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# Free Algebras in Varieties of Glivenko MTL-algebras Satisfying the Equation $2(x^2) = (2x)^2$

*Dedicated to the memory of Willem Johannes Blok*

**Abstract.** The aim of this paper is to give a description of the free algebras in some varieties of Glivenko MTL-algebras having the Boolean retraction property. This description is given (generalizing the results of [9]) in terms of weak Boolean products over Cantor spaces. We prove that in some cases the stalks can be obtained in a constructive way from free kernel DL-algebras, which are the maximal radical of directly indecomposable Glivenko MTL-algebras satisfying the equation in the title. We include examples to show how we can apply the results to describe free algebras in some well known varieties of involutive MTL-algebras and of pseudocomplemented MTL-algebras.

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## Introduction

The *monoidal t-norm based logic*, MTL for short, was introduced by Esteva and Godo in [16] as an attempt to formalize fuzzy logics in which the conjunction is interpreted by a left-continuous t-norm on the real segment $[0, 1]$ and the implication by its corresponding adjoint. The completeness of MTL with respect to left-continuous t-norms was proved by Jenei and Montagna [23]. Adding the double negation axiom one obtains the *involutive monoidal t-based logic*, IMTL, which corresponds to *involutive* left-continuous t-norms. [15]. These logics are algebraizable in the sense of Blok and Pigozzi [3], and the corresponding algebraic semantics are the varieties $\mathbb{MTL}$ and $\mathbb{IMTL}$ of MTL-algebras and of IMTL-algebras, respectively.

MTL is an axiomatic extension of Höhle Monoidal Logic [21]. Hence $\mathbb{MTL}$ is a variety of residuated lattices and $\mathbb{IMTL}$ is a variety of involutive residuated lattices or Girard monoids (see [21]).

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It turns out that Hájek Basic Fuzzy Logic (BL for short) [19], the logic of *continuous* t-norms [5] (see also [12]), is an axiomatic extension of MTL. The involutive basic fuzzy logic coincides with Łukasiewicz infinite-valued logic [19]. Hence the variety $\mathbb{BL}$ of BL-algebras is a subvariety of $\mathbb{MTL}$, and the variety $\mathbb{MV}$ of MV-algebras is a subvariety of $\mathbb{IMTL}$. Other interesting axiomatic extension of IMTL is the *minimum nilpotent logic*, which is the logic corresponding to Fodor's nilpotent minimum left-continuous t-norm (see [16]). The algebraic semantics of this logic is the variety $\mathbb{NM}$ of NM-algebras. This variety is investigated in [18].

We say that a variety $\mathbb{V}$ of MTL-algebras has the *Boolean retraction property* provided each algebra in $\mathbb{V}$ admits a homomorphism onto the subalgebra of its Boolean elements. Varieties of BL-algebras with the Boolean retraction property were investigated in [9], with the purpose of obtaining information on the structure of free algebras in such varieties. Our aim in this paper is to show how the results of [9] can be generalized to obtain information on the structure of free algebras in varieties of MTL-algebras. We also correct some erroneous statements in that paper.

It was shown in [9, Theorem 4.5] that a variety $\mathbb{V}$ of BL-algebras has the Boolean retraction property if and only the equation in the title, which was introduced by Di Nola and Lettieri [14] in the context of MV-algebras, holds in $\mathbb{V}$. This result remains true for varieties of MTL-algebras satisfying also the equation

$$\neg\neg(\neg\neg x \to x) = \top.$$

In the light of the results of [11] (see also [9]), we call the MTL-algebras satisfying this equation *Glivenko MTL-algebras*. Since BL-algebras, in contrast to MTL-algebras, are hoops (in the sense of [2]), $\mathbb{BL}$ is a subvariety of the variety of Glivenko MTL-algebras (see [9, Lemma 1.3]). We call DL-algebras the Glivenko MTL-algebras satisfying the Di Nola - Lettieri equation. The class of these algebras is a variety, denoted by $\mathbb{DL}$. Important examples of DL-algebras are the pseudocomplemented MTL-algebras, characterized by the equation $x \wedge \neg x = \bot$, and the involutive MTL-algebras satisfying Di Nola - Lettieri equation.

If $\boldsymbol{A}$ is a DL-algebra, then the kernel of the Boolean retract inherits a structure of a generalized MTL-algebra (roughly speaking, MTL-algebras without a bottom), equipped with a unary operation $\delta$ induced by the double negation. This leads us to define *kernel DL-algebras* as generalized MTL-algebras equipped with an extra unary operation $\delta$ satisfying some equations. The kernel DL-algebra corresponding to the kernel of the Boolean retract of a DL-algebra $\boldsymbol{A}$ is denoted $\boldsymbol{P}(\boldsymbol{A})$.

Generalizing the construction given in [9, Theorem 3.9], to each kernel DL-algebra $\boldsymbol{A}$ we associate a directly indecomposable DL-algebra $\boldsymbol{S}(\boldsymbol{A})$, such that $\boldsymbol{P}(\boldsymbol{S}(\boldsymbol{A})) = \boldsymbol{A}$. In particular, taking as $\delta$ the identity, we obtain an involutive DL-algebra $\boldsymbol{\iota}(\boldsymbol{A})$, and taking as $\delta$ the constant function $\top$, we obtain a pseudocomplemented MTL-algebra $\boldsymbol{\sigma}(\boldsymbol{A})$. To each variety $\mathbb{V}$ of DL-algebras, the kernel DL-algebras $\boldsymbol{A}$ such that $\boldsymbol{S}(\boldsymbol{A}) \in \mathbb{V}$ form a variety $\mathbb{V}^{\star}$.

The Boolean retraction property implies that free algebras in varieties of DL-algebras can be represented as weak Boolean products of directly indecomposable DL-algebras, over a Stone space. When $\mathbb{V}$ is a variety of either pseudocomplemented MTL-algebras or involutive DL-algebras, we show that the directly indecomposable algebras appearing in the Boolean product decomposition are of the form $\boldsymbol{S}(\boldsymbol{A})$, with $\boldsymbol{A}$ a free algebra in $\mathbb{V}^{\star}$. We apply these results to give explicit descriptions of free algebras in varieties of Gödel algebras, nilpotent minimum algebras, product algebras and some varieties of MV-algebras. In this last case, the results given here rectify the erroneous assertions of [9].

Although familiarity of the reader with residuated lattices and MTL algebras is assumed, some basic results, which are needed in the paper, are collected in Section 1, including the necessary background on Boolean products.

## 1. Preliminaries

An *integral residuated lattice-ordered commutative monoid*, or *residuated lattice* for short, is an algebra $\boldsymbol{A} = \langle A, *, \to, \vee, \wedge, \top\rangle$ of type $\langle 2, 2, 2, 2, 0\rangle$ such that $\langle A, *, \top\rangle$ is a commutative monoid, $\boldsymbol{L}(\boldsymbol{A}) = \langle A, \vee, \wedge, \top\rangle$ is a lattice with greatest element $\top$, and the following residuation condition holds:

$$x * y \leq z, \text{ iff } x \leq y \to z \tag{1}$$

where $x, y, z$ denote arbitrary elements of $A$ and $\leq$ is the order given by the lattice structure, which is called *the natural order of* $\boldsymbol{A}$.

It is well known that residuated lattices form a variety, that we shall denote by $\mathbb{RL}$. Indeed, the residuation condition can be replaced by the following equations:

$$x = x \wedge (y \to (x * y) \vee z)), \tag{2}$$

$$z = (y * (x \wedge (y \to z))) \vee z. \tag{3}$$

Following [25], residuated lattices satisfying the equation $x * y = x \wedge y$ will be called *generalized Heyting algebras*. They were called *Brouwerian algebras* in [13].

In the next lemma we list, for further reference, some well known consequences of (1) that will be used through this paper.

LEMMA 1.1. *The following properties hold true in any residuated lattice* $\boldsymbol{A}$*, where* $x, y, z$ *denote arbitrary elements of* $A$*:*

(i) $x \leq y$ *if and only if* $x \to y = \top$*,*

(ii) $\top \to x = x$*,*

(iii) $(x * y) \to z = x \to (y \to z)$*.* □

A *bounded residuated lattice* is an algebra $\boldsymbol{A} = \langle A, *, \to, \vee, \wedge, \top, \bot \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that $\langle A, *, \to, \vee, \wedge, \top \rangle$ is a residuated lattice and $\bot$ is the smallest element of the lattice $\boldsymbol{L}(\boldsymbol{A})$. The class of all bounded residuated lattice will be represented by $\mathbb{BRL}$

Notice that bounded generalized Heyting algebras are precisely the *Heyting algebras*, i. e., the algebras of intuitionistic logic (see, for instance [25]).

By a *residuated chain* (*bounded residuated chain*) we understand a residuated lattice (bounded residuated lattice) $\boldsymbol{A}$ whose natural order is total, i. e., given $a, b$ in $A$, $a \leq b$ or $b \leq a$.

On a bounded residuated lattice $\boldsymbol{A}$ we define a unary operation $\neg$ by the prescription:

$$\neg x = x \to \bot, \ \text{ for all } \ x \in A. \tag{4}$$

An *involutive residuated lattice*(or *integral, commutative Girard monoid* [21]) is a bounded residuated lattice that satisfies the *double negation equation*:

$$\neg\neg x = x. \tag{5}$$

It follows from Lemma 1.1.(iii) that in an involutive residuated lattice the operations $*$ and $\to$ are related as follows:

$$x * y = \neg(x \to \neg y), \tag{6}$$

$$x \to y = \neg(x * \neg y). \tag{7}$$

Notice that involutive Heyting algebras coincide with Boolean algebras.

On each bounded residuated lattice $\boldsymbol{A}$ we consider the term $x \oplus y := \neg(\neg x * \neg y)$. It follows that $\langle A, \oplus, \bot \rangle$ is a commutative monoid. Moreover when $\boldsymbol{A}$ is involutive, $\oplus$ is the dual of $*$, and the following equation holds in $\boldsymbol{A}$:

$$x \oplus y = \neg x \to y. \tag{8}$$

As usual, we shall write $2x$ in place of $x \oplus x$ and $x^2$ in place of $x * x$.

By an *implicative filter* or *i-filter* of a residuated lattice $\boldsymbol{A}$ we understand a subset $F \subseteq A$ satisfying the following two conditions:

**F1)** $\top \in F$,

**F2)** For all $x, y \in A$, if $x \in F$ and $x \leq y$, then $y \in F$.

**F3)** If $x, y$ are in $F$, then $x * y \in F$.

Alternatively, i-filters may be defined as subsets $F$ of $A$ satisfying F1) and F4): If $x$ and $x \to y$ belong to $F$, then $y \in F$. It follows that each i-filter $F$ is the universe of a subalgebra of $\boldsymbol{A}$, that we shall denote $\boldsymbol{F}$. For each $X \subseteq A$ we denote by $\langle X \rangle$ the i-filter generated by $X$, i.e., the intersection of all i-filters containing $X$. For each $x \in A$, we shall write $\langle x \rangle$ instead of $\langle \{x\} \rangle$.

An i-filter $F$ is *proper* provided $F \neq A$. When $\boldsymbol{A}$ is a bounded residuated lattice, an i-filter $F$ is proper if and only if $\perp \notin F$.

A *prime i-filter* is a proper i-filter such that $x \vee y \in F$ implies $x \in F$ or $y \in F$. The next result is well known, and can be proved by standard techniques.

Lemma 1.2. *Each proper i-filter of a residuated lattice $\boldsymbol{A}$ is an intersection of prime i-filters. In particular, $\{\top\}$ is the intersection of all prime i-filters of $\boldsymbol{A}$.* $\square$

Given an i-filter $F$ of a residuated lattice $\boldsymbol{A}$, the binary relation $\theta(F) := \{(x, y) \in A \times A : (x \to y) * (y \to x) \in F\}$ is a congruence relation on $\boldsymbol{A}$ such that $F$ is the congruence class of $\top$. As a matter of fact, the correspondence $F \mapsto \theta(F)$ is an order isomorphism from the set of i-filters of $\boldsymbol{A}$ onto the set of congruences of $\boldsymbol{A}$, with both sets ordered by inclusion. We will write simply $\boldsymbol{A}/F$ instead of $\boldsymbol{A}/\theta(F)$, and $x/F$ will denote the equivalence class determined by $x \in A$.

Let $\boldsymbol{A}$ be a residuated lattice and let $F$ be a proper i-filter of $\boldsymbol{A}$. It is easy to check that if the quotient $\boldsymbol{A}/F$ is a residuated chain, then $F$ is prime. If $\boldsymbol{A}$ satisfies the following *prelinearity equation*:

$$(x \to y) \vee (y \to x) = \top, \tag{9}$$

then one has that for each prime i-filter $F$ of $\boldsymbol{A}$, the quotient $\boldsymbol{A}/F$ is a residuated chain. Since residuated chains satisfy (9), taking into account Lemma 1.2 obtain the next lemma:

Lemma 1.3. *A residuated lattice $\boldsymbol{A}$ satisfies the prelinearity equation (9) if and only if it is isomorphic to a subdirect product of residuated chains.* $\square$

Bounded residuated lattices satisfying the prelinearity equation were baptized MTL-algebras by Esteva and Godo [16]. Clearly they are the bounded residuated lattices isomorphic to subdirect products of bounded residuated chains. For coherence, we shall call the (not necessarily bounded) residuated lattices satisfying (9) *generalized MTL-algebras*, or *GMTL-algebras* for short. Notice that they are termwise equivalent to *basic semihoops* as defined in [17], and to *prelinear semihoops* as defined in [26] . The variety of GMTL-algebras will be denoted by $\mathbb{GMTL}$. Heyting algebras satisfying (9) are called *linear Heyting algebras* in [20, 25]. But we prefer to follow Hájek's nomenclature [19], and call them *Gödel algebras*. Accordingly, we call *generalized Gödel algebras* the generalized Heyting algebras satisfying (9). The corresponding varieties will be denoted by $\mathbb{G}$ and $\mathbb{GG}$, respectively.

An *IMTL-algebra* [16] is an involutive MTL-algebra, i. e., an MTL-algebra satisfying the double negation equation (5). IMTL-algebras are the subdirect products of involutive residuated chains.

Given a bounded residuated lattice $\boldsymbol{A}$, $B(\boldsymbol{A})$ will denote the set of complemented elements of the bounded lattice $\boldsymbol{L}(\boldsymbol{A})$. That is, $x \in B(\boldsymbol{A})$ if and only if there is $y \in A$ such that $x \vee y = \top$ and $x \wedge y = \bot$. It is shown in [24] that $B(\boldsymbol{A})$ is the universe of a subalgebra of $\boldsymbol{A}$, denoted $\boldsymbol{B}(\boldsymbol{A})$, which is a Boolean algebra. If $x, y$ are in $B(\boldsymbol{A})$, then $x * y = x \wedge y$ and $x \to y = \neg x \vee y$.

In general, an algebra $\boldsymbol{A}$ is called *directly indecomposable* iff $A$ has more than one element and whenever it is isomorphic to a direct product of two algebras $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, then either $\boldsymbol{A}_1$ or $\boldsymbol{A}_2$ is the trivial algebra with just one element.

The next result is proved in [24, Proposition 1.5].

LEMMA 1.4. *A bounded residuated lattice $\boldsymbol{A}$ is directly indecomposable if and only if $\boldsymbol{B}(\boldsymbol{A})$ is the two-element Boolean algebra.* □

By a *Stone space* we understand a totally disconnected compact Hausdorff space $X$. As usual, a subset of $X$ that is simultaneously open and closed will be called *clopen*.

We recall that a *weak Boolean product* of a family $(A_x : x \in X)$ of algebras over a Stone space $X$ is a subdirect product $\boldsymbol{A}$ of the given family such that the following conditions hold:

(i) if $a, b \in A$, then $[\![a = b]\!] = \{x \in X : a(x) = b(x)\}$ is open,

(ii) if $a, b \in A$ and $Z$ is a clopen in $X$, then $a \restriction_Z \cup\, b \restriction_{X \setminus Z} \in A$.

An algebra $\boldsymbol{A}$ is *representable as weak Boolean product* when it is isomorphic to a weak Boolean product. As explained in [4], weak Boolean products are the global sections of (not necessarily Hausdorff) sheaves of algebras over Boolean spaces.

Since the variety of bounded residuated lattices admits 2/3-minority term (see[24]), then it is arithmetical and it has the Boolean Factor Congruence property. Therefore each nontrivial bounded residuated lattice can be represented as a weak Boolean product of directly indecomposable bounded residuated lattices [1]). We are going to give an explicit description of this representation. Firstly, note that the following lemma can be obtained in a standard way (see, for instance, [6]):

LEMMA 1.5. *Let* $\boldsymbol{A}$ *be a bounded residuated lattice, and let* $F$ *be a filter of the Boolean algebra* $\boldsymbol{B}(\boldsymbol{A})$*, then :*

$$\sim_F = \{(x,y) \in A^2 : x \wedge z = y \wedge z \text{ for some } z \in F\}$$

*is a congruence relation on* $\boldsymbol{A}$ *that coincides with the congruence relation given by the implicative filter* $\langle F \rangle$ *generated by* $F$*. Moreover, if* $F$ *is a prime filter (i.e., an ultrafilter) of* $\boldsymbol{B}(\boldsymbol{A})$*, then* $B(\boldsymbol{A}/\!\sim_F) = \{\bot/\!\sim_F, \top/\!\sim_F\}$. $\square$

We write $\mathbf{A}/\langle F \rangle$ in place of $\mathbf{A}/\sim_F$, and so $x/\langle F \rangle = x/\sim_F$ for the equivalence class of $x \in A$. We will represent by $Sp\,\boldsymbol{B}(\boldsymbol{A})$ the set of all prime filters (ultrafilters) of the Boolean algebra $\boldsymbol{B}(\boldsymbol{A})$. With these notations we have:

THEOREM 1.6. *Each nontrivial bounded residuated lattice* $\boldsymbol{A}$ *is representable as the weak Boolean product of the family* $(\boldsymbol{A}/\langle F \rangle : F \in Sp\,\boldsymbol{B}(\boldsymbol{A}))$ *over the Stone space* $\boldsymbol{Sp}\,\boldsymbol{B}(\boldsymbol{A})$. $\square$

## 2. Glivenko MTL-algebras and DL-algebras

It is shown in [11], that the variety of bounded residuated lattices is a natural expansion of the quasivariety of bounded BCK-algebras. Since $\mathbb{MTL}$ is subvariety of $\mathbb{BRL}$, the next theorem follows from [11, Theorem 46].

THEOREM 2.1. *For each* $\boldsymbol{A} \in \mathbb{BRL}$ *the following are equivalent:*

(i) *The equation* $\neg\neg(\neg\neg x \to x) = \top$ *holds in* $\boldsymbol{A}$*.*

(ii) *If we consider the set* $Reg(\boldsymbol{A}) := \{a \in A : \neg\neg a = a\}$*, and for each* $\star \in \{*, \wedge, \vee\}$*, the term* $u \star^* v =: \neg\neg(u \star v)$*, then:*

  (a) $\boldsymbol{Reg}(\boldsymbol{A}) = \langle Reg(\boldsymbol{A}), *^*, \to, \vee^*, \wedge^*, 1, 0\rangle$ *is an involutive residuated lattice, and*

  (b) $\neg\neg\colon A \to Reg(\boldsymbol{A}) : a \mapsto \neg\neg a$ *is a homomorphism from* $\boldsymbol{A}$ *onto* $\boldsymbol{Reg}(\boldsymbol{A})$. $\square$

A bounded residuated lattice satisfying the equation

$$\neg\neg(\neg\neg x \to x) = \top \tag{10}$$

will be called a *Glivenko residuated lattice.* If it is an MTL-algebra, then it will be called a *Glivenko MTL-algebra.*

In the case of Glivenko MTL-algebras, the operations $\vee^*$ and $\wedge^*$ defined in Theorem 2.1 coincide with $\vee$ and $\wedge$, respectively.

Examples of Glivenko MTL-algebras are the involutive MTL-algebras, and the pseudocomplemented MTL-algebras, i. e., MTL-algebras satisfying the equation

$$x \wedge \neg x = \bot. \tag{11}$$

If $\boldsymbol{A}$ is pseudocomplemented MTL-algebra, then for each $a \in A$, $\neg a$ is the pseudocomplement of $a$ in the lattice $\boldsymbol{L}(\boldsymbol{A})$.

BL-algebras, that is, the MTL-algebras satisfying the *hoop equation*:

$$x * (x \to y) = y * (y \to x). \tag{12}$$

are also examples of Glivenko MTL-algebras. Indeed, it is shown in [9](see also [10]) that they satisfy (10). Involutive BL-algebras coincide with MV-algebras [19].

Our next aim is to give a method to obtain Glivenko MTL-algebras starting from GMTL-algebras.

Let $\boldsymbol{A}$ be a GMTL-algebra. A map $\delta\colon A \to A$ is *dl-admissible* provided that for any $a, b \in A$ it satisfies:

| | | | |
|---|---|---|---|
| ($\boldsymbol{\delta}1$) | $a \to \delta(a) = \top$ | ($\boldsymbol{\delta}4$) | $\delta(a \wedge b) = \delta(a) \wedge \delta(b)$ |
| ($\boldsymbol{\delta}2$) | $\delta(\delta(a)) = \delta(a)$ | ($\boldsymbol{\delta}5$) | $\delta(a \vee b) = \delta(a) \vee \delta(b)$ |
| ($\boldsymbol{\delta}3$) | $\delta(a \to b) = a \to \delta(b)$ | ($\boldsymbol{\delta}6$) | $\delta(a * b) = \delta(\delta(a) * \delta(b))$. |

THEOREM 2.2. *Given a GMTL-algebra $\boldsymbol{A} = \langle A, *, \to, \vee, \wedge, \top\rangle$ and a dl-admissible $\delta\colon A \to A$, on the set $S(\boldsymbol{A}, \delta) := (A \times \{1\}) \cup (\delta[A] \times \{0\})$ define the binary operations $\odot$, $\Rightarrow$, $\sqcup$ and $\sqcap$ by the following prescriptions, where $a, b$ denote arbitrary elements of $A$:*

$$\langle a, i\rangle \sqcup \langle b, j\rangle = \langle b, j\rangle \sqcup \langle a, i\rangle = \begin{cases} \langle a \vee b, 1\rangle & \text{if } i = j = 1, \\ \langle a \wedge b, 0\rangle & \text{if } i = j = 0, \\ \langle b, 1\rangle & \text{if } i < j. \end{cases} \tag{13}$$

$$\langle a, i\rangle \sqcap \langle b, j\rangle = \langle b, j\rangle \sqcap \langle a, i\rangle = \begin{cases} \langle a \wedge b, 1\rangle & \text{if } i = j = 1, \\ \langle a \vee b, 0\rangle & \text{if } i = j = 0, \\ \langle a, 0\rangle & \text{if } i < j. \end{cases} \tag{14}$$

$$\langle a, i\rangle \odot \langle b, j\rangle = \langle b, j\rangle \odot \langle a, i\rangle = \begin{cases} \langle a * b, 1\rangle & \text{if } i = j = 1, \\ \langle \top, 0\rangle & \text{if } i = j = 0, \\ \langle b \to a, 0\rangle & \text{if } i < j. \end{cases} \tag{15}$$

$$\langle a, i\rangle \Rightarrow \langle b, j\rangle = \begin{cases} \langle a \to b, 1\rangle & \text{if } i = j = 1, \\ \langle b \to a, 1\rangle & \text{if } i = j = 0, \\ \langle \delta(a * b), 0\rangle & \text{if } i > j, \\ \langle \top, 1\rangle & \text{if } i < j. \end{cases} \tag{16}$$

*If* $\mathbf{1} = \langle \top, 1\rangle$ *and* $\mathbf{0} = \langle \top, 0\rangle$*, then the algebra*

$$\boldsymbol{S}(\boldsymbol{A}, \delta) = \langle S(\boldsymbol{A}, \delta) = (A \times \{1\}) \cup (\delta[A] \times \{0\}), \odot, \Rightarrow, \sqcup, \sqcap, \mathbf{1}, \mathbf{0}\rangle$$

*is a directly indecomposable Glivenko MTL-algebra, such that* $Reg(\boldsymbol{S}(\boldsymbol{A}, \delta)) = \delta[A] \times \{0, 1\}$*, and* $\boldsymbol{Reg}(\boldsymbol{S}(\boldsymbol{A}, \delta)) = \boldsymbol{S}(\boldsymbol{\delta}[\boldsymbol{A}], \delta)$*.*

Proof. First, observe that conditions $(\boldsymbol{\delta}\,1) - (\boldsymbol{\delta}\,6)$ guarantee that the operations are well defined. For instance, $(\boldsymbol{\delta}\,3)$ implies that $\langle a, 0\rangle \odot \langle b, 1\rangle \in \delta[A] \times \{0\}$. It is routine to verify that $\langle S(\boldsymbol{A}, \delta), \odot, \mathbf{1}\rangle$ is a commutative monoid and that $\langle S(\boldsymbol{A}, \delta), \sqcup, \sqcap, \mathbf{1}, \mathbf{0}\rangle$ is a bounded lattice whose partial order is given by $\langle a, i\rangle \sqsubseteq \langle b, j\rangle$ iff $\langle a, i\rangle \Rightarrow \langle b, j\rangle = \mathbf{1}$.

To prove the residual condition

$$(a, i) \odot (b, j) \leq (c, k) \text{ iff } (a, i) \leq (b, j) \Rightarrow (c, k)$$

we are going to check only the case $i = 0, j = 1, k = 0$, because the remaining cases are straightforward.

Firstly, note that from $(\boldsymbol{\delta}1), (\boldsymbol{\delta}2)$ and $(\boldsymbol{\delta}4)$ (or $(\boldsymbol{\delta}5)$), it follows that for $b \in A$ and $a \in \delta[A]$, $b \leq a$ iff $\delta(b) \leq a$. Now if $a, c \in \delta[A]$ and $b \in A$, then $\langle a, 0\rangle \odot \langle b, 1\rangle \sqsubseteq \langle c, 0\rangle$ iff $\langle b \to a, 0\rangle \sqsubseteq \langle c, 0\rangle$ iff $c \leq b \to a$ iff $c * b \leq a$ iff $\delta(c * b) \leq a$ iff $\langle a, 0\rangle \sqsubseteq \langle \delta(c * b), 0\rangle = \langle b, 1\rangle \Rightarrow \langle c, 0\rangle$.

It follows at once from the definitions of $\Rightarrow$ and $\sqcap$ that the prelinearity equation, which holds in $\boldsymbol{A}$, also holds in $\boldsymbol{S}(\boldsymbol{A}, \delta)$. Therefore $\boldsymbol{S}(\boldsymbol{A}, \delta)$ is a MTL-algebra.

If $\sim \langle x, i\rangle =: \langle x, i\rangle \Rightarrow \mathbf{0}$, then for $a \in A$, and $b \in \delta[A]$, we have that

$$\sim \langle a, 1\rangle = \langle \delta(a), 0\rangle, \quad \text{and} \quad \sim\sim \langle a, 1\rangle = \langle \delta(a), 1\rangle \tag{17}$$

$$\sim \langle b, 0\rangle = \langle b, 1\rangle \quad \text{and} \quad \sim\sim \langle b, 0\rangle = \langle b, 0\rangle \tag{18}$$

Hence it follows from $(\delta\,3)$ that $\boldsymbol{S}(\boldsymbol{A}, \delta)$ satisfies the equation

$$\neg\neg(x \to y) = x \to \neg\neg y \tag{19}$$

which in bounded BCK-algebras, and a fortiori, in bounded residuated lattices, is equivalent to (10) (see [11, Lemma 3.3]). It is clear $\mathbf{0}$ and $\mathbf{1}$ are the only boolean elements in $\boldsymbol{S}(\boldsymbol{A}, \delta)$, hence it is directly indecomposable.

Finally, observe that from (17) and (18) it follows that $\sim\sim \langle a, i\rangle = \langle a, i\rangle$ iff $a \in \delta[A]$, hence $Reg(\boldsymbol{S}(\boldsymbol{A}, \delta)) = \delta[A] \times \{0, 1\}$. ∎

Remark 2.3. The construction of $\boldsymbol{S}(\boldsymbol{A}, \delta)$ generalizes the construction given in [9, Theorem 3.9] in the context of BL-algebras. For $\delta = id_A$, it coincides with a construction of Jenei [22] (see also [26]) of disconnected rotations of ordered semigroups.

As an immediate consequence of the above theorem we obtain

Corollary 2.4. *Under the hypothesis of Theorem 2.2 we have:*

(1) $\boldsymbol{S}(\boldsymbol{A},\delta)$ *is an IMTL-algebra if and only if* $\delta = id_A$*, the identity function on* $A$*.*

(2) $\boldsymbol{S}(\boldsymbol{A},\delta)$ *is pseudocomplemented if and only if* $\delta = \delta_\top$*, where* $\delta_\top$ *is the constant function* $\top$*, i. e.,* $\delta_\top(x) = \top$ *for all* $x \in A$*.* □

For each GMTL-algebra $\boldsymbol{A}$ and each $\delta\colon A \to A$ dl-admissible, a straightforward computation shows that the equation

$$(2x)^2 = 2(x^2), \quad \text{i.e.,} \ (\neg x \to \neg\neg x)^2 = \neg(x^2) \to \neg\neg(x^2) \tag{20}$$

holds in $\boldsymbol{S}(\boldsymbol{A},\delta)$. We shall denote by $\mathbb{DL}$ the subvariety of the variety of Glivenko MTL-algebras determined by the equation (20). MTL-algebras satisfying (20) are called $\mathrm{BP}_0$ MTL-algebras in [27].

We call the algebras in $\mathbb{DL}$ *DL-algebras*, and the DL-algebras whose natural order is total, *DL-chains*. From Lemma 1.3 we obtain:

Lemma 2.5. *Each DL-algebra is isomorphic to a subdirect product of DL-chains.* □

Remark 2.6. Notice that in a DL-algebra $\boldsymbol{A}$, the equation (20) prevents the existence of an element $x \in A$ such that $x = \neg x$.

Example 2.7. Since the equations $(\neg x \to \neg\neg x)^2 = \neg\neg x = \neg(x^2) \to \neg\neg(x^2)$ hold in every pseudocomplemented chain, we have that *the variety of pseudocomplemented MTL-algebras is a subvariety of* $\mathbb{DL}$*.*

Example 2.8. The subvariety of the variety of MV-algebras generated by Chang's algebra (see [14]) is the variety of involutive DL-algebras determined by the hoop equation (5.11) This example will be considered in more detail in Section 5.2.

Example 2.9. The subvariety of the variety of Nilpotent Minimum algebras (NM-algebras for short) generated by the algebra $[0,1]^-$ (see [18]) is the subvariety of $\mathbb{DL}$ determined by the nilpotent minimum equation:

$$((x * y) \to \bot) \vee ((x \wedge y) \to (x * y)) = \top. \tag{21}$$

We shall return to this example in Section 5.1.

## 3. The operator $\nabla$ and kernel DL-algebras

On each DL-algebra $\boldsymbol{A}$ we define an operator $\nabla\colon \boldsymbol{A} \to \boldsymbol{A}$ by the prescription:

$$\nabla x = (2x)^2 = (\neg x \to \neg\neg x)^2 \ \text{ for all } \ x \in A.$$

Observe that for $a \in A$, $\nabla a = \nabla\neg\neg a$ and by (20), $\nabla a = \neg(a^2) \to \neg\neg a^2$. Notice also that when $\boldsymbol{A}$ is a pseudocomplemented MTL-algebra, $\nabla x = \neg\neg x$ for all $x \in A$ (see Example 2.7).

LEMMA 3.1. *The following properties hold in a DL-chain $\boldsymbol{C}$, where $a, b$ denote arbitrary elements of $C$:*

(i) $\nabla a \in \{\top, \bot\}$ *for each* $a \in C$,

(ii) $\nabla$ *is monotonic,*

(iii) $\nabla(a \vee b) = \nabla a \vee \nabla b$,

(iv) $\nabla(a \wedge b) = \nabla a \wedge \nabla b$,

(v) $\nabla(a * b) = \nabla a * \nabla b$,

PROOF. Notice that by Remark 2.6, for each $a \in C$, $\neg a < a \leq \neg\neg a$ or $a \leq \neg\neg a < \neg a$. If $\neg a < a$, then $\nabla a = (\neg a \to \neg\neg a)^2 = \top^2 = \top$, and if $a < \neg a$, then $a^2 = \bot$, and by (20), $\nabla a = \bot$. This proves (i). To prove (ii), suppose $a \leq b$. Hence $\neg b \leq \neg a$, and $\neg b * (\neg a \to \neg\neg a) \leq \neg a * (\neg a \to \neg\neg a) \leq \neg\neg a \leq \neg\neg b$. Then by (1), $\neg a \to \neg\neg a \leq \neg b \to \neg\neg b$, and we have proved (ii). Items (iii) and (iv) are immediate consequences of (ii). To prove (v), suppose first that $\nabla a * \nabla b = \top$. Hence $\nabla a = \nabla b = \top$, and by (i), $\neg a < a$ and $\neg b < b$, and by (20) we also have $\neg a^2 < \neg\neg a^2$ and $\neg b^2 < \neg\neg b^2$. If $a \leq b$, then $\neg(a * b) \leq \neg a^2 < a^2 \leq a * b$, and $\nabla(a * b) = \top$. In a similar way we show that $\nabla(a * b) = \top$ when $b \leq a$. Suppose now that $\nabla(a * b) = \top$. Then $\neg a \leq \neg(a * b) < a * b \leq a \leq \neg\neg a$, and similarly, $\neg b < \neg\neg b$. Hence $\nabla a * \nabla b = \top$. We have shown that $\nabla(a * b) = \top$ if and only if $\nabla a * \nabla b = \top$, and by (i), this implies (v). ■

THEOREM 3.2. *For each $\boldsymbol{A} \in \mathbb{DL}$, $\nabla$ is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{B}(\boldsymbol{A})$ satisfying $\nabla\nabla x = x$. In other words, $\nabla$ is a retract from $\boldsymbol{A}$ onto $\boldsymbol{B}(\boldsymbol{A})$.*

PROOF. By Lemmas 3.1 and 2.5, the equations $\nabla(x \vee y) = \nabla x \vee \nabla y$, $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, and $\nabla(x * y) = \nabla x * \nabla y$ hold in any DL-algebra. Let $a, b \in A$. Taking into account (20) and (19), we have

$$\nabla\neg a = \neg(\neg a)^2 \to \neg\neg(\neg a)^2 = \neg(\neg(\neg a)^2 * \neg(\neg a)^2) = \neg(2(a^2)) = \neg\nabla a.$$

Hence $\nabla$ also preserves $\neg$, and since

$$\nabla(a \to b) = \nabla\neg\neg(a \to b) = \nabla(a \to \neg\neg b) = \nabla\neg(a * \neg b) = \neg(\nabla a * \neg\nabla b),$$

$\nabla(x \to y) = \nabla x \to \nabla y$ also holds. Hence $\nabla$ is a homomorphism. By (i) of Lemma 3.1 the equation $\nabla x \vee \neg\nabla x = \top$ holds in all DL-chain, then again by Lemma 2.5, it also holds in $\boldsymbol{A}$. Hence $\nabla x \in B(\boldsymbol{A})$ for each $x \in A$. On the other hand, if $z \in B(\boldsymbol{A})$, then $\nabla z = (\neg z \to z)^2 = z^2 = z$. Hence $\nabla$ is onto $B(\boldsymbol{A})$, and the equation $\nabla\nabla x = \nabla x$ holds in $\boldsymbol{A}$. ■

REMARK 3.3. *If a variety $\mathbb{V}$ of MTL-algebras has the Boolean retraction property, then all algebras in $\mathbb{V}$ satisfy equation (20).* Indeed, let $\boldsymbol{C}$ be a residuated chain in $\mathbb{V}$. By hypothesis, there is a homomorphism $\nabla$ :

$\boldsymbol{C} \to \boldsymbol{B}(\boldsymbol{C}) = \{\bot, \top\}$. If $\nabla a = \top$, then $\neg a < a$. This implies that $a \oplus a = \neg(\neg a * \neg a) = \top$. Since $\nabla a = \top$ implies $\nabla a^2 = \top$, we also have $\neg(\neg a^2 * \neg a^2) = \top$. Therefore $(2a)^2 = 2(a^2) = \top$. If $\nabla a = \bot$, then $\nabla \neg a = \top$. Hence $a \le \neg\neg a < \neg a$, and $a^2 = \bot$. Since we also have $\nabla(\neg a)^2 = \top$, then $2a^2 = \neg(\neg a * \neg a) < \neg a * \neg a$. Therefore $(2a)^2 = \bot = 2(a^2)$. Hence (20) holds in $\boldsymbol{C}$. Since $\mathbb{V}$ is generated by the residuated chains that it contains, it follows that (20) holds in all algebras in $\mathbb{V}$.

Let $\boldsymbol{A}$ be a DL-algebra. Since $\nabla$ is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{B}(\boldsymbol{A})$, $\nabla^{-1}(\{\top\}) = \{x \in A : \nabla x = \top\}$ is an i-filter of $\boldsymbol{A}$, that we shall denote by $A^\top$. Consequently, it is the universe of a subalgebra $\boldsymbol{A}^\top$ of the GMTL-algebra given by the $\{\to, *, \wedge, \vee, \top\}$-reduct of $\boldsymbol{A}$. Moreover, one has that the quotient $\boldsymbol{A}/A^\top \cong \mathbf{B}(\boldsymbol{A})$. Notice that $A^\top = \{x \in A : \neg x \le \neg\neg x\}$.

The next result follows directly from Theorem 2.2.

Lemma 3.4. *For each GMTL-algebra $\boldsymbol{A}$ and each dl-admissible map $\delta\colon A \to A$, the correspondence $x \mapsto \langle x, 1\rangle$ defines an isomorphism $\varphi_{\boldsymbol{A}}$ from $\boldsymbol{A}$ onto $\boldsymbol{S}(\boldsymbol{A}, \delta)^\top$, such that $\varphi_{\boldsymbol{A}}(\delta(x)) = \neg\neg\varphi_{\boldsymbol{A}}(x)$.* □

Theorem 3.5. *Let $\boldsymbol{A}$ be a DL-algebra. Then*

(i) *The restriction of $\neg\neg$ to $A^\top$ is a dl-admissible map from $A^\top$ into $A^\top$.*

(ii) *The function $\psi_{\boldsymbol{A}}\colon S(\boldsymbol{A}^\top, \neg\neg) \to A$ defined for each $\langle x, i\rangle \in S(\boldsymbol{A}^\top, \neg\neg)$ by the prescription*

$$\psi_{\boldsymbol{A}}(x, i) = \begin{cases} x & \text{if } i = 1, \\ \neg x & \text{if } i = 0, \end{cases}$$

*is an injective homomorphism from $\boldsymbol{S}(\boldsymbol{A}^\top, \neg\neg)$ into $\boldsymbol{A}$. It is surjective (i. e., an isomorphism onto $\boldsymbol{A}$) iff $\boldsymbol{A}$ is directly indecomposable.*

Proof. Since any DL-algebra is a Glivenko MTL-algebra, (1) follows from Theorem 2.1 and the remarks following it. In order to prove (2), note first that if $a, b$ are in $A^\top$, then $\neg a < b$. Indeed, since $A^\top$ is an i-filter, $a * b \in A^\top$. Hence, taking into account Remark 2.6, one has $\neg a \le \neg(a * b) < a * b \le b$. Notice that since $b \le \neg\neg b$, this implies that $\neg a * \neg b = \bot$. Now it is routine to prove that $\psi_{\boldsymbol{A}}\colon \boldsymbol{S}(\boldsymbol{A}^\top, \neg\neg) \to \boldsymbol{A}$ is an injective homomorphism. If $\psi_{\boldsymbol{A}}$ is onto, then $\boldsymbol{B}(\boldsymbol{S}(\boldsymbol{A}^\top, \neg\neg))$ and $\boldsymbol{B}(\boldsymbol{A})$ are isomorphic Boolean algebras, and since $\boldsymbol{B}(\boldsymbol{S}(\boldsymbol{A}^\top, \neg\neg)) = \{\top, \bot\}$, by Lemma 1.4 it follows that $\boldsymbol{A}$ is directly indecomposable. Conversely, suppose that $\boldsymbol{A}$ is directly indecomposable. Then $\boldsymbol{B}(\boldsymbol{A}) = \{\top, \bot\}$, and hence $A = \nabla^{-1}(\{\top\}) \cup \nabla^{-1}(\{\bot\}) = \psi_{\boldsymbol{A}}(S(\boldsymbol{A}^\top, \neg\neg))$. ■

Remark 3.6. From the above theorem and Theorem 2.2 it follows that if $\boldsymbol{A}$ is a directly indecomposable DL-algebra, then $A \setminus A^\top \subseteq Reg(\boldsymbol{A})$.

We say that an algebra $\boldsymbol{H} = \langle H, *, \to, \vee, \wedge, \delta, \top\rangle$ of type $\langle 2, 2, 2, 2, 1, 0\rangle$ is a *kernel DL-algebra* (KDL-algebra for short) provided the reduct $\boldsymbol{H}^- = \langle H, *, \to, \vee, \wedge, \top\rangle$ is a GMTL-algebra and $\delta\colon A \to A$ is dl-admissible. Observe that any KDL-algebra $\boldsymbol{H}$ can be given as a pair $\boldsymbol{H} = (\boldsymbol{A}, \delta)$, with $\boldsymbol{H}^- = \boldsymbol{A} \in \mathbb{GMTL}$ and $\delta$ a dl-admissible unary operation. That the class $\mathbb{KDL}$ of kernel DL-algebras is a variety in the language $\{*, \to, \vee, \wedge, \delta, \top\}$ follows from the definition of dl-admissible map.

For each GMTL-algebra $\boldsymbol{A}$, $(\boldsymbol{A}, id_A)$ and $(\boldsymbol{A}, \delta_\top)$ are KDL-algebras. If $\boldsymbol{A}$ is the Boolean algebra with two atoms, say $a$ and $b$, and $\delta\colon A \to A$ is defined as $\delta(\bot) = \delta(a) = a$, and $\delta(b) = \delta(\top) = \top$, then $(\boldsymbol{A}, \delta)$ is an example of a KDL-algebra such that $id_A \neq \delta \neq \delta_\top$.

Let $(\boldsymbol{A}_1, \delta_1), (\boldsymbol{A}_2, \delta_2)$ be KDL-algebras, and let $h$ be a homomorphism from $(\boldsymbol{A}, \delta_1)$ into $(\boldsymbol{A}, \delta_2)$. By defining $\boldsymbol{S}(h)(x, i) = \langle h(x), i\rangle$ for all $\langle x, i\rangle \in S(\boldsymbol{A}_1, \delta_1)$ one obtains a homomorphism $\boldsymbol{S}(h)\colon \boldsymbol{S}(\boldsymbol{A}_1, \delta_1) \to \boldsymbol{S}(\boldsymbol{A}_2, \delta_2)$. It is easy to verify that $\boldsymbol{S}$ defines a functor from the category $\mathsf{KDL}$ of KDL-algebras and homomorphisms into the category $\mathsf{DL}$ of DL-algebras and homomorphisms.

For each DL-algebra $\boldsymbol{A}$, let $\boldsymbol{P}(\boldsymbol{A}) = (\boldsymbol{A}^\top, \neg\neg)$, and for each morphism $h\colon \boldsymbol{A}_1 \to \boldsymbol{A}_2$ in $\mathsf{DL}$, let $\boldsymbol{P}(h)$ be the restriction of $h$ to $\boldsymbol{A}_1^\top$. Since for each $x \in A^\top$, $\top = h(\nabla x) = \nabla h(x)$, and $h(\neg\neg x) = \neg\neg h(x)$, we have that $\boldsymbol{P}(h)$ is a homomorphism from $\boldsymbol{P}(\boldsymbol{A}_1)$ into $\boldsymbol{P}(\boldsymbol{A}_2)$. As a matter of fact, $\boldsymbol{P}$ is a functor from $\mathsf{DL}$ to $\mathsf{KDL}$.

Taking into account Lemma 3.4 it follows that if $h\colon (\boldsymbol{A}_1, \delta_1) \to (\boldsymbol{A}_2, \delta_2)$ denotes a morphism in $\mathsf{KDL}$, then the following diagram is commutative:

$$\begin{array}{ccc} (\boldsymbol{A}_1, \delta_1) & \xrightarrow{h} & (\boldsymbol{A}_2, \delta_2) \\ \varphi_{\boldsymbol{A}_1} \downarrow & & \downarrow \varphi_{\boldsymbol{A}_2} \\ \boldsymbol{P}(\boldsymbol{S}(\boldsymbol{A}_1, \delta_1)) & \xrightarrow{\boldsymbol{P}(\boldsymbol{S}(h))} & \boldsymbol{P}(\boldsymbol{S}(\boldsymbol{A}_2, \delta_2)). \end{array} \tag{22}$$

Then we have that the family of isomorphisms $\varphi$ defines an natural equivalence from $\boldsymbol{I}_{\mathsf{KDL}}$, the identity functor on $\mathsf{KDL}$, to $\boldsymbol{PS}$.

Analogously, taking into account Theorem 3.5, it follows that for each morphism $h\colon \boldsymbol{A}_1 \to \boldsymbol{A}_2$ in $\mathsf{DL}$, the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{A}_1 & \xrightarrow{h} & \boldsymbol{A}_2 \\ \psi_{\boldsymbol{A}_1} \uparrow & & \uparrow \psi_{\boldsymbol{A}_2} \\ \boldsymbol{S}(\boldsymbol{P}(\boldsymbol{A}_1)) & \xrightarrow{\boldsymbol{S}(\boldsymbol{P}(h))} & \boldsymbol{S}(\boldsymbol{P}(\boldsymbol{A}_2)). \end{array} \tag{23}$$

Then the family of monomorphisms $\psi$ defines a natural transformation from $\boldsymbol{SP}$ to $\boldsymbol{I}_{\mathsf{DL}}$. By Theorem 3.5, this natural transformation is an equivalence if we restrict the functor $\boldsymbol{P}$ to the full subcategory of $\mathsf{DL}$ whose objects are the directly indecomposable DL-algebras.

Summing up, we have that

*The categories of KDL-algebras and of directly indecomposable DL-algebras are equivalent.*

Remark 3.7. Observe that Corollary 2.4 implies that the functors $\boldsymbol{S}$ and $\boldsymbol{P}$ also define a natural equivalence between the categories of GMTL-algebras and involutive DL-algebras, and a natural equivalence between the categories of GMTL-algebras and pseudocomplemented MTL-algebras $\mathsf{PMTL}$. Consequently, the categories of involutive DL-algebras and pseudocomplemented MTL-algebras are equivalent.

Remark 3.8. Let $\boldsymbol{A}$ be a directly indecomposable DL-algebra, and let $X$ be a set of generators of $\boldsymbol{A}$. The set $Y := (A^{\top} \cap X) \cup (A^{\top} \cap \neg X)$ also generates $\boldsymbol{A}$. Indeed, let $\boldsymbol{H}$ be the subalgebra of $\boldsymbol{A}$ generated by $Y$. If $x \in X \cap A^{\top}$, then $x \in H$. If $x \notin A^{\top}$, then $\neg x \in H$, and by Remark 3.6, this implies that $x = \neg\neg x \in H$. Hence $X \subseteq H$, and $\boldsymbol{H} = \boldsymbol{A}$. Notice that, in particular, $Y$ generates the KDL-algebra $\boldsymbol{P}(\boldsymbol{A}) = (\boldsymbol{A}^{\top}, \neg\neg)$

With each class $\mathbb{K}$ of DL-algebras, we associate the class $\mathbb{K}^{\star}$ of KDL-algebras $(\boldsymbol{A}, \delta)$ such that $\boldsymbol{S}(\boldsymbol{A}, \delta)$ is in $\mathbb{K}$.

Theorem 3.9. *For each subvariety $\mathbb{V}$ of $\mathbb{DL}$, $\mathbb{V}^{\star}$ is a subvariety of $\mathbb{KDL}$.*

Proof. Let $\mathbb{V}$ be a subvariety of $\mathbb{DL}$. We need to prove that $\mathbb{V}^{\star}$ is closed under subalgebras, direct products and homomorphic images. By the construction of $\boldsymbol{S}$, $\mathbb{V}^{\star}$ is closed under subalgebras. If $h$ is a homomorphism from $(\boldsymbol{A}, \delta)$ onto a KDL-algebra $(\boldsymbol{U}, \delta')$, then $\mathbf{S}(h)$ is a homomorphism from $\boldsymbol{S}(\boldsymbol{A}, \delta)$ onto $\boldsymbol{S}(\boldsymbol{U}, \delta') \in \mathbb{V}$. Hence $(\boldsymbol{U}, \delta') \in \mathbb{V}^{\star}$. Therefore we have shown that $\mathbb{V}^{\star}$ is closed under subalgebras and homomorphic images.

Suppose now that $((\boldsymbol{A}_\lambda, \delta_\lambda) : \lambda \in \Lambda)$ is a family of algebras in $\mathbb{V}^{\star}$, and let $(\boldsymbol{A}, \delta) = \prod_{\lambda \in \Lambda} (\boldsymbol{A}_\lambda, \delta_\lambda)$. Since in $(\boldsymbol{A}, \delta)$, $\delta(x)(\lambda) = \delta_\lambda(x)$, the elements of $\boldsymbol{S}(\boldsymbol{A}, \delta)$ are pairs $\langle f, i \rangle$ with $i \in \{0, 1\}$ and

$$f \colon \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda \colon \lambda \mapsto \begin{cases} f(\lambda) \in A_\lambda & \text{if } i = 1, \\ f(\lambda) \in \delta[A_\lambda] & \text{if } i = 0. \end{cases}$$

Let $f_i^{\star} \colon \lambda \to \bigcup_{\lambda \in \Lambda} S(\boldsymbol{A}_\lambda, \delta_\lambda)$ be defined by the prescription $f_i^{\star}(\lambda) = \langle f(\lambda), i \rangle$. Then the correspondence $\langle f, i \rangle \mapsto f_i^{\star}$ is an embedding from $\boldsymbol{S}(\boldsymbol{A}, \delta)$ into $\prod_{\lambda \in \Lambda} \boldsymbol{S}(\boldsymbol{A}_\lambda, \delta_\lambda) \in \mathbb{V}$. Then $\boldsymbol{S}(\boldsymbol{A}, \delta)$ is in $\mathbb{V}$, and hence $(\boldsymbol{A}, \delta) \in \mathbb{V}^{\star}$. ∎

EXAMPLE 3.10 (BL-algebras). The variety $\mathbb{DBL}$ of DL-algebras given by the hoop equation (12), i. e., the variety of BL-algebras given by equation (20), has been investigated in [9]. From [9, Theorem 3.9] it follows:

*Let $(\boldsymbol{A}, \delta)$ be a KDL-algebra. Then $\boldsymbol{S}(\boldsymbol{A}, \delta)$ is a DBL-algebra if and only if the following hold:*

**(BLa)** *$\boldsymbol{A}$ is a generalized BL-algebra (GBL-algebra), i.e., a GMTL-algebra satisfying the hoop equation (12).*

**(BLb)** *The equations $\delta(x*y) = \delta(x)*\delta(y)$, $x \to \delta(x*y) = \delta(y)$ are satisfied.*

Now we recall from [2] that a *hoop* is an algebra $\boldsymbol{H} = \langle H, *, \to, \top\rangle$ such that $\langle H, *, \top\rangle$ is a commutative monoid that satisfies the following identities:

$\mathbf{HO}_1$ $x \to (y \to z) = (x * y) \to z$,

$\mathbf{HO}_2$ $x \to x = \top$,

$\mathbf{HO}_3$ $x * (x \to y) = y * (y \to x)$.

A hoop is *cancellative* provided it satisfies the identity:

**CHO** $y \to (y * x) = x$.

Each cancellative hoop becomes a GMTL-algebra by defining $x \wedge y = x * (x \to y)$, $x \vee y = ((x \to y) \to y)$ (see [2]). The variety of cancellative hoops will be denoted by $\mathbb{CHO}$. Summarizing the above we can state:

*Let $\boldsymbol{D}$ be a DL-algebra. Then $\boldsymbol{D}$ is a DBL-algebra if and only if $\boldsymbol{D}^\top$ is generalized BL-algebra, and $\boldsymbol{Reg}(\boldsymbol{D})$ is a subalgebra of $\boldsymbol{D}$ belonging to $\mathbb{CHO}$.*

We shall return to this example in Section 5.2.

## 4. Free algebras

In what follows given a variety $\mathbb{W}$ of algebras, we shall denote by $\mathfrak{F}_{\mathbb{W}}(X)$ the $|X|$-free algebra in $\mathbb{W}$, i. e., the free algebra in $\mathbb{W}$ over a set $X$ of free generators of cardinal $|X|$. Moreover, $\mathbb{B}$ will denote the variety of Boolean algebras.

Given a subvariety $\mathbb{V}$ of $\mathbb{DL}$, since by Theorem 3.2 the mapping $x \mapsto \nabla(x)$ is a retract from $\mathfrak{F}_{\mathbb{V}}(X)$ onto $\boldsymbol{B}(\mathfrak{F}_{\mathbb{V}}(X))$, taking into account that $\mathfrak{F}_{\mathbb{V}}(\emptyset) = \{\top, \bot\}$, we immediately obtain that (cf [9, Theorem 5.1]):

THEOREM 4.1. *For each subvariety $\mathbb{V}$ of $\mathbb{DL}$, $\boldsymbol{B}(\mathfrak{F}_{\mathbb{V}}(X))$ is the free Boolean algebra over the set $\nabla X = \{\nabla x : x \in X\}$, and the sets $X$ and $\nabla X$ have the same cardinal. That is $\boldsymbol{B}(\mathfrak{F}_{\mathbb{V}}(X))$ is isomorphic to the $|X|$-free Boolean algebra.* $\square$

Since the Stone space of the free Boolean algebra over $X$ is the Cantor space $2^X$, the ultrafilters of $\mathfrak{F}_{\mathbb{B}}(X)$ are in one-one correspondence with the subsets of $X$. Hence, as in [9, Corollary 5.2], we have:

COROLLARY 4.2. *Let $\mathbb{V}$ be a subvariety of $\mathbb{DL}$. If $X \neq \emptyset$, then the correspondence:*

$$U \mapsto S_U = \{x \in X : \nabla(x) \in U\}$$

*is a bijection from the set of ultrafilters of $\boldsymbol{B}(\mathfrak{F}_X(\mathbb{V}))$ into the power set of $X$. The inverse mapping is given by*

$$S \mapsto U_S = \textit{ultrafilter generated by } \nabla S \cup \{\neg\nabla(x) : x \in X \setminus S\}. \qquad \square$$

By Lemma 1.5, for each $S \subseteq X$, $\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle$ is directly indecomposable, hence by Theorem 3.5, $\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle \cong \boldsymbol{S}(\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle))$. Consequently, by Theorem 1.6, we obtain

THEOREM 4.3. *For each variety $\mathbb{V}$ of DL-algebras and for each set $X$, the free algebra $\mathfrak{F}_{\mathbb{V}}(X)$ is isomorphic to the weak Boolean product of the family $(\boldsymbol{S}(\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)) : S \subseteq X)$ over the Cantor space $2^X$.* $\square$

The remaining of the paper will be devoted to investigate the structure of the KDL-algebras $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle) = \big((\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)^\top, \neg\neg\big)$, where $\mathbb{V}$ is a variety of DL-algebras.

LEMMA 4.4. *For each $S \subseteq X$, the set $(S \cup \neg(X \setminus S))/\langle U_S\rangle$ generates the KDL-algebra $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)$, where for $Y \subseteq X \cup \neg X$, $Y/\langle U_S\rangle = \{y/\langle U_S\rangle : y \in Y\}$.*

PROOF. By the definition of $U_S$, one has $X/\langle U_S\rangle \cap (\mathfrak{F}_{\mathbb{V}}(X))/\langle U_S\rangle)^\top = S/\langle U_S\rangle$. Taking into account that $\nabla\neg x = \neg\nabla x \in U_S$ for each $x \in X \setminus S$, one also has $\neg(X/\langle U_S\rangle) \cap (\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)^\top = \neg(X \setminus S)/\langle U_S\rangle$. Since $X/\langle U_S\rangle$ generates $\mathbf{S}(\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)) = \mathbf{S}\left(\mathfrak{F}_X(\mathbb{V})/\langle U_S\rangle)^\top, \neg\neg\right)$, the result follows from Remark 3.8. ■

By Lemma 2.5, each variety $\mathbb{V}$ of DL-algebras is generated by the DL-chains contained in $\mathbb{V}$. Hence if $\mathbb{V} \neq \mathbb{B}$, $\mathbb{V}$ has to contain DL-chains with more than two elements. Any such chain $\boldsymbol{C} \in \mathbb{V}$ will be called *a test chain for* $\mathbb{V}$. In every test chain $\boldsymbol{C}$, we can find an element $a \in C$ such that $\neg a < a < \top$. Such $a$ will be called *a test element*.

With the notations of Corollary 4.2, for each variety $\mathbb{V}$ of DL-algebras we have:

LEMMA 4.5. *For each $S \subseteq X$ one has:*

(i) *For each $y \in X$, $y/\langle U_S\rangle \neq \top/\langle U_S\rangle$.*

(ii) *For each $y \in S$, $y/\langle U_S\rangle \neq \bot/\langle U_S\rangle$.*

(iii) *If $y, z$ are in $X$ and $y \neq z$, then $y/\langle U_S\rangle = z/\langle U_S\rangle$ implies that $y, z$ are in $X \setminus S$.*

PROOF. Observe that for each $\alpha$ in $\mathfrak{F}_{\mathbb{V}}(X)$, $\alpha \in \langle U_S\rangle$ if and only if there are finite sets $T \subseteq S$ and $W \subseteq X \setminus S$ such that $T \cup W \neq \emptyset$ and

$$\bigwedge_{t\in T} \nabla t \wedge \bigwedge_{w\in W} \neg\nabla w \leq \alpha. \tag{24}$$

Let $\boldsymbol{C}$ be a test algebra in $\mathbb{V}$, and let $a \in C$ be a test element. To prove (i) suppose that $y \in \langle U_S\rangle$ (absurdum hypothesis). Let $T, W$ as in (24), with $\alpha = y$, and let $f\colon X \to C$ be the function defined as follows:

$$f(x) = \begin{cases} a & \text{if } x \in X \setminus W, \\ \perp & \text{if } x \in W. \end{cases} \tag{25}$$

If there were a homomorphism $\widehat{f}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{C}$ extending $f$, then $\widehat{f}$ would assign the value $\top$ to the left member of (24), while $\widehat{f}(y) \in \{a, \perp\}$. Since this contradicts the inequality (24), $f$ cannot be extended to a homomorphism, in contradiction with the definition of free algebra. Hence we conclude that $y \notin \langle U_S\rangle$, and (i) holds. To prove (ii), suppose that $y \in S$ and $\neg y \in \langle U_S\rangle$. Let $T, W$ as in (24) with $\alpha = \neg y$. Since $y \notin W \subseteq X \setminus S$, we can show that the same function $f$ defined by (25) cannot be extended to a homomorphism $\widehat{f}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{C}$, and this proves (ii). To prove (iii), suppose that $y \to z \in \langle U_S\rangle$. Let $T, W$ as in (24), with $\alpha = y \to z$. If $y \notin W$, then the function $f\colon X \to C$ defined as follows:

$$g(x) = \begin{cases} a & \text{if } x \in X \setminus (\{y\} \cup W); \\ \top & \text{if } x = y, \\ \perp & \text{if } x \in W. \end{cases}$$

cannot be extended to a homomorphism $\widehat{g}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \mathbf{C}$. Hence if $(y \to z)$ and $(z \to y)$ are in $\langle U_S\rangle$, then $y$ and $z$ are in $X \setminus S$. ■

Taking into account Lemma 4.4, from the above result we obtain that if for each variety $\mathbb{V}$ of DL-algebras and each $S \subseteq X$, we define

$$\widetilde{S}_{\mathbb{V}} := \{x \in X \setminus S : x/\langle U_S\rangle \neq \perp/\langle U_S\rangle\}, \tag{26}$$

then we have that $(S \cup \neg\widetilde{S}_{\mathbb{V}})/\langle U_S\rangle$ generates $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)$.

Let $\mathbb{V}$ be a variety of pseudocomplemented MTL-algebras. It follows from the definition of $\langle U_S\rangle$ that if $x \in X \setminus S$, then $\neg x = \neg\neg\neg x = \neg\nabla x \in U_S$, i. e., $x/\langle U_S\rangle = \perp/\langle U_S\rangle$. Hence $\widetilde{S}_{\mathbb{V}} = \emptyset$.

THEOREM 4.6. *For each variety $\mathbb{V}$ of pseudocomplemented MTL-algebras, and for each set $X$, if $S \subseteq X$, then*

$$\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle) \cong \mathfrak{F}_{\mathbb{V}^\star}(S/\langle U_S\rangle)$$

*and the sets $S$ and $S/\langle U_S\rangle$ have the same cardinal.*

PROOF. Given $(\boldsymbol{A}, \delta) \in \mathbb{V}^\star$ and a function $f\colon S/\langle U_S\rangle \to A$, let $\bar{f} : X \to \boldsymbol{S}(\boldsymbol{A}, \delta)$ be defined by the prescription:

$$\bar{f}(x) = \begin{cases} \langle f(x/\langle U_S\rangle), 1\rangle & \text{if } x \in S, \\ \langle \top, 0\rangle & \text{if } x \in X \setminus S. \end{cases}$$

Since $\boldsymbol{S}(\boldsymbol{A}, \delta) \in \mathbb{V}$, there is a unique homomorphism $\bar{g}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{S}(\boldsymbol{A}, \delta)$ which extends $\bar{f}$. Since $\{\nabla x : x \in S\} \cup \{\neg\nabla x : x \in X \setminus S\} \subseteq \ker \bar{g} = \bar{g}^{-1}(\{\top\})$, it follows that $\langle U_S\rangle \subseteq \ker(\bar{g})$. Therefore the correspondence $\alpha/\langle U_S\rangle \mapsto \bar{g}(\alpha)$ gives a homomorphism $h\colon \mathfrak{F}_X(\mathbb{V})/\langle U_S\rangle \to \boldsymbol{S}(\boldsymbol{A}, \delta)$, and $\boldsymbol{P}(h)$ gives a homomorphism from $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)$ into $\boldsymbol{A}$ that extends $f$. Since $\widetilde{S}_{\mathbb{V}} = \emptyset$, $S/\langle U_S\rangle$ generates $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)$. Therefore $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle)$ is the free algebra in $\mathbb{V}^\star$ over $S/\langle U_S\rangle$, and it follows from Lemma 4.5 that $S$ and $S/\langle U_S\rangle$ have the same cardinal. ■

To simplify the notation, given a GMTL-algebra $\boldsymbol{A}$, in what follows we shall write $\boldsymbol{\sigma}(\boldsymbol{A})$ to represent the pseudocomplemented MTL-algebra $\boldsymbol{S}(\boldsymbol{A}, \delta_\top)$ (see Corollary 2.4). Moreover, $|S|$ will denote the cardinal of a set $S$. With these notations, and having into account that free algebras in a variety are isomorphic if and only if their sets of free generators have the same cardinal, from Theorems 4.3 and 4.6, we have:

COROLLARY 4.7. *For each set $X$, and for each nontrivial variety $\mathbb{V}$ of pseudocomplemented MTL-algebras the free algebra $\mathfrak{F}_{\mathbb{V}}(X)$ is isomorphic to the weak Boolean product of the family $(\boldsymbol{\sigma}(\mathfrak{F}_{\mathbb{V}^*}(|S|)) : S \in 2^X)$ over the Cantor space $2^X$.* □

Since weak Boolean products over a finite set coincide with direct products, we have:

COROLLARY 4.8. *For each variety $\mathbb{V}$ of pseudocomplemented MTL-algebras, and for each finite cardinal $k \geq 1$, $\mathfrak{F}_{\mathbb{V}}(k) = \prod_{r \leq k} \boldsymbol{\sigma}(\mathfrak{F}_{\mathbb{V}^\star}(r))^{\binom{k}{r}}$.* □

LEMMA 4.9. *Let $\mathbb{V}$ be a subvariety of $\mathbb{DL}$. If $\mathbb{V}$ is not a variety of pseudocomplemented MTL-algebras, then for each $S \subseteq X$, $\widetilde{S}_{\mathbb{V}} = X \setminus S$, and the sets $X$ and $(S \cup \neg\widetilde{S})/\langle U_S\rangle$ have the same cardinal.*

PROOF. Suppose that $y \in X \setminus S$ and that $y/\langle U_S\rangle = \bot/\langle U_S\rangle$. Since $\mathbb{V}$ is not a variety of pseudocomplemented MTL-algebras, there is a non pseudocomplemented test chain $\boldsymbol{C} \in \mathbb{V}$, and a test element $a \in C$ such that

$\bot < \neg a < a \leq \neg\neg a < \top$. Let $T, W$ be as in (24) of the proof of Lemma 4.5 with $\alpha = \neg y$, and define $f\colon X \to C$ by the prescription

$$f(x) = \begin{cases} a & \text{if } x \in X \setminus W, \\ \neg a & \text{if } x \in W. \end{cases}$$

If there were a homomorphism $\widehat{f}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{C}$ extending $f$, then $\widehat{f}$ would assign the value $\top$ to the left member of (24), while $\widehat{f}(\neg y) \in \{a, \neg a\}$, contradicting the inequality (24). Hence $f$ cannot be extended to a homomorphism, in contradiction with the definition of free algebra. Therefore $\widetilde{S}_{\mathbb{V}} = X \setminus S$. In the proof of (iii) in Lemma 4.5 we have shown that for $y, z$ in $X$, $y \neq z$, if $y \to z \in \langle U_S \rangle$, then $y \in X \setminus S$. Let us see now that $z \in S$. Indeed, let $T, W$ be as in (24), with $\alpha = y \to z$, and let $a$ be the same test element as before. If $z \notin T$, then the function $g\colon X \to C$ defined as follows

$$g(x) = \begin{cases} a & \text{if } x \in T, \\ \bot & \text{if } x = z, \\ \neg a & \text{if } x \in X \setminus (T \cup \{z\}). \end{cases}$$

cannot be extended to a homomorphism $\widehat{g}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \mathbf{C}$. Hence $y \to z \in \langle U_S \rangle$ implies that $z \in T \subseteq S$. Consequently, $y/\langle U_S \rangle = z/\langle U_S \rangle$ would imply that $y, z$ are simultaneously in $S$ and $X \setminus S$, absurdum. Similar arguments show that $y, z$ in $S$ and $y \neq z$ imply that $\neg y/\langle U_S \rangle \neq \neg z/\langle U_S \rangle$ and $y/\langle U_S \rangle \neq \neg z/\langle U_S \rangle$. Hence $|X| = |(S \cup \neg\widetilde{S})/\langle U_S \rangle|$. ∎

Remark 4.10. In the proof of [9, Lemma 5.6] it is erroneously stated that for each variety of BL-algebras with the Boolean retraction property and for each $S \subseteq X$, $S/\langle U_S \rangle$ generates $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$. Hence the results of §5 of that paper apply only to varieties of pseudocomplemented BL-algebras. The results corresponding to varieties of MV-algebras should be replaced by those given in Section 5.2 of the present paper.

Theorem 4.11. *For each variety $\mathbb{V}$ of involutive DL-algebras, and for each set $X$, if $S \subseteq X$, then*

$$\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle) \cong \mathfrak{F}_{\mathbb{V}^\star}((S \cup \neg\widetilde{S}_{\mathbb{V}})/\langle U_S \rangle)$$

*and the sets $X$ and $(S \cup \neg\widetilde{S_{\mathbb{V}}})/\langle U_S \rangle$ have the same cardinal.*

Proof. Let $(\boldsymbol{A}, \delta) \in \mathbb{V}^\star$. Since $\delta = id_A$, given a function $f\colon (S/\langle U_S \rangle) \to A$, we can define a function $\bar{f}\colon \to X \to S(\boldsymbol{A}, \delta)$ by the prescription:

$$\bar{f}(x) = \begin{cases} \langle f(x/\langle U_S \rangle), 1 \rangle & \text{if } x \in S, \\ \langle f(\neg x/\langle U_S \rangle), 0 \rangle & \text{if } x \in \widetilde{S}_{\mathbb{V}} = X \setminus S. \end{cases}$$

Since $\boldsymbol{S}(\boldsymbol{A}, \delta) \in \mathbb{V}$, there is a unique homomorphism $\bar{g}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{S}(\boldsymbol{A}, \delta)$ which extends $\bar{f}$. Since $\{\nabla x : x \in S\} \cup \{\neg\nabla x : x \in X \setminus S\} \subseteq \ker \bar{g} =$

$\bar{g}^{-1}(\{\top\})$, it follows that $\langle U_S \rangle \subseteq \ker(\bar{g})$. Therefore the correspondence $\alpha/\langle U_S \rangle \mapsto \bar{g}(\alpha)$ gives a homomorphism $h\colon \mathfrak{F}_X(\mathbb{V})/\langle U_S \rangle \to \boldsymbol{S}(\boldsymbol{A}, \delta)$, and $\boldsymbol{P}(h)$ gives a homomorphism from $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$ into $\boldsymbol{A}$ that extends $f$. Since $(S \cup \widetilde{S}_{\mathbb{V}})/\langle U_S \rangle$ generates $\boldsymbol{P}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$, it is the free algebra in $\mathbb{V}^\star$ over $(S \cup \widetilde{S}_{\mathbb{V}})/\langle U_S \rangle$. Finally, it follows from Lemma 4.9 that the sets $X$ and $(S \cup \widetilde{S}_{\mathbb{V}})/\langle U_S \rangle$ have the same cardinal. ■

If for each GMTL-algebra $\boldsymbol{A}$, $\boldsymbol{\iota}(\boldsymbol{A})$ denotes the involutive DL-algebra $\boldsymbol{S}(\boldsymbol{A}, id_{\boldsymbol{A}})$, then we have:

Corollary 4.12. *For each non trivial variety $\mathbb{V}$ of involutive DL-algebras and for each cardinal $\kappa \geq 1$, the free algebra $\mathfrak{F}_{\mathbb{V}}(\kappa)$ is isomorphic to the weak Boolean power of the family $(\boldsymbol{\iota}(\mathfrak{F}_{\mathbb{V}^\star}(\kappa)) : S \in 2^\kappa)$ over the Cantor space $2^\kappa$.* □

Corollary 4.13. *For each variety $\mathbb{V}$ of involutive DL-algebras and for each finite cardinal $k \geq 1$, $\mathfrak{F}_{\mathbb{V}}(k) \cong (\boldsymbol{\iota}(\mathfrak{F}_{\mathbb{V}^\star}(k))^{2^k}$.* □

## 5. Examples

In this section we shall apply the results of the previous sections to describe the free algebras in varieties of DL-algebras obtained by applying the construction given in Theorem 2.2 to Gödel algebras and to cancellative hoops.

### 5.1. From Gödel algebras

Recall that $\mathbb{GG}$ represents the variety of generalized Gödel algebras and $\mathbb{G}$ the variety of Gödel algebras.

It is well known that each totally ordered set $\langle C, \leq \rangle$ with greatest element $\top$ admits a unique generalized Heyting algebra structure, which is given by the operations $x * y = \min(x, y)$, and $x \to y = \begin{cases} \top & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$. The chain $C$ endowed this structure will be denoted by $\boldsymbol{C}$. If $C$ is infinite, then $\boldsymbol{C}$ generates the variety $\mathbb{GG}$, and $\boldsymbol{\sigma}(\boldsymbol{C})$ generates the variety $\mathbb{G}$.

Since generalized Gödel algebras satisfy the hoop equation (12), they satisfy condition (**BLa**) of Example 3.10, and since the constant function $\delta_\top$ trivially satisfies condition (**BLb**), we have

Lemma 5.1. *Let $\boldsymbol{A}$ be a GMTL-algebra. Then $\boldsymbol{\sigma}(\boldsymbol{A})$ is a Gödel algebra if and only if $\boldsymbol{A}$ is a generalized Gödel algebra.* □

Note that from the above lemma we deduce that the categories of generalized Gödel algebras and of directly indecomposable Gödel algebras are equivalent.

For each integer $n \geq 1$, let $C_n = [1, n] \cap Z$, where $Z$ denotes the set of integers. $\mathbb{GG}_n$ will denote the subvariety of $\mathbb{GG}$ generated $\boldsymbol{C_n}$. For each $n \geq 1$, consider the equation

$$(\mathbf{E}_n) \qquad \bigwedge_{i<n} ((x_i \to x_{i+1}) \to x_{i+1}) \to \bigvee_{i<n+1} x_i = \top.$$

It is well known (see, for instance [13]) that $\mathbb{GG}_n$ is the subvariety of $\mathbb{GG}$ characterized by $(\mathbf{E}_n)$. Observe, that for each $n > 1$, $\boldsymbol{C}_n$ is a Gödel chain by taking $1 = \bot$, and $\boldsymbol{\sigma}(\boldsymbol{C}_{n-1}) \cong \boldsymbol{C}_n$, i.e., $\boldsymbol{C}_n^\top = \boldsymbol{C}_{n-1}$. Then if $\mathbb{G}_n$ represents the subvariety of $\mathbb{G}$, generated by the Gödel chain of cardinal $n > 1$,we have

THEOREM 5.2. *$\mathbb{G}^\star = \mathbb{GG}$, and for each integer $k \geq 1$, $\mathbb{G}^\star_{k+1} = \mathbb{GG}_k$.* □

The next result follows from the above theorem and Corollary 4.7.

THEOREM 5.3. *For each set $X$, $\mathfrak{F}_{\mathbb{G}}(X)$ is isomorphic to the weak Boolean product of the family $(\boldsymbol{\sigma}(\mathfrak{F}_{\mathbb{GG}}(S)) : S \in 2^X)$ over the Cantor space $2^X$.* □

The next corollary is a consequence of Corollary 4.8 and Theorem 5.2. The same description of $\mathfrak{F}_{\mathbb{G}_n}(k)$ for $n \geq 2$ and $k$ a finite cardinal was obtained by Davey [13, Theorem 5.5] by a different method.

COROLLARY 5.4. *For each finite cardinal $k \geq 1$, one has*

(i) $\mathfrak{F}_{\mathbb{G}_2}(k) = \boldsymbol{\sigma}(\boldsymbol{C_1})^{2^k}$.

(ii) *For each $n \geq 2$,* $\mathfrak{F}_{\mathbb{G}_{n+1}}(k) = \prod_{r \leq k} \boldsymbol{\sigma}(\mathfrak{F}_{\mathbb{GG}_n}(r))^{\binom{k}{r}}$. □

REMARK 5.5. To obtain a full description of finitely generated free algebras in $\mathbb{G}_n$, we can use the following characterization of finitely generated free algebras in $\mathbb{GG}_n$ that appears in [13, Theorem 5.3]:

*(i)* $\mathfrak{F}_{\mathbb{GG}_2}(k) = (\boldsymbol{C}_2)^{2^k - 1}$, *for all finite cardinals $k$.*

*(ii) For $n \geq 3$, $\mathfrak{F}_{\mathbb{GG}}(0) = \boldsymbol{C}_1$, and for a finite $k \geq 1$,*

$$\mathfrak{F}_{\mathbb{GG}_n}(k) = \prod_{s=0}^{k-1} \boldsymbol{\sigma}^*(\mathfrak{F}_{\mathbb{GG}_{n-1}}(s))^{\binom{k}{s}},$$

*where $\boldsymbol{\sigma}^*(\boldsymbol{A})$ denotes the $\{\to, \vee, \wedge, \top\}$-reduct of $\boldsymbol{\sigma}(\boldsymbol{A})$.*

Our next aim is to analyse the class of involutive DL-algebras obtained from generalized Gödel algebras. We start by recalling that a *nilpotent minimum algebra* (*NM-algebra* for short) is an IMTL-algebra $\boldsymbol{A}$ that satisfies the nilpotent minimum equation (21) of Example 2.9. We will represent by $\mathbb{DLN}$ the variety of DL-algebras that satisfy the nilpotent minimum equation (21), i. e., the variety of nilpotent minimum algebras generated by the algebra $[\mathbf{0}, \mathbf{1}]^-$ (see Example 2.9).

LEMMA 5.6. *Let $\boldsymbol{A}$ be a Generalized MTL-algebra. Then $\boldsymbol{\iota}(\boldsymbol{A})$ is a NM-algebra if and only if $\boldsymbol{A}$ is a generalized Gödel algebra.*

Proof. Observe for each $a, b \in A$,

$$(((\langle a,1\rangle \odot \langle b,1\rangle) \Rightarrow \mathbf{0}) \sqcup (((\langle a,1\rangle \sqcap \langle b,1\rangle) \Rightarrow (\langle a,1\rangle \odot \langle b,1\rangle)) = \mathbf{1}$$

holds if and only if $(a \wedge b) \to (a * b) = \top$, i. e., if and only if $\boldsymbol{A}$ is a generalized Heyting algebra. Consequently, if $\boldsymbol{\iota}(\boldsymbol{A})$ is a NM-algebra, then $\boldsymbol{A}$, is a generalized Gödel algebra. Suppose now that $\boldsymbol{A}$ is a generalized Gödel algebra, then $\boldsymbol{\iota}(\boldsymbol{A})$ is a involutive DL-algebra and for any $a, b \in A$,

$$(((\langle a,i\rangle \odot \langle b,j\rangle) \Rightarrow \mathbf{0}) \sqcup (((\langle a,i\rangle \sqcap \langle b,j\rangle) \Rightarrow (\langle a,i\rangle \odot \langle b,j\rangle)) = \mathbf{1}$$

holds when $i = j = 1$, and when $i = j = 0$. When $i < j$, it holds if and only it satisfies $(b \to a) \vee ((b \to a) \to a) = \top$. Since this last equality holds for any $a, b$ in any generalized Heyting chain, it holds in all generalized Gödel algebras. Hence the nilpotent minimum equation (21) holds in $\boldsymbol{\iota}(\boldsymbol{A})$. ■

Remark 5.7. It follows from the above lemma that the restrictions of the functors $\boldsymbol{S}$ $(\boldsymbol{\iota})$ and $\boldsymbol{P}$ define an equivalence between the categories of directly indecomposable DL-algebras satisfying (21) and of generalized linear Heyting algebras.

When $\boldsymbol{A}$ is the semi-closed real interval $(\frac{1}{2}, 1]$ endowed with is natural structure of generalized Heyting algebra, then the NM-algebra $\boldsymbol{\iota}(\boldsymbol{A})$ coincide with the algebra $[\mathbf{0}, \mathbf{1}]^{-}$, that Gispert [18] showed generates the variety $\mathbb{DLN}$. For each integer $n \geq 1$, $\mathbb{DLN}_n$ will represent the subvariety of $\mathbb{DLN}$ generated by the NM-algebra $\boldsymbol{\iota}(\boldsymbol{C_n})$. The algebras $\boldsymbol{\iota}(\boldsymbol{C_n})$ coincide with the algebras $\boldsymbol{A_{2n}}$ defined by Gispert in [18], who proved that $\mathbb{DLN}_n \subseteq \mathbb{DLN}_{n+1}$, and that $\bigcup_{n \geq 1} \mathbb{DLN}_n = \mathbb{DLN}$. It follows from [18, Theorem 3], that equation $(\mathbf{E}_n)$ also characterizes $\mathbb{DLN}_n$ as a subvariety of $\mathbb{DLN}$.

Theorem 5.8. *$\mathbb{DLN}^\star = \mathbb{GG}$, and for each integer $n \geq 1$, $\mathbb{DLN}_n^\star = \mathbb{GG}_n$.*

Proof. Let $\mathbb{V}$ be a subvariety of $\mathbb{DLN}$. By Theorem 3.9 and Lemma 5.6 one has that $\mathbb{V}^\star$ is a variety of generalized Gödel algebras. Since for any generalized Gödel chain $\boldsymbol{C}$, $\boldsymbol{\iota}(\boldsymbol{C}) \in \mathbb{DLN}$, all generalized Gödel chains are in $\mathbb{DLN}^\star$, hence $\mathbb{DLN}^\star = \mathbb{GG}$. Let $n \geq 1$ and let $\boldsymbol{C}$ be a generalized Gödel chain. Since equation $(\mathrm{E}_n)$ holds in $\boldsymbol{\iota}(\boldsymbol{C})$ if and only if it holds in $\boldsymbol{\iota}(\boldsymbol{C})^\top \cong \boldsymbol{C}$, it follows that $\boldsymbol{C} \in \mathbb{GLH}_n^\star$ if and only if $\boldsymbol{C} \in \mathbb{GLH}_n$. ■

Theorem 5.9. *For each cardinal $\kappa \geq 1$, the free algebra $\mathfrak{F}_{\mathbb{G}}(\kappa)$ is isomorphic to the weak Boolean power of the family $(\boldsymbol{\iota}(\mathfrak{F}_{\mathbb{GG}}(\kappa)) : S \in 2^\kappa)$ over the Cantor space $2^\kappa$.* □

Combining Corollary 4.13, Theorem 5.8 and [13, Theorem 5.3] (see Remark 5.5), we obtain:

Corollary 5.10. *For each finite cardinal $k$ one has:*

(i) $\mathfrak{F}_{\mathbb{DLN}_1}(k) = \boldsymbol{\iota}(\boldsymbol{C_1})^{2^k}$.

(ii) $\mathfrak{F}_{\mathbb{DLN}_2}(k) = \boldsymbol{\iota}(\boldsymbol{C_2}^{2^k-1})^{2^k}$.

(iii) *For each $n \geq 3$,*

$$\mathfrak{F}_{\mathbb{DLN}_n}(k) = (\boldsymbol{\iota}(\mathfrak{F}_{\mathbb{GG}_n}(k))^{2^k} = \left(\boldsymbol{\iota}\left(\prod_{s=0}^{k-1} \boldsymbol{\sigma}^*(\mathfrak{F}_{\mathbb{GG}_{n-1}}(s)\right)^{\binom{k}{s}}\right)^{2^k}.$$

Notice that since $\boldsymbol{\sigma}(\boldsymbol{C_1})$ and $\boldsymbol{\iota}(\boldsymbol{C_1})$ are, up isomorphisms, the two-element Boolean algebra, (i) of corollaries 5.4 and 5.10, give the well known structure of Boolean algebras with $k$ free generators.

### 5.2. From Cancellative Hoops

We recall from example 3.10, that in a each cancellative hoop $\boldsymbol{A} = \langle A, *, \rightarrow, \top\rangle$ hoop becomes a residuated lattice with the meet and join defined as follows $x \wedge y = x * (x \rightarrow y)$, $x \vee y = ((x \rightarrow y) \rightarrow y)$. From [9, Theorem 3.9] (see example 3.10) we deduce that for all $\boldsymbol{A} \in \mathbb{CHO}$, $\boldsymbol{\sigma}(\boldsymbol{A})$ and $\boldsymbol{\iota}(\boldsymbol{A})$ are BL-algebras. Moreover, it is a routine to proof (cf. [22, Theorem 6]):

Theorem 5.11. *Let $\boldsymbol{A}$ be a MTL-algebra, then the following are equivalent:*

(i) *$\boldsymbol{A}$ is a cancellative hoop*

(ii) *$\boldsymbol{\sigma}(\boldsymbol{A})$ is a product algebra, that is, a Pseudocomplemented BL-algebra (= BL-algebra satisfying* (11)*) in which the following equation holds:*

$$(\neg\neg x * (x \rightarrow (x * y)) \rightarrow y = \top \tag{27}$$

(iii) *$\boldsymbol{\iota}(\boldsymbol{A})$ is a MV-algebra*

In what follows, $\mathbb{PL}$ represents the class of product algebras and $\mathbb{CMV}$ represents class MV-algebras that satisfies the equation (20). It is well known that $\mathbb{B}$ is the only nontrivial proper subvariety of $\mathbb{PL}$ and of $\mathbb{CMV}$. Then we have

Corollary 5.12. *If $\mathbb{K} \in \{\mathbb{BL}, \mathbb{CMV}\}$, then $\mathbb{K}^\star = \mathbb{CHO}$.* □

Let $\boldsymbol{G}$ be a lattice ordered abelian group, $\ell$-group for short. The negative cone $G^- := \{x \in G : x \leq 0\}$ is a cancellative hoop $\boldsymbol{G}^-$ under the operations $x*y = x+y$, $x \rightarrow y = (y-x)\wedge 0$, and $\top = 0$. Negative cones of $\ell$-groups are the most general examples of cancellative hoops. Indeed, every cancellative hoop is isomorphic to the negative cone of an $\ell$-group, that is unique up to isomorphisms (see [2]). It is also known that if $\mathbf{Z}$ denote the additive group of the integers with its usual order, then $\boldsymbol{\sigma}(\mathbf{Z}^-)$ generates the variety $\mathbb{PL}$ (see [8]), and $\boldsymbol{\iota}(\mathbf{Z}^-)$ coincides with the MV-algebra $\boldsymbol{S}_2^\omega$ that, as shown in [14], generates the variety $\mathbb{CMV}$.

Taking into account Corollaries 4.7 and 4.8, we obtain (cf [8]):

Corollary 5.13.

1. *The free algebra* $\mathfrak{F}_{\mathbb{PL}}(X)$ *is isomorphic to the weak Boolean product of the family* $(\boldsymbol{\sigma}(\mathfrak{F}_{\mathbb{CHO}}(S)) : S \in 2^X)$ *over the Cantor space* $2^X$.
2. *for each finite cardinal* $k \leq 1$ $\mathfrak{F}_{\mathbb{PL}}(k) \cong \prod_{r \leq k} \boldsymbol{\sigma}((\mathfrak{F}_{\mathbb{CHO}}(r))^{\binom{k}{r}}$. □

Taking into account Corollaries 4.12 and 4.13, we obtain

Corollary 5.14. *For each cardinal* $\kappa \geq 1$,

1. *The free algebra* $\mathfrak{F}_{\mathbb{CMV}}(\kappa)$ *is isomorphic to the weak Boolean power of the family* $(\boldsymbol{\iota}(\mathfrak{F}_{\mathbb{CHO}}(\kappa)) : S \in 2^{\kappa})$ *over the Cantor space* $2^{\kappa}$.
2. *If* $\kappa$ *is finite cardinal, then* $\mathfrak{F}_{\mathbb{CMV}}(\kappa) \cong (\boldsymbol{\iota}(\mathfrak{F}_{\mathbb{CHO}}(\kappa)))^{2^{\kappa}}$ . □

By dualizing the results of [7], we obtain a description of free cancellative hoops with $\kappa$ free generators in terms of piecewise linear functions from $(Z^-)^{\kappa}$ into $Z^-$. In particular, $\mathfrak{F}_{\mathbb{CHO}}(1) = \mathbf{Z}^-$. Hence $\mathfrak{F}_{\mathbb{PL}}(1) = \boldsymbol{C_2} \times \boldsymbol{\sigma}(\mathbf{Z}^-)$ and $\mathfrak{F}_{\mathbb{CMV}}(1) = \boldsymbol{\sigma}(\mathbf{Z}^-)^2$.

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