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T-structures on the Bounded Derived Category of the Kronecker Algebra

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Abstract We study suspended subcategories of the bounded derived category $\mathbf{D^b}(\text{mod}\, H)$, where H is a tame hereditary k-algebra. First, we consider $\mathcal{U}_M$ the smallest suspended subcategory containing M, where M is a brick. We give necessary and sufficient conditions for $\mathcal{U}_M$ to be an aisle and we show that it occurs when M is a silting object. Then we concentrate on the case that H is the path algebra of the Kronecker quiver. In that context, we classify all the suspended subcategories having Ext-projective objects. We prove that these are aisles and we give their description. Finally, we determine which suspended subcategories generated by an object are aisles and we describe them.

Keywords T-structure · Bounded derived category · Hereditary k-algebra

Mathematics Subject Classifications (2010) 18E30 · 18E40 · 18G25

1 Introduction

The concept of t-structure on a triangulated category $\mathcal{T}$ has been introduced in the work [4] at the beginning of the eighties. One of the reasons why the study of t-structures is interesting is that a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ gives a pair of functors. These functors resemble the usual truncation functors in derived categories and provide an homological functor in $\mathcal{T}$ which takes values in an abelian full subcategory of $\mathcal{T}$.

The subcategories $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ are generally not triangulated subcategories, they are usually right or left triangulated. In fact, each subcategory $\mathcal{T}^{\leq 0}$ has the

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structure of suspended category, in the sense of Keller and Vossieck (see [10]). The main difference from triangulated categories is that the shift functor in a suspended category may not have an inverse, and therefore some objects cannot be shifted back.

In [11], they studied suspended subcategories of a triangulated category and found necessary and sufficient conditions for the inclusion of a suspended subcategory to have a right adjoint functor. Suspended subcategories satisfying these conditions are called aisles in $\mathcal{T}$. They also proved that there is a bijective correspondence between aisles and t-structures on $\mathcal{T}$. Moreover, if the triangulated category is Krull-Schmidt, they characterize the aisles as the contravariantly finite suspended subcategories of $\mathcal{T}$.

The motivation for our work comes from the representation theory of finite dimensional algebras. In this context, *t*-structures are useful due to their relationship with tilting theory. Happel's interpretation of tilting theory involving derived categories provides an important theoretical development of this subject.

A tilting A-module (and, in general, a tilting complex) induces an equivalence between $\mathbf{D^b}(\text{mod}\,A)$ and $\mathbf{D^b}(\text{mod}\,\text{End}(T))$, the bounded derived categories of finitely generated modules over the algebras A and End(T) (see [8] and [12]). In the case that the algebra A has finite global dimension, there exists an equivalence between the natural aisle in $\mathbf{D^b}(\text{mod}\,A)$ and its essential image $\mathcal{U}_T$, the smallest suspended subcategory containing T, The construction of a *t*-structure, independently of the existence of an equivalence of derived categories, is an important question. In this direction, some authors studied suspended subcategories generated by an object and looked for sufficient conditions that imply the existence of a t-structure (see [7–9, 11]).

In [1], the authors have proved that any complex generates an aisle in the unbounded derived category $\mathbf{D}(\text{Mod}\,A)$. To decide whether $\mathcal{U}_X$, the smallest suspended subcategory containing a perfect complex X, is an aisle in $\mathbf{D}^b(\text{mod}\,A)$ is still an open problem. We are going to show that, in general, it is not true that every perfect complex generates an aisle in the bounded derived category of the finitely generated modules. Our first theorem will show that, even in the hereditary case, it is not true that every complex generates an aisle in $\mathbf{D^b}(\text{mod}\,H)$.

Another natural question is whether an arbitrary aisle in $\mathbf{D}^b(\text{mod}\,A)$ is generated by an object or it is not. We are going to show that not every aisle is determined by an object even in the case that we are working in the hereditary context. We will construct aisles in $\mathbf{D^b}(\text{mod}\,H)$ which are not generated by an object.

Both problems were first discussed in [11] in the context of the bounded derived category of the path algebra of a Dynkin quiver. There, they introduced the concept of "silting complex" which generalizes the notion of tilting complex.

Recently, in [3] the authors have studied aisles in the bounded derived category $\mathbf{D^b}(\mathcal{H})$ where $\mathcal{H}$ is an hereditary abelian category. They proved that the number of indecomposable Ext-projective objects in a suspended subcategory of $\mathbf{D^b}(\mathcal{H})$ is bounded by the rank of the Grothendieck group $K_0(\mathcal{H})$ (see Theorem 2.3 in [3]). They also showed that silting complexes generate aisles in $\mathbf{D^b}(\mathcal{H})$. Moreover, they proved that aisles having exactly $s = \text{r}k K_0(\mathcal{H})$ isomorphism classes of indecomposable Ext-projective objects are in bijective correspondence with complete silting complexes (i.e, silting complexes having s isomorphism classes of indecomposable summands).

In this paper, we consider the case that $\mathcal{H}$ is the category of finitely generated modules over a tame hereditary finite dimensional k-algebra (we assume that k is an algebraically closed field).

We recall that an object X in a suspended subcategory $\mathcal{U}$ is called *Ext-projective* in $\mathcal{U}$ if and only if $\mathrm{Hom}_{\mathcal{T}}(X, Y[1]) = 0$ for all $Y \in \mathcal{U}$. An object M in a triangulated category $\mathcal{T}$ is called a *silting object* if $\mathrm{Hom}_{\mathcal{T}}(M, M[l]) = 0$ for all $l > 0$. Given a full subcategory $\mathcal{U}$ of $\mathcal{T}$, the right orthogonal of $\mathcal{U}$, $\mathcal{U}^{\perp}$ consists of the objects $Y \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for every $X \in \mathcal{U}$.

We are in a position to state now our first result:

Theorem 1.1 *Let H be a tame hereditary finite dimensional k-algebra and $X = M[t]$ be an object in $\mathbf{D^b}(\mathrm{mod}\, H)$ $(t \in \mathbb{Z})$ such that $\mathrm{End}_{\mathbf{D^b}(\mathrm{mod}\, H)}(X) = k$. The following statements are equivalent:*

(1) *$\mathcal{U}_X$ is an aisle in $\mathbf{D^b}(\mathrm{mod}\, H)$.*
(2) *X is a silting complex.*
(3) *There exists $X^n \to D(H)[t] \to L \to X^n[1]$ a triangle in $\mathbf{D^b}(\mathrm{mod}\, H)$ with $L \in \mathcal{U}_X^{\perp}$ and $n \geq 1$.*
(4) *X is an Ext-projective object in $\mathcal{U}_X$.*

Moreover, in this case, $\mathrm{ind}\, \mathcal{U}_X = \{X[i],\ i \geq 0\}$.

This result shows that it is not true that any complex determines an aisle in $\mathbf{D^b}(\mathrm{mod}\, H)$. For example, in the case of hereditary algebras of Euclidean type, the simple homogeneous modules are not silting complexes in the bounded derived category. That is the first difference with the case of the bounded derived category of hereditary algebra of Dynkin type, which was considered in [10], where all the indecomposable objects are silting complexes.

Next, we focus our attention on the bounded derived category $\mathbf{D^b}(\mathrm{mod}\, H)$ where $H = kQ$ is the path algebra of the Kronecker quiver. This algebra is useful example of an hereditary algebra of infinite representation type. In this case, we give the classification of all the suspended subcategories having Ext-projective objects in $\mathbf{D^b}(\mathrm{mod}\, kQ)$. We also show that any suspended subcategory having Ext-projectives objects is an aisle in $\mathbf{D^b}(\mathrm{mod}\, kQ)$. Furthermore, we describe all these suspended subcategories (see Theorem 3.3).

We denote by $\mathcal{L}_X$ the smallest triangulated subcategory containing a given object X. Recall from [4] that, if $\mathcal{U}, \mathcal{V}$ are suspended subcategories, then $\mathcal{W} = \mathcal{U} * \mathcal{V}$ denotes the full additive subcategory consisting of all the objects Y' such that there exists an object $Y = Y' \oplus Y''$ and a triangle $X \to Y \to Z \to X[1]$ with $X \in \mathcal{U}$ and $Z \in \mathcal{V}$.

Our second theorem is the following:

Theorem 1.2 *Let $\mathcal{U}$ be a suspended subcategory of $\mathbf{D^b}(\mathrm{mod}\, H)$ having Ext-projective objects. Then*

(a) *If $\mathcal{U}$ has only one indecomposable Ext-projective object, then $\mathcal{U}$ is one of the following suspended subcategories:*

   (i) $\mathcal{U} = \mathcal{U}_{E[t]}$
   (ii) $\mathcal{U} = \mathcal{U}_{E[t]} * \mathcal{L}_{E^p}$

where $E[t]$ is the Ext-projective in $\mathcal{U}$ with E an indecomposable non-regular H-module, $t \in \mathbb{Z}$ and E^p is the immediate predecessor of E with respect to the order given by the morphims.

(b) *If $\mathcal{U}$ has 2 indecomposable Ext-projective objects then*

$$\mathcal{U} = \mathcal{U}_{E[t] \oplus E^p[l]}$$

where $\{E[t], E^p[l]\}$ is the set of indecomposable Ext-projective objects in $\mathcal{U}$ with E an indecomposable non-regular H-module, E^p its immediate predecessor with respect to the order given by the morphims and $l \leq t$.

In either case, $\mathcal{U}$ is an aisle.

Finally, we describe all the suspended subcategories generated by an object in $\mathbf{D^b}(\text{mod}\, H)$. We decide which of them are aisles. We say that we have a *splitting* aisle $\mathcal{U}$ if every indecomposable object either belongs to $\mathcal{U}$ or to $\mathcal{U}^\perp$. We denote by $\mathcal{P}$ (respectively, $\mathcal{R}$, $\mathcal{I}$) the set consisting of all indecomposable postprojective (respectively, regular, preinjective) H-modules. Recall that $\mathcal{R}$ is the family of the regular components, that is $\mathcal{R}$ is a disjoint union of tubes $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(k)$ (where $\mathbb{P}_1(k)$ denotes the projective line). We state now our third theorem:

Theorem 1.3 *Let X be a complex in $\mathbf{D^b}(\text{mod}\, H)$. Then $\mathcal{U}_X$ is the shift of one of the suspended subcategories:*

(1) *$\mathcal{U}$ is an aisle with only one Ext-projective object of the form $\mathcal{U} = \mathcal{U}_{E_0}$ where E_0 is an indecomposable non-regular H-module.*
(2) *$\mathcal{U} = \mathcal{U}_T$ is an aisle with 2 Ext-projective objects and T is a tilting module.*
(3) *$\mathcal{U} = \mathcal{U}_{E_0 \oplus E_0^s[j]}$ is an aisle with 2 Ext-projective objects and $E_0 \oplus E_0^s[j]$ is a silting complex where E_0^s is the immediate successor of E_0 and $j > 0$.*
(4) *$\mathcal{U}$ is a splitting aisle without Ext-projective objects and it is of the form*

$$\text{ind}\, \mathcal{U} = \cup_{\lambda \in L} \mathcal{T}_\lambda \cup \mathcal{I} \cup_{n>0} \mathcal{H}[n]$$

where L is a finite set.
(5) *$\mathcal{U}$ is not an aisle and it consists of a disjoint finite union of shifted tubes.*
(6) *$\mathcal{U}$ is such that* ind $\mathcal{U}$ *is the union of all indecomposable objects from the above suspended subcategories.*

The paper is organized as follows: In Section 2 we start with some preliminary results about suspended subcategories, aisles and t-structures on the bounded derived category of an hereditary algebra. At the end of the section, we prove Theorem 1.

In Section 3 we study suspended subcategories of the bounded derived category of a Kronecker algebra having Ext-projective objects. We prove Theorem 2 by classifying these suspended subcategories in terms of the number of indecomposable Ext-projective objects that they have. We also prove that all of them are aisles.

In Section 4 we study suspended subcategories of the bounded derived category of a Kronecker algebra generated by an object. We prove Theorem 3 by classifying all these suspended subcategories and we decide which ones are aisles.

2 Aisles in Derived Hereditary Categories

In this section we start with some preliminary results on t-structures, suspended subcategories and aisles. We study suspended subcategories of the bounded derived category of a tame hereditary algebra. Our first theorem will show that even in the hereditary case it is not true that every complex generates an aisle in $\mathbf{D^b}(\text{mod } H)$. We characterize the indecomposable objects which generate an aisle in $\mathbf{D^b}(\text{mod } H)$. We start by recalling some definitions and notations that we will use in this paper.

2.1 Suspended Subcategories and Aisles

We assume that k is an algebraically closed field. Let $\mathcal{T}$ be a triangulated k-category. Given a full subcategory $\mathcal{U}$ of $\mathcal{T}$, the *right* and the *left orthogonal* of $\mathcal{U}$ are defined by

$$\mathcal{U}^{\perp} = \left\{ Y \in \mathcal{T} | \text{Hom}_{\mathcal{T}}(X, Y) = 0 \text{ for every } X \in \mathcal{U} \right\}$$

$$^{\perp}\mathcal{U} = \left\{ Y \in \mathcal{T} | \text{Hom}_{\mathcal{T}}(Y, X) = 0 \text{ for every } X \in \mathcal{U} \right\}.$$

A full additive subcategory $\mathcal{U}$ of $\mathcal{T}$, closed under direct summands, is *suspended* if it is closed under positive shifts and extensions, that is,

(1) if $X \in \mathcal{U}$, then $X[i] \in \mathcal{U}$, for every $i > 0$, and
(2) if $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{T}$ with $X, Z \in \mathcal{U}$ then $Y \in \mathcal{U}$.

We recall that a *t-structure* on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that

(a) For any X in $\mathcal{T}^{\leq 0}$ and Y in $\mathcal{T}^{\geq 0}[-1]$, we have $\text{Hom}_{\mathcal{T}}(X, Y) = 0$.
(b) $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}[-1]$ and $\mathcal{T}^{\geq 0} \supseteq \mathcal{T}^{\geq 0}[-1]$.
(c) For any Y on $\mathcal{T}$, there exist X in $\mathcal{T}^{\leq 0}$, Z in $\mathcal{T}^{\geq 0}[-1]$ and a triangle

$$X \to Y \to Z \to X[1].$$

A suspended subcategory $\mathcal{U}$ of $\mathcal{T}$ is called an *aisle* if the inclusion functor has a right adjoint. It is shown in [11, (1.1)(1.3)] that the following conditions are equivalent for a suspended subcategory $\mathcal{U}$ of $\mathcal{T}$:

(a) $\mathcal{U}$ is an aisle.
(b) The pair $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a t-structure.
(c) For any object Y in $\mathcal{T}$, there is a triangle $X \to Y \to Z \to X[1]$ with X in $\mathcal{U}$ and Z in $\mathcal{U}^{\perp}$.

We denote by $\mathcal{U}_M$ (respectively, $\mathcal{L}_M$) the smallest suspended (respectively, triangulated) subcategory containing a given object M. We will say $\mathcal{U}_M$ (respectively, $\mathcal{L}_M$) is the suspended (respectively, triangulated) subcategory generated by M. Its orthogonal has the following easy characterization: $X \in \mathcal{U}_M^{\perp}$ (or $X \in \mathcal{L}_M^{\perp}$) if and only if $\text{Hom}_{\mathcal{T}}(M[j], X) = 0$ for all $j \geq 0$ (for all $j \in \mathbb{Z}$, respectively).

We will need to use the property that if $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a t-structure then $\mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp})$ (see [4, (1.3.1)]).

2.2 Ext-projective and Silting Complexes

Let $\mathcal{U}$ be a full additive subcategory of $\mathcal{T}$ closed under extensions. An object X in $\mathcal{U}$, is called *Ext-projective* in $\mathcal{U}$ if and only if $\mathrm{Hom}_{\mathcal{T}}(X, Y[1]) = 0$ for all $Y \in \mathcal{U}$.

We consider $\mathcal{U}$ a full additive subcategory closed under extensions of a Krull-Schmidt triangulated category $\mathcal{T}$ with Serre duality, and $X \in \mathcal{U}$ is an indecomposable object. We recall from [3] the characterization of Ext-projective objects X in $\mathcal{U}$ as the ones having $\tau X \in \mathcal{U}^{\perp}$.

Silting complexes were introduced in [11] in order to study t-structures on the derived category of an hereditary algebra (see also [5, 14]). An object M in a triangulated category $\mathcal{T}$ is called a *silting object* if $\mathrm{Hom}_{\mathcal{T}}(M, M[l]) = 0$ for all $l > 0$. Note that, if M is an indecomposable H-module and $X = M[j] \in \mathbf{D^b}(\mathrm{mod}\, H)$, then X is a silting complex in $\mathbf{D^b}(\mathrm{mod}\, H)$ if and only if $\mathrm{Ext}^1_H(M, M) = 0$.

From now on, H will be a tame hereditary finite dimensional k-algebra and $\mathcal{H} = \mathrm{mod}\, H$. We denote by $\mathbf{D^b}(\mathrm{mod}\, H)$ the bounded derived category of H-modules and by D the usual duality.

We recall that the indecomposable complexes are simply shifted copies of indecomposable modules. That is, they are of the form $X = M[i]$ where [ ] is the shift and M is an indecomposable H-module.

An H-module M is called a *partial tilting* if $\mathrm{Ext}^1_H(M, M) = 0$. It is a *generator* if $\mathrm{Hom}_H(M, X) = 0$ and $\mathrm{Ext}^1_H(M, X) = 0$ imply $X = 0$, and it is a *tilting object* if it is both partial tilting and generator.

Our first result characterizes the bricks (indecomposable objects with trivial endomorphisms rings) that generate aisles in $\mathbf{D^b}(\mathrm{mod}\, H)$. We will show that, such indecomposable object generates an aisle if and only if it is a silting complex. More precisely:

Theorem 2.1 *Let H be a tame hereditary finite dimensional k-algebra and $X = M[t]$ be an object in $\mathbf{D^b}(\mathrm{mod}\, H)$ $(t \in \mathbb{Z})$ such that $\mathrm{End}_{\mathbf{D^b}(\mathrm{mod}\, H)}(X) = k$. The following statements are equivalent:*

(1) *$\mathcal{U}_X$ is an aisle in $\mathbf{D^b}(\mathrm{mod}\, H)$.*
(2) *X is a silting complex.*
(3) *There exists $X^n \to D(H)[t] \to L \to X^n[1]$ a triangle in $\mathbf{D^b}(\mathrm{mod}\, H)$ with $L \in \mathcal{U}_X^{\perp}$ and $n \geq 1$.*
(4) *X is an Ext-projective object in $\mathcal{U}_X$.*

Moreover, in this case, $\mathrm{ind}\, \mathcal{U}_X = \{X[i],\ i \geq 0\}$.

Proof (1) implies (2). By definition of aisle, there exists $U \to D(H)[t] \to L \to U[1]$ a triangle in $\mathbf{D^b}(\mathrm{mod}\, H)$ with $U \in \mathcal{U}_X$ and $L \in \mathcal{U}_X^{\perp}$. Applying the cohomological functor $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, -)$ to the triangle above, we get the following exact sequence

$$\cdots \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, L) \to \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, U[1]) \to \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, D(H)[t+1]) \to \cdots .$$

Since $L \in \mathcal{U}_X^{\perp}$, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, L) = 0$. Moreover, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, D(H)[t+1]) = Ext^1_H(M, D(H)) = 0$ because $D(H)$ is an injective module. We conclude that $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, U[1]) = 0$.

If $U[-t]$ were not a module, U would have a non-zero summand N whose (-t)-th homology would vanish. Since $D(H)$ is injective, there would be no non-zero map from N to $D(H)[t]$. This would mean that the canonical morphism $U \to D(H)[t]$ would not be right minimal, which is a contradiction.

Assume that (2) is false. Since H is a tame hereditary finite dimensional k-algebra, we conclude that M is indecomposable regular of quasi-length n, where n is the rank of the tube $\mathcal{T}$ to which M belongs.

Now, using that tubes are closed under the Auslander-Reiten translation τ and the facts that τ commutes with shifts, direct summands and extensions, we get $\mathcal{U}_X \subset \cup_{j\geq t}\mathcal{T}[j]$. Therefore, $U[-t]$ is a finite sum of indecomposable objects, $U[-t] = \oplus_{i_j\in F} M_{ij}$ where M_{i_j} denotes an indecomposable regular module either in the ray starting at $\tau^s M$ or in the coray ending with $\tau^s M$ for $s = 1, \cdots, n$.

The mesh relation in the tube leads us to the following contradiction:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, U[1]) &= \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[t], U[1])\\ &= \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, U[-t+1])\\ &= \mathrm{E}xt^1_H\left(M, \oplus_{i_j\in F} M_{i_j}\right)\\ &= \oplus_{i_j\in F} D\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}\left(M_{i_j}, \tau M\right) \neq 0.\end{aligned}$$

(2) implies (3). First, we see that $\mathcal{U}_X$ consists of complexes of the form $\oplus_{i\in F} M^{n_i}[i]$ with $n_i \geq 0$ and $F \subset \mathbb{N}$ a finite set. In fact, since M is indecomposable and $\mathrm{E}xt^1_H(M, M) = 0$ then $X = M[t]$ is a partial tilting complex and there is an embeding

$$\mathbf{D^b}\left(\mathrm{E}nd_{\mathbf{D^b}(\mathrm{mod}\, H)}(X)\right) \hookrightarrow \mathbf{D^b}(\mathrm{mod}\, H).$$

Furthermore, we have an equivalence between $\mathcal{U}_X$ and the canonical aisle in $\mathbf{D^b}(\mathrm{E}nd_{\mathbf{D^b}(\mathrm{mod}\, H)}(X))$, it follows that $\mathcal{U}_X$ is also an aisle. Here, $\mathrm{E}nd_{\mathbf{D^b}(\mathrm{mod}\, H)}(X) = k$ and the equivalence is $-\otimes_k X$. Thus we get our statement since the objects in $\mathbf{D^b}(\mathrm{m}od\, k)$ are of the form $\oplus_{j\in F} V_j[j]$ where $V_j = k^{m_j}$ is a finite dimensional k-vector space.

Since $\mathcal{U}_X$ is an aisle, we can consider $U \to D(H)[t] \to L \to U[1]$ the triangle in $\mathbf{D^b}(\mathrm{mod}\, H)$ with $U \in \mathcal{U}_X$ and $L \in \mathcal{U}_X^{\perp}$. We know by the argument above that $U = \oplus_{i\geq t} M^{n_i}[i]$ with $n_i \geq 0$. If $i > t$ then $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[i], D(H)[t]) = 0$. So we finish.

(3) implies (4). Applying the cohomological functor $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, -)$ to the triangle $X^n \to D(H)[t] \to L \to X^n[1]$, we get the following exact sequence

$$\cdots \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, L) \to \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, X^n[1]) \to \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, D(H)[t+1]) \to \cdots .$$

We infer $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, X^n[1]) = 0$ because $L \in \mathcal{U}_X^{\perp}$ and $D(H)$ is an injective H-module. So $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, X[1]) = 0$ and X is an Ext-projective in $\mathcal{U}_X$ (see [3, (3.1)]).

(4) implies (1). If X is Ext-projective in $\mathcal{U}_X$ then, in particular, for all $j > 0$, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, X[j]) = 0$. By [6] we know $\mathcal{U}_X$ is an aisle. □

Remark 2.2 Note that with the notation in the theorem and a similar argument as in the proof above we get that ind $\mathcal{L}_X = \{X[j], j \in \mathbb{Z}\}$.

3 Aisles in $\mathbf{D^b}$(mod H) where H is the Kronecker Algebra

From now on, $H = kQ$ will denote the path algebra of the Kronecker quiver Q and $\mathcal{H} = \text{mod}\,H$. Recall that the Auslander-Reiten quiver, Γ_H, of mod H is the disjoint union of infinitely many connected components: the postprojective, the regular and the preinjective component.

We denote by Pred_X (respectively, Suc_X) the set consisting of all predecessors (respectively, successors) of X including X. For E an indecomposable non-regular H-module we denote by E^p its immediate predecessor and by E^s its immediate successor.

Recall from [4] that, if $\mathcal{U}, \mathcal{V}$ are suspended subcategories, then $\mathcal{W} = \mathcal{U} * \mathcal{V}$ denotes the full additive subcategory consisting of all the objects Y' such that there exists an object $Y = Y' \oplus Y''$ and a triangle $X \to Y \to Z \to X[1]$ with $X \in \mathcal{U}$ and $Z \in \mathcal{V}$.

Now we give a description of silting complexes in $\mathbf{D^b}$(mod H) that we will need later.

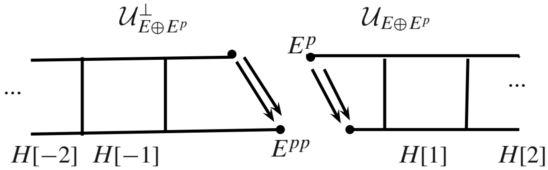
Lemma 3.1 *Let $X = \oplus_{i=1}^m X_i$ be a complex in $\mathbf{D^b}$(mod H) with the X_i indecomposable. If X is a silting complex then there are an indecomposable non-regular H-module E and a number $t \in \mathbb{Z}$, such that*

(a) *There is some i such that $X_i = E[t]$.*
(b) *If $j \neq i$, either $X_j = X_i$ or $X_j = E^p[l]$ where $l \leq t$.*

Proof By [3, (2.3) and (3.1)], the number of indecomposable pairwise non-isomorphic summands, is at most $2 = rk\text{K}_0(H)$. We know that, each X_i is of the form $X_i = M_i[k_i]$ where M_i is an indecomposable object in mod H and $k_i \in \mathbb{Z}$. By definition of silting object, for all i, $\text{Hom}_{\mathbf{D^b}(\mathcal{H})}(M_i[k_i], M_i[k_i+1]) = \text{Ext}^1_H(M_i, M_i) = 0$ and hence, M_i is an indecomposable non-regular H-module. Moreover, for all i, j and $n > 0$, $\text{Hom}_{\mathbf{D^b}(\mathcal{H})}(M_i[k_i], M_j[k_j][n]) = 0$. If $k_i = k_j$, in particular, $\text{Ext}^1_H(M_i, M_j) = 0 = \text{Ext}^1_H(M_j, M_i)$. If $k_i > k_j$ in particular, $\text{Ext}^1_H(M_i, M_j) = 0 = \text{Hom}_H(M_i, M_j)$. Hence, $M_j = M_i^p$ is the immediate predecessor of the indecomposable non-regular H-module M_i. The other case is analogous. □

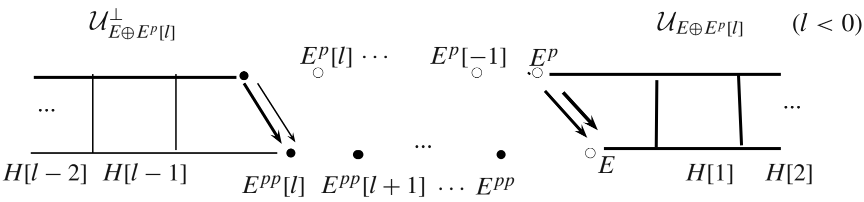
Proposition 3.2 *Let E be an indecomposable non-regular H-module, $t \in \mathbb{Z}$ and $\mathcal{H} =$ mod H.*

(1) *If $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[t]}$ then*
$\text{ind}\,\mathcal{U} = \text{Suc}_{E^p}[t] \cup \cup_{n>t}\mathcal{H}[n]$ *and*
$\text{ind}\,\mathcal{U}^\perp = \cup_{n<t}\mathcal{H}[n] \cup \text{Pred}_{E^{pp}}[t]$.



(2) *If $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[t+1]}$ then $\mathcal{U} = \mathcal{U}_{E\oplus E^s}[t]$ is the shift of the aisle generated by the tilting module $T = E \oplus E^s$.*

(3) *If* $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[l]}$ *with* $l < t$,
ind $\mathcal{U} = \{E^p[i],\ l \leq i < t\} \cup \mathrm{Suc}_{E^p}[t] \cup \cup_{n>t}\mathcal{H}[n]$ *and*
ind $\mathcal{U}^{\perp} = \cup_{n<l}\mathcal{H}[n] \cup \mathrm{Pred}_{E^{pp}}[l] \cup \{E^{pp}[i],\ l < i \leq t\}$.



Proof

(1) Note that $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[t]} = \mathcal{U}_{E\oplus E^p}[t]$ is the shift of the aisle generated by the tilting module $T = E \oplus E^p$. We know that $\mathcal{U}_T^{\perp}$ consists of the complexes Y such that $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(T[i], Y) = 0$ for all $i \geq 0$. We claim that the subcategory $\mathcal{U}_T$ consists of the complexes Y such that $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(T[i], Y) = 0$ for all $i < 0$. In fact, we know that if $Y \in \mathcal{U}_T$, then $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(T[i], Y) = 0$ for all $i < 0$. For the converse, we consider an object Y satisfying $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(T[i], Y) = 0$ for all $i < 0$. Since $\mathcal{U}_T$ is an aisle, there exists a triangle $L \to Y \to U \to L[1]$ with $L \in \mathcal{U}_T$ and $U \in \mathcal{U}_T^{\perp}$. Applying $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(T[i], -)$ for all $i < 0$, we get that $U \in \mathcal{L}_T^{\perp}$. Since T is a tilting module, we have that $\mathcal{L}_T^{\perp} = 0$. It follows that $Y \in \mathcal{U}_T$, and the claim is proved. Using this characterization of $\mathcal{U}_T$ and the Auslander-Reiten formula, we get $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(Y, (E^{pp} \oplus E^{ppp})[i]) = 0$ for all $i \leq 0$. Then, if $j > 0$, all indecomposable complexes $M[j]$ belong to $\mathcal{U}_T$ and if $j < 0$, all indecomposable complexes $M[j]$ belong to $\mathcal{U}_T^{\perp}$. Note that $M \in \mathcal{U}$ if and only if $\mathrm{Hom}_H(M, E^{pp} \oplus E^{ppp}) = 0$. That is exactly the case if $M \in \mathrm{Suc}_{E^p}$. On the other hand, $M \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E \oplus E^p, M) = 0$. That happens exactly when $M \in \mathrm{Pred}_{E^{pp}}$.

(2) Using the fact that $\mathcal{U}_{E\oplus E^s}[t]$ is an aisle, we have $\mathcal{U}_{E\oplus E^s}[t] = {}^{\perp}(\mathcal{U}^{\perp}_{E\oplus E^s}[t])$. By (i), ind $\mathcal{U}^{\perp}_{E\oplus E^s}[t] = \cup_{n<t}\mathcal{H}[n] \cup \mathrm{Pred}_{E^p}[t]$.
We see that $\mathcal{U}^{\perp}_{E\oplus E^s}[t] = \mathcal{U}^{\perp}$ or equivalently, ind $\mathcal{U}^{\perp} = \cup_{n<t}\mathcal{H}[n] \cup \mathrm{Pred}_{E^p}[t]$. In fact, we have, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E[i], M[j]) = 0$ for all $i \geq t$ and $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E^p[i], M[j]) = 0$ for all $i \geq t+1$. For all $j < t$, $M[j] \in \mathcal{U}^{\perp}$ using the fact that H is hereditary. The indecomposable $M[t]$ belongs to $\mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E, M) = 0$. That is exactly the case if $M \in \mathrm{Pred}_{E^p}$. If $j > t$, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E, M) = 0 = \mathrm{Ext}^1_H(E, M)$ and $\mathrm{Hom}_H(E^p, M) = 0 = \mathrm{Ext}^1_H(E^p, M)$, but this is not possible.
Then, we get $\mathcal{U}_{E\oplus E^s}[t] = {}^{\perp}(\mathcal{U}^{\perp}_{E\oplus E^s}[t]) = {}^{\perp}(\mathcal{U}^{\perp})$.
Since $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E[i], E^p[j]) = 0$ for all i, j then, $\mathcal{U}_{E^p[t+1]} \subset \mathcal{U}^{\perp}_{E[t]}$. By [11, (1.4)], $U_{E[t]} * \mathcal{U}_{E^p[t+1]} = \mathcal{U}_{E[t]\oplus E^p[t+1]}$ is an aisle, therefore $\mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp})$.

(3) Now, assume $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[l]}$ with $l < t$.
We have, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E[i], M[j]) = 0$ for all $i \geq t$ and $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E^p[i], M[j]) = 0$ for all $i \geq l$.
For all $j < l$, $M[j] \in \mathcal{U}^{\perp}$ using the fact that H is hereditary. Now, the indecomposable $M[l] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E^p, M) = 0$. That is $M \in \mathrm{Pred}_{E^{pp}}$.

If $l+1 \leq j < t$, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E^p, M) = 0 = \mathrm{Ext}^1_H(E^p, M)$. This happens exactly when $M = E^{pp}$.
If $j > t$, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E, M) = 0 = \mathrm{Ext}^1_H(E, M)$ and $\mathrm{Hom}_H(E^p, M) = 0 = \mathrm{Ext}^1_H(E^p, M)$, but this is not possible. Finally, $M[t]$ belongs to $\mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E, M) = 0$ and $\mathrm{Hom}_H(E^p, M) = 0 = \mathrm{Ext}^1_H(E^p, M)$ if and only if $M = E^{pp}$.
Then we conclude that $\mathrm{ind}\,\mathcal{U}^{\perp} = \cup_{n<l}\mathcal{H}[n] \cup \mathrm{Pred}_{\mathrm{E^{pp}}}[\mathrm{l}] \cup \{\mathrm{E^{pp}[i]},\ \mathrm{l} < \mathrm{i} \leq \mathrm{t}\}$.
Now, $T = E \oplus E^p$ is a tilting H-module and $T[t] \in \mathcal{U}$ since $t > l$. Then, using Proposition 3.2 we get $\mathrm{Suc}_{\mathrm{E^p}}[\mathrm{t}] \cup \cup_{\mathrm{n>t}}\mathcal{H}[\mathrm{n}] = \mathcal{U}_{\mathrm{T}}[\mathrm{t}] \subset \mathcal{U}$.
It remains to study when $N[j]$ (with $l \leq j \leq t$) belongs to $\mathcal{U}$.
Since $\mathcal{U}$ is an aisle, we have $\mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp})$. Note that $M[t] \in {}^{\perp}(\mathcal{U}^{\perp})$ if and only if $M[t] \in \mathrm{Suc}_{\mathrm{E^p}}[\mathrm{t}]$.
If $l \leq j < t$, $M[j] \in {}^{\perp}(\mathcal{U}^{\perp})$ if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,H)}(M[j], E^{pp}[i]) = 0$, for all $l < i \leq t$. That is equivalent to $\mathrm{Hom}_H(M, E^{pp}) = 0 = \mathrm{Ext}^1_H(M, E^{pp})$. This happens exactly when $M = E^p$. □

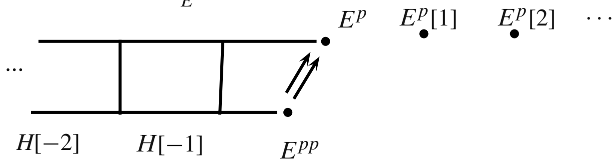
The next result gives a classification of all suspended subcategories of $\mathbf{D^b}(\mathrm{mod}\,H)$ having Ext-projective objects. By [3, 2.3], the number of indecomposable Ext-projective objects is $s \leq 2 = rk\mathrm{K}_0(H)$.

Theorem 3.3 *Let $\mathcal{U}$ be a suspended subcategory of $\mathbf{D^b}(\mathrm{mod}\,H)$ having $s \neq 0$ Ext-projective objects. Then*

(a) *If $\mathcal{U}$ has only one indecomposable Ext-projective object, then $\mathcal{U}$ is one of the following suspended subcategories:*

(i) $\mathcal{U} = \mathcal{U}_{E[t]}$,
$\mathrm{ind}\,\mathcal{U} = \{E[i],\ i \geq t\}$ *and*
$\mathrm{ind}\,\mathcal{U}^{\perp} = \cup_{n<t}\mathcal{H}[n] \cup \mathrm{Pred}_{\mathrm{E^p}}[\mathrm{t}] \cup \{\mathrm{E^p[i]},\ \mathrm{i} \in \mathbb{Z}\}$

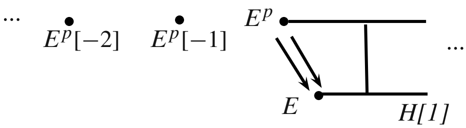
$\mathcal{U}_E^{\perp}$:



(ii) $\mathcal{U} = \mathcal{U}_{E[t]} * \mathcal{L}_{E^p}$,
$\mathrm{ind}\,\mathcal{U} = \{E^p[i],\ i \in \mathbb{Z}\} \cup \mathrm{Suc}_{\mathrm{E^p}}[\mathrm{t}] \cup \cup_{\mathrm{n>t}}\mathcal{H}[\mathrm{n}]$ *and*
$\mathrm{ind}\,\mathcal{U}^{\perp} = \{E^{pp}[i],\ i \leq t\}$

where $E[t]$ is the Ext-projective in $\mathcal{U}$ with E an indecomposable non-regular H-module, $t \in \mathbb{Z}$ and E^p is the immediate predecessor of E with respect to the order given by the morphims.

$\mathcal{U}_E * \mathcal{L}_{E^p}$:



(b) *If $\mathcal{U}$ has 2 indecomposable Ext-projective objects then*

(i) $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[l]}$ *with* $l = t$, *is the shift of the aisle generated by the tilting module* $T = E \oplus E^p$.

(ii) $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[l]}$ *with* $l < t$,

where $\{E[t], E^p[l]\}$ *is the set of indecomposable Ext-projective objects in* $\mathcal{U}$ *with* E *an indecomposable non-regular H-module,* E^p *its immediate predecessor with respect to the order given by the morphims and* $l \le t$.

In either case, $\mathcal{U}$ *is an aisle.*

Proof (b) follows from Lemma 3.1, Proposition 3.2 and [3, 4.5].

(a) Now, assume $X = E[t]$ is the unique indecomposable Ext-projective in $\mathcal{U}$. By [3, (4.2)], $\mathcal{U} = \mathcal{U}_X * (\mathcal{U} \cap \mathcal{L}_X^{\perp})$. If $\mathcal{U} \cap \mathcal{L}_X^{\perp} = \{0\}$ then we get $\mathcal{U} = \mathcal{U}_{E[t]}$. In the other case, there is a non zero indecomposable H-module and a integer $j \in \mathbb{Z}$ such that $M[j] \in \mathcal{U} \cap \mathcal{L}_X^{\perp}$. By other hand we have that $M[j] \in \mathcal{L}_X^{\perp}$ if and only if, for all integer $i \in \mathbb{Z}$, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X[i], M[j]) = 0$. This is equivalent to $\mathrm{Hom}_H(E, M) = 0 = \mathrm{Ext}_H^1(E, M)$ and the only M which satisfies this condition is $M = E^p$, the predecessor of E. Then ind $(\mathcal{U} \cap \mathcal{L}_X^{\perp}) \subset \{E^p[j],\ j \in \mathbb{Z}\} = \mathrm{ind}\,\mathcal{L}_{E^p}$. We are going to see that $\mathcal{U} \cap \mathcal{L}_X^{\perp} = \mathcal{L}_{E^p}$ and then we get ii).

In case that $\mathcal{U} \cap \mathcal{L}_X^{\perp} \neq \mathcal{L}_{E^p}$, there is $i \in \mathbb{Z}$ such that $E^p[i] \notin \mathcal{U} \cap \mathcal{L}_X^{\perp}$. Using the definition of suspended subcategory, we can take

$$l = min\left\{j > i,\ E^p[j] \in \mathcal{U} \cap \mathcal{L}_X^{\perp}\right\}$$

and conclude that $\mathcal{U} \cap \mathcal{L}_X^{\perp} = \mathcal{U}_{E^p[l]}$. Therefore, $\mathcal{U} = U_{E[t]} * \mathcal{U}_{E^p[l]} = \mathcal{U}_{E[t]\oplus E^p[l]}$.

Note that by (b) we have that $l > t$. Moreover, by Proposition 3.2, $l \neq t + 1$.

We will see $\mathcal{U}$ is an aisle. Any Ext-projective object is a silting object so, by [3, (3.2)], we conclude $\mathcal{U}_{E[t]}$, $\mathcal{U}_{E^p[l]}$ and $\mathcal{L}_{E^p}$ are aisles. Since $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E[i], E^p[j]) = 0$ for all i, j then, $\mathcal{L}_{E^p} \subset \mathcal{U}_{E[t]}^{\perp}$ and $\mathcal{U}_{E^p[l]} \subset \mathcal{U}_{E[t]}^{\perp}$. Hence, by [11, (1.4)], $\mathcal{U}_{E[t]} * \mathcal{L}_{E^p}$ and $U_{E[t]} * \mathcal{U}_{E^p[l]} = \mathcal{U}_{E[t]\oplus E^p[l]}$ are aisles.

Now, we are going to prove that if $l > t + 1$ then $\mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[l]}$ is the aisle generated by the silting complex $E[t] \oplus E^s[l-1]$, $\mathcal{U}_{E[t]\oplus E^s[l-1]}$, and then we get a contradiction since it has two indecomposable Ext-projective objects.

First, we see that ind $\mathcal{U} = \{E[i],\ i \ge t\} \cup \mathrm{Suc_E}[l-1] \cup \cup_{n > l-1}\mathcal{H}[n]$ and ind $\mathcal{U}^{\perp} = \cup_{n<t}\mathcal{H}[n] \cup \mathrm{Pred_{E^p}}[t] \cup \{E^p[i],\ i \le l-1\}$.

In fact, note that in this case, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E[i], M[j]) = 0$ for all $i \ge t$ and $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(E^p[i], M[j]) = 0$ for all $i \ge l$. Then, $M[j] \in \mathcal{U}^{\perp}$, for all $j < t$.

If $j = t$, we have that, $M[t] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E, M) = 0$, that is $M \in \mathrm{Pred_{E^p}}$.

If $l - 1 \ge j > t$, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\mathrm{Hom}_H(E, M) = 0 = \mathrm{Ext}_H^1(E, M)$. That is $M = E^p$. Note that if $j > l$ then $M[j] \notin \mathcal{U}^{\perp}$.

Now, we use $\mathcal{U}$ is an aisle and then $\mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp})$.

If $t \le j \le l-2$, $N[j] \in \mathcal{U}$ if and only if $\mathrm{Hom}_H(N, E^p) = 0 = \mathrm{Ext}_H^1(N, E^p)$. This happens exactly when $N = E$.

We have that $N[l-1] \in \mathcal{U}$ if and only if $\mathrm{Hom}_H(N, E^p) = 0$. That is exactly the case if $M \in \mathrm{Suc_E}$.

We conclude because if $j > l - 1$ then $N[j] \in \mathcal{U}$.

Note that $E^s[l-1] \in \mathcal{U}$ implies $\mathcal{U}_{E[t]\oplus E^s[l-1]} \subset \mathcal{U} = \mathcal{U}_{E[t]\oplus E^p[l]}$. Using Proposition 3.2(3), we have $E^p[l] \in \mathcal{U}_{E[t]\oplus E^s[l-1]}$ and we get $\mathcal{U}_{E[t]\oplus E^p[l]} \subset \mathcal{U}_{E[t]\oplus E^s[l-1]}$.

Finally, we see the description of ind $\mathcal{U}$ when $\mathcal{U}$ is an aisle having Ext-projective objects. We denote by $Y = M[j]$ ($j \in \mathbb{Z}$) an arbitrary indecomposable complex in $\mathbf{D^b}(\text{mod } H)$.

a)i) From Theorem 2.1 we know that ind $\mathcal{U}_{E[t]} = \{E[i],\ i \geq t\}$. Then, we have that, $M[j] \in \mathcal{U}^{\perp}_{E[t]}$ if and only if $\text{Hom}_{\mathbf{D^b}(\mathcal{H})}(E[i], M[j]) = 0$ for all $i \geq t$. Since H is an hereditary algebra we get that, $M[j] \in \mathcal{U}^{\perp}_{E[t]}$, for all $j < t$. If $j > t$, $M[j] \in \mathcal{U}^{\perp}_{E[t]}$ if and only if $\text{Hom}_H(E, M) = 0 = \text{Ext}^1_H(E, M)$. This implies that $M = E^p$. We have that $M[t] \in \mathcal{U}^{\perp}_{E[t]}$ is equivalent to $\text{Hom}_H(E, M) = 0$. Finally, the indecomposable modules M satisfying $\text{Hom}_H(E, M) = 0$ exactly coincide with the predecessors of E^p.

a)ii) First, we have $M[j] \in \mathcal{U}^{\perp}$ if and only if $\text{Hom}_{\mathbf{D^b}(\mathcal{H})}(E[i], M[j]) = 0$ for all $i \geq t$ and $\text{Hom}_{\mathbf{D^b}(\mathcal{H})}(E^p[i], M[j]) = 0$ for all $i \in \mathbb{Z}$.
If $j \leq t$, $M[j] \in \mathcal{U}^{\perp}$ is equivalent to $\text{Hom}_H(E, M) = 0 = \text{Hom}_H(E^p, M) = \text{Ext}^1_H(E^p, M)$ and the only M which satisfies this condition is $M = E^{pp}$.
If $j > t$, $M[j] \in \mathcal{U}^{\perp}$ if and only if $\text{Hom}_H(E, M) = 0 = \text{Ext}^1_H(E, M)$ and $\text{Hom}_H(E^p, M) = 0 = \text{Ext}^1_H(E^p, M)$, but this is not possible.
Using the fact that $\mathcal{U}$ is an aisle we have $\mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp})$. We conclude that, $X \in {}^{\perp}(\mathcal{U}^{\perp})$ if and only if $\text{Hom}_{\mathbf{D^b}(\text{mod } H)}(X, E^{pp}[i]) = 0$, for all $j \leq t$. The indecomposable modules satisfying the previous conditions are exactly the ones which belong to the following union $\{E^p[i],\ i \in \mathbb{Z}\} \cup \text{Suc}_{\text{E}^\text{p}}[\text{t}] \cup \cup_{\text{n}>\text{t}} \mathcal{H}[\text{n}]$.

□

4 Aisles Generated by a Kronecker Complex

In this section we turn to the investigation of the aisles generated by a complex in $\mathbf{D^b}(\text{mod } H)$ where H is the Kronecker algebra.

Following [2, 13] we denote by $\mathcal{P}$ (respectively, $\mathcal{R}$, $\mathcal{I}$) the set consisting of all indecomposable postprojective (respectively, regular, preinjective) H-modules. Recall that $\mathcal{R}$ is the family of the regular components, that is $\mathcal{R}$ is a disjoint union of pairwise orthogonal standard stable homogeneous tubes $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(k)$ (where $\mathbb{P}_1(k)$ denotes the projective line). For each $\lambda \in \mathbb{P}_1(k)$, the indecomposable regular H-modules in the tube $\mathcal{T}_\lambda$ are denoted by $R_{\lambda,t}$ where $t \geq 0$ and, up isomorphisms, they are the indecomposable modules liying on the ray starting at the simple regular module $R_\lambda = R_{\lambda,0}$.

We will use the following result.

Lemma 4.1 *Let $\mathcal{U}$ be a suspended subcategory of* $\mathbf{D^b}(\text{mod } H)$ *and $\mathcal{T}_\lambda$ a tube. If $\mathcal{U} \cap \mathcal{T}_\lambda \neq \emptyset$ then $\mathcal{T}_\lambda \subset \mathcal{U}$.*

Proof We have a short exact sequence in mod H, $R_{\lambda,t} \to R_{\lambda,t+1} \oplus R_{\lambda,t-1} \to R_{\lambda,t}$ for each indecomposable regular module $R_{\lambda,t}$ in $\mathcal{T}_\lambda$, with $\lambda \in \mathbb{P}_1(k)$ and $t \geq 0$ (we consider $R_{\lambda,-1} = 0$ for a simple regular module $R_{\lambda,0}$). Then we get triangles in $\mathbf{D^b}(\text{mod } H)$ $R_{\lambda,t} \to R_{\lambda,t+1} \oplus R_{\lambda,t-1} \to R_{\lambda,t} \to R_{\lambda,t}[1]$. Consequently, if $\mathcal{U}$ is a suspended subcategory and $R_{\lambda,t} \in \mathcal{U}$ then $R_{\lambda,j} \in \mathcal{U}$ for all $j \geq 0$. □

The following result gives a description of all suspended subcategories of $\mathbf{D^b}(\text{mod } H)$ generated by a decomposable Kronecker module $M = \oplus_{i=1}^{b} M_i\,(b > 1)$ with the indecomposable pairwise non-isomorphic summands. We assume the indecomposable objects M_i in $\mathcal{P}$ and the ones in $\mathcal{I}$ are ordered as follows, $i < j$ if and only if M_i is a predecessor of M_j.

Theorem 4.2 *Let $M = \oplus_{i=1}^{b} M_i\,(b > 1)$ be an H-module with indecomposable pairwise non-isomorphic summands. Then the smallest suspended subcategory $\mathcal{U}_M$ containing M is one of the following:*

(1) *$\mathcal{U}_M = \mathcal{U}_T$ is an aisle generated by a tilting module T.*
(2) *$\mathcal{U}_M$ is a splitting aisle without Ext-projective objects and*

$$\text{ind}\,\mathcal{U}_M = \cup_{\lambda\in F}\mathcal{T}_\lambda \cup \mathcal{I} \cup \cup_{n>0}\mathcal{H}[n]$$

with F a finite
(3) *$\mathcal{U}_M$ is not an aisle and it is a suspended subcategory without Ext-projective objects and*

$$\text{ind}\,\mathcal{U}_M = \cup_{j\geq 0} \cup_{\lambda\in F} \mathcal{T}_\lambda[j]$$

with F a finite set.

Proof The module category can be tilted so we can assume the H-module M has no preinjective summands.

If M has some postprojective indecomposable summands, take $l = min\{j, M_j \in \mathcal{P}\}$. By Proposition 3.2 we know that ind $\mathcal{U}_{M_l\oplus M_l^s} = \text{Suc}_{\text{M}_\text{l}} \cup \cup_{\text{n}>0}\mathcal{H}[\text{n}]$. We are going to show that $\mathcal{U}_M = \mathcal{U}_{M_l\oplus M_l^s}$. Note that $M_j \in \text{Suc}_{\text{M}_\text{l}}$ for all j, then $\mathcal{U}_M \subset \mathcal{U}_{M_l\oplus M_l^s}$. Now, if $M_j = M_l^s$ for some j then $\mathcal{U}_M = \mathcal{U}_{M_l\oplus M_l^s}$. Suppose that no M_j for $j \neq l$ belongs to $\mathcal{P}$ and some M_j belongs to $\mathcal{T}_\lambda$. Then for each postprojective module N which is a successor of M_l, there exists a triangle $M_l \to N \to R_{\lambda,t} \to M_l[1]$. It follows that $N \in \mathcal{U}_M = \mathcal{U}_{M_l\oplus M_l^s}$.

Otherwise, if $M_j \in \mathcal{P}$, for each $\lambda \in \mathbb{P}_1(k)$, there is a triangle $M_l \to M_l^s \to R_{\lambda,t} \to M_l[1]$ and so, $\mathcal{U}_{M_l\oplus M\,j} = \mathcal{U}_{M_l\oplus R_{\lambda,t}}$ by definition of suspended subcategory. Then $\cup\mathcal{T}_\lambda \subset \mathcal{U}_{M_l\oplus M_j} \subset \mathcal{U}_M$. Consider $N \in \text{SucM}_\text{l}$. If $N \in \mathcal{P}$ using the triangle $M_l \to N \to R_{\lambda,t} \to M_l[1]$ we conclude that $N \in \mathcal{U}_{M_l\oplus R_{\lambda,t}} \subset \mathcal{U}_M$. If $N \in \mathcal{I}$ we get that $N \in \mathcal{U}_M$ using the triangle $M_l \to R_{\lambda,t} \to N \to M_l[1]$ and the fact that $\mathcal{U}_M$ is a suspended subcategory. Then we get $\text{Suc}_{\text{M}_\text{l}} \subset \mathcal{U}_\text{M}$. Using the description of $\mathcal{U}_{M_l\oplus M_l^s}$, we get that $\mathcal{U}_{M_l\oplus M_l^s} \subset \mathcal{U}_M$. Then $\mathcal{U}_M$ is generated by the tilting H-module $M_l \oplus M_l^s$ and we are in case 1.

If $M_i \in \mathcal{I}$ for all i. There exists a derived equivalent hereditary algebra H' such that M is a postprojective H'-module and we fall in the above case. It follows that $\mathcal{U}_M$ is generated by a tilting H'-module.

If M has no postprojective indecomposable summands, but has regular and preinjective indecomposable summands, there exists $M_j = R_{\lambda,t}$ with $t \geq 0$, then by Lemma 4.1 we get that $\mathcal{T}_\lambda \subset \mathcal{U}_M$. Consider one preinjective summand M_n, we prove that all the predecessors of M_n in $\mathcal{I}$ belong to $\mathcal{U}_M$ using the triangle $R_{\lambda,t} \to N \to M_n \to R_{\lambda,t}[1]$. Since $\mathcal{T}_\lambda[1] \subset \mathcal{U}_M$ we get that all the successors of M_n belong to $\mathcal{U}_M$ using the triangle $M_n \to N \to R_{\lambda,t}[1] \to M_n[1]$. Then we get that ind $\mathcal{U}_M = \cup_{\lambda\in F}\mathcal{T}_\lambda \cup \mathcal{I} \cup \cup_{n>0}\mathcal{H}[n]$, where F is a finite set. In this case $\mathcal{U}_M$ is a splitting aisle, has not Ext-projective objects because the tubes are closed under the Auslander-Reiten translation. We fall in case 2.

If M has only regular indecomposable summands. Then we have that ind $\mathcal{U}_M = \cup j \geq 0 \cup_{\lambda \in F} \mathcal{T}_\lambda[j]$ where $j \geq 0$ (see proof of Theorem 2.1). We fall in case 3. □

Finally, we describe all the suspended subcategories generated by a complex in $\mathbf{D^b}(\mathrm{mod}\, H)$. In order to do that, we first introduce the following notation.

Notation Let $X = \oplus_{i=a}^{b} X_i$ be a decomposable complex in $\mathbf{D^b}(\mathrm{mod}\, H)$. We denote by $X_i = C_i \oplus R_i$, where $C_i = \oplus_j E_{i_j}[i]$ (respectively, $R_i = \oplus_j R_{i_j}[i]$) is the direct summand of X concentrated in i-degree such that either $E_{i_j} = 0$ (respectively, $R_{i_j} = 0$) or $E_{i_j} \neq 0$ is a non-regular (respectively, $R_{i_j} \neq 0$ is a regular) indecomposable H-module. We assume that the indecomposable direct summands are pairwise non isomorphic and they are ordered as follows, $i_v < i_w$ if and only if E_{i_v} is a predecessor of E_{i_w}.

We denote by E_j the indecomposable H-module such that $E_j[j] \neq 0$ is a direct summand of C_j and if $E_i[j]$ is another one, then E_j is a predecessor of E_i.

Theorem 4.3 *Let X be a complex in $\mathbf{D^b}(\mathrm{mod}\, H)$. Then $\mathcal{U}_X$ is the shift of one of the following suspended subcategories:*

(1) *$\mathcal{U}$ is an aisle with only one Ext-projective object of the form $\mathcal{U} = \mathcal{U}_{E_0}$ where E_0 is an indecomposable non-regular H-module.*
(2) *$\mathcal{U} = \mathcal{U}_T$ is an aisle with 2 Ext-projective objects generated by a tilting module T.*
(3) *$\mathcal{U} = \mathcal{U}_{E_0 \oplus E_0^s[j]}$ is an aisle with 2 Ext-projective objects generated by a silting complex of the form $E_0 \oplus E_0^s[j]$ where E_0^s is the immediate successor of E_0 and $j > 0$.*
(4) *$\mathcal{U}$ is a splitting aisle without Ext-projective objects and it is of the form*

$$\mathrm{ind}\, \mathcal{U} = \cup_{\lambda \in L} \mathcal{T}_\lambda \cup \mathcal{I} \cup_{n>0} \mathcal{H}[n]$$

where L is a finite set.
(5) *$\mathcal{U}$ is not an aisle and it consists of a finite disjoint union of shifted tubes.*
(6) *$\mathcal{U}$ is such that* ind $\mathcal{U}$ *is the union of all indecomposable objects from the above suspended subcategories.*

Proof For a complex X we consider the notation described above. We suppose, for the sake of simplicity, that $a = 0$ and $X_0 \neq 0$. Note that for each complex Z there is another Y such that $\mathcal{U}_Z = \mathcal{U}_Y$ and verifies the following condition: if $E[i]$ is a direct summand of Y then $E[t]$ is not a direct summand of Y for all $t > i$.

We can assume, up to derived equivalences, that all C_i are postprojectives.

In the case that X is an indecomposable object, it can be considered a module, and we can apply Theorem 4.2.

We consider X a decomposable complex. We prove the result in several steps:

a) First, we assume $C_0 \neq 0$. Note that if X_0 is a decomposable H-module then by the proof of Theorem 4.2 and Proposition 3.2 we have $\mathcal{U}_{E_0 \oplus E_0^s} = \mathcal{U}_{X_0} \subset \mathcal{U}_X$ and $X \in \mathcal{U}_{E_0 \oplus E_0^s}$. Then $\mathcal{U}_X \subset \mathcal{U}_{E_0 \oplus E_0^s} \subset \mathcal{U}_X$ and we conclude that $\mathcal{U}_X$ is generated by a tilting module, that is case 2.
b) Now, we assume $C_0 \neq 0$ and X_0 is an indecomposable H-module.

Note that $X_0 = C_0 = E_0$. Take X_j the first non zero summand of X with $j > 0$. We have to study two cases:

Case 1 If $C_j \neq 0$.

We consider in the derived category the order given by the morphisms. Take $X_{1j} = E_j[j]$ the first direct summand in C_j. By the assumption at the beginning, E_j is different from E_0.
Since $E_0[j], E_j[j] \in \mathcal{U}_X$ we get $\mathcal{U}_{E_0 \oplus E_j}[j] = \mathcal{U}_{E_0[j] \oplus E_j[j]} \subset \mathcal{U}_X$, but $E_0^s \in \mathcal{U}_{E_0 \oplus E_j}$ by Theorem 4.2. Then $E_0 \oplus E_0^s[j] \in \mathcal{U}_X$ and $\mathcal{U}_{E_0 \oplus E_0^s[j]} \subset \mathcal{U}_X$.
Assume first that E_j is a successor of E_0. By Proposition 3.2 we know that ind $\mathcal{U}_{E_0 \oplus E_0^s[j]} = \{E_0[i],\ 0 \le i < j\} \cup \mathrm{Suc_{E_0}[j]} \cup \cup_{\mathrm{n>j}} \mathcal{H}[\mathrm{n}]$. Recall that $X_t = 0$ for all $t \neq 0$, $t < j$. Then we have $X \in \mathcal{U}_{E_0 \oplus E_0^s[j]}$ and $\mathcal{U}_X \subset \mathcal{U}_{E_0 \oplus E_0^s[j]}$, but we also know that $\mathcal{U}_{E_0 \oplus E_0^s[j]} \subset \mathcal{U}_X$. Then we get that $\mathcal{U}_X$ is generated by a silting complex and we fall in case 3.
Now, if E_j is a predecessor of E_0. Since $E_0[j], E_j[j] \in \mathcal{U}_X$ we get $\mathcal{U}_{E_0 \oplus E_j}[j] = \mathcal{U}_{E_0[j] \oplus E_j[j]} \subset \mathcal{U}_X$, but $E_0^p \in \mathcal{U}_{E_0 \oplus E_j}$ by Theorem 4.2. Then $E_0 \oplus E_0^p[j] \in \mathcal{U}_X$ and $\mathcal{U}_{E_0 \oplus E_0^p[j]} \subset \mathcal{U}_X$. With similar arguments as above, if $j > 1$ we get case 3 and if $j = 1$ we fall in case 2 using Theorem 3.3.

Case 2 If $C_j = 0$.

We have $R_j \neq 0$, that is there exists a regular module $R_{\lambda,t}[j]$ which is a direct summand of X_j. Using the triangle $E_0[j] \to E_0^s[j] rightarrow R_{\lambda,t}[j] \to E_0[j+1]$ and the definition of suspended subcategory, we infer $E_0^s[j] \in \mathcal{U}_{R_{\lambda,t}[j] \oplus E_0[j]} \subset \mathcal{U}_X$ and then $\mathcal{U}_{E_0 \oplus E_0^s[j]} \subset \mathcal{U}_X$. Again by Proposition 3.2 b)iii), ind $\mathcal{U}_{E_0 \oplus E_0^s[j]} = \{E_0[i],\ 0 \le i < j\} \cup \mathrm{Suc_{E_0}[j]} \cup \cup_{\mathrm{n>j}} \mathcal{H}[\mathrm{n}]$. Then we have $X \in \mathcal{U}_{E_0 \oplus E_0^s[j]}$ and $\mathcal{U}_X = \mathcal{U}_{E_0 \oplus E_0^s[j]}$. We get that $\mathcal{U}_X$ is generated by a silting complex and we fall in case 3.

c) Finally, we study the case $C_0 = 0$.
We are assuming $X_0 \neq 0$ then $R_0 \neq 0$. By Lemma 4.1, $\mathcal{T}_\lambda \subset \mathcal{U}_X$ for all λ such that $R_{\lambda,t}$ is a direct summand of R_0.
If for all $i > 0$ either $X_i = 0$ or $X_i \in R_i$ it follows from Theorem 4.2 and the fact that the tubes are pairwise orthogonal that $\mathcal{U}_X = \cup_i \mathcal{U}_{R_i}$ is a disjoint union of tubes

$$\mathrm{ind}\, \mathcal{U}_X = \cup_{j \ge l_j} \cup_{\lambda \in L} \mathcal{T}_\lambda[j]$$

where L is a finite set, $l_j \in F$ and $F \subset \mathbb{N}$ and we fall in case 5.
If $C_1 \neq 0$ then, for each preinjective N there is a triangle $E_1 \to R_{\lambda,t} \to N \to E_1[1]$ so, $\mathcal{U}_{R_{\lambda,t} \oplus E_1[1]} = \mathcal{U}_{R_{\lambda,t} \oplus N} \subset \mathcal{U}_X$, but $X \in \cup_{\lambda \in F} \mathcal{T}_\lambda \cup \mathcal{I} \cup \cup_{n>0} \mathcal{H}[n]$ and then $X \in \mathcal{U}_{R_{\lambda,t} \oplus N}$ by Theorem 4.2(2). Consequently, $\mathcal{U}_X = \mathcal{U}_{R_{\lambda,t} \oplus N}$ is an aisle without Ext-projective objects and we get case 4.
If $X_i \in R_i$ for all $i = 1, \cdots, j$ and $C_j \neq 0$, analyzing C_j in place of C_0, we get that $\mathcal{U}_X$ is an union of shifts of the suspended subcategories described above, that means case 6. □

As we have seen earlier, suspended subcategories having Ext-projective objects are aisles (see Theorem 3.3). As a consequence of Theorem Theorem 4.3, we will conclude the existence of suspended subcategories (generated by objects) which are not aisles. Moreover, in case that a suspended subcategory has not Ext-projectives, we find that some are aisles and some are not.

Example 4.4 The following is an example of a suspended subcategory which is not an aisle, is not generated by an object and does not have Ext-projectives objects. Consider the following subcategory $\mathcal{U}$

$$\operatorname{ind}\mathcal{U} = \cup_{j\geq 0} \cup_{\lambda\in\mathbb{P}_1} \mathcal{T}_\lambda[j].$$

Note that $\mathcal{U}$ is not generated by an object because there exist infinitely many pairwise orthogonal tubes in the family and we are dealing with finitely generated modules. This suspended subcategory is not an aisle, since for the shifts of the postprojective objects X, there is not a triangle $U \to X \to L \to U[1]$ with U in $\mathcal{U}$ and L in $\mathcal{U}^{\perp}$. The subcategory $\mathcal{U}$ has not Ext-projectives objects because ind $\mathcal{U}$ is formed for unions of complete Auslander-Reiten components.

We summarize in the following result.

Corollary 4.5 *Let* $X = \oplus_{i=0}^{b} X_i$ *be a complex in* $\mathbf{D^b}(\mathrm{mod}\, H)$, $X_0 \neq 0$ *and keep the above notation.*

(1) $\mathcal{U}_X$ *is an aisle in* $\mathbf{D^b}(\mathrm{mod}\, H)$ *if and only if either* $C_0 \neq 0$ *or* $C_0 = 0,\ R_0 \neq 0$ *and* $C_1 \neq 0$.
(2) $\mathcal{U}_X$ *is an aisle without Ext-projectives if and only if* $C_0 = 0,\ R_0 \neq 0$ *and* $C_1 \neq 0$.

Proof

(1) By Theorem 4.3, $\mathcal{U}_X$ is an aisle if either $C_0 \neq 0$ or $C_0 = 0,\ R_0 \neq 0$ and $C_1 \neq 0$. Now, assume $R_0 \neq 0$, and there is $j > 1$ such that $C_i = 0$, for all $0 \leq i < j$ and $C_j \neq 0$. Note that the only modules in $\mathcal{U}_X$ are regular.

 If $\mathcal{U}_X$ is an aisle then there is a triangle $U \xrightarrow{f} D(H) \xrightarrow{g} L \xrightarrow{h} U[1]$ with $U \in \mathcal{U}_X$ and $L \in \mathcal{U}_X^{\perp}$. First, $L \neq 0$ because in other case $D(H)$ will be a direct summand of U and then regular. On the other hand, if $U = 0$ then $D(H)$ will be a direct summand of L and then $D(H) \in \mathcal{U}_X^{\perp}$, but $\mathrm{Hom}_H(R_0, D(H)) \neq 0$.
 Observe that U must be concentrated in 0-degree because $D(H)$ is an injective module and then U is a regular module. Moreover, L is also an H-module because U and $D(H)$ are. We get a contradiction since there is not any triangle with this form. Then $\mathcal{U}_X$ is not an aisle.
(2) follows from (1) and Theorem 4.3 □

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