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Kleene-Isomorphic σ-Complete MV-Algebras with Product are Isomorphic

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We prove that the Kleene structure in a σ-complete MV-algebra with product is sufficient to recover the MV-structure.

1 INTRODUCTION

A natural problem is to find classes of MV-algebras that are characterized by their underlying order structure, in the sense that order isomorphic algebras are isomorphic.

As a well known example we mention the class of finite MV-algebras. Indeed, since a finite chain admits only one MV-algebra structure, and finite MV-algebras are direct products of finitely many finite chains, it follows that order isomorphic finite MV-algebras are isomorphic. In [6] this result was generalized to the class of liminary MV-algebras, i.e., MV-algebras with prime lattice filters occurring in disjoint finite chains.

The underlying lattice of an MV-algebra admits an involution, given by negation. More precisely, it is a Kleene algebra (see §1). Since a finite chain admits a unique structure of a Kleene algebra, the underlying lattices of the MV-algebras considered above admit a unique Kleene algebra structure.

But an infinite chain may admit non-isomorphic Kleene algebra structures. Therefore, in the absence of finiteness conditions it seems more appropriate to look for classes of MV-algebras that are characterized by their underlying Kleene algebra structure (see [9, 10]). In this note we show that σ-complete MV-algebras with product are characterized by their underlying Kleene algebra structure. This example is interesting because these algebras play an important role in MV-algebraic probability theory [14].

2 BASIC NOTIONS

Kleene algebras

A *De Morgan algebra* is an algebra $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ such that $\mathbf{L}(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with a smallest element 0 and a greatest element 1, and the operation $\neg\colon A \to A$ satisfies the identities $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. A *Kleene algebra* is a De Morgan algebra satisfying the identity

$$x \wedge \neg x \;\leq\; y \vee \neg y. \tag{K}$$

Kleene algebras coincide with those De Morgan algebras that can be represented as subdirect products of totally ordered De Morgan algebras. Notice that Boolean algebras are the De Morgan algebras that satisfy the identity $x \vee \neg x = 1$. A systematic study of De Morgan and Kleene algebras can be found in [1, Chapter XI].

We say that a Kleene algebra A is *centered* if there is a $z \in A$ such that $\neg z = z$. It follows at once from condition (K) that this z is unique. It is called the *center* of A. Recall that a homomorphism between Kleene algebras is a lattice homomorphism which preserves the bottom, the top, and the negation $\neg$. We state, for further reference, the following property:

2.1 *All homomorphic images of a centered Kleene algebra are centered Kleene algebras.* $\square$

Given a distributive lattice L with 0 and 1, we say that an element $x \in L$ is *boolean* iff it is *complemented*, in the sense that there is a (unique) $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$. The set of boolean elements of L is the universe of a sublattice of L which turns out to be a boolean algebra. This boolean algebra is called *the boolean skeleton of* L, and will be denoted by $\mathbf{B}(L)$.

Since every Kleene algebra A is a subdirect product of totally ordered algebras, then for each $x \in \mathbf{B}(\mathbf{L}(A))$, the element $\neg x$ coincides with the complement of x, i.e., $x \vee \neg x = 1$ and $x \wedge \neg x = 0$. This property is crucial in the proof of the following well-known result (see [4, Lemma 4.3]):

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2.2 *Let A be a Kleene algebra and let J be an ideal of the boolean algebra $\mathbf{B}(\mathbf{L}(A))$. Let the binary relation $\equiv_J$ be defined by the stipulation $a \equiv_J b$ iff there is $j \in J$ such that $a \vee j = b \vee j$. Then $\equiv_J$ is a congruence of A.* □

The quotient algebra $A/{\equiv_J}$ will be denoted by A/J, and the equivalence class of an element a by a/J. With these notations we have:

2.3 *If h is an isomorphism of a Kleene algebra A onto a Kleene algebra B, then the restriction $\bar{h}$ of h to $\mathbf{B}(\mathbf{L}(A))$ is an isomorphism of the boolean algebra $\mathbf{B}(\mathbf{L}(A))$ onto the boolean algebra $\mathbf{B}(\mathbf{L}(B))$, and for each ideal J in $\mathbf{B}(\mathbf{L}(B))$, the correspondence $a/\bar{h}^{-1}(J) \mapsto h(a)/J$ defines an isomorphism from $A/\bar{h}^{-1}(J)$ onto B/J.* □

MV-algebras

An *MV-algebra* $A = \langle A, 0, \oplus, \neg\rangle$ is an algebra where the operation $\oplus\colon A \times A \to A$ is associative and commutative with 0 as the neutral element, the operation $\neg\colon A \to A$ satisfies the identities $\neg\neg x = x$ and $x \oplus \neg 0 = \neg 0$, and, in addition,

$$y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y). \tag{1}$$

The real unit interval $[0, 1]$ equipped with the operations $x \oplus y = \min(1, x + y)$ and $\neg x = 1 - x$ is an MV-algebra, called the *standard MV-algebra*. The defining equations of MV-algebras express all equational properties of the standard MV-algebra (Chang completeness theorem [3], [5, 2.5]). Equation (1), in particular, states that the maximum operation over $[0, 1]$ is commutative. Upon adding $x \oplus x = x$ to the equations of MV-algebras one obtains the variety of boolean algebras. Thus MV-algebras may be regarded as a non-idempotent equational generalization of boolean algebras. Following common usage, for any elements x, y of an MV-algebra, we shall use the abbreviations

$$1 = \neg 0, \quad x \odot y = \neg(\neg x \oplus \neg y), \quad x \ominus y = x \odot \neg y,$$
$$x \vee y = x \oplus \neg(x \oplus \neg y), \quad x \wedge y = x \odot \neg(x \odot \neg y).$$ [1]

For every MV-algebra A, its lattice reduct $\mathbf{L}(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with 1 and 0 as top and bottom elements, respectively, and $\mathbf{K}(A) := \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Kleene algebra. For any MV-algebra A, we shall write $\mathbf{B}(A)$ for $\mathbf{B}(\mathbf{L}(A))$. It is a subalgebra of A. Given $z \in \mathbf{B}(A)$, we have $z \oplus x = z \vee x$ and $z \odot x = z \wedge x$ for each $x \in A$.

An MV-algebra A is *σ-complete* (resp., *complete*) iff $\mathbf{L}(A)$ is σ-complete (resp., complete). Equivalently, every sequence (resp., every family) of elements of A has a supremum in A.

[1] Unless otherwise specified, all MV-algebras in this paper shall be nontrivial, i.e., $0 \neq 1$.

Remark 1 *Since the lattice operations of an MV-algebra A are defined from the operations $\oplus$ and $\neg$, it follows that for every subalgebra S of A, $\mathbf{L}(S)$ is also a sublattice of $\mathbf{L}(A)$, with preservation of* 0 *and* 1. *However, a subalgebra S of a σ-complete MV-algebra A need not be closed under the denumerable lattice operations. We say that S is a σ-subalgebra of A iff $\bigvee_{n\in\mathbf{N}} a_n \in S$ for every denumerable family $\{a_n\}_{n\in\mathbf{N}}$ of elements of S. Analogously, given σ-complete MV-algebras A, B, a homomorphism $h: A \to B$ (i.e., $h(0) = 0$, $h(x \oplus y) = h(x) \oplus h(y)$, and $h(\neg x) = \neg h(x)$)) is also a lattice homomorphism. In case h also preserves the denumerable lattice operations, we shall say that h is a σ-homomorphism. If, in addition, h is an isomorphism (i.e., h is injective and surjective), then it is an order-isomorphism. It follows that h preserves all existing suprema and infima, whence h is a σ-isomorphism.*

An easy adaptation of the proof of [5, Lemma 6.6.4] shows that if the supremum (or the infimum) of a family of elements of $\mathbf{B}(A)$ exists in A, then it belongs to $\mathbf{B}(A)$. Then we have:

2.4 *If A is a σ-complete MV-algebra, then $\mathbf{B}(A)$ is a σ-complete boolean algebra, and the σ-infinitary operations of $\mathbf{B}(A)$ agree with the restrictions of the corresponding operations of A.* □

If G is a lattice-ordered abelian group (ℓ-group for short) and u is a strong order unit of G, $\Gamma(G, u)$ denotes the MV-algebra $\langle [0, u], \oplus, \neg, 0\rangle$ where $[0, u] := \{x \in G \mid 0 \le x \le u\}$ and the operations $\oplus$ and $\neg$ are defined by $x \oplus y := u \wedge (x + y)$, and $\neg x := u - x$. MV-algebras of the form $\Gamma(G, u)$ are the most general examples of MV-algebras. Indeed, it was shown in [12] (see also [5, Section 7]) that Γ defines an equivalence between the category of MV-algebras and homomorphisms, and the category whose objects are ℓ-groups with a distinguished strong order unit and whose morphisms are unit preserving ℓ-group homomorphisms. In particular we have $\langle [0, 1], \min(1, x + y), \neg, 0\rangle = \Gamma(\mathbf{R}, 1)$.

For each $n = 1, 2, 3, \ldots$, the symbol $\mathbf{L}_{n+1}$ will denote the finite subalgebra of the standard MV-algebra $[0, 1]$ formed by the fractions with denominator n:

$$\mathbf{L}_{n+1} = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\} \subseteq [0, 1]. \tag{2}$$

We have $\mathbf{L}_{n+1} = \Gamma(\frac{1}{n}\mathbf{Z}, 1)$, where $\mathbf{Z}$ denotes the additive group of the integers with its usual order.

Following [8], we say that an ℓ-group G is *Dedekind σ-complete* if every nonempty denumerable subset of G which is bounded above in G has a supremum in G. We have:

2.5 *An MV-algebra A is σ-complete iff $A \cong \Gamma(G, u)$ for a Dedekind σ-complete ℓ-group G with strong order unit u.* □

Functional representation of σ-complete MV-algebras

Given an MV-algebra A, $\mathcal{X}(A)$ will denote the set of homomorphisms of A into the standard MV-algebra $[0, 1]$. It is well known that the map $\chi \mapsto Ker\ \chi = \{a \in A \mid \chi(a) = 0\}$ is a one-one correspondence between $\mathcal{X}(A)$ and the set of maximal ideals of A. In particular, $\mathcal{X}(A) \neq \emptyset$ for every (nontrivial) A (see [5, Corollary 1.2.15]). The set $\mathcal{X}(A)$ becomes a compact Hausdorff space with the topology inherited from the product space $[0, 1]^A$, where $[0, 1]$ is endowed with its usual topology as a subspace of $\mathbf{R}$. An open subbasis for the topology of $\mathcal{X}(A)$ is given by the sets of the form: $W_{a,U} = \{\chi \in \mathcal{X}(A) \mid \chi(a) \in U\}$, for $a \in A$ and U open in $[0, 1]$.

Given a compact Hausdorff space X, we denote by $\mathcal{C}(X)$ the MV-algebra of all $[0, 1]$-valued continuous functions on X, with the pointwise operations. It follows from [8, Lemma 9.10] and 2.5 that $\mathcal{C}(X)$ is σ-complete iff X is *basically disconnected*, i.e., the closure of every open F_σ-set in X is open. As is well known, basically disconnected compact Hausdorff spaces are precisely the dual spaces of σ-complete boolean algebras.

Let A be a σ-complete MV-algebra. For each $a \in A$, let $\overline{a} = \bigvee_{n \in \mathbf{N}} n.a$, where $n.a$ is defined inductively by $0.a = 0$ and $(n + 1).a = a \oplus n.a$. It follows from [8, Lemma 9.8] that $\overline{a} \in \mathbf{B}(A)$. Moreover, in the proof of [8, Theorem 9.9] it is shown that $a \wedge b = 0$ implies $a \wedge \overline{b} = 0$. Thus, if $a \wedge b = 0$ there is a $z \in \mathbf{B}(A)$ such that $a \leq \neg z$ and $b \leq z$. As a consequence we have (see [7, Theorem 3.3 (iii)] or [8, Theorem 8.1]):

2.6 *Let A be a σ-complete MV-algebra. Given a maximal ideal M of the boolean algebra $\mathbf{B}(A)$, there is a unique $\chi = \chi_M \in \mathcal{X}(A)$ such that $Ker\ \chi \cap \mathbf{B}(A) = M$.* □

Let A be a σ-complete MV-algebra. By 2.4, $\mathbf{B}(A)$ is a σ-complete boolean algebra. Hence the set of maximal ideals of $\mathbf{B}(A)$ equipped with the Stone topology is a basically disconnected compact Hausdorff space that we denote by $\mathcal{M}(\mathbf{B}(A))$. We say that $M \in \mathcal{M}(\mathbf{B}(A))$ is *discrete* if $\chi_M(A)$ is a finite subalgebra of the standard algebra $[0, 1]$. We shall denote by $Disc(A)$ the set of all discrete maximal ideals of $\mathbf{B}(A)$. To each $a \in A$, we associate the function $\hat{a} \colon \mathcal{M}(\mathbf{B}(A)) \to [0, 1]$ defined by the stipulation: $\hat{a}(M) = \chi_M(a)$, for each $M \in \mathcal{M}(\mathbf{B}(A))$. Since the correspondence $M \mapsto \chi_M$ defines a homeomorphism of $\mathcal{M}(\mathbf{B}(A))$ onto $\mathcal{X}(A)$, it follows that $\hat{a} \in \mathcal{C}(\mathcal{M}(\mathbf{B}(A)))$. More precisely, we have the following MV-algebraic version of the Goodearl–Handelman–Lawrence Theorem for Dedekind σ-complete ℓ-groups [8, Corollary 9.14] (see [13, Lemma 6.1]):

Theorem 1 *Let A be a σ-complete MV-algebra, and let $\hat{A}$ be the σ-subalgebra of $\mathcal{C}(\mathcal{M}(\mathbf{B}(A)))$ defined as follows:*

$$\hat{A} = \{f \in \mathcal{C}(\mathcal{M}(\mathbf{B}(A))) \mid f(M) \in \chi_M(A) \text{ for all } M \in \mathcal{M}(\mathbf{B}(A))\}.$$

Then the correspondence $a \mapsto \hat{a}$ defines an isomorphism of A onto $\hat{A}$.

Notice that by Remark 1, the isomorphism $a \mapsto \hat{a}$ preserves denumerable suprema and infima.

3 σ-COMPLETE MV-ALGEBRAS WITH PRODUCT

Definition 1 [11] An *MV-algebra A with product* is an algebra $\langle A, \oplus, \cdot, \neg, 0\rangle$ such that $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra and $\cdot$ is a binary operation (called *product*) which is commutative, associative, and satisfies the following conditions for all $a, b, c \in A$:

- $1 \cdot a = a$,
- $a \cdot (b \odot \neg c) = (a \cdot b) \odot \neg(a \cdot c)$.

The standard MV-algebra $[0, 1]$ endowed with the ordinary multiplication of real numbers is an example of an MV-algebra with product. In [14, 3.1.3] it is shown that multiplication is the only binary operation satisfying the conditions of Definition 1 that can be defined on subalgebras of the standard algebra $[0, 1]$. Hence $\mathbf{L}_2$ is the only finite subalgebra of $[0, 1]$ which admits a product.

Lemma 1 *Let A be an MV-algebra with product. Every MV-algebra homomorphism $\chi : A \to [0, 1]$ preserves the product, and $\chi(A)$ is a subalgebra with product of* $[0, 1]$. □

Proof: By [11, Lemma 2.11], all MV-congruences preserve the product. Hence $\chi(a) = \chi(a')$ and $\chi(b) = \chi(b')$ imply that $\chi(a \cdot b) = \chi(a' \cdot b')$. Therefore the stipulation $\chi(a) * \chi(b) = \chi(a \cdot b)$ defines a binary operation $*$ on the image $\chi(A)$ which satisfies all the identities of Definition 1. Consequently, $*$ must coincide with the usual multiplication of real numbers. □

Lemma 2 *Let A be a σ-complete MV-algebra with product and $\chi : A \to [0, 1]$ be an MV-algebra homomorphism. Then the image $\chi(A)$ is either the MV-algebra $\mathbf{L}_2$ or the whole standard MV-algebra* $[0, 1]$. □

Proof: By Theorem 1, A is isomorphic to the σ-subalgebra $\hat{A}$ of the σ-complete MV-algebra $\mathcal{C}(\mathcal{M}(\mathbf{B}(A)))$. By Lemma 1, for all $M \in Disc(A)$, $\chi_M(A) = \mathbf{L}_2$. Since $\{0, 1\}$ is a closed subset of $[0, 1]$, then $Disc(A)$ is closed in $\mathcal{M}(\mathbf{B}(A))$. Let $M_0 \in \mathcal{M}(\mathbf{B}(A)) \setminus Disc(A)$, and let $\alpha \in \chi_{M_0}(A)$.

For any clopen set U such that $M_0 \in U \subseteq \mathcal{M}(\mathbf{B}(A)) \setminus Disc(A)$, let the function $f\colon \mathcal{M}(\mathbf{B}(A)) \to [0, 1]$ be defined by

$$f(M) = \begin{cases} \alpha & \text{if } M \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Then f belongs to $\hat{A}$. Hence there is an $a \in A$ such that $\hat{a} = f$, and $\alpha = \hat{a}(M_0) = \chi_{M_0}(a) \in \chi_{M_0}(A)$. □

Theorem 2 *Let A_1, A_2 be σ-complete MV-algebras with product. If their Kleene reducts $\mathbf{K}(A_1)$ and $\mathbf{K}(A_2)$ are isomorphic, then A_1 and A_2 are isomorphic.*

Proof: Let h be a (Kleene algebra) isomorphism of $\mathbf{K}(A_1)$ onto $\mathbf{K}(A_2)$. Since for every MV-algebra A, $\mathbf{B}(\mathbf{L}(\mathbf{K}(A))) = \mathbf{B}(\mathbf{L}(A)) = \mathbf{B}(A)$, with the notation of 2.3 it follows that the correspondence $M \mapsto \psi(M) = \bar{h}^{-1}(M)$ defines a homeomorphism ψ from $\mathcal{M}(\mathbf{B}(A_2))$ onto $\mathcal{M}(\mathbf{B}(A_1))$. Further, the quotient Kleene algebras $\mathbf{K}(A_1)/\psi(M)$ and $\mathbf{K}(A_2)/M$ are isomorphic. Since MV-algebra homomorphisms are automatically Kleene algebra homomorphisms, from $M \subseteq Ker\ \chi_M$ it follows that $\chi_{\psi(M)}(\mathbf{K}(A_1))$ and $\chi_M(\mathbf{K}(A_2))$ are homomorphic images of $\mathbf{K}(A_1)/\psi(M)$ and of $\mathbf{K}(A_2)/M$, respectively. Suppose that $\chi_{\psi(M_0)}(A) = [0, 1]$. As in the proof of Lemma 2, there is a clopen U such that $\psi(M_0) \in U$ and the function $f\colon \mathcal{M}(\mathbf{B}(A_2)) \to [0, 1]$ defined by

$$f(M) = \begin{cases} \frac{1}{2} & \text{for } M \in U, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $\hat{A}$. Let $k\colon \mathcal{M}(\mathbf{B}(A_2)) \to [0, 1]$ be defined by

$$k(M) = \begin{cases} 0 & \text{if } M \in U, \\ 1 & \text{otherwise.} \end{cases}$$

We have $k = \hat{b}$ for some $b \in M$, and $k \vee f = k \vee \neg f$. Hence the equivalence class $f/\psi(M_0)$ is the center of the Kleene algebra $\mathbf{K}(A_1)/\psi(M_0)$. Therefore the isomorphic Kleene algebra $\mathbf{K}(A_2)/M_0$ is also centered, and by 2.1, $\chi_{M_0}(\mathbf{K}(A_2))$ is centered. Hence, by Lemma 2 we get $\chi_{M_0}(A_2) = [0, 1]$. One similarly proves that if $\chi_{M_0}(A_2) = [0, 1]$ then $\chi_{\psi(M_0)}(A_1) = [0, 1]$, whence $\chi_{\psi(M)}(A_1) = \chi_M(A_2)$ for each $M \in \mathcal{M}(\mathbf{B}(A_2))$. Consequently, for each $a \in A_1$, the composite map $\hat{a}\psi$ belongs to $\hat{A}_2$. Since, by Lemma 1, MV-homomorphisms χ_M also preserve the product, we now easily conclude that the correspondence $a \mapsto \hat{a}\psi$ defines an isomorphism of A_1 onto A_2. □

Remark 2 *The above theorem remains true if instead of assuming that* A_1, A_2 *are* σ*-complete MV-algebras with product, we require that they are* σ*-complete MV-algebras such that for each* $M \in \mathcal{M}(\mathbf{B}(A_i))$*, and* $i = 1, 2$, $\chi_M(A_i)$ *is either the standard MV-algebra* $[0, 1]$ *or the finite chain* $\mathbf{L}_n$*, for some fixed even number* n*.*

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