tag{7.2}$$

Lemma 7.1. *Let $0 < \alpha < 1$ and ν be a complex-valued measure such that its total variation $|\nu|$ is a Carleson measure. If $1/p_1 + 1/q_1 = 1$, where $q_1 > 1$ is close enough to 1 so that $q_1\alpha < 1$ and $q_1(n+1-\alpha) < n+1$, then there is a constant $C_{p_1} > 0$ such that*

$$\int_{\mathbb{B}} \frac{|(T_\nu K_z)(w)|}{(1-|w|^2)^\alpha} \,\mathrm{d}v(w) \leq \frac{C_{p_1}\|T_{\nu_z}1\|_{p_1}}{(1-|z|^2)^\alpha} \tag{7.3}$$

for all $z \in \mathbb{B}$.

Proof. If $z \in \mathbb{B}$, a straightforward calculation from (7.1) gives

$$(J\varphi_z)[(T_\nu J\varphi_z) \circ \varphi_z] = T_{\nu_z} 1,$$

and consequently $(-1)^n(1-|z|^2)^{(n+1)/2} T_\nu K_z = T_\nu J\varphi_z = [(T_{\nu_z}1) \circ \varphi_z](J\varphi_z)$. Thus

$$\begin{aligned}
&\int_{\mathbb{B}} \frac{|(T_\nu K_z)(w)|}{(1-|w|^2)^\alpha} \,\mathrm{dv}(w) \\
&\quad = \frac{1}{(1-|z|^2)^{(n+1)/2}} \int_{\mathbb{B}} \frac{|(T_{\nu_z}1)(\varphi_z(w))|\,|J\varphi_z(w)|}{(1-|w|^2)^\alpha} \,\mathrm{dv}(w) \\
&\quad = \frac{1}{(1-|z|^2)^\alpha} \int_{\mathbb{B}} \frac{|(T_{\nu_z}1)(\lambda)|}{(1-|\lambda|^2)^\alpha\,|1-\langle\lambda,z\rangle|^{(n+1)-2\alpha}} \,\mathrm{dv}(\lambda) \\
&\quad \le \frac{\|T_{\nu_z}1\|_{p_1}}{(1-|z|^2)^\alpha} \left(\int_{\mathbb{B}} \frac{d\mathrm{v}(\lambda)}{(1-|\lambda|^2)^{\alpha q_1}|1-\langle\lambda,z\rangle|^{q_1(n+1-2\alpha)}} \right)^{1/q_1} \\
&\quad \le C_{p_1} \frac{\|T_{\nu_z}1\|_{p_1}}{(1-|z|^2)^\alpha},
\end{aligned}$$

where the second equality follows from the substitution $w = \varphi_z(\lambda)$, and the last inequality from Lemma 2.4 and our conditions on q_1. □

Lemma 7.2. *Let $1 < p < \infty$ and ν be a measure as in Lemma 7.1. If $1/p_1 + 1/q_1 = 1$, where q_1 satisfies the conditions of Lemma 7.1 for both $\alpha = 1/p$ and $1/q$, where $q = p/(p-1)$, then*

$$\|T_\nu\|_{\mathfrak{L}(A^p)} \le C_{p_1} \Big(\sup_{z\in\mathbb{B}} \|T_{\nu_z}1\|_{p_1} \Big)^{1/p} \Big(\sup_{z\in\mathbb{B}} \|T^*_{\nu_z}1\|_{p_1} \Big)^{1/q}, \tag{7.4}$$

where C_{p_1} is the constant of Lemma 7.1.

Proof. Let $f \in A^p$ and $w \in \mathbb{B}$. Since $(T_\nu K_\lambda)(w) = \overline{(T^*_\nu K_w)(\lambda)}$, we have

$$(T_\nu f)(w) = \langle T_\nu f, K_w\rangle = \langle f, T^*_\nu K_w\rangle = \int_{\mathbb{B}} f(\lambda)(T_\nu K_\lambda)(w)\,\mathrm{dv}(\lambda).$$

Letting $\Phi(\lambda, w) = |(T_\nu K_\lambda)(w)| = |(T^*_\nu K_w)(\lambda)|$ and $h(\lambda) = (1-|\lambda|^2)^{-1/pq}$, (7.3) with $\alpha = 1/q$ yields

$$\int_{\mathbb{B}} \Phi(\lambda, w) h(w)^p \,\mathrm{dv}(w) \le C_{p_1} \sup_{z\in\mathbb{B}} \|T_{\nu_z}1\|_{p_1} h(\lambda)^p,$$

and (7.3) with $\alpha = 1/p$ gives

$$\int_{\mathbb{B}} \Phi(\lambda, w) h(\lambda)^q \,\mathrm{dv}(\lambda) \le C_{p_1} \sup_{z\in\mathbb{B}} \|T^*_{\nu_z}1\|_{p_1}\, h(w)^q.$$

Therefore (7.4) follows from Lemma 2.6. □

If ν is a Carleson measure, the formula $B_k(\nu) = C_{n,k}\int |J\varphi_z|^2(1-|\varphi_z|^2)^k\,d\nu$ shows that $\|B_k(\nu)\|_\infty \leq C_{n,k}\|B_0(\nu)\|_\infty = C_{n,k}\|\nu\|_*$ for all $k \geq 0$, and since [14, Theorem 2.11] says that $B_k(\nu)$ is Lipschitz with respect to the pseudohyperbolic metric, it follows that $B_k(\nu) \in \mathcal{A}$ for all $k \geq 0$. Hence, the same holds for a complex measure ν such that $|\nu|$ is Carleson. If ν is absolutely continuous, so $\nu = a\,\mathrm{dv}$, with $a \in L^1(d\mathrm{v})$, the k-Berezin transform of ν will be simply denoted $B_k(a)$. In this case, the change of variable $w = \varphi_z(\xi)$ in the integral defining $B_k(a)$ yields

$$(B_k a)(z) = \binom{n+k}{n}\int_{\mathbb{B}}(1-|\xi|^2)^k a(\varphi_z(\xi))\,\mathrm{dv}(\xi).$$

Since $\binom{n+k}{n}(1-|w|^2)^k\,\mathrm{dv}$ are probability measures whose masses tend to concentrate at 0 as k increases, it is clear that if $a \in \mathcal{A}$, then $\|B_k(a) - a\|_\infty \to 0$ when $k \to \infty$.

Theorem 7.3. *Let $1 < p < \infty$ and ν be a complex-valued measure such that $|\nu|$ is a Carleson measure. Then $T_{B_k(\nu)} \to T_\nu$ in the norm of $\mathfrak{L}(A^p)$. In particular, $\mathfrak{T}_p$ is the closed algebra generated by $\{T_a : a \in \mathcal{A}\}$.*

Proof. By the linearity of B_k it is enough to prove the theorem for a Carleson measure ν. In [1, Proposition 2.6] it is shown that $B_0B_k(\nu) = B_kB_0(\nu)$ for an absolutely continuous measure ν, but the proof works in general. Since $B_0(\nu) \in \mathcal{A}$,

$$\begin{aligned}||B_0(B_k(\nu)\,\mathrm{dv} - d\nu)||_\infty &= ||B_0B_k(\nu) - B_0(\nu)||_\infty \\ &= ||B_kB_0(\nu) - B_0(\nu)||_\infty \to 0\end{aligned}$$

as $k \to \infty$. Consequently,

$$\begin{aligned}||B_k(\nu)\,\mathrm{dv}||_* + \|\nu\|_* &= ||B_0B_k(\nu)||_\infty + ||B_0(\nu)||_\infty \\ &\leq C(\nu),\end{aligned} \tag{7.5}$$

which together with Lemma 2.1 says that $\|T_{B_k(\nu)} - T_\nu\|_{\mathfrak{L}(A^2)}$ is bounded independently of k. Under these conditions, [21, Lemma 5.5] for $n = 1$ and [14, Lemma 3.4] for a general n, say that

$$\sup_{z\in\mathbb{B}} |T_{(B_k(\nu)\,\mathrm{dv} - d\nu)_z} 1| \to 0 \tag{7.6}$$

uniformly on compact sets as $k \to \infty$. Let $\varepsilon > 0$ and write $F_{k,z} = T_{(B_k(\nu)\,\mathrm{dv}-d\nu)_z}1$. If $0 < r < 1$ and $1 < p_1 < \infty$ is big enough so that (7.4) holds for our value of p, split the integral $\|F_{k,z}\|_{p_1}^{p_1} = \|F_{k,z}\chi_{(r\mathbb{B})^c}\|_{p_1}^{p_1} + \|F_{k,z}\chi_{r\mathbb{B}}\|_{p_1}^{p_1}$. The Cauchy-Schwarz's inequality gives

$$\begin{aligned}||F_{k,z}\chi_{(r\mathbb{B})^c}||_{p_1}^{p_1} &\le ||F_{k,z}||_{2p_1}^{p_1}\,\|\chi_{(r\mathbb{B})^c}\|_2 = ||F_{k,z}||_{2p_1}^{p_1}(1-r^{2n})^{1/2}\\ &\le C_{2p_1}(\|(B_k(\nu)\,\mathrm{d}\nu)_z\|_* + \|d\nu_z\|_*)^{p_1}(1-r^{2n})^{1/2}\\ &\le C_{2p_1}C(\nu)^{p_1}(1-r^{2n})^{1/2} < \varepsilon\end{aligned}$$

if r is chosen close enough to 1, where the second inequality follows from Lemma 2.1 and the last one from (7.2) and (7.5). Once we have fixed such r, (7.6) says that $F_{k,z}(w)\chi_{r\mathbb{B}}(w)$ tends to 0 uniformly on $z, w \in \mathbb{B}$ when $k \to \infty$. Henceforth,

$$\sup_{z\in\mathbb{B}}\|F_{k,z}\|_{p_1} = \sup_{z\in\mathbb{B}}\|T_{(B_k(\nu)\,\mathrm{d}\nu - d\nu)_z}1\|_{p_1} \to 0$$

as $k \to \infty$, and since $T^*_{(B_k(\nu)\,\mathrm{d}\nu - d\nu)_z} = T_{(B_k(\bar\nu)\,\mathrm{d}\bar\nu - d\bar\nu)_z}$, the theorem follows from (7.4). ◻

8. Maps from $M_{\mathcal{A}}$ into $\mathfrak{L}(A^p)$

If $z, w \in \mathbb{B}$ and α is any real number, we shall write

$$J_z^\alpha(w) = \frac{(1-|z|^2)^{\alpha(n+1)/2}}{(1-\langle w,z\rangle)^{\alpha(n+1)}},$$

where the argument of $(1-\langle w,z\rangle)$ used to define its $\alpha(n+1)$-root varies within the open interval $(-\pi,\pi)$. In particular, for $\alpha = 1$ we get $J_z = (-1)^n J\varphi_z$, where we recall that $J\varphi_z$ is the Jacobian of the map φ_z. It follows from $(J\varphi_z)(\varphi_z)(J\varphi_z) = 1$ that $(J_z^\alpha \circ \varphi_z)J_z^\alpha = 1$ for any real number α. For $1 < p < \infty$, $z \in \mathbb{B}$ and $f \in A^p$, consider the map

$$\begin{aligned}U_z^p f(w) &= (f\circ\varphi_z)(w)J_z^{2/p}(w)\\ &= f(\varphi_z(w))\frac{(1-|z|^2)^{(n+1)/p}}{(1-\langle w,z\rangle)^{2(n+1)/p}}.\end{aligned}$$

Keep in mind that the p of U_z^p is an index, not a power. A change of variables and the identity $(J_z^{2/p}\circ\varphi_z)J_z^{2/p} = 1$ show that $\|U_z^p f\|_p = \|f\|_p$ for all $f \in A^p$ and $U_z^p U_z^p = I_{A^p}$. Also,

$$U_z^p = T_{J_z^{2/p-1}}U_z^2 = U_z^2 T_{J_z^{1-2/p}},$$

and consequently for $q = p/(p-1)$,

$$(U_z^q)^* = U_z^2 T_{\bar J_z^{2/q-1}} = T_{\bar J_z^{1-2/q}}U_z^2.$$

Thus,

$$(U_z^q)^* U_z^p = T_{\bar{J}_z^{1-2/q}} U_z^2 U_z^2 T_{J_z^{1-2/p}} = T_{\bar{J}_z^{1-2/q} J_z^{1-2/p}} = T_{b_z}$$

and

$$U_z^p (U_z^q)^* = T_{J_z^{2/p-1}} U_z^2 U_z^2 T_{\bar{J}_z^{2/q-1}} = T_{J_z^{2/p-1}} T_{\bar{J}_z^{2/q-1}} = T_{b_z}^{-1},$$

where

$$b_z(w) = \bar{J}_z^{1-2/q}(w) J_z^{1-2/p}(w) = \frac{(1-\overline{\langle w,z\rangle})^{(n+1)(1/q-1/p)}}{(1-\langle w,z\rangle)^{(n+1)(1/q-1/p)}}. \tag{8.1}$$

Definition. For $S \in \mathfrak{L}(A^p)$ and $z \in \mathbb{B}$ define $S_z = U_z^p S (U_z^q)^*$.

It should be kept in mind that the definition of S_z depends on p. Consider the map $\Psi_S : \mathbb{B} \to \mathfrak{L}(A^p)$ given by $\Psi_S(z) = S_z$. We will study the possibility to extend Ψ_S continuously to $M_{\mathcal{A}}$ when $\mathfrak{L}(A^p)$ is provided with the weak or the strong operator topologies (WOT and SOT, respectively). The inclusion $C(\bar{\mathbb{B}}) \subset \mathcal{A}$ induces by transposition a natural projection $\pi : M_{\mathcal{A}} \to M_{C(\bar{\mathbb{B}})}$. If $x \in M_{\mathcal{A}}$, let

$$b_x(w) = \frac{(1-\overline{\langle w,\pi(x)\rangle})^{(n+1)(1/q-1/p)}}{(1-\langle w,\pi(x)\rangle)^{(n+1)(1/q-1/p)}}.$$

It is clear that when (z_α) is a net in $\mathbb{B}$ that tends to x in $M_{\mathcal{A}}$, then $z_\alpha = \pi(z_\alpha) \to \pi(x)$ in the Euclidean metric. Therefore $b_{z_\alpha} \to b_x$ uniformly on compact sets of $\mathbb{B}$ and boundedly. Thus,

$$(U_{z_\alpha}^q)^* U_{z_\alpha}^p = T_{b_{z_\alpha}} \xrightarrow{\text{SOT}} T_{b_x} \quad \text{and} \quad (U_{z_\alpha}^p)^* U_{z_\alpha}^q = T_{\bar{b}_{z_\alpha}} \xrightarrow{\text{SOT}} T_{\bar{b}_x} \tag{8.2}$$

in $\mathfrak{L}(A^p)$ and $\mathfrak{L}(A^q)$, respectively. If $a \in \mathcal{A}$, Lemma 6.3 says that $(a \circ \varphi_{z_\alpha}) \to (a \circ \varphi_x)$ uniformly on compact sets of $\mathbb{B}$, and the above argument shows that

$$T_{(a\circ\varphi_{z_\alpha})b_{z_\alpha}} \xrightarrow{\text{SOT}} T_{(a\circ\varphi_x)b_x} \tag{8.3}$$

in $\mathfrak{L}(A^p)$. The following theorem for the disk is in [21, Theorem 4.1], but the proof works word by word for a general n.

Theorem 8.1. *Let (E,d) be a metric space and $f : \mathbb{B} \to E$ be a continuous map. Then f admits a continuous extension from $M(\mathcal{A})$ into E if and only if f is uniformly (ρ,d) continuous and $\overline{f(\mathbb{B})}$ is compact.*

We recall that if $1 < p < \infty$ and $k_\xi^{(p)} = (1-|\xi|^2)^{(n+1)/q} K_\xi$, where $\xi \in \mathbb{B}$ and $1/p + 1/q = 1$, there is a constant $c_p > 0$ such that $c_p^{-1} \le \|k_\xi^{(p)}\|_p \le c_p$ for all $\xi \in \mathbb{B}$. It is clear that

$$(1-|\xi|^2)^{(n+1)/p} J_z(\xi)^{2/p} = (1-|\varphi_z(\xi)|^2)^{(n+1)/p} \frac{|1-\langle \xi,z\rangle|^{2(n+1)/p}}{(1-\langle \xi,z\rangle)^{2(n+1)/p}},$$

where the unimodular function at the end of the formula will be denoted $\lambda_p(\xi, z)$. If $f \in A^p$,

$$\begin{aligned}\langle f, (U_z^p)^* k_\xi^{(q)} \rangle &= \langle U_z^p f, k_\xi^{(q)} \rangle = \langle (f \circ \varphi_z) J_z^{2/p}, k_\xi^{(q)} \rangle \\ &= f(\varphi_z(\xi))(1-|\xi|^2)^{(n+1)/p} J_z(\xi)^{2/p} \\ &= f(\varphi_z(\xi))(1-|\varphi_z(\xi)|^2)^{(n+1)/p} \lambda_p(\xi, z) \\ &= \langle f, \overline{\lambda_p(\xi,z)} k_{\varphi_z(\xi)}^{(q)} \rangle,\end{aligned}$$

meaning that

$$(U_z^p)^* k_\xi^{(q)} = \lambda_p(z, \xi) k_{\varphi_z(\xi)}^{(q)}. \tag{8.4}$$

Lemma 8.2. *Let $\xi \in \mathbb{B}$ be a fixed point. Then the map $z \mapsto (U_z^p)^* k_\xi^{(q)}$ is uniformly continuous from $(\mathbb{B}, \rho)$ into $(A^q, \| \|_q)$.*

Proof. By (8.4) it suffices to prove that the maps $z \mapsto \lambda_p(z, \xi)$ and $z \mapsto k_{\varphi_z(\xi)}^{(q)}$ are uniformly continuous from $(\mathbb{B}, \rho)$ into $(\mathbb{C}, | |)$ and $(A^q, \| \|_q)$, respectively.

For the first of these maps the assertion is obvious (actually, the map can be extended continuously to the closure of $\mathbb{B}$ in $\mathbb{C}^n$). Since Lemma 6.2 says that $z \mapsto \varphi_z(\xi)$ is uniformly continuous from $(\mathbb{B}, \rho)$ into itself, the proof for the second map reduces to show the uniform continuity of $w \mapsto k_w^{(q)}$. That is, we want to prove that given $\varepsilon > 0$, there is $\delta > 0$ such that $\sup_{z \in \mathbb{B}} \|k_z^{(q)} - k_{\varphi_z(\alpha)}^{(q)}\|_q < \varepsilon$ if $|\alpha| < \delta$. For z, $\alpha \in \mathbb{B}$, the isomorphism $(A^p)^* \simeq A^q$ implies

$$\begin{aligned}&\|k_z^{(q)} - k_{\varphi_z(\alpha)}^{(q)}\|_q \\ &\sim \sup_{f \in A^p : \|f\|_p = 1} \left| (1-|z|^2)^{(n+1)/p} f(z) - (1-|\varphi_z(\alpha)|^2)^{(n+1)/p} f(\varphi_z(\alpha)) \right|,\end{aligned} \tag{8.5}$$

where for $f \in A^p$ of norm 1, the modulus in the above expression is bounded by

$$\begin{aligned}&(1-|z|^2)^{(n+1)/p} |f(z) - f(\varphi_z(\alpha))| \\ &\quad + (1-|\varphi_z(\alpha)|^2)^{(n+1)/p} \, |f(\varphi_z(\alpha))| \left| 1 - \frac{(1-|z|^2)^{(n+1)/p}}{(1-|\varphi_z(\alpha)|^2)^{(n+1)/p}} \right| \\ &\qquad \le |g_z(0) - g_z(\alpha)| + c_q \|f\|_p \left| 1 - \frac{|1-\langle \alpha, z\rangle|^{2(n+1)/p}}{(1-|\alpha|^2)^{(n+1)/p}} \right|,\end{aligned} \tag{8.6}$$

where

$$g_z(w) = (1-|z|^2)^{(n+1)/p}(f\circ\varphi_z)(w) = (1-\langle w,z\rangle)^{2(n+1)/p}(U_z^p f)(w).$$

and the last inequality holds because

$$(1-|\varphi_z(\alpha)|^2)^{(n+1)/p}\;|f(\varphi_z(\alpha))| = |\langle f, k^{(q)}_{\varphi_z(\alpha)}\rangle| \le \|f\|_p\;||k^{(q)}_{\varphi_z(\alpha)}||_q.$$

Since $\|f\|_p = 1$ and U_z^p is an isometry, $\|g_z\|_p \le 4^{(n+1)/p}$. The second summand in (8.6) can be made $< \varepsilon/2$ independently of f and z if $|\alpha|$ is small. So, if we denote by s the supremum in (8.5) and take α as small as before,

$$\begin{aligned} s &\le 4^{(n+1)/p} \sup_{g\in A^p:\|g\|_p=1} |g(\alpha)-g(0)| + \frac{\varepsilon}{2} \\ &\le 4^{(n+1)/p} \sup_{g\in A^p:\|g\|_p=1} \|g\|_p\;\|K_\alpha - K_0\|_\infty + \frac{\varepsilon}{2}, \end{aligned}$$

which can be made as small as wished by taking α small enough. □

Proposition 8.3. *Let $S\in\mathfrak{L}(A^p)$. Then the map $\Psi_S:\mathbb{B}\to(\mathfrak{L}(A^p),\mathrm{WOT})$ extends continuously to $M_{\mathcal{A}}$.*

Proof. Bounded sets in $\mathfrak{L}(A^p)$ are metrizable and have compact closure with the weak operator topology. Since $\Psi_S(\mathbb{B})$ is bounded, Theorem 8.1 reduces the problem to show that Ψ_S is uniformly continuous from the ball with the pseudo-hyperbolic metric into $\mathfrak{L}(A^p)$ with the weak operator topology. This amounts to see that for every $f\in A^p$ and $g\in A^q$, the function $z\mapsto\langle S_z f, g\rangle$ is uniformly continuous from $(\mathbb{B},\rho)$ into $(\mathbb{C},|\;|)$. For $z_1, z_2\in\mathbb{B}$ we have

$$\begin{aligned} U^p_{z_1}S(U^q_{z_1})^* - U^p_{z_2}S(U^q_{z_2})^* &= U^p_{z_1}S[(U^q_{z_1})^* - (U^q_{z_2})^*] + [U^p_{z_1} - U^p_{z_2}]S(U^q_{z_2})^* \\ &= A + B. \end{aligned}$$

Then

$$\begin{aligned} |\langle Af,g\rangle| &\le ||U^p_{z_1}S||\;||[(U^q_{z_1})^* - (U^q_{z_2})^*]f||_p\;||g||_q\,, \\ |\langle Bf,g\rangle| = |\langle f,B^*g\rangle| &\le ||f||_p\;||U^q_{z_2}S^*||\;||\,[(U^p_{z_1})^* - (U^p_{z_2})^*]g\,||_q. \end{aligned}$$

Interchanging p and q, it is enough to deal with the last expression. Since $\|(U_z^p)^*\|\le C_p$ for every z, we can assume that g is in a dense subset of A^q, and since the linear span of $\{k_\xi^{(q)}:\xi\in\mathbb{B}\}$ is dense in A^q, it is enough to see that for every $\xi\in\mathbb{B}$, $\|[(U^p_{z_1})^* - (U^p_{z_2})^*]k_\xi^{(q)}\|_q$ can be made small as long as $\rho(z_1,z_2)$ is small enough (depending on ξ). This is precisely the statement of Lemma 8.2. □

Lemma 8.4. *If (z_α) is a net in $\mathbb{B}$ converging to $x \in M_{\mathcal{A}}$, then T_{b_x} is invertible and $T^{-1}_{b_{z_\alpha}} \xrightarrow{\text{SOT}} T^{-1}_{b_x}$ in $\mathfrak{L}(A^p)$.*

Proof. By Proposition 8.3 applied to the identity, we know that $U^p_{z_\alpha}(U^q_{z_\alpha})^* = T^{-1}_{b_{z_\alpha}}$ has a WOT-limit in $\mathfrak{L}(A^p)$, say Q. The Banach-Steinhaus Theorem then says that there is a constant C_0 such that $\|T^{-1}_{b_{z_\alpha}}\| \le C_0$ for all α. Given $f \in A^p$ and $g \in A^q$, (8.2) says that $\|(T_{\bar{b}_{z_\alpha}} - T_{\bar{b}_x})g\|_q \to 0$. Thus

$$\begin{aligned}\langle T_{b_x} Qf, g\rangle = \langle Qf, T_{\bar{b}_x} g\rangle &= \lim_\alpha [\langle T^{-1}_{b_{z_\alpha}} f, (T_{\bar{b}_x} - T_{\bar{b}_{z_\alpha}})g\rangle + \langle T^{-1}_{b_{z_\alpha}} f, T_{\bar{b}_{z_\alpha}} g\rangle]\\ &= \lim_\alpha \langle T^{-1}_{b_{z_\alpha}} f, (T_{\bar{b}_x} - T_{\bar{b}_{z_\alpha}})g\rangle + \langle f, g\rangle,\end{aligned}$$

where

$$\begin{aligned}|\langle T^{-1}_{b_{z_\alpha}} f, (T_{\bar{b}_x} - T_{\bar{b}_{z_\alpha}})g\rangle| &\le \|T^{-1}_{b_{z_\alpha}}\|\, \|f\|_p\, \|(T_{\bar{b}_x} - T_{\bar{b}_{z_\alpha}})g\|_q\\ &\le C_0 \|f\|_p\, \|(T_{\bar{b}_x} - T_{\bar{b}_{z_\alpha}})g\|_q \to 0.\end{aligned}$$

This proves that $T_{b_x} Q = I_{A^p}$. Since taking adjoints is continuous with respect to the weak operator topologies, $T^{-1}_{\bar{b}_{z_\alpha}} \xrightarrow{\text{WOT}} Q^*$ in $\mathfrak{L}(A^q)$. So, interchanging the roles of p and q we obtain that $T_{\bar{b}_x} Q^* = I_{A^q}$, which in turn proves that $QT_{b_x} = I_{A^p}$. Thus, $Q = T^{-1}_{b_x}$ and $T^{-1}_{b_{z_\alpha}} \xrightarrow{\text{WOT}} T^{-1}_{b_x}$ in $\mathfrak{L}(A^p)$. Since

$$T^{-1}_{b_{z_\alpha}} - T^{-1}_{b_x} = T^{-1}_{b_{z_\alpha}} (T_{b_x} - T_{b_{z_\alpha}}) T^{-1}_{b_x},$$

where $\|T^{-1}_{b_{z_\alpha}}\| \le C_0$ and $T_{b_x} - T_{b_{z_\alpha}} \xrightarrow{\text{SOT}} 0$ in $\mathfrak{L}(A^p)$, then $T^{-1}_{b_{z_\alpha}} - T^{-1}_{b_x} \xrightarrow{\text{SOT}} 0$ in $\mathfrak{L}(A^p)$, as claimed. □

Observe that for any operators $S^1, \ldots, S^m \in \mathfrak{L}(A^p)$,

$$\begin{aligned}(8.7)\quad (S^1 \cdots S^m)_z &=\\ &= [U^p_z S^1 (U^q_z)^*](U^q_z)^* U^p_z [U^p_z S^2 (U^q_z)^*] \cdots (U^q_z)^* U^p_z [U^p_z S^m (U^q_z)^*]\\ &= S^1_z (U^q_z)^* U^p_z S^2_z \cdots (U^q_z)^* U^p_z S^m_z = S^1_z T_{b_z} S^2_z \cdots T_{b_z} S^m_z.\end{aligned}$$

Proposition 8.5. *If $S \in \mathfrak{T}_p$ and (z_α) is a net in $\mathbb{B}$ that tends to $x \in M_{\mathcal{A}}$, then $S_{z_\alpha} \xrightarrow{\text{SOT}} S_x$ in $\mathfrak{L}(A^p)$. Thus, $\Psi_S : \mathbb{B} \to (\mathfrak{L}(A^p), \text{SOT})$ extends continuously to $M_{\mathcal{A}}$.*

Proof. If $S \in \mathfrak{T}_p$ and $\varepsilon > 0$, Theorem 7.3 assures that there is a finite sum of finite products of Toeplitz operators with symbols in $\mathcal{A}$, denoted R, such that

$\|S - R\| < \varepsilon$. Then $\|S_z - R_z\| < C_p\varepsilon$ for every $z \in \mathbb{B}$, and since except for a multiplicative constant, WOT limits do not increment the norm, $\|S_x - R_x\| < C'_p\varepsilon$ for every $x \in M_{\mathcal{A}}$. Thus, it is enough to prove the proposition for R, and by linearity, it is enough to assume that $R = T_{a_1} \cdots T_{a_m}$, where $a_j \in \mathcal{A}$ for $1 \le j \le m$. Since for $a \in \mathcal{A}$, $U_z^2 T_a U_z^2 = T_{a\circ\varphi_z}$,

$$\begin{aligned}(T_a)_z &= U_z^p (U_z^q)^* (U_z^q)^* T_a U_z^p U_z^p (U_z^q)^* \\ &= U_z^p (U_z^q)^* T_{\bar{J}_z^{1-2/q}} U_z^2 T_a U_z^2 T_{J_z^{1-2/p}} U_z^p (U_z^q)^* \\ &= U_z^p (U_z^q)^* T_{(a\circ\varphi_z)\bar{J}_z^{1-2/q} J_z^{1-2/p}} U_z^p (U_z^q)^* \\ &= T_{b_z}^{-1} T_{(a\circ\varphi_z)b_z} T_{b_z}^{-1},\end{aligned}$$

which together with (8.7) gives

$$\begin{aligned}(T_{a_1} \cdots T_{a_m})_z &= (T_{a_1})_z T_{b_z} (T_{a_2})_z \cdots T_{b_z} (T_{a_m})_z \\ &= T_{b_z}^{-1} T_{(a_1\circ\varphi_z)b_z} T_{b_z}^{-1} T_{(a_2\circ\varphi_z)b_z} \cdots T_{b_z}^{-1} T_{(a_m\circ\varphi_z)b_z} T_{b_z}^{-1}.\end{aligned}$$

Since the product of SOT convergence nets is SOT convergent, Lemma 8.4 and (8.3) imply that when $z_\alpha \to x$,

$$(T_{a_1} \cdots T_{a_m})_{z_\alpha} \xrightarrow{\text{SOT}} T_{b_x}^{-1} T_{(a_1\circ\varphi_x)b_x} T_{b_x}^{-1} T_{(a_2\circ\varphi_z)b_x} \cdots T_{b_x}^{-1} T_{(a_m\circ\varphi_x)b_x} T_{b_x}^{-1}$$

in $\mathfrak{L}(A^p)$. The second assertion of the proposition now follows from a simple diagonal argument. ◻

9. THE ESSENTIAL NORM VIA S_x FOR $1 < p < \infty$

Lemma 9.1. *Let $S \in \mathfrak{L}(A^p)$. Then $B(S)(z) \to 0$ when $|z| \to 1$ if and only if $S_x = 0$ for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$.*

Proof. If $z, \xi \in \mathbb{B}$, by (8.4)

$$\begin{aligned}B(S_z)(\xi) &= \langle S(U_z^q)^* k_\xi^{(p)}, (U_z^p)^* k_\xi^{(q)} \rangle \\ &= \lambda_q(z,\xi)\overline{\lambda_p(z,\xi)} \langle S k_{\varphi_z(\xi)}^{(p)}, k_{\varphi_z(\xi)}^{(q)} \rangle \\ &= \lambda_q(z,\xi)\overline{\lambda_p(z,\xi)} B(S)(\varphi_z(\xi)).\end{aligned}$$

Thus, $|B(S_z)(\xi)| = |B(S)(\varphi_z(\xi))|$. If $x \in M_{\mathcal{A}} \setminus \mathbb{B}$, (z_α) is a net in $\mathbb{B}$ that tends to x, and $\xi \in \mathbb{B}$ is fixed, Proposition 8.3 assures that

$$B(S_{z_\alpha})(\xi) = \langle S_{z_\alpha} k_\xi^{(p)}, k_\xi^{(q)} \rangle \to \langle S_x k_\xi^{(p)}, k_\xi^{(q)} \rangle = B(S_x)(\xi).$$

Therefore,

$$|B(S)(\varphi_{z_\alpha}(\xi))| \to |B(S_x)(\xi)|. \tag{9.1}$$

Since $x \in M_{\mathcal{A}} \setminus \mathbb{B}$ and $z_\alpha \to x$, then $|z_\alpha| \to 1$, and consequently $|\varphi_{z_\alpha}(\xi)| \to 1$. So, if $B(S)$ vanishes on $\partial\mathbb{B}$, (9.1) says that $B(S_x)(\xi) = 0$, and since $\xi \in \mathbb{B}$ is arbitrary and B is one-to-one, $S_x = 0$.

Reciprocally, if there is a sequence $\{z_k\} \subset \mathbb{B}$ such that $|z_k| \to 1$ and $|B(S)(z_k)| \geq \delta > 0$, the compactness of $M_{\mathcal{A}}$ implies that there is a subnet (z_α) of $\{z_k\}$ that converges in $M_{\mathcal{A}}$ to some point $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. Taking $\xi = 0$ in (9.1) we get that $|B(S_x)(0)| \geq \delta$, and consequently $S_x \neq 0$. ◻

The following result follows immediately from a theorem of Berndtsson [3].

Lemma 9.2. *Suppose that $\varrho > 0$, $0 < r < 1$ and $w_k \in r\mathbb{B}$, for $k = 1, \ldots, m$, are points such that $\beta(w_k, w_j) \geq \varrho$ if $j \neq k$. Then for any $1 \leq k_0 \leq m$ there is $g_{k_0} \in H^\infty(\mathbb{B})$ such that*

$$g_{k_0}(w_k) = \delta_{k_0,k} \quad \text{and} \quad \|g_{k_0}\|_\infty \leq C(\varrho, r),$$

where $\delta_{k_0,k}$ denotes Kronecker's delta.

Proof. Since $\rho(w_k, w_j) \geq \tanh\varrho$ for $j \neq k$ and $|w_j| \leq r$ for all $1 \leq j \leq m$, there is an integer M depending only on ϱ and r such that $m \leq M$. Thus

$$\inf_k \prod_{j \neq k} \rho(w_j, w_k) \geq (\tanh\varrho)^{M-1}.$$

By [3, Theorem 2] there is $g_{k_0} \in H^\infty(\mathbb{B})$ satisfying the interpolation, with $\|g_{k_0}\|_\infty \leq C$, a constant depending only on $(\tanh\varrho)^M$. ◻

Theorem 9.3. *There exists a constant $C_p > 0$ such that if $S \in \mathfrak{T}_p$,*

$$C_p^{-1} \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \|S_x\| \leq \|S\|_e \leq C_p \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \|S_x\|. \tag{9.2}$$

Proof of the Theorem and of (5.10). If $S \in \mathfrak{L}(A^p)$ is compact,

$$|B(S)(\xi)| = |\langle Sk_\xi^{(p)}, k_\xi^{(q)}\rangle| \leq \|Sk_\xi^{(p)}\|_p \, \|k_\xi^{(q)}\|_q \to 0 \quad \text{as } |\xi| \to 1, \tag{9.3}$$

because $\|k_\xi^{(q)}\|_q \leq c_q$ independently of $\xi \in \mathbb{B}$ and $k_\xi^{(p)} \to 0$ weakly in A^p when $|\xi| \to 1$. Hence, Lemma 9.1 says that $S_x = 0$ for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$.

Now assume that $S \in \mathfrak{L}(A^p)$ is arbitrary. Let $Q \in \mathfrak{L}(A^p)$ be a compact operator and $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. Take a net $(z_\alpha) \subset \mathbb{B}$ that converges to x. Since $U_{z_\alpha}^p$ and $U_{z_\alpha}^q$ are isometries on A^p and A^q, respectively, we have $\|S_{z_\alpha} + Q_{z_\alpha}\| \leq C_p\|S + Q\|$. Since, except for a multiplicative constant, WOT limits do not increase the norm, the convergence $S_{z_\alpha} + Q_{z_\alpha} \xrightarrow{\text{WOT}} S_x + Q_x = S_x$ implies that $\|S_x\| \le C'_p \liminf \|S_{z_\alpha} + Q_{z_\alpha}\|$. Thus

$$\|S_x\| \le C''_p \|S + Q\|, \quad \text{for all } x \in M_{\mathcal{A}} \setminus \mathbb{B} \text{ and } Q \in \mathfrak{L}(A^p) \text{ compact.}$$

Taking infimum at the right side and supremum at the left side we get the first inequality in (9.2). Observe that this holds for any bounded operator S.

Now assume that $S \in \mathfrak{T}_p$. Since (5.9) tells us that $\|S\|_e \le G'_p \, \alpha_S$, we only need to prove the second inequality in (9.2) with $\|S\|_e$ replaced by α_S. This and the first inequality in (9.2) will also prove (5.10), therefore finishing the proof of Theorem 5.2. Since $\alpha_S(r)$ is an increasing function of r that tends to α_S when $r \to \infty$, we must show that there is a constant $C_p > 0$ such that

$$\alpha_S(r) \le C_p \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \|S_x\|, \quad \text{for } r > 0.$$

So, fix $r > 0$. By definition of $\alpha_S(r)$, there is a sequence $\{z_j\} \subset \mathbb{B}$ tending to $\partial\mathbb{B}$ and a normalized sequence $f_j \in T_{\chi_{D(z_j,r)}\mu} A^p$ such that $\|Sf_j\| \to \alpha_S(r)$. Thus, there are $h_j \in A^p$ such that

$$\begin{aligned} f_j(w) = T_{\chi_{D(z_j,r)}\mu} h_j(w) &= \sum_{w_m \in D(z_j,r)} \frac{\mathrm{v}(D_m) h_j(w_m)}{(1 - \langle w, w_m \rangle)^{n+1}} \\ &= \sum_{w_m \in D(z_j,r)} a_{j,m} \frac{(1 - |w_m|^2)^{(n+1)/q}}{(1 - \langle w, w_m \rangle)^{n+1}}, \end{aligned}$$

where $a_{j,m} = \mathrm{v}(D_m) h_j(w_m)(1 - |w_m|^2)^{-(n+1)/q}$. That is,

$$f_j = \sum_{w_m \in D(z_j,r)} a_{j,m} k^{(p)}_{w_m}.$$

If we write $w_{j,m} = \varphi_{z_j}(w_m)$, (8.4) gives

$$(U^q_{z_j})^* f_j = \sum_{w_m \in D(z_j,r)} a_{j,m} \lambda_q(z_j, w_m) k^{(p)}_{\varphi_{z_j}(w_m)} = \sum_{w_{j,m} \in D(0,r)} a'_{j,m} k^{(p)}_{w_{j,m}},$$

where $a'_{j,m} = a_{j,m} \lambda_q(z_j, w_m)$ and $|w_{j,m}| = |\varphi_{z_j}(w_m)| \le s_r = \tanh r$. For each j arrange the points $w_{j,m}$ (for $m \ge 1$) such that $|w_{j,m}| \le |w_{j,m+1}|$ and $\arg w_{j,m} \le \arg w_{j,m+1}$. Since (a) and (b) of Lemma 2.3 say that $\beta(w_{j,m}, w_{j,k}) = \beta(w_m, w_k) \ge \varrho/4$ when $m \ne k$, there are only N_j points $w_{j,m}$, where for each j, $N_j \le M(\varrho, r)$, a bound that depends only on ϱ and r. Taking a subsequence we can assume that $N_j = M$, a quantity independent of j. Fix j and $1 \le m_0 \le M$.

By Lemma 9.2 there is $g = g_{j,m_0} \in H^\infty(\mathbb{B})$, with $\|g\|_\infty \le C(\varrho/4, s_r)$, such that $g(w_{j,m}) = \delta_{m_0,m}$ for $1 \le m \le M$. Therefore,

$$\begin{aligned}\langle (U^q_{z_j})^* f_j, g\rangle &= \sum_{w_{j,m}\in D(0,r)} a'_{j,m}(1-|w_{j,m}|^2)^{(n+1)/q} g(w_{j,m})\\ &= a'_{j,m_0}(1-|w_{j,m_0}|^2)^{(n+1)/q},\end{aligned}$$

and consequently

$$\begin{aligned}|a'_{j,m_0}| &\le (1-|w_{j,m_0}|^2)^{-(n+1)/q} |\langle (U^q_{z_j})^* f_j, g\rangle|\\ &\le (1-s_r^2)^{-(n+1)/q} \|(U^q_{z_j})^*\| \, \|f_j\|_p \, \|g\|_q \le C_0,\end{aligned}$$

where $C_0 = C_0(n,p,\varrho,r) > 0$ is independent of j and m_0. Hence, the sequence

$$(w_{j,1},\dots,w_{j,M}, a'_{j,1},\dots,a'_{j,M}) \in \mathbb{C}^{2M}$$

is bounded. Taking another subsequence we can also assume that this sequence converges in $\mathbb{C}^{2M}$ to a point $(v_1,\dots,v_M,a'_1,\dots,a'_M)$, where $|v_i| \le s_r$ and $|a'_i| \le C_0$. Thus,

$$(U^q_{z_j})^* f_j \to h \stackrel{\text{def}}{=} \sum_{i=1}^M a'_i k^{(p)}_{v_i} \quad \text{in } L^p\text{-norm},$$

where $\|h\|_p = \lim \|(U^q_{z_j})^* f_j\|_p \le \|(U^q_{z_j})^*\| \, \|f_j\|_p \le C_p$. Since $U^p_{z_j}$ is isometric, $(U^q_{z_j})^*(U^q_{z_j})^* = I_{A^p}$, and $\|S_{z_j}\|$ is bounded independently of j, we get

$$\alpha_S(r) = \lim \|Sf_j\| = \lim \|S_{z_j}(U^q_{z_j})^* f_j\| = \lim \|S_{z_j} h\|.$$

By the compactness of $M_{\mathcal{A}}$ there is a subnet (z_β) of the sequence $\{z_j\}$ that converges to some point $x \in M_{\mathcal{A}} \setminus \mathbb{B}$ ($x \notin \mathbb{B}$ because $|z_j| \to 1$). Consequently, Proposition 8.5 says that $S_{z_\beta} h \to S_x h$ in A^p-norm, which leads to

$$\alpha_S(r) = \lim \|S_{z_\beta} h\| = \|S_x h\| \le \|S_x\| C_p \le C_p \sup_{u \in M_{\mathcal{A}}\setminus\mathbb{B}} \|S_u\|.$$

This proves the theorem and (5.10). □

Corollary 9.4. *Let $1 < p < \infty$ and $S \in \mathfrak{T}_p$. Then*

$$\|S\|_{\mathrm{e}} \sim \sup_{\|f\|_p=1} \limsup_{|z|\to 1} \|S_z f\|_p.$$

Proof. Proposition 8.5 and the compactness of $M_{\mathcal{A}}$ imply that

$$\sup_{x\in M_{\mathcal{A}}\setminus\mathbb{B}} \|S_x f\|_p = \limsup_{|z|\to 1} \|S_z f\|_p$$

for every $f \in A^p$. Taking supremum over the functions $f \in A^p$ of norm 1 and commuting the two suprema in the first member of the equality we get

$$\sup_{x\in M_{\mathcal{A}}\setminus\mathbb{B}} \|S_x\| = \sup_{\|f\|_p=1} \limsup_{|z|\to 1} \|S_z f\|_p.$$

The result follows from Theorem 9.3. □

Theorem 9.5. *Let $1 < p < \infty$ and $S \in \mathfrak{L}(A^p)$. Then S is compact if and only if $S \in \mathfrak{T}_p$ and $B(S) \equiv 0$ on $\partial\mathbb{B}$.*

Proof. If S is compact, $B(S) \equiv 0$ on $\partial\mathbb{B}$ by (9.3). When $p = 2$, the inclusion of the compact operators in $\mathfrak{T}_2$ follows from [4] or [8], both results being stronger than this easy fact. For $1 < p < \infty$ we give here a short proof. It is well-known that L^p has the bounded approximation property, meaning that there exists a constant $C > 0$ such that for every compact set $K \subset L^p$ and $\varepsilon > 0$, there is a finite rank operator $T \in \mathfrak{L}(L^p)$ such that $\|T\| \le C$ and $\|Tf - f\| < \varepsilon$ for all $f \in K$ (see [23, pp. 69–70]). It follows that every compact operator $Q \in \mathfrak{L}(L^p)$ can be approximated by operators of finite rank. Since A^p is a projection of L^p, the same holds for A^p. Thus, it is enough to prove that the operators of rank 1 are in $\mathfrak{T}_p$. Every operator of rank 1 has the form $f \otimes g$, where $f \in A^p$, $g \in A^q$ and $(f \otimes g)h = \langle h, g\rangle f$ for $h \in A^p$. Since $\|f \otimes g\|$ is equivalent to $\|f\|_p \, \|g\|_q$ and the polynomials are dense in A^p and A^q, it is enough to assume that f and g are polynomials. In such case, $f \otimes g = T_f (1 \otimes 1) T_{\bar{g}}$, and the problem reduces to show that $1 \otimes 1 \in \mathfrak{T}_p$. This follows from Theorem 7.3 by noticing that $1 \otimes 1 = T_{\delta_0}$, where δ_0 is the Dirac measure with mass concentrated at 0.

Now suppose that $B(S) \equiv 0$ on $\partial\mathbb{B}$. Lemma 9.1 then says that $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. If in addition $S \in \mathfrak{T}_p$, Theorem 9.3 says that S is compact. □

10. THE CASE $p = 2$

Let $S \in \mathfrak{L}(A^p)$, where $1 < p < \infty$. Since $(S_z)^* = (S^*)_z$ for $z \in \mathbb{B}$ and the adjoints of a WOT convergent net is WOT convergent, then $(S_x)^* = (S^*)_x$ for all $x \in M_{\mathcal{A}}$.

If $p = 2$, (8.1) shows that $b_z = 1$ for all $z \in \mathbb{B}$. Thus, $(ST)_z = S_z T_z$ for S, $T \in \mathfrak{L}(A^2)$ and $z \in \mathbb{B}$. When $z \to x \in M_{\mathcal{A}}$, the first member tends WOT to $(ST)_x$ and each of the factors of the second member tends WOT to S_x and T_x, respectively. But since the product of two WOT-convergent nets is not necessarily WOT-convergent, we could have $(ST)_x \neq S_x T_x$. Indeed, if $Sf(z) = f(-z)$, it is clear that $(S^2)_x = I_x = I$, but since $SK_z = K_{-z}$,

$$B(S)(z) = (1 - |z|^2)^{n+1} \langle K_{-z}, K_z\rangle = [(1 - |z|^2)/(1 + |z|^2)]^{n+1},$$

and Lemma 9.1 implies that $S_x = 0$ for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. However, since the product of a WOT-convergent net by a SOT-convergent net is WOT-convergent, Propositions 8.3 and 8.5 imply that if $T \in \mathfrak{L}(A^2)$ and $S \in \mathfrak{T}_2$, then $T_z S_z \xrightarrow{\text{WOT}} T_x S_x$ when $z \to x$. In particular, $(TS)_x = T_x S_x$ in this case. Furthermore, since $\mathfrak{T}_2$ is a self-adjoint algebra, the above equality applied to the adjoints gives $(T^*S^*)_x = (T^*)_x (S^*)_x$ for all $x \in M_{\mathcal{A}}$ whenever $T \in \mathfrak{L}(A^2)$ and $S \in \mathfrak{T}_2$. Now taking adjoints we also get $(ST)_x = S_x T_x$. Summing up,

$$(T_x)^* = (T^*)_x, \quad (TS)_x = T_x S_x, \quad \text{and} \quad (ST)_x = S_x T_x \tag{10.1}$$

for all $x \in M_{\mathcal{A}}$, $T \in \mathfrak{L}(A^2)$ and $S \in \mathfrak{T}_2$. Also, observe that for any $S \in \mathfrak{L}(A^2)$, $\|S_z\| = \|S\|$ for all $z \in \mathbb{B}$, and since WOT limits in $\mathfrak{L}(A^2)$ do not increase the norm, then $\|S_x\| \le \|S\|$ for all $x \in M_{\mathcal{A}}$.

Let $\mathcal{K} \in \mathfrak{L}(A^2)$ be the ideal of compact operators. The Calkin algebra is the C^*-algebra $\mathfrak{L}(A^2)/\mathcal{K}$. We shall denote by $\sigma(S)$ the spectrum of $S \in \mathfrak{L}(A^2)$ and by $\sigma_{\mathrm{e}}(S)$ the essential spectrum of S, which is defined as the spectrum of $S + \mathcal{K}$ in $\mathfrak{L}(A^2)/\mathcal{K}$. The spectral radius of $S \in \mathfrak{L}(A^2)$ is $r(S) = \sup\{|\lambda| : \lambda \in \sigma(S)\}$, and its essential spectral radius is $r_{\mathrm{e}}(S) = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{e}}(S)\}$. Theorem 9.3 can be improved considerably when $p = 2$, as the next result shows.

Theorem 10.1. *If $S \in \mathfrak{T}_2$, then*

$$\|S\|_{\mathrm{e}} = \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \|S_x\| \tag{10.2}$$

and

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} r(S_x) \le \lim_{k \to \infty} \Big(\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \|S_x^k\|^{1/k} \Big) = r_{\mathrm{e}}(S), \tag{10.3}$$

with equality if S is essentially normal.

Proof. Let k be a positive integer. Since by (10.1) $(S_x)^k = (S^k)_x$, (9.2) implies that

$$C_2^{-1/k} \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \|(S_x)^k\|^{1/k} \le \|S^k\|_{\mathrm{e}}^{1/k} \le C_2^{1/k} \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \|(S_x)^k\|^{1/k}.$$

The equality in (10.3) follows by taking limits when $k \to \infty$ and the inequality holds because $r(T) \le \|T^k\|^{1/k}$ for every operator T and $k \ge 1$ (see [6, Theorem 2.38]). If S is essentially normal (i.e., $SS^* - S^*S$ is compact), then

$$S_x S_x^* - S_x^* S_x = (SS^* - S^*S)_x = 0$$

for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. That is, S_x is normal, and consequently $\|(S_x)^k\|^{1/k} = r(S_x)$ for every $k \ge 1$ (see [6, Theorem 4.30]). Finally, applying (10.3) with equality to the self-adjoint operator S^*S, we get

$$\|S\|_e^2 = \|S^*S\|_e = r_e(S^*S) = \sup_{x\in M_{\mathcal{A}}\setminus\mathbb{B}} r(S_x^*S_x)$$
$$= \sup_{x\in M_{\mathcal{A}}\setminus\mathbb{B}} \|S_x^*S_x\| = \sup_{x\in M_{\mathcal{A}}\setminus\mathbb{B}} \|S_x\|^2,$$

proving (10.2). ◻

Corollary 10.2. *Let $R \in \mathfrak{T}_2$ be a self-adjoint operator and $\gamma, \delta \in \mathbb{R}$ such that $\gamma I \le R_x \le \delta I$ for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. Then given $\varepsilon > 0$ there is a compact self-adjoint operator K such that $(\gamma - \varepsilon)I \le R + K \le (\delta + \varepsilon)I$.*

Proof. Since $\gamma I \le R_x \le \delta I$, then

$$-\left(\frac{\delta-\gamma}{2}\right) I \le R_x - \left(\frac{\delta+\gamma}{2}\right) I \le \left(\frac{\delta-\gamma}{2}\right) I$$

for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. Since the spectral radius of a self-adjoint element in a C^*-algebra coincides with its norm, Theorem 10.1 says that $\|R - (\delta+\gamma)2^{-1}I\|_e \le (\delta-\gamma)2^{-1}$, and consequently there is a compact operator K such that

$$\|R - (\delta+\gamma)2^{-1}I + K\| \le (\delta-\gamma)2^{-1} + \varepsilon.$$

We can assume that K is self-adjoint by taking $2^{-1}(K + K^*)$ instead of K. This means that

$$-\left(\frac{\delta-\gamma}{2} + \varepsilon\right) I \le R + K - \left(\frac{\delta+\gamma}{2}\right) I \le \left(\frac{\delta-\gamma}{2} + \varepsilon\right) I,$$

and the result follows by adding $(\delta+\gamma)2^{-1}I$ to all the members of the inequality. ◻

Theorem 10.3. *Let $S \in \mathfrak{T}_2$. The following statements are equivalent.*

(1) $\lambda \notin \sigma_e(S)$,
(2) $\lambda \notin \bigcup_{x\in M_{\mathcal{A}}\setminus\mathbb{B}} \sigma(S_x)$ *and* $\sup_{x\in M_{\mathcal{A}}\setminus\mathbb{B}} \|(S_x - \lambda I)^{-1}\| < \infty$,
(3) *there is $\gamma > 0$ depending only on λ, such that*

$$\|(S_x - \lambda I)f\| \ge \gamma\|f\| \quad \text{and} \quad \|(S_x^* - \bar{\lambda} I)f\| \ge \gamma\|f\|$$

for all $f \in A^2$ and $x \in M_{\mathcal{A}} \setminus \mathbb{B}$.

Proof. Replacing S by $S - \lambda I$, there is no loss of generality if we assume $\lambda = 0$. Suppose that $0 \notin \sigma_e(S)$. This means that there is $Q \in \mathfrak{L}(A^2)$ such that both $QS - I$ and $SQ - I$ are compact operators. Let $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. Since $S \in \mathfrak{T}_2$, we have $(SQ)_x = S_x Q_x$ and $(QS)_x = Q_x S_x$, and since $K_x = 0$ for $K \in \mathfrak{L}(A^2)$ compact,

$$Q_x S_x - I = 0 = S_x Q_x - I.$$

Hence, S_x is invertible and $Q_x = (S_x)^{-1}$. So, $\|(S_x)^{-1}\| = \|Q_x\| \leq \|Q\|$ for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$ and (2) holds.

Now assume that (2) holds with $\lambda = 0$. Hence, S_x is invertible and there is $\gamma^{-1} > 0$ such that

$$\|(S_x^*)^{-1}\| = \|(S_x)^{-1}\| \leq \gamma^{-1} \quad \text{for all } x \in M_{\mathcal{A}} \setminus \mathbb{B}.$$

Then $\gamma^{-1}\|S_x f\| \geq \|S_x^{-1} S_x f\| = \|f\|$ for all $f \in A^2$ and $x \in M_{\mathcal{A}} \setminus \mathbb{B}$, and since the same holds for S_x^*, (3) follows.

Finally, suppose that (3) holds for $\lambda = 0$. Thus, $\|S_x f\| \geq \gamma\|f\|$ for every $f \in A^2$ and $x \in M_{\mathcal{A}} \setminus \mathbb{B}$, meaning that

$$\gamma^2 I \leq S_x^* S_x \leq \|S\|^2 I.$$

So, given ε, with $0 < \varepsilon < \gamma^2$, Corollary 10.2 tells us that there is a self-adjoint compact operator K such that

$$(\gamma^2 - \varepsilon)I \leq S^* S + K \leq (\|S\|^2 + \varepsilon)I.$$

Since $\gamma^2 - \varepsilon > 0$, $S^*S + K$ is invertible, and consequently there is $Q \in \mathfrak{L}(A^2)$ such that $(QS^*)S + QK = I$. This means that $S + \mathcal{K}$ is left-invertible in the Calkin algebra. Since (3) also says that $\|S_x^* f\| \geq \gamma\|f\|$ for every $f \in A^2$ and $x \in M_{\mathcal{A}} \setminus \mathbb{B}$, the above argument applied to S^* gives that $S^* + \mathcal{K}$ is left-invertible in the Calkin algebra, or equivalently, that $S + \mathcal{K}$ is right-invertible in the Calkin algebra. Therefore $S + \mathcal{K}$ is invertible in the Calkin algebra and $0 \notin \sigma_e(S)$. □

Corollary 10.4. *If $S \in \mathfrak{T}_2$, then*

$$\overline{\bigcup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \sigma(S_x)} \subset \sigma_e(S),$$

with equality if S is essentially normal.

Proof. Suppose that $0 \notin \sigma_e(S)$. It follows from Theorem 10.3 that S_x is invertible and there is $\gamma > 0$ such that $\|(S_x)^{-1}\| \leq \gamma^{-1}$ for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. Thus

$$r((S_x)^{-1}) \leq \|(S_x)^{-1}\| \leq \gamma^{-1}.$$

Since

$$\sigma(S_x) = \{\xi^{-1} : \xi \in \sigma((S_x)^{-1})\}, \tag{10.4}$$

it follows that $|\xi| \geq \gamma$ for all $\xi \in \sigma(S_x)$. This means that the open ball centered at the origin of radius γ does not meet $\sigma(S_x)$ for any $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. Therefore $0 \notin \overline{\bigcup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \sigma(S_x)}$.

If S is essentially normal, S_x is normal for every $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. If

$$0 \notin \overline{\bigcup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}} \sigma(S_x)},$$

there is some $\gamma > 0$ such that the open ball of center 0 and radius γ does not meet $\sigma(S_x)$ for any $x \in M_{\mathcal{A}} \setminus \mathbb{B}$. The spectral equality (10.4) then says that $r((S_x)^{-1}) \leq \gamma^{-1}$. Since $(S_x)^{-1}$ is normal and the spectral radius of a normal operator coincides with its norm, we have $\|(S_x)^{-1}\| \leq \gamma^{-1}$. Theorem 10.3 then says that $0 \notin \sigma_{\mathrm{e}}(S)$. ◻

For a general $S \in \mathfrak{L}(A^2)$ it could happen that none of the sets of the Corollary is contained in the other, as our all-purpose counterexample shows. If $Sf(z) = f(-z)$, we saw that $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}$, but $\sigma_{\mathrm{e}}(S) = \{-1, 1\}$.

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