MINIMAL 3×3 HERMITIAN MATRICES

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Abstract. Given a Hermitian matrix $M \in M_3(\mathbb{C})$ we describe the real diagonal matrices D_M such that

$$\|M + D_M\| \leqslant \|M + D\|$$

for all real diagonal matrices $D \in M_3(\mathbb{C})$, where $\| \ \|$ denotes the operator norm. Moreover, we generalize our techniques to some $n \times n$ cases.

1. Introduction

Let $M_3(\mathbb{C})$ and $D_3(\mathbb{R})$ be, respectively, the algebras of complex and real diagonal 3×3 matrices. Given a fixed Hermitian matrix $M \in M_3(\mathbb{C})$ we study the diagonals D_M that realize the quotient norm

$$\|M + D_M\| = |||\,[M]\,||| = \min_{D \in D_3(\mathbb{R})} \|M + D\| = \operatorname{dist}(M, D_3(\mathbb{R})),$$

or equivalently

$$\|M + D_M\| \leqslant \|M + D\|, \text{ for all } D \in D_3(\mathbb{R})$$

where $\| \ \|$ denotes the operator norm.

The matrices $M + D_M$ will be called minimal. These matrices appeared in the study of minimal length curves in the flag manifold $\mathcal{P}(n) = \mathcal{U}(M_n(\mathbb{C})) / \mathcal{U}(\mathcal{D}_n(\mathbb{C}))$, where $\mathcal{U}(\mathcal{A})$ denotes the unitary matrices of the algebra $\mathcal{A}$, when $\mathcal{P}(n)$ is endowed with the quotient Finsler metric of the operator norm [5]. Minimal length curves δ in $\mathcal{P}(n)$ are given by the left action of $\mathcal{U}(M_n(\mathbb{C}))$ on $\mathcal{P}(n)$. Namely

$$\delta(t) = \left[e^{itM} U\right],$$

where M is minimal and $[V]$ denotes the class of V in $\mathcal{P}(n)$. Moreover, the natural questions and some particular examples that appear from the geometric description of these objects are related to problems that appear in other contexts: problems of minimization of operators related with optimization and control ([6, 8]), matrix analysis ([4, 7]), Leibnitz seminorms ([9, 10]) and unitary stochastic matrices ([2]).

Previous attempts to describe minimal matrices and their properties were done in [1] and for 3×3 matrices. In that paper, all 3×3 minimal matrices were parametrized. We stress that there are no known results showing which is the minimizing diagonal of a given Hermitian matrix M (except on trivial cases).

Several recent approaches have been made to describe the closest diagonal matrix to a given Hermitian matrix (see for instance [9], [2] and [1]). These papers give qualitative properties that describe properties of these matrices and even parametrize all the solutions. Nevertheless the problem

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of finding the diagonal matrix or matrices closest to a concrete Hermitian matrix remained open even for the first non trivial case: 3×3.

Our goal in the present paper is to study this problem for 3×3 minimal matrices and some $n \times n$ cases where the 3×3 techniques can be extended.

2. Preliminaries and notation

Let $M_n(\mathbb{C})$ denote the algebra of square $n \times n$ complex matrices, $M_n^h(\mathbb{C})$ the real subspace of Hermitian complex matrices, and $D_n(\mathbb{R})$ the real subalgebra of the diagonal real matrices. The symbol $\sigma(A)$ denotes here the spectrum of A, that is the (unordered) set of eigenvalues of A. We denote with $\|A\|$ the usual operator norm or spectral norm of $A \in M_n(\mathbb{C})$ and with $\|C\|_2$ the euclidean norm for $C \in \mathbb{C}^n$.

We denote with $\{e_i\}_{i=1}^n$ the canonical basis of $\mathbb{C}^n$. Given a matrix $A \in M_n(\mathbb{C})$, we denote with $A_{i,j}$ the i,j entry of A and we write $A = [A_{i,j}]$ for $i, j = 1, \ldots, n$.

For $M \in M_n(\mathbb{C})$ we denote with MN the usual matrix product, with $\operatorname{tr}(M)$ the usual (non-normalized) trace of M and with $C_i(M)$ the vector given by the i^{th} column of M.

For $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ we denote with $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ the diagonal matrix of $M_n^h(\mathbb{R})$ with $(a_1, a_2, \ldots, a_n)$ in its diagonal. Nevertheless, if $M \in M_n(\mathbb{C})$, then $\operatorname{Diag}(M)$ denotes the diagonal matrix defined by the principal diagonal of M.

Observe that if $M \in M_n^h(\mathbb{C})$ and $D \in D_n(\mathbb{R})$ then $(M + D) \in M_n^h(\mathbb{C})$. Let us consider the quotient $M_n^h(\mathbb{C})/D_n(\mathbb{R})$ and the quotient norm

$$||| \, [M] \, ||| = \min_{D \in D_n(\mathbb{R})} \|M + D\| = \operatorname{dist}(M, D_n(\mathbb{R}))$$

for $[M] = \{M + D : D \in D_n(\mathbb{R})\} \in M_n^h(\mathbb{C})/D_n(\mathbb{R})$. The minimum is clearly attained.

Definition 1. *A matrix $M \in M_n^h(\mathbb{C})$ is called* **minimal** *if*

$$\|M\| \leqslant \|M + D\| \quad \text{for all } D \in D_n(\mathbb{R}),$$

or equivalently, if $\|M\| = ||| \, [M] \, ||| = \min_{D \in D_n(\mathbb{R})} \|M + D\| = dist(M, D_n(\mathbb{R}))$.

Definition 2. *Let $M \in M_n^h(\mathbb{C})$ and $D \in D_n(\mathbb{R})$ such that $M + D$ is minimal. Then D is a* **minimizing diagonal** *of M.*

For a matrix $M \in M_3^h(\mathbb{C})$ with at least two non zero off-diagonal entries this minimizing matrix is unique (see [1, Theorem 3.15] for a proof):

Proposition 1. *If $M \in M_3^h(\mathbb{C})$ is a minimal matrix and at least two of $M_{1,2}$, $M_{1,3}$ and $M_{2,3}$ are non zero then the values of its minimizing diagonal are unique.*

Remark 1. *Note that if $M \in M_n^h(\mathbb{C})$ is a minimal matrix then its spectrum is centered in the sense that $\|M\|$, $-\|M\| \in \sigma(M)$. In general, for a given matrix $A \in M_n^h(\mathbb{C})$, $\pm\|A\| \in \sigma(A)$ if and only if $\|A\| = \min_{\lambda \in \mathbb{R}} \|A + \lambda I\|$ if and only if $\lambda_{min}(A) + \lambda_{max}(A) = 0$. Note that this implies that if $M \in M_3^h(\mathbb{C})$ is a minimal matrix then in particular $\sigma(M) = \{\|M\|, \mu, -\|M\|\}$ for $|\mu| \leqslant \|M\|$, $\mu = tr(M)$.*

Throughout the paper, for a given non-zero minimal matrix $M \in M_3^h(\mathbb{C})$, we denote with $\sigma(M) = \{\lambda, \mu, -\lambda\}$ the spectrum of M, for $0 < \lambda = \|M\|$, $|\mu| \leqslant \lambda$ and $\mu = \operatorname{tr}(M)$.

Given $v = (v_2, v_2, v_3) \in \mathbb{C}^3$, $v \otimes v$ denotes the matrix such that $(v \otimes v)_{i,j} = v_i \overline{v_j}$ for $i, j = 1, 2, 3$.

For $M \in M_3^h(\mathbb{C})$ and $v \in \mathbb{C}^n$ we write $\overline{M}$ and $\overline{v}$ to denote the matrix and vector obtained from M and v by conjugation of its coordinates.

If $M, N \in \mathbb{C}^{n\times m}$ we denote with $M \circ N$ the Schur or Hadamard product of these matrices defined by $(M \circ N)_{i,j} = M_{i,j}N_{i,j}$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Therefore, if $v \in \mathbb{C}^3$, with coordinates in the canonical basis given by $v = (v_1, v_2, v_3)$,

$$v \circ \overline{v} = (|v_1|^2, |v_2|^2, |v_3|^2) = \sum_{j=1}^{3} |v_j|^2 e_j \in \mathbb{R}_+^n.$$

If $A \in \mathbb{C}^{n\times m}$ we denote with $A^t \in \mathbb{C}^{m\times n}$ its transpose, with $\operatorname{ran}(A)$ the range of the linear transformation A and with $\ker(A)$ its kernel.

3. Minimal 3×3 matrices with zero entries

Proposition 2. *If $x, y, z \in \mathbb{C}$, $c \in \mathbb{R}$, $|c| \leqslant |x|$, $b \in \mathbb{R}$, $|b| \leqslant |y|$, $a \in \mathbb{R}$, $|a| \leqslant |z|$, then the matrices*

$$M_x = \begin{pmatrix} 0 & x & 0 \\ \overline{x} & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \quad M_y = \begin{pmatrix} 0 & 0 & \overline{y} \\ 0 & b & 0 \\ y & 0 & 0 \end{pmatrix} \quad M_z = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & z \\ 0 & \overline{z} & 0 \end{pmatrix}$$

are minimal. Moreover, these are all the possible diagonals that make them minimal.

Proof. Let $v \in \mathbb{C}^3$ with $\|v\| = 1$. It is easy to prove that $\|M_x v\| \leqslant |x|$ for all $c \in \mathbb{R}$ such that $|c| \leqslant |x|$. Since $\|M_x e_2\| = |x|$ then $\|M_x\| = |x|$. Moreover, if we consider

$$M = \begin{pmatrix} \alpha & x & 0 \\ \overline{x} & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

with $\alpha \neq 0$, then $\|Me_1\| = \|(\alpha, \overline{x}, 0)\| > |x|$. Therefore, $\|M\| > \|M_x\|$. Something similar happens if $\beta \neq 0$ checking $\|Me_2\|$. If $|\gamma| > |x|$ then $\|M\| > \|M_x\|$ and therefore M_x is minimal (with $|c| \leqslant |x|$).

The proof for the matrices M_y and M_z is similar. $\square$

The following theorem is proved in [1, Theorem 3.8]. We restate it here for the sake of clarity.

Theorem 1. *Let $M_{3\times3}^h(\mathbb{C})$ with $\|M\| = \lambda > 0$. Then M is minimal if and only if there exist two eigenvectors v_+ corresponding to the eigenvalue λ and v_- corresponding to the eigenvalue $-\lambda$, such that their coordinates have the same module. That is, if for every e_i then $|\langle v_+, e_i\rangle| = |\langle v_-, e_i\rangle|$ or equivalently $v_+ \circ \overline{v_+} = v_- \circ \overline{v_-}$.*

Using the above theorem we can prove the following proposition.

Proposition 3. *Let x, y, z non-zero complex numbers. Then the matrices*

$$M_{xy} = \begin{pmatrix} 0 & x & \overline{y} \\ \overline{x} & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \quad M_{yz} = \begin{pmatrix} 0 & 0 & \overline{y} \\ 0 & 0 & z \\ y & \overline{z} & 0 \end{pmatrix} \quad M_{xz} = \begin{pmatrix} 0 & x & 0 \\ \overline{x} & 0 & z \\ 0 & \overline{z} & 0 \end{pmatrix}$$

are minimal. These are the only Hermitian minimal matrices with four non-zero entries outside the diagonal.

Proof. A direct calculation proves that the eigenvalues of M_{xy} are $\{\sqrt{|x|^2+|y|^2}, -\sqrt{|x|^2+|y|^2}, 0\}$ and their corresponding eigenvectors are

$$v_+ = \left(\frac{1}{\sqrt{2}}, \frac{\overline{x}}{\sqrt{2(|x|^2+|y|^2)}}, \frac{y}{\sqrt{2(|x|^2+|y|^2)}}\right),$$

$$v_- = \left(-\frac{1}{\sqrt{2}}, \frac{\overline{x}}{\sqrt{2(|x|^2+|y|^2)}}, \frac{y}{\sqrt{2(|x|^2+|y|^2)}}\right)$$

and

$$v_0 = \left(0, -\frac{y}{\sqrt{|x|^2+|y|^2}}, \frac{\overline{x}}{\sqrt{|x|^2+|y|^2}}\right).$$

Then using Theorem 1 M_{xy} is minimal.

This diagonal is unique (see Proposition 1).

Similar results can be proved for M_{xz} and M_{yz}. □

4. Minimal 3×3 matrices with non-zero entries

The following theorem describes minimizing diagonals for matrices M with real nonzero entries.

Theorem 2. ***Real (symmetric) minimal matrices***

Let $x, y, z \in \mathbb{R}$, $x, y, z \neq 0$.

- *Case 1: if*

$$x^2y^2 > z^2(x^2+y^2) \tag{4.1}$$

then

$$M = \begin{pmatrix} 0 & x & y \\ x & -\frac{yz}{x} & z \\ y & z & -\frac{xz}{y} \end{pmatrix} \text{ is minimal.}$$

- *Case 2: if $x^2z^2 > y^2(x^2+z^2)$ then $M = \begin{pmatrix} -\frac{yz}{x} & x & y \\ x & 0 & z \\ y & z & -\frac{xy}{z} \end{pmatrix}$ is minimal.*
- *Case 3: if $y^2z^2 > x^2(y^2+z^2)$ then $M = \begin{pmatrix} -\frac{xz}{y} & x & y \\ x & -\frac{xy}{z} & z \\ y & z & 0 \end{pmatrix}$ is minimal.*
- *Case 4: if none of the previous cases hold, that is*

$$-x^2z^2 + y^2(x^2+z^2) \geqslant 0 \ \wedge \ -x^2y^2 + z^2(x^2+y^2) \geqslant 0 \ \wedge \ -y^2z^2 + x^2(y^2+z^2) \geqslant 0, \tag{4.2}$$

then

$$M = \begin{pmatrix} \frac{1}{2}\left(+\frac{xy}{z} - \frac{xz}{y} - \frac{zy}{x}\right) & x & y \\ x & \frac{1}{2}\left(-\frac{xy}{z} + \frac{xz}{y} - \frac{zy}{x}\right) & z \\ y & z & \frac{1}{2}\left(-\frac{xy}{z} - \frac{xz}{y} + \frac{zy}{x}\right) \end{pmatrix} \text{ is minimal.}$$

Proof. Let us consider first case 1. Note that $\|C_1(M)\|_2^2 = x^2 + y^2 > \|C_i(M)\|_2^2$ with $i = 2, 3$:

$$\|C_2(M)\|_2^2 = x^2 + \frac{y^2z^2}{x^2} + z^2 = x^2 + \frac{y^2z^2 + x^2z^2}{x^2} = x^2 + \frac{z^2(x^2+y^2)}{x^2} < x^2 + y^2 = \|C_1(M)\|_2^2,$$

$$\|C_3(M)\|_2^2 = y^2 + z^2 + \frac{x^2z^2}{y^2} = y^2 + \frac{y^2z^2 + x^2z^2}{y^2} = y^2 + \frac{z^2(x^2+y^2)}{y^2} < x^2 + y^2 = \|C_1(M)\|_2^2.$$

Observe that $\|M\| \geqslant \|C_1(M)\|_2 = \sqrt{x^2+y^2}$. Moreover, direct calculations show that $\lambda = \sqrt{x^2+y^2}$ is an eigenvalue with corresponding eigenvector $v_+ = \left\{\frac{1}{\sqrt{2}}, \frac{x}{\sqrt{2}\sqrt{x^2+y^2}}, \frac{y}{\sqrt{2}\sqrt{x^2+y^2}}\right\}$, and $-\lambda$ is an eigenvalue with corresponding eigenvector $v_- = \left\{\frac{1}{\sqrt{2}}, -\frac{x}{\sqrt{2}\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{2}\sqrt{x^2+y^2}}\right\}$.

If we consider $v_\mu = \left\{0, -\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right\}$ it is apparent that v_μ is the corresponding eigenvector of $\mu = -\frac{(x^2+y^2)z}{xy}$. Then, using (4.1)

$$\mu^2 = \frac{\left(x^2+y^2\right)^2 z^2}{x^2y^2} < \left(x^2+y^2\right) = \lambda^2.$$

Therefore v_+ and v_- satisfy the condition of Theorem 1 and M is minimal.

Cases 2 and 3 are proved in a similar way.

Let us consider now case 4. Note that in this case it can be computed the spectrum $\sigma(M) = \left\{\pm\frac{x^2y^2+x^2z^2+y^2z^2}{2xyz}\right\}$. The eigenvalue $\frac{x^2y^2+x^2z^2+y^2z^2}{2xyz}$ has multiplicity one and its eigenspace is generated by $v = (xy, xz, yz)$. The eigenvector $\frac{1}{\|v\|}v$ is triangular in the sense of [1, Definition 3.2] because it satisfies inequalities (4.2). That is, the coordinates of $v \circ \overline{v}$ can form the sides of a triangle (any coordinate is bigger than the sum of the two others). Under these hypothesis there is another triangular vector w orthogonal to v (see [1, Proposition 3.5]). Therefore, w belongs to the dimension two eigenspace of $-\frac{x^2y^2+x^2z^2+y^2z^2}{2xyz}$. Then using Theorem 1 M is minimal.

□

Remark 2. *From the proof of the previous theorem follows that in the first three cases the norm of the matrix is the norm of the column that has a zero entry being this the column with greatest norm.*

The first three cases verify that $|\mu| < \lambda$ and the fourth that $|\mu| = \lambda$.

Theorem 3. *If $x, y, z \in \mathbb{R}$, $x, y, z \neq 0$, then*

$$M = \begin{pmatrix} 0 & x\,i & -y\,i \\ -x\,i & 0 & z\,i \\ y\,i & -z\,i & 0 \end{pmatrix}$$

is minimal.

Proof. The eigenvalues of M are: $\pm\sqrt{x^2+y^2+z^2}$ and $\mu = 0$. Then

$$v_+ = \left(-\frac{x\sqrt{x^2+y^2+z^2} + iyz}{\sqrt{2}\left(z\sqrt{x^2+y^2+z^2} - ixy\right)}, -\frac{x^2+z^2}{\sqrt{2}\left(xy + iz\sqrt{x^2+y^2+z^2}\right)}, \frac{1}{\sqrt{2}}\right)$$

is an eigenvector associated to $\sqrt{x^2+y^2+z^2}$, and

$$v_- = \left(-\frac{x\sqrt{x^2+y^2+z^2} - iyz}{\sqrt{2}\left(z\sqrt{x^2+y^2+z^2} + ixy\right)}, -\frac{x^2+z^2}{\sqrt{2}\left(xy - iz\sqrt{x^2+y^2+z^2}\right)}, \frac{1}{\sqrt{2}} \right)$$

an eigenvector associated to $-\sqrt{x^2+y^2+z^2}$. Clearly v_+ and v_- satisfy Theorem 1 and therefore M is minimal. $\square$

Remark 3. *Let $x, y, z \in \mathbb{R}_{\geqslant 0}$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Then if*

$$M = \begin{pmatrix} a & x\ e^{i\alpha} & y\ e^{-i\beta} \\ x\ e^{-i\alpha} & b & z\ e^{i\gamma} \\ y\ e^{i\beta} & z\ e^{-i\gamma} & c \end{pmatrix} \tag{4.3}$$

is a minimal matrix, its characteristic polynomial is

$$\begin{aligned} P_M[t] = -t^3 + t^2(a+b+c) + t\left(-ab - ac - bc + x^2 + y^2 + z^2\right) + \\ + abc - az^2 - by^2 - cx^2 + 2xyz\cos(\alpha+\beta+\gamma). \end{aligned} \tag{4.4}$$

Therefore, if $\cos(\theta) = \cos(\alpha+\beta+\gamma)$ (where we can chose $0 \leqslant \theta \leqslant \pi$) then the following matrix

$$M_\theta = \begin{pmatrix} a & x\ e^{i\theta} & y \\ x\ e^{-i\theta} & b & z \\ y & z & c \end{pmatrix} \tag{4.5}$$

is also minimal. The reason of this fact is that $P_M[t] = P_{M_\theta}[t]$ including how the terms x, y, z, a, b, c and $\cos(\theta) = \cos(\alpha+\beta+\gamma)$ appear (this implies that changing a, b, c the norm of M_θ cannot be made smaller without contradicting that M is minimal). Therefore M_θ is also minimal with the same minimizing diagonal as M.

This is also obvious if we see that $M_\theta = UMU^*$ for U the unitary diagonal matrix

$$U = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i(\alpha-\beta-\gamma)} & 0 \\ 0 & 0 & e^{i(\alpha-\beta)} \end{pmatrix}. \tag{4.6}$$

Proposition 4. *Let x, y, $z \in \mathbb{R}_{>0}$ and $\theta \in [0, \pi]$ such that $M_\theta = \begin{pmatrix} a & x\ e^{i\theta} & y \\ x\ e^{-i\theta} & b & z \\ y & z & c \end{pmatrix}$ is minimal. Then the following self-adjoint matrices are minimal as well as their transposes (with a, b, c the same as those of the diagonal of M_θ): $\begin{pmatrix} b & x\ e^{i\theta} & z \\ x\ e^{-i\theta} & a & y \\ z & y & c \end{pmatrix}$, $\begin{pmatrix} c & z\ e^{i\theta} & y \\ z\ e^{-i\theta} & b & x \\ y & x & a \end{pmatrix}$, $\begin{pmatrix} b & z\ e^{i\theta} & x \\ z\ e^{-i\theta} & c & y \\ x & y & a \end{pmatrix}$, $\begin{pmatrix} a & y\ e^{i\theta} & x \\ y\ e^{-i\theta} & c & z \\ x & z & b \end{pmatrix}$, $\begin{pmatrix} c & y\ e^{-i\theta} & z \\ y\ e^{i\theta} & a & x \\ z & x & b \end{pmatrix}$. Moreover, the factor $e^{i\theta}$ can be in any of the x, y, z entries above the diagonal of the previous matrices (completed conjugated below the diagonal) without changing the minimizing diagonal.*

Proof. The proof follows after similar considerations as the ones done about the characteristic polynomials of the matrices in the previous Remark 3. It can also be proved using conjugation of M_θ by permutation matrices or permutations of the unitary diagonals $\begin{pmatrix} e^{\pm i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\pm i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\pm i\theta} \end{pmatrix}$ and using that those conjugations produce matrices whose eigenvectors are permutations of the coordinates of eigenvectors of M_θ or permutations with one coordinate multiplied by $e^{\pm i\theta}$. The proof is completed using this property relating the modulus of the coordinates of the eigenvectors of minimal matrices (see Theorem 1), the fact that they correspond to the same eigenvalues of M_θ and the fact that M is minimal if and only if M^t is minimal. □

Remark 4. *Observe that using the above Remark 3 and the Proposition 4 we can suppose that $M = M_\theta$ as in (4.5), since any other matrix has its minimizing diagonal equal to one of this type or at least a permutation of its diagonal. Moreover since minimizing diagonals have been described in the cases an off-diagonal entry of the matrix is zero (see Propositions 2, 3) and in the real case (see Theorem 2) we can also suppose that*

- $0 < \theta < \pi$ *(because the cases $\theta = 0$ or $\theta = \pi$ have the same minimizing diagonals that the real symmetric matrices and for other $\theta \notin (0, \pi)$ is enough to consider the case of $\theta_1 \in (0, \pi)$ such that $\cos(\theta_1) = \cos(\theta)$) and that*
- $x \geqslant y \geqslant z > 0$ *(in view of Proposition 4).*

Note that the above Proposition 4 and the previous Remark 3 prove that if two matrices have its off-diagonal entries with equal module (even if they are permuted in their positions) and if $\cos(\theta) = \cos(\alpha + \beta + \gamma)$ (with α, β, γ as in (4.3) and θ as in (4.5)) then its minimizing diagonals coincide (with the corresponding permutations if necessary).

Remark 5. The unique minimizing diagonals are continuous functions of x, y, z and θ.

If $M_\theta \in M_n^h(\mathbb{C})$ is a minimal matrix as 4.5 with x, y, z not null and $\theta \in \mathbb{R}$, then its diagonal is unique (see Proposition 1). Denote with $\mathcal{O}(M_\theta) = M_\theta - Diag(M_\theta)$ the matrix with the same off-diagonal entries than M_θ and zero diagonal (recall that $Diag(M_\theta)$ is the diagonal matrix with the same diagonal of M_θ). Suppose that $M_m \in M_n^h(\mathbb{C})$ (for $m \in \mathbb{N}$), are minimal matrices with non zero off diagonal entries and $\lim_{m\to\infty} \mathcal{O}(M_m) = \mathcal{O}(M_\theta)$. Eventually choosing a subsequence we can suppose that $Diag(M_m)$ converges to a real diagonal D_0 (this follows considering that the sequence $Diag(M_n)$ must be bounded and therefore has an accumulation point).

Suppose that $D_0 \neq Diag(M_\theta)$. Then given $\varepsilon > 0$ and choosing m_0 such that $\|\mathcal{O}(M_\theta) - \mathcal{O}(M_m)\| < \varepsilon$ and $\|Diag(M_m) - D_0\| < \varepsilon$ for all $m \geqslant m_0$. Then

$$\begin{aligned} \|\mathcal{O}(M_\theta) + D_0\| &= \|\mathcal{O}(M_\theta) \pm M_m + D_0\| \\ &= \|\mathcal{O}(M_\theta) - \mathcal{O}(M_m) + M_m + D_0 - Diag(M_m))\| \\ &< \varepsilon + \|M_m\| + \varepsilon \\ &\leqslant 2\varepsilon + \|\mathcal{O}(M_m) + D\| = 2\varepsilon + \|\mathcal{O}(M_m) \pm \mathcal{O}(M_\theta) + D\| \\ &< 3\varepsilon + \|\mathcal{O}(M_\theta) + D\| \end{aligned}$$

for every real diagonal D and $\varepsilon > 0$. This contradicts the uniqueness of the minimal diagonal of $\mathcal{O}(M_\theta)$. Therefore $\lim_{m\to\infty} Diag\,(M_m) = D_0 = Diag\,(M_\theta)$. This proves the continuity of the map that carries x, y, z, θ to the entries of the unique diagonal of the minimal matrix corresponding to $\mathcal{O}(M_\theta)$.

Corollary 1. *Let $x \in \mathbb{R}_{>0}$ and $0 < \theta < \pi$, then* $M = \begin{pmatrix} a & x\, e^{i\theta} & x \\ x\, e^{-i\theta} & b & x \\ x & x & c \end{pmatrix}$ *is minimal if and only if* $a = b = c = -x\cos\left(\frac{\theta+\pi}{3}\right)$.

Proof. The equality $a = b = c$ follows from the previous considerations. If we set $a = b = c = -x\cos\left(\frac{\theta+\pi}{3}\right)$ the eigenvalues and eigenvectors of M can be explicitly computed. Then using Theorem 1 it can be proved that M is a minimal matrix with that choice of a, b and c. This is the only possible choice because the minimizing diagonal is unique (see Proposition 1). □

Proposition 5. *Let M be a matrix as in (4.3) with $x, y, z \in \mathbb{R}_{>0}$, $\alpha, \beta, \gamma, a, b, c \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $\alpha + \beta + \gamma = k\pi + \frac{\pi}{2}$ *with $k \in \mathbb{Z}$ and $a = b = c = 0$,*
(ii) *M is minimal and $\sigma(M) = \{\lambda, -\lambda, 0\}$, for $\lambda = \|M\|$.*

Proof. (i)⇒ (ii). If $\alpha + \beta + \gamma = k\pi + \frac{\pi}{2}$ and $a = b = c = 0$ it can be checked that the eigenvalues of M are $\pm\lambda = \pm\sqrt{x^2 + y^2 + z^2}$ and 0 and that there are corresponding eigenvectors of $\pm\lambda$ that satisfy Theorem 1. Therefore (ii) holds.

(ii)⇒(i). If M satisfies (ii) then its characteristic polynomial is $P_M[t] = -t^3 + t^2(a + b + c) + t\left(-ab - ac - bc + x^2 + y^2 + z^2\right) + abc - az^2 - by^2 - cx^2 + 2xyz\cos(\theta)$ where $\theta = \alpha + \beta + \gamma$. In this case since $\sigma(M) = \{\lambda, -\lambda, 0\}$ then $P_M[t] = -t^3 + t\lambda^2$ (see [1, 3.3] for details). The condition (ii) implies that $\mathrm{tr}(M) = 0 = a + b + c$ and that

$$\|M\|^2 = \lambda^2 = -ab - ac - bc + x^2 + y^2 + z^2. \tag{4.7}$$

Note that

$$\begin{aligned} \mathrm{tr}(M^2) &= 2\lambda^2 = (a^2 + x^2 + y^2) + (x^2 + b^2 + z^2) + (y^2 + z^2 + c^2) \\ &= 2(-ab - ac - bc + x^2 + y^2 + z^2) \end{aligned}$$

and then $a^2 + b^2 + c^2 = 2(-ab - ac - bc) \geqslant 0$. Therefore using (4.7) if $\|M\|$ is a minimum for a, b, c, then $a = b = c = 0$. Then, the coefficient of P_M given by $abc - az^2 - by^2 - cx^2 + 2xyz\cos(\theta) = 2xyz\cos(\theta) = 0$. Therefore $\cos(\theta) = 0$ and $\alpha + \beta + \gamma = k\pi + \frac{\pi}{2}$ with $k \in \mathbb{Z}$. □

Corollary 2. *Let M be a matrix as in (4.3) with $x, y, z \in \mathbb{R}_{>0}$, $\alpha, \beta, \gamma, a, b, c \in \mathbb{R}$. If M is minimal then the following statements are equivalent:*

(i) $\alpha + \beta + \gamma = k\pi + \frac{\pi}{2}$, *for $k \in \mathbb{Z}$,*
(ii) $a = b = c = 0$,
(iii) $\sigma(M) = \{\lambda, -\lambda, 0\}$, *for $\lambda = \|M\|$.*

Proof. The proof is direct using Remark 3, Propositions 1 and 5. □

Proposition 6. *If M_θ as in (4.5) is a minimal not null matrix such that $\sigma(M) = \{\lambda, \mu, -\lambda\}$ with $|\mu| = \lambda$, then x, y, z must be nonzero and $\theta = k\pi$, with $k \in \mathbb{Z}$.*

Proof. It is easy to prove in this case that $M_\theta^2 = \lambda^2 I$ and then the columns of M_θ are orthogonal vectors of norm λ. Easy calculations then prove that if one of the off-diagonal entries of M_θ is zero then all the others must be zero. Then it must be $x \neq 0$, $y \neq 0$ and $z \neq 0$.

Then it is apparent that $axe^{i\theta} + bxe^{i\theta} + yz = 0$ and then $ia\sin(\theta)x + ib\sin(\theta)x = 0$ and if we suppose $\sin(\theta) \neq 0$ it implies that $a = -b$.

In the same way we can prove that $aye^{-i\theta}+cye^{-i\theta}+xz=0$, and then $a=-c$, and $bze^{i\theta}+cze^{i\theta}+xy=0$ which proves that $b=-c$. Therefore $a=-b=-(-c)=-a$ and then $a=b=c=0$. Nevertheless $a+b+c=\mu\neq 0$, then it must be $\sin(\theta)=0$, and then $\theta=k\pi\ k\in\mathbb{Z}$. □

Theorem 4. *If $M\in M_3^h(\mathbb{C})$ is a minimal matrix with nonzero off diagonal entries and spectrum $\{\lambda,\mu,-\lambda\}$ ($\|M\|=\lambda\geqslant|\mu|$), then there exist corresponding orthogonal eigenvectors v_λ, $v_{-\lambda}$ and v_μ such that*

$$M=\lambda\ (v_\lambda\otimes v_\lambda)-\lambda\ (v_{-\lambda}\otimes v_{-\lambda})+\mu\ (v_\mu\otimes v_\mu)\,,$$

$N=\lambda\ (v_\lambda\otimes v_\lambda)-\lambda\ (v_{-\lambda}\otimes v_{-\lambda})$ is minimal and Diag($\mu\ (v_\mu\otimes v_\mu)$) =Diag($M$).

Proof. Let us suppose first that $|\mu|<\lambda$. Then all eigenspaces have dimension one and any choice of the eigenvectors v_λ, $v_{-\lambda}$ corresponding to λ and $-\lambda$ verify Theorem 1. Then using the same theorem N is minimal, and using Proposition 5 then Diag(N) = 0. Therefore Diag($\mu\ (v_\mu\otimes v_\mu)$) =Diag($M$).

If $|\mu|=\lambda$ then one of the eigenspaces corresponding to λ or $-\lambda$ has dimension two. Since M is minimal there exist eigenvectors v_λ and $v_{-\lambda}$ corresponding to the eigenvalues λ and $-\lambda$ such that $v_\lambda\circ\overline{v_\lambda}=v_{-\lambda}\circ\overline{v_{-\lambda}}$ (Theorem 1). Pick this eigenvectors and any other v_μ orthogonal to both of them. Then it can be proved similarly as above that they satisfy the claims of the theorem. □

Proposition 7. *Let $M_0,M_1\in\mathbb{C}^{3\times 3}$ be two minimal matrices with the same diagonal and eigenvalues $\{\lambda,\mu,-\lambda\}$, with $0\neq|\mu|\leqslant\lambda$, given by*

$$M_0=\begin{pmatrix} a & x_0\ e^{\alpha_0 i} & y_0\ e^{-\beta_0 i}\\ x_0\ e^{-\alpha_0 i} & b & z_0\ e^{\gamma_0 i}\\ y_0\ e^{\beta_0 i} & z_0\ e^{-\gamma_0 i} & c\end{pmatrix}\ \text{ and }\ M_1=\begin{pmatrix} a & x_1\ e^{\alpha_1 i} & y_1\ e^{-\beta_1 i}\\ x_1\ e^{-\alpha_1 i} & b & z_1\ e^{\gamma_1 i}\\ y_1\ e^{\beta_1 i} & z_1\ e^{-\gamma_1 i} & c\end{pmatrix},$$

with $x_0,y_0,z_0,x_1,y_1,z_1,\in\mathbb{R}_{>0}$.

Then $x_0=x_1$, $y_0=y_1$, $z_0=z_1$ and $\cos(\alpha_0+\beta_0+\gamma_0)=\cos(\alpha_1+\beta_1+\gamma_1)$.

Proof. M_0 and M_1 are matrices of *non-extremal* type in the sense of definition 3.5 of [1]. Note that $\mu=a+b+c\neq 0$. Following the same notations of (3.9) and (3.10) of that paper for α, β, γ, $(n_{12})_0$, $(m_{12})_0$ (for M_0) and $(n_{12})_1$, $(m_{12})_1$ (for M_1), then it must be $\alpha=\frac{a}{2(a+b+c)}$, $\beta=\frac{b}{2(a+b+c)}$ and $\gamma=\frac{c}{2(a+b+c)}$. Then, considering all the cases, it can be proved that $x_0=|x_0|=|\mu\,(n_{12})_0+\lambda\,(m_{12})_0|=|\mu\,(n_{12})_1+\lambda\,(m_{12})_1|=|x_1|=x_1$. The same reasoning could be done to prove $y_0=y_1$ and $z_0=z_1$.

Finally $\cos(\alpha_0+\beta_0+\gamma_0)=\cos(\alpha_1+\beta_1+\gamma_1)$ because the coefficients of the characteristic polynomial of each matrix are determined by $\{\lambda,\mu,-\lambda\}$ and using (4.4) we obtain that $-\lambda\mu=abc-az^2-by^2-cx^2+2xyz\cos(\alpha_0+\beta_0+\gamma_0)=abc-az^2-by^2-cx^2+2xyz\cos(\alpha_1+\beta_1+\gamma_1)$. □

We state here the following result that was already mentioned in Remark 4.

Proposition 8. *Let M_0 and M_1 be matrices with the structure of those of Proposition 7. If their off-diagonal entries have equal modulus $x_0=x_1$, $y_0=y_1$, $z_0=z_1$, and $\cos(\alpha_0+\beta_0+\gamma_0)=\cos(\alpha_1+\beta_1+\gamma_1)$, then both matrices have the same minimizing diagonal.*

Proof. The proof follows reducing each matrix to one like M_θ as in Remark 3 and then applying Proposition 4. □

Theorem 5. *Let $x,y,z\in\mathbb{R}_{>0}$, $\theta\in\mathbb{R}$ and $M=\begin{pmatrix} a & x\ e^{i\theta} & y\\ x\ e^{-i\theta} & b & z\\ y & z & c\end{pmatrix}$ be a minimal matrix.*

Then there exist α, β, $\gamma\in[0,\pi]$ such that:

(i) $\cos(\alpha+\beta+\gamma) = \cos(\theta)$.

(ii) *The matrices N, S defined by*

$$N = \begin{pmatrix} 0 & i\,x\,\sin\alpha & -i\,y\,\sin\beta \\ -i\,x\,\sin\alpha & 0 & i\,z\,\sin\gamma \\ i\,y\,\sin\beta & -i\,z\,\sin\gamma & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} a & x\,\cos\alpha & y\,\cos\beta \\ x\,\cos\alpha & b & z\,\cos\gamma \\ y\,\cos\beta & z\,\cos\gamma & c \end{pmatrix} \tag{4.8}$$

satisfy that:

a) *$Diag(N+S) = Diag(M)$,*

b) *If $v \in \ker(N)$ with $\|v\| = 1 \;\Rightarrow\; S = (a+b+c)\,(v\otimes v)$,*

c) *$M_0 = N+S$ is minimal,*

d) *M_0 is unitarily equivalent to M or to M^t by means of unitary diagonals.*

Proof. Let us suppose that $\sigma(M) = \{\lambda, \mu, -\lambda\}$ with $|\mu| \leqslant \lambda = \|M\|$. Then, using Theorem 4, it can be proved that there exist v_λ, $v_{-\lambda}$ and v_μ orthonormal eigenvectors of λ, $-\lambda$ and μ respectively, such that $M = N + S$, with $N = \lambda(v_\lambda \otimes v_\lambda) - \lambda(v_{-\lambda}\otimes v_{-\lambda})$ a minimal matrix with $\mathrm{Diag}(N) = 0$ and $S = \mu(v_\mu \otimes v_\mu)$ satisfies $\mathrm{Diag}(S) = \mathrm{Diag}(M)$ (even in the case $|\mu| = \lambda$). Let $v_\mu = (r,s,t)$, then it is apparent that $a = \mu\,|r|^2$, $b = \mu\,|s|^2$, $c = \mu\,|t|^2$. Furthermore defining $\xi = |r|$, $\psi = |s|$ and $\zeta = |t|$, the matrix $N_1 = \lambda \begin{pmatrix} 0 & i\,\zeta & -i\,\psi \\ -i\,\zeta & 0 & i\,\xi \\ i\,\psi & -i\,\xi & 0 \end{pmatrix}$ is a minimal matrix and $\|N_1\| = \lambda$. Moreover, $v = (\xi,\psi,\zeta)$ is an eigenvector corresponding to the eigenvalue 0 of N_1.

Let $S_1 = \mu(v\otimes v) = \mu \begin{pmatrix} \xi^2 & \xi\psi & \xi\zeta \\ \psi\xi & \psi^2 & \psi\zeta \\ \zeta\xi & \zeta\psi & \zeta^2 \end{pmatrix}$.

By construction N_1 is minimal with $\sigma(N_1) = \{\lambda, 0, -\lambda\}$ and $\sigma(S_1) = \{\mu, 0\}$. Then,

$$M_1 = N_1 + S_1 = \begin{pmatrix} \mu\xi^2 & \mu\xi\psi + i\,\lambda\zeta & \mu\xi\,\zeta - i\,\lambda\psi \\ \mu\psi\xi - \,i\,\lambda\zeta & \mu\psi^2 & \mu\psi\zeta + i\,\lambda\xi \\ \mu\zeta\xi + i\,\lambda\psi & \mu\zeta\psi - i\,\lambda\xi & \mu\zeta^2 \end{pmatrix}$$

has the same diagonal than M and $\sigma(M_1) = \sigma(M)$.

If $\mu = 0$ then the diagonal of M must be zero and, using Proposition 5, it must be $\theta = k\pi + \frac{\pi}{2}$ for $k\in\mathbb{Z}$ and $\lambda = \sqrt{x^2+y^2+z^2}$. Then, choosing $\zeta = x/\lambda$, $\psi = y/\lambda$, $\xi = z/\lambda$, and $\alpha=\beta=\gamma=\pi/2$ if $\theta = (2k+1)\pi + \pi/2$, with $k\in\mathbb{Z}$, or $\alpha=\beta=\gamma=-\pi/2$ if $\theta = 2k\pi+\pi/2$, with $k\in\mathbb{Z}$, follows easily that N_1 is unitarily equivalent by means of diagonal matrices to M, and therefore the theorem is proved in this case.

If $\mu\neq 0$, using Proposition 7, then it must be $x = |\mu\xi\psi + i\,\lambda\zeta|$, $y = |\mu\xi\zeta - i\,\lambda\psi|$ and $z = |\mu\psi\zeta + i\,\lambda\xi|$. If we consider $0 \leqslant \arg(z) < 2\pi$ and define

$$\alpha = \arg(\mu\xi\psi + i\,\lambda\zeta)\ ,\ \beta = 2\pi - \arg(\mu\xi\zeta - i\,\lambda\psi)\ ,\ \gamma = \arg(\mu\psi\zeta + i\,\lambda\xi), \tag{4.9}$$

and $\theta_1 = \alpha+\beta+\gamma$, then $\alpha,\beta,\gamma \in [0,\pi]$ and from Proposition 7 follows that $\cos(\theta) = \cos(\theta_1)$.

Moreover M_1 is unitarily equivalent by means of unitary diagonals to M_{θ_1} (see (4.5) and (4.6)). Since $M_{\theta_1} = M_\theta$, or $M_{\theta_1} = M_{-\theta} = (M_\theta)^t$, then M_1 is unitary equivalent (by means of unitary diagonals) to M_θ or to its transpose. Choosing α, β and γ as have been defined before and putting $N = N_1$ and $S = S_1$ the items (i) and (ii) of the theorem follow. ☐

Remark 6. *With the same notations and hypothesis as those of Theorem 5 and its proof and using the fact that $M_0 = N+S$ is a minimal matrix we can consider different cases*

- *Two of the numbers ζ, ξ, ψ cannot be zero simultaneously because otherwise M_0 would not be equivalent to M by means of diagonal unitary matrices with $x, y, z \in \mathbb{R}_{>0}$.*
- *If only one of ζ, ξ, ψ is zero, M_0 is equivalent to a real matrix by means of diagonal unitary matrices (see (4.9) and Remark 3).*
- *If ζ, ξ and ψ are all not null and $\mu \neq 0$ ($\theta \neq k\pi + \pi/2$, $k \in \mathbb{Z}$), since we are supposing $\lambda = \|M\| = \|M_0\|$ follows that $Im((M_0)_{1,2}) = x\sin\alpha = \lambda\zeta \neq 0$, $Im((M_0)_{1,3}) = y\sin\beta = \lambda\psi \neq 0$ and $Im((M_0)_{2,3}) = z\sin\gamma = \lambda\xi \neq 0$. Therefore, in this case (since $x, y, z \in \mathbb{R}_{>0}$) $\sin\alpha \neq 0$, $\sin\beta \neq 0$ and $\sin\gamma \neq 0$ and also (since $\mu \neq 0$) $\cos\alpha \neq 0$, $\cos\beta \neq 0$ and $\cos\gamma \neq 0$.*

  Then it can be verified that

$$v_\mu = \frac{1}{\sqrt{(x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2}}\,(z\sin\gamma, y\sin\beta, x\sin\alpha) \tag{4.10}$$

  is an eigenvevector of M_0. Therefore by construction

$$a = \frac{(a+b+c)(z\sin\gamma)^2}{\lambda^2} \quad , \quad b = \frac{(a+b+c)(y\sin\beta)^2}{\lambda^2} \quad , \quad c = \frac{(a+b+c)(x\sin\alpha)^2}{\lambda^2}$$

  and $\lambda^2 = (x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2$.

As we proved before the cases $\theta = k\pi/2$ ($k \in \mathbb{Z}$) are equivalent to $\mu = 0$ or M_θ being equivalent to a real matrix by means of diagonal unitary matrices (see Remark 3). Then considering $\theta \neq k\pi/2$ ($k \in \mathbb{Z}$) it can also be proved that only the following corresponding disjoint cases can occur:

a) *If $\theta \in (\pi, \frac{3}{2}\pi)$ then $0 < \alpha < \pi/2$, $0 < \beta < \pi/2$ and $0 < \gamma < \pi/2$,*
b) *If $\theta \in (\frac{3}{2}\pi, 2\pi)$ then $\pi/2 < \alpha < \pi$, $\pi/2 < \beta < \pi$ and $\pi/2 < \gamma < \pi$.*

Proposition 9. *If $\theta \neq k\pi/2$ with $k \in \mathbb{Z}$ and with the hypothesis and notations of Theorem 5 and its proof, then the angles α, β and γ, and the matrices M and $M_0 = N + S$ fulfill the following conditions:*

1) $\cos\alpha \neq 0$, $\cos\beta \neq 0$ *and* $\cos\gamma \neq 0$
2) $x^2\sin(2\alpha) = y^2\sin(2\beta) = z^2\sin(2\gamma)$
3) $\|M\|^2 = \|M_0\|^2 = (x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2$
4) $Diag(M_0) = Diag(S) = Diag(M) = \left(\frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)}, \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)}, \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)}\right)$
5) $(x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2 \geqslant \left(\frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)} + \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)} + \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)}\right)^2$

Proof. Item 1) has been proved in the preceding Remark 6 in the case μ, ζ, ξ and ψ all not null.

Pick v_μ as in (4.10). Then $S_{1,2} = x\cos\alpha = ((a+b+c)\,(v_\mu \otimes v_\mu))_{1,2} = \frac{\mu zy\sin\gamma\sin\beta}{\lambda^2}$. Thus $\frac{\mu}{\lambda^2} = \frac{x\cos\alpha}{zy\sin\gamma\sin\beta}$. Similarly, considering $S_{1,3}$ we obtain $\frac{\mu}{\lambda^2} = \frac{y\cos\beta}{zx\sin\gamma\sin\alpha}$ and therefore $\frac{x\cos\alpha}{zy\sin\gamma\sin\beta} = \frac{y\cos\beta}{zx\sin\gamma\sin\alpha}$. Reordering we obtain

$$x^2\sin 2\alpha = y^2\sin 2\beta.$$

Using $S_{1,3}$ we obtain $\frac{\mu}{\lambda^2} = \frac{z\cos\gamma}{xy\sin\alpha\sin\beta}$ and reasoning as before we can prove 2).

From (ii) d) of Theorem 5, is apparent that M_0 and M have the same norm (that of N) and diagonal (that of S). The norm of N is $\sqrt{(x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2}$ which proves 3).

Using again the same v_μ as in (4.10) we obtain

$$\begin{aligned} S_{1,1} &= \mu(z\sin\gamma)^2/\lambda^2 \\ &= \frac{\left(\mu(\frac{z\sin\gamma}{\lambda})(\frac{y\sin\beta}{\lambda})\right)\left(\mu(\frac{x\sin\alpha}{\lambda})(\frac{z\sin\gamma}{\lambda})\right)}{\left(\mu(\frac{x\sin\alpha}{\lambda})(\frac{y\sin\beta}{\lambda})\right)} = \frac{S_{1,2}\ S_{1,3}}{S_{2,3}} \\ &= \frac{(x\cos\alpha)(y\cos\beta)}{(z\cos\gamma)} = \frac{xy\cos\alpha\cos\beta}{z\cos\gamma}. \end{aligned}$$

The formulas for $S_{2,2}$ and $S_{3,3}$ are obtained similarly which proves 4).

Items 3) and 4) imply 5) because since M is minimal, then $\operatorname{tr}(M)$ is an eigenvalue of M and therefore $\operatorname{tr}(M)^2 \leqslant \|M\|^2$.

□

Proposition 10. *If α, β, $\gamma \in \mathbb{R}$, $\alpha, \beta, \gamma \neq k\pi/2$ with $k \in \mathbb{Z}$, and $x, y, z \in \mathbb{R}_{>0}$, $M_0 = N + S$, with*

$$N = \begin{pmatrix} 0 & i\ x\ \sin\alpha & -i\ y\ \sin\beta \\ -i\ x\ \sin\alpha & 0 & i\ z\ \sin\gamma \\ i\ y\ \sin\beta & -i\ z\ \sin\gamma & 0 \end{pmatrix} \quad \textit{and}$$

$$S = \begin{pmatrix} \frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)} & x\ \cos\alpha & y\ \cos\beta \\ x\ \cos\alpha & \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)} & z\ \cos\gamma \\ y\ \cos\beta & z\ \cos\gamma & \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)} \end{pmatrix}$$

and α, β, γ, x, y, z satisfy:

1) $x^2\sin(2\alpha) = y^2\sin(2\beta) = z^2\sin(2\gamma)$
2) $(x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2 \geqslant \left(\frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)} + \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)} + \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)}\right)^2$

then, $NS = SN = 0$ and $M_0 = N + S$ is minimal.

Proof. Using 1) follows that $NS = 0$. Furthermore S has rank one and N rank two. Then $\operatorname{ran}(S) = \ker(N)$ and $\ker(S) = \operatorname{ran}(N)$ and $\sigma(S) = \{0, \operatorname{tr}(S)\}$. Therefore if we call

$$\lambda = \sqrt{x^2\sin^2(\alpha) + y^2\sin^2(\beta) + z^2\sin^2(\gamma)}$$

follows that $\sigma(N) = \{0, \lambda, -\lambda\}$. Then $\sigma(N + S) = \{\operatorname{tr}(S), \lambda, -\lambda\}$, and using 2) then $M_0 = N + S$ verifies $\|M_0\| = \|N\| = \lambda = \sqrt{x^2\sin^2(\alpha) + y^2\sin^2(\beta) + z^2\sin^2(\gamma)}$. Furthermore the eigenvectors of M_0 corresponding to the eigenvalues $\pm\lambda$ are the same than that of N (that is a minimal matrix because of Proposition 5) and therefore they verify the conditions of Theorem 1. Therefore M_0 is minimal. □

Theorem 6. *Given a minimal matrix of the form*

$$M = \begin{pmatrix} a & x\ e^{i\theta} & y \\ x\ e^{-i\theta} & b & z \\ y & z & c \end{pmatrix} \ \textit{with}\ x \geqslant y \geqslant z > 0\ \textit{and}\ \theta \in \left(\frac{3}{2}\pi, 2\pi\right) \tag{4.11}$$

then there exist unique $\alpha \in (\pi/2, \frac{3}{4}\pi]$, $\beta \in (\pi/2, \frac{3}{4}\pi]$, $\gamma \in (\pi/2, \pi)$ which are continuous functions of θ, x, y, z such that:

(1) $\alpha + \beta + \gamma = \theta$

(2) *The matrices N, S defined by*

$$N = \begin{pmatrix} 0 & i\ x\ \sin\alpha & -i\ y\ \sin\beta \\ -i\ x\ \sin\alpha & 0 & i\ z\ \sin\gamma \\ i\ y\ \sin\beta & -i\ z\ \sin\gamma & 0 \end{pmatrix} \tag{4.12}$$

and

$$S = \begin{pmatrix} a & x\ \cos\alpha & y\ \cos\beta \\ x\ \cos\alpha & b & z\ \cos\gamma \\ y\ \cos\beta & z\ \cos\gamma & c \end{pmatrix} \tag{4.13}$$

verify that:

a) $Diag(N+S) = Diag(M)$

b) *If* $v \in \ker(N)$ *with* $v \in \mathbb{R}^3$ *and* $\|v\| = 1 \Rightarrow S = (a+b+c)(v \otimes v)$

c) $M_0 = N + S$ *is minimal.*

d) M_0 *is unitarily equivalent to* M *or to* M^t *by means of diagonal unitaries.*

and

1') $x^2\sin(2\alpha) = y^2\sin(2\beta) = z^2\sin(2\gamma)$

2') $\|M\|^2 = \|M_0\|^2 = (x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2$

3') $Diag(M_0) = Diag(M) = \left(\frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)}, \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)}, \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)}\right)$

4') $(x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2 \geqslant \left(\frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)} + \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)} + \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)}\right)^2$

Proof. It only remains to prove that for θ, x, y, z fixed the angles α, β and γ that fulfill the conditions of the Theorem are unique, that they can be chosen in the specified intervals and that they are continuous functions of θ.

Analyzing the signs of the real and imaginary parts of the complexes such that their arguments define the angles α, β and γ that appear in the proof of the Theorem 5 we can conclude that in this case, (since we can prove that $\mu \leqslant 0 \Leftrightarrow \theta \in [\frac{3}{2}\pi, 2\pi]$) we can choose $\alpha,\beta,\gamma \in [\pi/2, 2\pi]$. If we consider $\mu < 0$ ($\mu = 0$ corresponds to $\theta = 3\pi/2$ that as Corollary 2 states it has the same minimizing diagonals than those considered in Theorem 3), then we can suppose that (for α, β, γ from Theorem 5) $x_c = x\cos\alpha$, $x_s = x\sin\alpha$, $y_c = y\cos\beta$, $y_s = y\sin\beta$, $z_c = z\cos\gamma$ and $z_s = z\sin\gamma$ are all non zero (as it was discussed in Remark 6). Then using the inequality 4') we obtain

$$z_c^2 y_c^2 x_c^2 \left(x_s^2 + y_s^2 + z_s^2\right) \geqslant \left(x_c^2 y_c^2 + x_c^2 z_c^2 + y_c^2 z_c^2\right)^2$$

and with 1'), if we denote with $k = x_c x_s = y_c y_s = z_c z_s$ we can prove that

$$k^2 \geqslant \left(x_c^2 y_c^2 + x_c^2 z_c^2 + y_c^2 z_c^2\right) \tag{4.14}$$

We will prove first that $\alpha \notin (\frac{3}{4}\pi, \pi)$. Suppose that $\alpha \in (\frac{3}{4}\pi, \pi)$ and consider two cases:

a) $\beta \in (\alpha, \pi)$: in this case since $x_c x_s = y_c y_s \wedge y \leqslant x$ then $\sin(\beta) < \sin(\alpha)$, $y_s < x_s$ then $x_s \leqslant |x_c| < |y_c|$ and then

$$k^2 = x_s^2 x_c^2 < y_c^2 x_c^2 < \left(x_c^2 y_c^2 + x_c^2 z_c^2 + y_c^2 z_c^2\right),$$

which contradicts (4.14).

b) $\beta \in (\pi/2, \alpha]$:

(i) if $\beta \in [3/4\pi, \alpha]$ then $|y_s| \leqslant |y_c| \leqslant |x_c|$ and then

$$k^2 = y_s^2 y_c^2 \leqslant x_c^2 y_c^2 < \left(x_c^2 y_c^2 + x_c^2 z_c^2 + y_c^2 z_c^2\right),$$

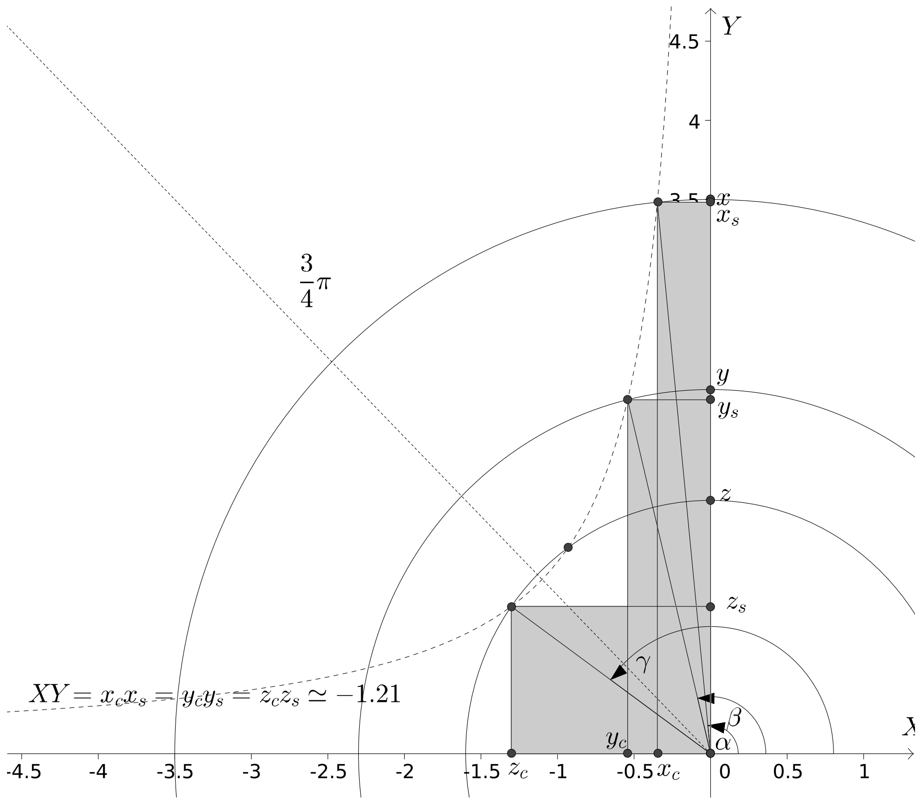

FIGURE 1. The corresponding α, β and γ for $\theta = 6.$, $x = 3.5$, $y = 2.3$ and $z = 1.6$.

which contradicts (4.14).

(ii) if $\beta \in (\pi/2, 3/4\pi)$ we will compare $|x_c|$ with y_s

(ii$_1$) If $|x_c| \geqslant y_s$ then

$$k^2 = y_s^2 y_c^2 \leqslant x_c^2 y_c^2 < \left(x_c^2 y_c^2 + x_c^2 z_c^2 + y_c^2 z_c^2\right),$$

which contradicts (4.14).

(ii$_2$) If $|x_c| < y_s$ then (recall that $x_c, y_c < 0$) $y_s + x_c > 0$. Moreover $x_s^2 + x_c^2 = x^2 \geqslant y^2 = y_s^2 + y_c^2$, then $(x_s + x_c)^2 = x_s^2 + 2x_s x_c + x_c^2 \geqslant y_s^2 + 2y_s y_c + y_c^2 = (y_s + y_c)^2$, and then $|x_s + x_c| \geqslant |y_s + y_c|$, but $0 < x_s < |x_c|$ and $0 < |y_c| < y_s$, which proves that $-x_s - x_c \geqslant y_s + y_c$. Then $-x_s - y_c \geqslant y_s + x_c > 0$ and hence $-y_c > x_s$ holds and

$$k^2 = x_s^2 x_c^2 < y_c^2 x_c^2 < \left(x_c^2 y_c^2 + x_c^2 z_c^2 + y_c^2 z_c^2\right),$$

which contradicts (4.14).

Then $\alpha \notin (\frac{3}{4}\pi, \pi)$ holds and if $\theta \in (\frac{3}{2}\pi, 2\pi)$ then $\alpha \in (\pi/2, \frac{3}{4}\pi]$.

Similarly, comparing $|y_c|$ with $|z_c|$ it can be proved that $\beta \notin (\frac{3}{4}\pi, \pi)$ and therefore $\beta \in (\pi/2, \frac{3}{4}\pi]$, and that $\gamma \in [\beta, \frac{3}{2}\pi - \beta] \subset [\pi/2, \pi]$ (see Figure 1).

Uniqueness:

The angles α and β are unique in this intervals because they must fulfill the conditions $x_c x_s = y_c y_s = k$, $\pi/2 \leqslant \alpha \leqslant \frac{3}{4}\pi$ and $\pi/2 \leqslant \beta \leqslant \frac{3}{4}\pi$. If there are two different angles γ and γ' in $(\pi/2, \pi)$ that fulfill the conditions of Theorem 5 and Proposition 9, then the only posible case is that one belongs to $(\beta, \frac{3}{4}\pi)$ and the other one to $(\frac{3}{4}\pi, \frac{3}{2}\pi - \beta)$. Suppose that $\beta < \gamma \leqslant \frac{3}{4}\pi$ and $\frac{3}{4}\pi < \gamma' \leqslant \frac{3}{2}\pi - \beta$. then only γ' satisfies the conditions of Proposition 9. This is because, if both satisfy the minimality conditions of that proposition, then $\lambda^2 = \|M\| = (x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma)^2 = (x\sin\alpha)^2 + (y\sin\beta)^2 + (z\sin\gamma')^2$, a contradiction because $\sin\gamma' < \sin\gamma$.

If x, y, z are fixed, we denote with $\alpha = \alpha(\theta)$, $\beta = \beta(\theta)$ and $\gamma = \gamma(\theta)$ the angles that are uniquely determined by θ in the corresponding intervals. If we trace the definition of these angles given in 4.9 of Theorem 5, it can be seen that it is a continuous application with respect to θ (and also with respect to x,y,z).

The sum of α, β, and γ gives θ:

Considering that $\theta \in (\frac{3}{2}\pi, 2\pi)$, $\alpha(\theta), \beta(\theta) \in (\pi/2, \frac{3}{4}\ \pi)$, and $\gamma(\theta) \in \big(\beta(\theta), \frac{3}{2}\pi - \beta(\theta)\big)$ then $3/2\ \pi \leqslant \alpha(\theta) + \beta(\theta) + \gamma(\theta) \leqslant 9/4\ \pi$. Therefore, since $\cos(\alpha(\theta) + \beta(\theta) + \gamma(\theta)) = \cos(\theta)$, then by continuity and uniqueness arguments $\alpha(\theta) + \beta(\theta) + \gamma(\theta) = \theta$ holds for every $\theta \in (\frac{3}{2}\pi, 2\pi)$.

□

Remark 7. *Given a minimal matrix M_θ as in 4.11 with $\theta = \frac{3}{2}\pi$, $x \geqslant y \geqslant z > 0$ then $\sigma(M_{3\pi/2}) = \{\lambda, 0, \lambda\}$ (see Corollary 2). M_θ has the same minimizing diagonals than those matrices considered in Theorem 3. Then we can define $\alpha(\pi/2) = \beta(\pi/2) = \gamma(\pi/2) = \pi/2$ and they satisfy 1), 2) and 1') through 4') of Theorem 6. As we will see this definition makes α, β and γ continuous in terms of $\theta \in (\pi, 2\pi)$.*

In the case $\theta \in (\pi, \frac{3}{2}\pi)$ let us consider $\theta' = 3\pi - \theta$. Then $\theta' \in (\frac{3}{2}\pi, 2\pi)$ and if we denote with α', β' and γ' the solutions which existence was proved in Theorem 6, then it is enough to take $\alpha = \pi - \alpha'$, $\beta = \pi - \beta'$ and $\gamma = \pi - \gamma'$ and verify that these angles α, β and $\gamma \in (0, \pi/2)$ satisfy all the required conditions 1), 2) and 1') through 4') of Theorem 6.

If $\theta \in (\frac{3}{2}\pi, 2\pi)$ it is apparent that if θ is close to $\frac{3}{2}\pi$ then $\alpha(\theta), \beta(\theta)$ and $\gamma(\theta)$ from the previous Theorem must be close to $\pi/2$. Then α, β and γ are right continuous in $\theta = \frac{3}{2}\pi$, i.e. $\lim_{\theta\to 3\pi/2^+} \alpha(\theta) = \lim_{\theta\to 3\pi/2^+} \beta(\theta) = \lim_{\theta\to 3\pi/2^+} \gamma(\theta) = \pi/2$. Similarly it can be proved that α, β and γ are left continuous in $\theta = \frac{3}{2}\pi$.

If $\theta \in (\pi, \frac{3}{2}\pi)$ then similar considerations as the ones made before (using the proven uniqueness, continuity and sum of α, β, γ of the previous case) prove that also in this case $\alpha + \beta + \gamma = \theta$.

If $\theta = \frac{3}{2}\pi$ choosing $\alpha = \beta = \gamma = \pi/2$, then obviously $\alpha + \beta + \gamma = \theta$, and because of the previous considerations α, β and γ are continuous functions of θ in the whole interval $(\pi, 2\pi)$.

Remark 8. Algorithm. *Given a generic Hermitian matrix it can be conjugated with diagonal unitary matrices (see remark 4 and Proposition 4) to obtain a matrix with the structure* $M_\theta = \begin{pmatrix} a & x\,e^{i\theta} & y \\ x\,e^{-i\theta} & b & z \\ y & z & c \end{pmatrix}$ *with $x \geqslant y \geqslant z > 0$ and $\theta \in [0, 2\pi)$.*

We discuss next how to find the diagonal matrices that added to M_θ give a minimal matrix.

Case $\theta = 0$ or $\theta = \pi$:

In this case the minimizing diagonal coincides with that of the real case (see Proposition 6) and therefore it was computed exactly in Theorem 2.

Case $0 < \theta < \pi$:

This case corresponds to the transpose of a matrix from the case where $0 \leqslant \theta < 2\pi$ that has the same minimizing diagonal. That is, if $\theta_0 \in [0, \pi)$, then $-\theta_0 \in (-\pi, 0]$ and the minimizing diagonal *corresponding to θ_0 is the same that the one corresponding to $-\theta_0$ that is described in the following case.*

Case $\pi < \theta < 2\pi$:

Using the results and notations of the previous theorem for a minimal matrix M with the structure of (4.11) and considering the cases $\mu \in (-\lambda, 0)$ (that is equivalent to $\theta \in (\frac{3}{2}\pi, 2\pi)$), or $\mu \in (0, \lambda)$ (that is equivalent to $\theta \in (\pi, \frac{3}{2}\pi)$), or $\mu = 0$ (that is equivalent to $\theta = \frac{3}{2}\pi$), then it can be proved that the unique angles $\alpha \in (\pi/2, \frac{3}{4}\pi)$, $\beta \in (\pi/2, \frac{3}{4}\pi)$, $\gamma \in (\beta, \frac{3}{2}\pi - \beta)$ from Theorem 6 must satisfy

$$\alpha + \beta + \gamma = \theta \quad , \quad \alpha = \frac{1}{2}\left(\pi - \arcsin\left(\frac{z^2 \sin(2\gamma)}{x^2}\right)\right) \quad , \quad \beta = \frac{1}{2}\left(\pi - \arcsin\left(\frac{z^2 \sin(2\gamma)}{y^2}\right)\right)$$

Observe that the uniqueness of these angles in the specified intervals for each θ and under the conditions

$$\alpha + \beta + \gamma = \theta$$
$$x^2 \sin(2\alpha) = y^2 \sin(2\beta) = z^2 \sin(2\gamma)$$
$$(x \sin\alpha)^2 + (y \sin\beta)^2 + (z \sin\gamma)^2 \geqslant \left(\frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)} + \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)} + \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)}\right)^2$$

imply that the root of

$$\frac{1}{2}\left(2\pi - \arcsin\left(\frac{z^2\sin(2\gamma)}{x^2}\right) - \arcsin\left(\frac{z^2\sin(2\gamma)}{y^2}\right)\right) + \gamma - \theta = 0$$

closer to $\gamma = \frac{3}{4}\pi$ is the wanted solution and is easily approximated by a standard numerical method. After obtaining γ with the desired precision α, β and the wanted minimizing diagonal

$$\mathrm{Diag}\left(\frac{xy\cos(\alpha)\cos(\beta)}{z\cos(\gamma)}, \frac{xz\cos(\alpha)\cos(\gamma)}{y\cos(\beta)}, \frac{yz\cos(\beta)\cos(\gamma)}{x\cos(\alpha)}\right).$$

can also be approximated as much as needed.

To obtain the minimal matrix corresponding to the original matrix M, then the inverse conjugation with diagonal unitary matrices used to obtain M_θ may be required. This inverse conjugation applied to the minimizing diagonal of M_θ gives the minimizing diagonal of M. Note that this operation can only change the order of the diagonal entries.

5. Some $n \times n$ cases

In this section we describe the minimizing diagonals for some particular $n \times n$ Hermitian matrices.

Theorem 7. *If $M \in \mathbb{C}^{n\times n}$ is a Hermitian matrix such that diag$(M) = 0$ and Re$(M_{i,j}) = 0$ for all i, j, then M is minimal.*

Proof. Let us suppose that v_λ is an eigenvector of $\lambda = \|M\|$. Then, it is apparent that $-\lambda \in \sigma(M)$ and that the vector $\overline{v_\lambda}$ is an eigenvector of $-\lambda$. Since $|(v_\lambda)_i| = |(\overline{v_\lambda})_i|$ for every i, a generalization of Theorem 1 (see [2, Corollary 3]) proves that M is minimal. □

In the next theorem for $M \in \mathbb{C}^{n\times n}$ we denote with $C_j(M)$ the j^{th} column of M, with $M_{\breve{k}}$ the matrix in $\mathbb{C}^{(n-1)\times(n-1)}$ resulting after taking off the k^{th} column and row of M and with $v_{\breve{k}}$ the element of $\mathbb{C}^{n-1}$ obtained after taking off the k^{th} entry of $v \in \mathbb{C}^n$.

Theorem 8. *Let $N \in \mathbb{R}^{n\times n}$ be a symmetric matrix such that*

- *its k^{th} column $C_k(N)$ satisfies that $C_k(N)_i = N_{i,k} \neq 0, \quad \forall i \neq k$,*
- $N_{i,i} = \begin{cases} 0 & \textit{if } j = k \\ -\frac{C_j(N) \cdot C_k(N)}{N_{j,k}} & \textit{if } j \neq k. \end{cases}$
- $\|N_{\breve{j}}\| \leqslant \|C_k(N)\|_2$.

Then N is a minimal matrix with $\|N\| = \|C_k(N)\|_2$. Moreover, this is the only possible diagonal that makes N a minimal matrix.

Proof. Note that $C_j(N) \perp C_k(N)$ for all $j \neq k$. Let us denote with $c_k = \|C_k(N)\|_2$.

Let $\{e_i\}_{i=1,\dots,n}$ be the canonical basis of $\mathbb{C}^n$ and define

$$v_+ = \frac{1}{\sqrt{2}\ c_k}\left(C_k(N) + c_k\ e_k\right) \quad \text{and} \quad v_- = \frac{1}{\sqrt{2}\ c_k}\left(-C_k(N) + c_k\ e_k\right).$$

Direct calculations show that $\|v_+\|_2 = \|v_-\|_2 = 1$, $Nv_+ = c_k\ v_+$, $Nv_- = -c_k\ v_-$ and $v_+ \cdot v_- = 0$.

Let v be an eigenvector of N, with $\|v\|_2 = 1$ and eigenvalue $\sigma \neq c_k$. Then it is apparent that v is orthogonal to v_+, v_-, $e_k = \frac{1}{\sqrt{2}}(v_+ + v_-)$ and $C_k(N) = c_k\sqrt{2}\ v_+ - c_k\ e_k$. Let $v_{\breve{k}} \in \mathbb{C}^{n-1}$ be the vector v without its k^{th} coordinate. Then $|\sigma| = \|Nv\|_2 = \|N_{\breve{k}}\ v_{\breve{k}}\|_2 \leqslant \|N_{\breve{k}}\| \leqslant c_k$. Therefore $\|N\| = c_k = \|C_k(N)\|_2$ and since $|v_+ \cdot e_i| = |v_- \cdot e_i|$ for all $i = 1, \dots, n$, then N is a minimal matrix (by [2, Corollary 3]). Moreover, a direct computation proves that if we choose a diagonal i^{th} entry different from $N_{i,i}$ and call with N' this new matrix, then $\|N' C_k(N)\|_2 > \|C_k(N)\|_2$, which proves that the diagonal of N is the only possible diagonal that makes it minimal. $\square$

Note that the column $C_k(N) = C_k(N)$ of the previous theorem must verify $\|C_k(N)\| \geqslant \|C_j(N)\|$ for all j.

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