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Effective field theory and integrability in two-dimensional Mott transition

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ABSTRACT

We study the Mott transition in a two-dimensional lattice spinless fermion model with nearest neighbors density–density interactions. By means of a two-dimensional Jordan–Wigner transformation, the model is mapped onto the lattice *XXZ* spin model, which is shown to possess a quantum group symmetry as a consequence of a recently found solution of the Zamolodchikov tetrahedron equation. A projection (from three to two space–time dimensions) property of the solution is used to identify the symmetry of the model at the Mott critical point as $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$, with deformation parameter $q = -1$. Based on this result, the low-energy effective field theory for the model is obtained and shown to be a lattice double Chern–Simons theory with coupling constant $k = 1$ (with the standard normalization). By further employing the effective filed theory methods, we show that the Mott transition that arises is of topological nature, with vortices in an antiferromagnetic array and matter currents characterized by a *d*-density wave order parameter. We also analyze the behavior of the system upon weak coupling, and conclude that it undergoes a quantum gas–liquid transition which belongs to the Ising universality class.

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1. Introduction

In spite of many substantial advances in recent years, the study of some problems in the physics of strongly correlated electrons continues to provide stimulating challenges. Among the several reasons for this, we could mention the still incomplete understanding of all of the properties of strongly correlated electron systems and the lack of many reliable techniques to study them. One of these paradigmatic problems is the Mott transition, loosely defined as a metal–insulator transition driven by correlations. As early as in 1939, Mott argued that if the electronic density in metallic systems was lowered enough, the Coulomb repulsion would dominate over the kinetic energy and the system could undergo a transition to an insulating state [1].

On the one hand, many widely employed theoretical descriptions of the Mott transition are based on the study of the microscopic dynamics of the electronic system: one starts by writing down a model Hamiltonian for the electrons, sometimes coupled to external fields o to some other degrees of freedom, and then tries to solve it within some approximation or through the use of numerical methods. However, numerical methods like exact diagonalization are restricted to small clusters, and if the interaction among electrons is strong enough, the typical approximation schemes based on the resummation of some class of Feynman diagrams are not completely reliable. For these reasons, it is a difficult task to find solutions displaying the Mott transition, even in the simplest models, like the Hubbard one. As of today, there exist two successful approaches for study this transition based on microscopic dynamics: one is the dynamical mean field theory (DMFT) method [2], valid in the limit of infinite spatial dimensions, which neglects the spatial correlations; the second approach consists in finding analytic expressions for physical observables in integrable models (mainly in one-dimensional systems) by means of the Bethe Ansatz or bosonization methods.

On the other hand, systems like conventional superconductors or quantum Hall systems have universal properties that are well described by field theories which do not deal with the microscopic degrees of freedom, but rather with fields representing *effective* degrees of freedom. These two cases are classic examples of the more general framework of the effective field theories (EFT) approach, which has its roots in Landau's ideas for condensed matter systems and which is widely and successfully used in high energy physics. This approach can be considered as 'way of thinking' which emphasizes the symmetries of the systems and that naturally incorporates Wilson's renormalization group ideas. In this approach, the study of a system starts by wisely choosing the *effective degree of freedom*, which are the relevant ones at a given energy scale, and then one proceeds to write down the most general second quantized action compatible with the characteristic symmetries of these degrees of freedom, retaining only the marginal and relevant terms, *i.e.*, terms that are non-decreasing in the low energy (long-distance) limit [3]. During the last years, several exotic states that contain droplets of approximately constant density have been found experimentally in electronic systems considered to be close to the Mott transition. These states that may seem to be surprising and difficult to explain from the point of view of the free electrons, are good candidates to be understood following the EFT approach. For example, the appearance of effective gauge forces arising from the dynamics, and which have not been included in the microscopic electron Hamiltonian can be properly taken into account within the EFT framework [4].

The goal of this article is to construct and consider an EFT for a two-dimensional square lattice system which displays the Mott transition, which implies that we shall focus our attention on the symmetry aspects of this transition. Specifically, we will consider a simple model of electrons with nearest neighbors density–density interaction which has also been previously studied, with the goal of identifying its effective degrees of freedom and their characteristic symmetries. Since the scope of this article is to make it readable to both condensed matter and field theory physicists, we shall also review (without pretending to be exhaustive) some basic aspects of bosonization, conformal field theory, and integrable models.

The paper is organized as follows: in Section 2, we present the model of strongly coupled fermions on a (two-dimensional) square lattice that we shall consider in the paper. We review some known properties of the corresponding one-dimensional version of this model and we also discuss the relationship among different approaches for treating the one-dimensional Mott transition. We also apply a two-dimensional bosonization prescription as considered in [5] for the two-dimensional fermion model.

After reviewing some basic properties of integrability in statistical mechanics models, we discuss in the Section 3 the integrability of the specific two-dimensional fermion model considered in this paper. We show that in the strong coupling regime, the system defined by the ground and low-lying states of the model satisfies the Zamolodchikov tetrahedron equation, and is characterized by a novel family of solution to the tetrahedron equation recently found by Bazhanov et al. [6,7]. We review these solutions for the sake of completeness and discuss the three-dimensional structure of an underlying quantum group algebraic structure. This analysis allows us to identify the symmetry of the model at the Mott transition point as given by the quantum group $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$. The identification of the symmetry and the corresponding effective degrees of freedom allows us to write down the EFT for the model, which is done in Section 4. With the EFT at hand, we then analyze the order parameter and the universality class of the transition. We find that it is given by a Kosterlitz–Thouless type transition, with vortices in an anti-ferromagnetic array. We also discuss how this order is modified by doping, and that this procedure induces an Ising-like phase transition. Finally, we present our conclusions.

2. The two-dimensional fermion lattice model

In this section we introduce the model that we shall be considering throughout the paper. Let us consider a spinless fermions system with nearest neighbors interaction on a square lattice, with Hamiltonian

$$H_{2d} = -\frac{t}{2}\sum_{x,\mu}[\psi^\dagger(x+ae_\mu)e^{iA_\mu}\psi(x) + \text{h.c.}] + U\sum_{x,\mu}\rho(x)\rho(x+ae_\mu), \tag{2.1}$$

where $\psi(x)$ is the fermionic field, x labels the lattice sites and e_μ are the unit lattice vectors pointing to the nearest neighbors of a given site, a is the lattice spacing, t is the hopping parameter, U is the (constant) Coulomb potential, $\rho(x)$ is the charge density (normal-ordered with respect to the half-filling ground state), $\rho(x) = [:\psi^\dagger(x)\psi(x): -1/2]$ and A_μ is an Abelian statistical gauge field defined on the links of the lattice.

2.1. Review of the one-dimensional model

In order to proceed in our study of this model, we first would like to review the physics of the one-dimensional model analog of (2.1), given by the Hamiltonian

$$H_{1d} = -\frac{t}{2}\sum_{x}[\psi^\dagger(x+a)\psi(x) + \text{h.c.}] + U\sum_{x}\rho(x)\rho(x+a), \tag{2.2}$$

where the sums are taken over the lattice sites. Note that the gauge field is unimportant in this case, as it should, given that there are no statistical Gauge fields in one spatial dimension (see, *e.g.*, [8]). This model has an interesting history that begins with the work of Luther and Peschel [9] and which has later on been studied in detail by several authors, including Shankar [10]. The Mott transition in one-dimensional systems has been discussed not only in the context of Hamiltonian models like (2.2), but also within the scope of Luttinger liquids. In the following, we review and relate both of these approaches with the scope of setting up a framework suitable for further generalizations and for finding the effective degrees of freedom for the simplest one-dimensional case.

As it is well-known, the model (2.2) can be mapped onto the (one-dimensional) *XXZ* model through the Jordan–Wigner transformation:

$$S^+(i) = \psi_i^\dagger \exp\left(i\pi\sum_{i<j}\psi_j^\dagger\psi_j\right), \tag{2.3}$$

$$S^-(i) = \exp\left(-i\pi\sum_{i<j}\psi_j^\dagger\psi_j\right)\psi_i, \tag{2.4}$$

$$S_z(i) = \psi^\dagger(i)\psi(i) - 1/2, \tag{2.5}$$

where i labels the lattice sites. The Hamiltonian goes onto

$$H^{1d}_{xxz} = \sum_i \left[-\left(S^x_i S^x_{i+1} + S^y_i S_{i+1}\right) + \Delta S^z(i) S^z(i+1) \right]. \tag{2.6}$$

In the thermodynamical limit, when the total number of sites is even, this Hamiltonian corresponds to a the system at half-filling, which has the property:

$$H_{xxz}(\Delta) = -H_{xxz}(-\Delta). \tag{2.7}$$

Moreover, the *XXZ* model is integrable and its spectrum and other analytic properties have been found in [11] using the Bethe Ansatz. Its ground state energy is:

$$E_0 = \begin{cases} \frac{1}{4}\cosh\lambda - \frac{1}{4}\sinh\lambda \left[\lambda + 2\lambda \sum_n (1 + e^{2n\lambda})^{-1} \right] & \text{if } \Delta = \cosh\lambda > 1, \\ 1/4 - \ln 2 & \text{if } \Delta = 1, \\ \frac{1}{4}\cos\mu - \sin^2\mu \int_{-\infty}^{\infty} dx/[2\cosh\pi x(\cosh 2\mu x - \cos\mu)] & \text{if } \Delta = \cos\mu < 1. \end{cases} \tag{2.8}$$

Note that E_0 is an analytic function of Δ in the range $1 < \Delta < \infty$, so that the singularity at $\Delta = 1$ signals a phase transition. This fact has been used in [10] to show that the transition at $\Delta = 1$ is identified as a Mott one. The argument relies on the duality between two opposite regimes for the system, ranging from the insulator behavior for $\Delta \to \infty$ to the metallic one for $\Delta = 0$. Moreover, it is known that for $\Delta = 1 + \epsilon$ the spins are in a Neel state, and therefore the system must be in a charge density wave (CDW) state.

An alternative description of the system (2.2) is given by bosonization of its fermionic degrees of freedom. For $\Delta < 1$, the action of the system is given by

$$S = \frac{g}{4\pi} \int dz d\bar{z}\ \partial_z \phi(z)\ \partial_{\bar{z}} \bar{\phi}(\bar{z}), \tag{2.9}$$

where g is a self-coupling parameter and we have defined complex space–time coordinates $z = x + it$ and $\bar{z} = x - it$, and the normal ordered charge density is given by $\rho(z) = i\partial_z \phi(z)$. The model has effective degrees of freedom which are bosonic fields representing charge density waves. Eq. (2.9) defines a conformal field theory (CFT) whose energy–momentum tensor has holomorphic and anti-holomorphic components given by:

$$T(z) = -g\partial_z \phi(z), \tag{2.10}$$

$$\overline{T}(\bar{z}) = -g\partial_{\bar{z}} \bar{\phi}(\bar{z}). \tag{2.11}$$

Moreover, the Fourier modes of the fields, defined by

$$T(z) = \sum_n L_n z^{-n-2}, \tag{2.12}$$

$$\rho(z) = \sum_n \rho_n z^{-n-1}, \tag{2.13}$$

satisfy the following chiral algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}(n^3 - n), \tag{2.14}$$

$$[\rho_n, \rho_m] = n\delta_{n+m,o}, \tag{2.15}$$

$$[L_m, \rho_n] = -m\rho_{n+m}. \tag{2.16}$$

These three lines define a current algebra: the first line is the Virasoro algebra for the generators L_n with central charge $c = 1$. The second one is the $\widehat{u(1)}$ current (or Kac-Moody) algebra for the charge modes. The third is required for consistency among the other two. The $\widehat{u(1)}$ current algebra is used to define the Luttinger model, which in Hamiltonian form is usually written as:

$$H = \frac{1}{2\pi} \int dx \left[uK(\pi\Pi(x))^2 + \frac{u}{K}(\partial_x \phi)^2 \right], \tag{2.17}$$

where u, K are called the Luttinger parameters. It is straightforward to regain the action (2.9) with coupling constant $g = K$ starting from the Hamiltonian (2.17), by transforming the arguments of the fields into imaginary time. The parameters of the model in the different representations are related by (for a detailed discussion see [12]):

$$K = \frac{\pi}{2[\pi - \arccos(\Delta)]}. \tag{2.18}$$

Hence, the Mott transition in the one-dimensional lattice fermion system (2.2) is characterized by $\Delta = 1$ or $K = g = 1/2$. The Luther–Emery transformation [13] allows us to rewrite the Hamiltonian (2.17) as $H = H_0 + H_1$, where:

$$H_0 = v \int p\left[\psi_+^\dagger(p)\psi_+(p) - \psi_-^\dagger(p)\psi_-(p)\right] dp, \tag{2.19}$$

$$H_1 = \frac{\pi u}{L}\sinh(2\theta)\left[\int \left(2\rho_+\rho_- + f_1 \sum_{\alpha=\pm} :\rho_\alpha(p)\rho_\alpha(-p):\right) dp\right], \tag{2.20}$$

where $v = u(\cosh 2\theta + f_1 \sinh 2\theta)$, $\exp(2\theta) = 1/(2K)$ and f_1 is an arbitrary constant. Note that for the Luther–Emery line, which coincides with Mott transition, *i.e.*, $K = 1/2$, one has that H_1 vanishes so that the theory consists of two free decoupled chiral fermions. Therefore, we identify the effective degrees of freedom of the theory at the Mott transition as two free fermionic currents of opposite chirality.

Next we would like to discuss the characteristic symmetry of the degrees of freedom and, therefore, of the system at Mott transition point. In order to do this, it would be more convenient to switch to the spin representation (*i.e.*, the *XXZ* model). Note that the relation (2.7) allows us to write

$$H_{xxz}^{1d} = \sum_i \left[\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}\right) - \frac{(q+q^{-1})}{2} S_i^z S_{i+1}^z\right], \tag{2.21}$$

where $q = \exp i\gamma$, $\gamma = \cos\Delta$ *i.e.*, $\Delta = -(q+q^{-1})/2$. For $q = -1$, we have the isotropic anti-ferromagnetic spin chain, which is a critical system with an explicit $SU(2)$ symmetry.

It is well-known that the corresponding low energy effective field theory is the Wess–Zumino–Witten (WZW) model with coupling constant $k = 1$ [14] and action:

$$S_{WZW} = \frac{k}{4\pi}\int d^2x \mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}] + \frac{k}{12\pi}\int_{B(D)} d^3x \mathrm{Tr}[\epsilon^{\mu\nu\lambda} g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g], \tag{2.22}$$

where $g(z)$ is a field in the group manifold of $SU(2)$ (*i.e.*, g is a $SU(2)$-valued matrix). The first integral in (2.22) is defined over a compactified two-dimensional domain D and the second is done over a three-dimensional ball with boundary D. For the sake of completeness, we will sketch here the procedure leading to this EFT. Following [14], one makes a transformation from the spin variables S_i to a fermionic system ψ_i^α that preserves the $SU(2)$ symmetry defined by $\mathbf{S}_i = 1/2\sum_{\alpha,\beta}\psi_i^\alpha \boldsymbol{\sigma}\psi_i^\beta$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and σ_i are the Pauli matrices. The standard commutation relations reproduce the correct spin commutators, however the Hilbert space of the fermion system is too large and one must restrict it by projecting out the states with one particle by site

$$\left(\psi_n^\dagger\right)^\alpha(\psi_n)_\alpha = 1. \tag{2.23}$$

In the low energy regime, the only excitations that should be taken into account are localized around the two Fermi points of the one-dimensional Fermi surface. One transforms to a set of new fermionic degrees of freedom (we will label the lattice sites by the integer n to avoid confusion):

$$\psi_\alpha(n) = \sqrt{a}[i^n\psi_{\alpha L}(n \pm 1/2) + (-i)^n\psi_{\alpha R}(n \pm 1/2)]. \tag{2.24}$$

A final redefinition of variables takes us to current operators:

$$J = i :\left(\psi_L^\dagger\right)^\alpha(\psi_L)_\alpha : \quad J^i = i :\left(\psi_L^\dagger\right)^\alpha \sigma^i(\psi_L)_\alpha : \tag{2.25}$$

$$G = \psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L \quad G^i = i :\psi_L^\dagger\sigma^i\psi_R : + : \psi_R^\dagger\sigma^i\psi_L :, \tag{2.26}$$

where R and L denote the left and right chiral components. The spin operators and the constraint (2.23) become:

$$J+\bar{J}=G=0, \tag{2.27}$$

$$S^i/a=J^i+\bar{J}^i+(-1)^n G^i. \tag{2.28}$$

In the continuum limit, the Hamiltonian (2.21) of the spin chain becomes:

$$H=\frac{a}{2}\int d^2x[J^i(x)J^i(x)+\bar{J}^i(x)\bar{J}^i(x)+2J^i\bar{J}^i(x)], \tag{2.29}$$

where the x variable is the continuum limit of the lattice position. The last term in (2.29) is irrelevant in the renormalization group sense and the effective Hamiltonian becomes the Hamiltonian of the WZW model.

2.2. Bosonization of the two-dimensional fermion model

We now turn our attention back to the two-dimensional fermionic model on the square lattice with Hamiltonian (2.1). It can be bosonized as discussed, *e.g.* in [15,16], by applying a two-dimensional Jordan–Wigner transformation. Let us first consider the case $U=0$. As it is known from the one-dimensional case, the Jordan–Wigner transformation owns its existence to a natural ordering of the particles along the line. This ordering is lost in on two-dimensional lattice, but the mapping could still be defined by adding extra degrees of freedom, in the form of attached statistical fluxes to the particles, *i.e.*, by introducing branch-cuts on the otherwise analytic fermionic field operators. Equivalently, one considers the Hamiltonian (2.1) with the additional Gauss law constrain:

$$\rho(x)-\theta B(r)=0, \tag{2.30}$$

where $\rho(x)$ is the charge density and $B(r)$ is the magnetic field defined on sites of the dual lattice, *i.e.*, a lattice obtained from the original (direct) one by translating its set of vertices to the centers of each plaquette of the direct lattice:

$$B=\epsilon_{ij}\Delta_i A_j, \tag{2.31}$$

$$\Delta_i A_j=A_j(x+e_i)-A(x). \tag{2.32}$$

The Gauss law constraint (2.30) implies that for each fermion on the site x of the direct lattice, there is also a quantum flux (or vortex) in the corresponding site r of the dual lattice. It can be implemented at the field theory level by coupling the fermions to an Abelian statistical Chern–Simons (CS) Gauge field. In order to show how to do it, let us consider the Lagrangian:

$$L_{2d}=\sum_x \psi^\dagger(x)iD_0\psi(x)-t\sum_{x,j=1,2}\left[\psi^\dagger(x)e^{iA_j}\psi(x+e_j)+\mathrm{hc}\right]+\frac{\theta}{4}\sum_x \epsilon_{\mu,\nu,\lambda}A^\mu(x)F^{\nu\lambda}(x), \tag{2.33}$$

where

$$D_0=\partial_0-iA_0, \tag{2.34}$$

$$F_{ij}=\Delta_i A_j-\Delta_j A_i, \tag{2.35}$$

$$F_{0i}=\partial_0 A_i-\Delta_i A_0. \tag{2.36}$$

The canonical quantization of the above Lagrangian in the Gauge $A_0=0$ imposes the constraint (2.30) at the level of the Hilbert space [17]. The classical solutions of the constraint can be written as follows:

$$A_j=\Delta_j\phi(x)=\frac{1}{\theta}\sum_{x'}[\Theta(x+e_j,r')-\Theta(x,r')]\rho(x'), \tag{2.37}$$

where $\Theta(x,r)$ is the lattice-angle function [18] which contains a branch cut from r to ∞ satisfying: $\oint_\Gamma \Delta\theta=+1$ for any closed curve Γ that encloses the point r of the dual lattice. Moreover, the operators:

$$a(x) = e^{i\phi(x)}\psi(x), \tag{2.38}$$

$$a^\dagger(x) = \psi^\dagger(x)e^{-i\phi(x)} \tag{2.39}$$

satisfy bosonic commutations relations for $1/2\theta = m\pi$ and due to the Pauli principle, the operator (2.38) are hard-core bosons representing charge density waves (CDW) states of the underlying electrons.So that the mapping defined by Eqs. (2.38) and (2.39), relate the fermionic degree of freedom to bosonic ones, and therefore the spinless fermion system can be mapped onto the following Hamiltonian:

$$H^{2d'}_{Bos} = -t\sum_{x,\mu=1,2}[a^\dagger(x)a(x+e_\mu) + \text{h.c.}] + U\sum_x \rho(x)\rho(x+a). \tag{2.40}$$

The identifications:

$$S^+(x) = a^\dagger(x), \tag{2.41}$$

$$S^-(x) = a(x), \tag{2.42}$$

$$S^z(x) = \rho(x) - 1/2, \tag{2.43}$$

where the operators $S^+(x)$ and $S^-(x)$ are the raising and lowering spin operators for a $s = 1/2$ spin particle, allows us to map the original fermion system onto the two-dimensional *XXZ*-spin model:

$$H^{2d}_{XXZ} = \sum_{\langle ij\rangle}\left[-\left(S^x_iS^x_j + S^y_iS^y_j\right) + \Delta S^z_iS^z_j\right], \tag{2.44}$$

where i, j denote lattice sites and the sum is taken over nearest neighbors, and we have defined $\Delta = t/U$.

Summarizing, fermionic two-dimensional systems can be bosonized by attaching fluxes to particles, which is achieved at quantum level by constraining the Hilbert space states after imposing the Gauss Law. The dynamics of these systems may be described at the Hamiltonian level in terms of these bosonic degrees of freedom, which physically represent charge density waves. Alternatively, the charge density waves may be replaced by another set of degrees of freedom, such as the spin waves in the so-called *XXZ* spin model.

3. Integrability of the two-dimensional model

As we have seen in the previous section, the one-dimensional fermionic model with nearest neighbors interaction displays a Mott transition. This property can be established through the integrability of the (one-dimensional) *XXZ* spin chain, which is equivalent to the linearity of the associated free boson system. Our strategy for discussing the Mott transition in the corresponding two-dimensional spinless fermionic model on the square lattice (2.1) is to show that the property of integrability could also be extended to encompass this case. The discussion of the Mott transition in this system could then follow the line of reasoning of the previous Section. For reasons that will be clear latter, we will first consider the Six-vertex model [20]), and later show its relation to the lattice fermionic model. The Six-vertex model is a statistical model on a two-dimensional lattice, on which a classical electric current defined on each link can interact with other currents at the lattice sites. Each site of the lattice (which we will refer to as a vertex) may be in one of the six possible configurations shown in Fig. 1. The energy ϵ_v associated with a given vertex depends on the four current states at the edges only. If we further impose a parity (Z_2) invariance, we are left with three possible vertices:

$$a = w_1 = w_2 = e^{-\beta\epsilon_1}, \tag{3.1}$$

$$b = w_3 = w_4 = e^{-\beta\epsilon_2}, \tag{3.2}$$

$$c = w_5 = w_6 = e^{-\beta\epsilon_3}. \tag{3.3}$$

Following Refs. [19,20], one defines a vector space V_i for each vertical link, another one in a given horizontal row V_a (the so-called *auxiliary space*) and a *vertex operator* $\mathbf{R}$ such that its matrix elements are interpreted as the Boltzmann weights of a given vertex according to:

$$\langle\mu_{i+1}, \alpha_i|\mathbf{R}|\mu_i, \beta_i\rangle = W(\alpha_i\beta_i, \mu_{i+1}\mu_i), \tag{3.4}$$

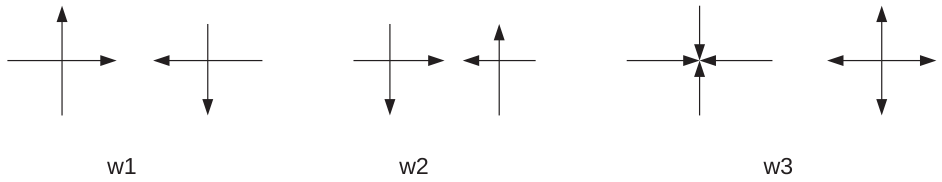

Fig. 1. Boltzmann weights of the Six-vertex model.

where $|a_i\rangle$, $|b_i\rangle$ denote the states on the vertical links, and $|\mu_{i+1}\rangle$, $|\mu_i\rangle$ the ones on the horizontal links. In this basis the R-matrix of the Six-vertex model reads:

$$\mathbf{R} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{bmatrix}. \tag{3.5}$$

Following Baxter, we introduce the row-to-row transfer matrix which plays the role of a discrete-time evolution operator (the time variable is taken as flowing upwards from one row to the next). More precisely, the transfer matrix is an endomorphism:

$$T : H_N \equiv V_1 \otimes V_2 \otimes \cdots \otimes V_N \to V_1 \otimes V_2 \otimes \cdots \otimes V_N, \tag{3.6}$$

defined by:

$$T = \mathrm{Tr}_a[\mathbf{R}_{aN}\mathbf{R}_{a(N-1)} \cdots \mathbf{R}_{a1}], \tag{3.7}$$

where the trace Tr_a is taken on the auxiliary space. The integrability of the model is guaranteed by the existence of a set of mutual commuting row-to-row transfer matrices. In fact, it has been shown [20] that for two sets of Boltzmann weights, (a, b, c) and (a', b', c') the transfer matrices of the Six-vertex model satisfy:

$$[T(a, b, c), T(a', b', c')] = 0 \Longleftrightarrow \Delta_{6v} = \Delta'_{6v}, \tag{3.8}$$

where $\Delta_{6v} = (a^2 + b^2 - c^2)/(2ab)$ (Δ'_{6v} is defined in an analogous way) is the so-called *anisotropy parameter*. This condition is equivalent to the existence of solutions of the Yang–Baxter equation [20]:

$$\mathbf{R}_{12}(u)\mathbf{R}_{13}(u + v)\mathbf{R}_{23}(v) = \mathbf{R}_{23}(v)\mathbf{R}_{13}(u + v)\mathbf{R}_{12}(u), \tag{3.9}$$

where u is the spectral (uniformization) parameter, and each operator $\mathbf{R}_{ij}$ acts non-trivially on $V_i \otimes V_j$. This equation can also be written in the form of the commutation relations for the quantum L-operators

$$\mathbf{R}_{12}(u - v)\mathbf{L}_{13}(u)\mathbf{L}_{23}(v) = \mathbf{L}_{23}(u)\mathbf{L}(v)_{13}\mathbf{R}_{12}(u - v). \tag{3.10}$$

As it is known [19], the transfer matrix of the Six-vertex model is related to the Hamiltonian of the *XXZ* model by:

$$T(\mu) = e^{-\mu H_{xxz}(\Delta)}, \tag{3.11}$$

which also shows that the Yang–Baxter equation implies the integrability of the *XXZ* spin model.

Moreover, Eq. (3.9) may be considered as an equation among operators, whose solutions define integrable planar lattice models, *i.e.*, $(2 + 0)$ dimensional statistical systems or $(1 + 1)$ dimensional quantum systems. Within this approach, the Six-vertex model may be considered as one specific solution of the Yang–Baxter equation when the space V_i is the representation space of spin 1/2 particles. Although a classification of the solutions of (3.9) is not known, some solutions have been found. These solutions are related to the quantum deformations of Lie algebras or, more precisely to the quantum deformations of the universal enveloping Lie algebras also called *quantum groups*. In fact, the universal Yang–Baxter equation:

$$\mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{23} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}, \tag{3.12}$$

which is the Yang–Baxter equation independent of the spectral parameter, has the so-called universal R-matrices as solutions. For example, when V_i is the space representation of spin 1/2, Drinfeld [21] has given a solution:

$$R = q^{H\otimes H/2}\sum_{n=0}^{\infty}\frac{(1-q^{-2})^n}{[n]_q!}q^{\frac{n(1-n)}{2}}q^{nH/2}(X_+)^n\otimes q^{-nH/2}(X_-)^n, \tag{3.13}$$

where q is a complex parameter, $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ is a q-number and the generators X_+, X_- and H satisfy the commutation relations:

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \tag{3.14}$$

$$[H, X_+] = 2X_+, \tag{3.15}$$

$$[H, X_-] = 2X_-, \tag{3.16}$$

with co-products:

$$\Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2}\otimes X_\pm, \tag{3.17}$$

$$\Delta(H) = H\otimes 1 + 1\otimes H, \tag{3.18}$$

which define the quantum group $U_q(sl(2))$ (note that the co-multiplication operator Δ should not be confused with the anisotropy parameter, as is clear from the context). Furthermore, we define the operators [19]:

$$E_1 = e^{\mu}S^+, \quad F_1 = e^{-\mu}S^-, \quad H_1 = 2S^z, \tag{3.19}$$

$$E_o = e^{\mu}S^-, \quad F_0 = e^{-\mu}S^+, \quad H_o = -2S^z, \tag{3.20}$$

where $S^\pm$ are the raising (lowering) operators of the spin-1/2 particle and $x = e^{\mu}$ is the affinization parameter. These operators define an irreducible representation $(e^{\mu}, 1/2)$ of the affine algebra $\widehat{sl(2)}$. In this context, the R-matrix act as an *intertwiner* between the tensor product of two representations:

$$R(e^{\mu_1}, e^{\mu_2})\Delta(g) = \Delta'(g)R(e^{\mu_1}, e^{\mu_2}), \tag{3.21}$$

where g is any element of the quantum group and Δ' is the inverse co-product, *i.e.*, the co-product composed with the operator permuting vector spaces. This R-matrix has the form:

$$R(e^{\mu_1}, e^{\mu_2}) = \begin{bmatrix} qx - q^{-1}x^{-1} & 0 & 0 & 0 \\ 0 & x - x^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & x - x^{-1} & 0 \\ 0 & 0 & 0 & qx - q^{-1}x^{-1} \end{bmatrix}, \tag{3.22}$$

where $x = e^{\mu}$, $\mu = \mu_1 - \mu_2$. This R-matrix coincides with that of the Six-vertex model for the parametrization $a = \sinh(u + i\gamma)$, $b = \sinh(u)$, $c = i\sin\gamma q = \exp(i\gamma)$ (for details see [19]), so that the Six-vertex model possesses symmetry $U_q(\widehat{sl(2)})$. Moreover, it have been shown in [22] that the Hamiltonian of the *XXZ* model in the thermodynamic limit commutes with the Affine quantum group $U_q(\widehat{sl(2)})$ and that the space of states is identified with the tensor product of level *1* highest and level (-1) lowest representations of $U_q(\widehat{sl(2)})$. Besides, the corresponding L-operator is a q-deformation of the fundamental L-operator of the *XXX* (Heisenberg) spin chain, given by [23]:

$$L^{xxz}_{n,a} = \begin{bmatrix} xq^{S^z_n} - x^{-1}q^{-S^z_n} & (q - q^{-1})S^-_n \\ (q - q^{-1})S^+_n & xq^{-S_z} - x^{-1}q^{S^z_n} \end{bmatrix}. \tag{3.23}$$

This Lax operator, together with the R-matrix (3.22) satisfies the $LLR = RLL$ condition (3.10)). It is possible to rewrite this condition introducing

$$\widetilde{L}(x) = Q(x)L(x)Q^{-}1(x) \quad \widetilde{R} = Q(x)Q(y)R(x/y)Q^{-1}(x)Q^{-1}(y), \tag{3.24}$$

where

$$Q(x) = \begin{bmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{bmatrix}, \tag{3.25}$$

which yields:

$$\widetilde{R} = \begin{bmatrix} qx - q^{-1}x^{-1} & 0 & 0 & 0 \\ 0 & x - x^{-1} & x^{-1}(q - q^{-1}) & 0 \\ 0 & x(q - q^{-1}) & x - x^{-1} & 0 \\ 0 & 0 & 0 & qx - q^{-1}x^{-1} \end{bmatrix}. \tag{3.26}$$

3.1. Three-dimensional structure of quantum groups and vertex models

The (2 + 1)-dimensional analogue of the Six-vertex model, is a 'quantum-vertex model' where the classical weights $e^{-\beta\epsilon_a}$, $e^{-\beta\epsilon_b}$ $e^{-\beta\epsilon_c}$, should be replaced by 'quantum vertex operators' defined on the lattice Hilbert space. In order to define this vertex model, we need first to consider a quantum lattice [24] defined as follows: for each lattice site (x_i, y_j), we define a Fock space F_{ij} and a set of representation spaces of spin 1/2 particles V_i, V_j, V_{i+1} and V_{j+1} on each link joining two lattice sites. The states of the link in the lattice are arrows (as in the Six-vertex model), and the states in the Fock space F_{ij} are labeled by the number of particles in the site $|n_{ij}\rangle$ as shown in Fig. 2. Then, we assign a scattering amplitude (and a vertex operator) to each lattice site by:

$$S^{\alpha'_i,\beta'_j}_{\alpha_i,\beta_j} = \langle \alpha'_i \beta'_j n'_{ij} | \mathbf{L}_{V_i,V_j,F_{ij}} | \alpha_i, \beta_j n_{ij} \rangle = L^{i'j',n'}_{i,j,n}, \tag{3.27}$$

where $\mathbf{L}_{V_i,V_j,F_{ij}}$ is a 'three dimensional Lax operator' acting on the spaces $V_i \otimes V_j \otimes F$. Furthermore, it is possible to define layer-to-layer transfer matrices $T_{m\ n}(\{\lambda\},\{\mu\})$, where the pair (m,n) labels the rows and columns of a given layer and $(\{\lambda\},\{\mu\})$ are the spectral parameters, associated to the rows and columns, respectively, by:

$$T_{mn} = \mathrm{Tr}_{V_x \otimes V_y} \left[\prod_i \prod_j \mathbf{L}_{V_i,V_j,F,}(\lambda_i, \mu_j) \right] = T_{mn}(\{\lambda\},\{\mu\}), \tag{3.28}$$

where $V_x = \otimes_i^n V_i$ and $V_y = \otimes_j^m V_j, \{\lambda\} = \{\lambda_1, \lambda_2, \ldots \lambda_n\}\{\mu\} = \{\mu_1, \ldots, \mu_m\}$. Here, the layer-to-layer transfer matrix plays the role of a temporal evolution operator in an unitary time step, where the temporal axis coincides with the direction perpendicular to the layer. For quantum systems in (2 + 1)-

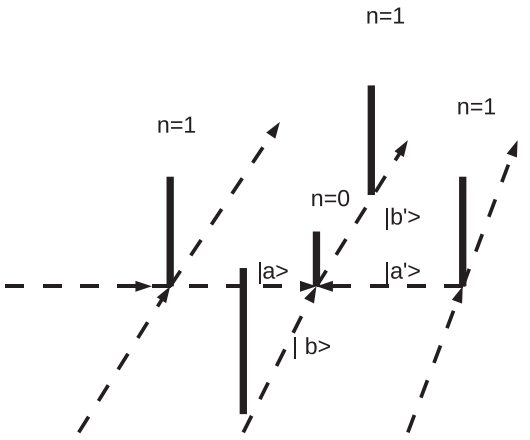

Fig. 2. Graphical representation of the sates $|n = 0\rangle$ and $|n = 1\rangle$ belonging to the Fock spaces at each lattice site. The arrows correspond to the states $|a_i\rangle$, $|a'_i\rangle$, $|b_j\rangle$, $|b'_j\rangle$.

dimensions (and also for three-dimensional statistical systems), the integrability is guaranteed by the commutativity of the layer-to-layer transfer matrices. The integrability is also encoded in the so-called Zamolodchikov tetrahedron equation (TE) [27], which is the three-dimensional analogue of the Yang–Baxter equation, or equivalently in terms of the Lax operators, through the three-dimensional analogue of the *LLR–RLL* equation:

$$\mathbf{R}_{abc}\mathbf{R}_{ade}\mathbf{R}_{bdf}\mathbf{R}_{cef} = \mathbf{R}_{cef}\mathbf{R}_{bdf}\mathbf{R}_{adc}\mathbf{R}_{abc}, \tag{3.29}$$

$$\mathbf{L}_{12,a}\mathbf{L}_{13,b}\mathbf{L}_{23,c}\mathbf{R}_{abc} = \mathbf{R}_{abc}\mathbf{L}_{23c}\mathbf{L}_{13,c}\mathbf{L}_{13,b}, \tag{3.30}$$

where the operators R_{ijk} define the mapping R_{abc}: $F_a \otimes F_b \otimes F_c \to F_a \otimes F_b \otimes F_c$, and the operators L act on $V = V_1 \otimes V_2 \otimes V_3 \otimes F_a \otimes F_b \otimes F_c$. If the Fock space F_a is considered as the representation space of some algebra $\mathcal{A}$, then the operators L can be represented as operator-valued matrices acting on $V_i \otimes V_j$ such that their coefficients are given in terms of the generators v_a of the algebra $\mathcal{A}$ and some complex parameters s_a, so that Eq. (3.30) takes the form of a local-Yang–Baxter-equation:

$$L_{12}(\mathbf{v}_a, s_a)L_{13}(\mathbf{v}_b, s_b)L_{23}(\mathbf{v}_c, s_c) = L_{23}(\mathbf{v}'_c, s_c)L_{13}(\mathbf{v}'_b, s_b)L_{1a}(\mathbf{v}'_a, s_a). \tag{3.31}$$

For classical systems, the algebra $\mathcal{A}$ is chosen to be the Poisson algebra P. In this case, the matrix L which solves the local Yang–Baxter equation (3.31) is given by

$$L_{1,2}(k_a, a_a, a_a^*) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_a & a_a^* & 0 \\ 0 & -a_a & k_a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.32}$$

where the indices in the second space enumerate the two-dimensional blocks, while those for the first space enumerate the elements inside the blocks and $k = 1 - a^*a$. Moreover,

$$\{a_i^*, a_j\}_{PB} = 2\delta_{ij} \quad \{k_i, a_j\}_{PB} = \delta_{ij}k_i a_j \quad \left\{k_i, a_j^*\right\}_{PB} = -\delta_{ij}k_i a_j^*, \tag{3.33}$$

where $\{,\}_{PB}$ denote the Poisson brackets. It was shown in [6] that Eq. (3.31) defines a canonical transformation (automorphism) of the triple tensor product of the Poisson algebra. This solution correspond to the classical three-wave problem, *i.e.*, the linear propagation of three-dimensional waves along to three mutually perpendicular axes.

In quantum systems, we expect that the Algebra $\mathcal{A}$ will be the either bosonic or fermionic. This is actually the case for free Boson or fermion systems. However, we are interested in *interacting solutions* of the *TE*. Recently, a new solution to the *TE* associated with the three-dimensional structure of the affine quantum group $U_q(\widehat{sl(n)})$ has been found in [6] [7]. The new solution may be understood as the quantization of either the classical three-dimensional wave problem or the quantization of fluctuations of extended spatial objects. It amounts to taking L-operators as block matrices with two-dimensional blocks, in which matrix indices in the second space enumerate the blocks while those for the first space enumerate the elements inside the blocks.

$$L_{i,j}(\mathcal{A}_v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_v k_v & a_v^\dagger & 0 \\ 0 & -q^{-1}\lambda_v \mu a_v & \mu_v k_v & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda_v \mu_v, \end{bmatrix}. \tag{3.34}$$

with $i, j = 1,2,3$, $v = a,b,c$ and where now $(k, a^\dagger, a)$ are quantum operators acting on the Hilbert space and the algebra $\mathcal{A}$ is the q-oscillator algebra defined by [6]:

$$qa^\dagger a - q^{-1}aa^\dagger = q - q^{-1}, \quad [h, a^\dagger] = a^\dagger \quad [h, a] = -a, \tag{3.35}$$

$$k^2 = (1 - a^\dagger a), \quad k = q^h. \tag{3.36}$$

The above solution allows one to define a 'quantum vertex model' by assigning 'quantum vertex operators' f_j at each vertex on the lattice, according the rules shown in the Fig. 3, where $v^2 = -q^{-1}\lambda\mu$. We now map the square lattice onto a torus, and let C_a and C_b be the two basic homotopy cycles on

that torus, such that each homology cycle corresponds to a path along one coordinate axis of the square lattice. Any path on the torus belongs to a given homotopy class $P \sim nC_a + mC_b$, and it is possible to define [25]:

$$T_{n,m} = \sum_P \sum_{j \in P} f_j, \tag{3.37}$$

where the sum over P means the sum over different paths, which is exactly Eq. (3.28). The commutativity of the layer-to-layer transfer matrices follows from another (related) Tetrahedron equation [26] [6]:

$$\begin{aligned} &M_{ii'}(H_0\mu/\mu')M_{jj'}(H_0\lambda/\lambda')L_{ij}(A_v,\lambda,\mu)L_{i'j'}(A_v,\lambda,\mu) \\ &\quad = L_{i'j'}(A_v,\lambda,\mu)L_{ij}(A_v,\lambda,\mu)M_{jj'}(H_0\lambda/\lambda')M_{ii'}(H_0\mu/\mu'), \end{aligned} \tag{3.38}$$

where the matrix elements of $M(H_0, \zeta)$ belong to an additional copy of the q-oscillator algebra denoted by H_0 and:

$$M_{i,j}(H_0) = \begin{bmatrix} \zeta^{h_0} & 0 & 0 & 0 \\ 0 & \lambda_0(-q\zeta)^{h_0} & \nu_0\zeta^{-1/2+h_0}a_0^\dagger & 0 \\ 0 & \mu_0\zeta^{1/2+h_0}a_0 & \mu_0(-q\zeta)^{h_0} & 0 \\ 0 & 0 & 0 & \mu_0\zeta^{h_0} \end{bmatrix}, \tag{3.39}$$

with $\nu_0^2 = q^{-1}\lambda_0\mu_0$. Eq. (3.38) can be verified directly from the operator (3.39), and the commutativity of the layer-to-layer transfer matrices follows from its definition and the use of Eq. (3.38). In the cubic lattice with boundary conditions (where the third dimension corresponds to the temporal axis), the solutions of the *TE* given in 3.34, 3.39 have the following properties:

- For $q < 1$, they yield the Fock space representation of the q-oscillator algebra:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{(q^2;q^2)}}|0\rangle, \quad k|n\rangle = q^{n+1/2}|n\rangle, \tag{3.40}$$

where $(x;q^2) = (1-x)(1-q^2x)\cdots(1-q^2x)$.
It is possible to construct the elements of the R-matrix in the basis of the q-oscillator algebra $\langle n_1', n_2', n_3'|R|n_1, n_2, n_3\rangle$. The resulting R-matrix is non-degenerate in $F^{\otimes^3}$.
- The above solution implies that the standard Yang–Baxter equation is satisfied. One defines the operators

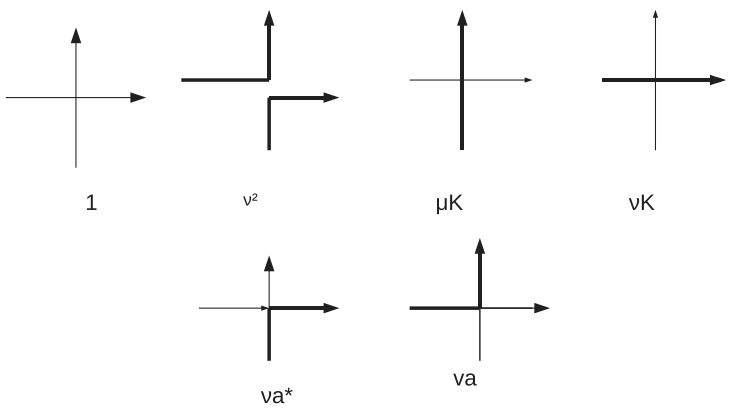

Fig. 3. Weights of the quantum-vertex-model.

$$R_{bc} = Tr_{F_a}[R_{ab_1,c_1} R_{ab_2,c_2} \cdots R_{ab_n,c_n}], \tag{3.41}$$
$$L_{vb} = Tr_{V_0}[L_{0,1,b_1} L_{0,2,b_2} \cdots L_{0,n,b_n}], \tag{3.42}$$

where $F_b = F_{b_1} \otimes F_{b_2} \otimes \cdots F_{b_n}$, $F_c = F_{c_1} \otimes F_{b_2} \otimes \cdots F_{c_n}$, which involves the product of n operators along the 'third direction, *i.e.*, the direction labeled by 'a'. Due to the fact that the R-matrix is non-degenerate in $F_a \otimes F_b \otimes F_c$, the *TE* implies that these operators satisfy

$$R_{bc} R_{bd} R_{cd} = R_{cd} R_{bd} R_{bc}, \tag{3.43}$$
$$L_{Vb} L_{Vc} R_{bc} = R_{bc} L_{Vc} L_{vb}. \tag{3.44}$$

This construction yields a *projection* from the *TE* to the *YB* ones. Using a similar projection, Eq. (3.39) implies that there also exists another Yang–Baxter equation (for details see [6]):

$$R_{V_i,V_j} L_{V_i,b} L_{V_j,b} = L_{V_i,b} L_{V_j,b} R_{V_i,V_j}. \tag{3.45}$$

- The new solution is associated with the affine quantum group $U_q(\widehat{sl(n)})$, where n is the range of the third dimension, *i.e.*, n is the number of two-dimensional layers.

For simplicity we now take $\lambda = 1$, $\mu = 1$ and the indices 1, 2, 3 denoting the quantum spaces a, b, c. Inserting the operator L (3.34) in the local Yang–Baxter equation (3.31), we obtain the explicit mapping R_{123} given by:

$$\begin{aligned}
&k_2' (a_1^\dagger)' = k_3 a_1^\dagger - k_1 a_2^\dagger a_3, \quad k_2' a_1' = k_3 a_1 - k_1 a_2 a_3^\dagger, \\
&(a_2^\dagger)' = a_2^\dagger a_3^\dagger + k_1 k_3 a_2^\dagger, \quad a_2' = a_2 a_3 + k_1 k_3 a_2, \\
&k_2' (a_3^\dagger)' = k_1 a_3^\dagger - k_3 a_1 a_2^\dagger, \quad k_2' a_3' = k_1 a_3 - k_3 a_1^\dagger a_2.
\end{aligned} \tag{3.46}$$

Eq. (3.46) are the Heisenberg equations of motion for the quantum operators $(a_i, a_i^\dagger, k_i)$, where we have denoted with primes those operators that evolve forward in time $t' = t + \Delta t$ which describe the time evolution of the quantum three-wave problem.

3.2. Integrability and the two-dimensional Mott transition

In this section we will come back to the two-dimensional fermion model defined by (2.1) to study its symmetries and integrability. We first recall (Section 1) that it is equivalent to a two-dimensional *XXZ* spin model:

$$H_F = H_{XXZ} = -\sum_{ij} \left[S_i^x S_J^x + S_i^y S_J^y - \Delta S_i^z S_j^z \right].$$

On the other hand, the quantum vertex model defined in the previous Section has the remarkable property that can be projected from three to two dimensions. This means that the equations:

$$L_{Vb} L_{Vc} R_{bc} = R_{bc} L_{Vc} L_{Vb}, \tag{3.47}$$
$$L_{V_i,b} L_{V_j,b} R_{V_i,V_j} = R_{V_i,V_j} L_{V_i,b} L_{V_j,b}, \tag{3.48}$$

can be interpreted as arising from a two-dimensional system. Moreover, two remarkable properties of the quantum vertex model and the associated three-dimensional structure of quantum groups have been discussed in [6]:

$$L_{Vb} = \otimes_{i=1}^{n} \mathcal{L}(\omega_k, \lambda, \{\mu_i\}) \tag{3.49}$$
$$T_m = Tr_{\pi_{\omega_k}}[\mathcal{L}(\omega_k, \lambda_m \{\mu_i\}) \cdots \mathcal{L}(\omega_k, \lambda_1, \{\mu_i\})]. \tag{3.50}$$

The first equation shows that the operator L_{Vb} decomposes into a direct sum of the fundamental $\mathcal{L}$-operators $\mathcal{L}^{sl(n)}$ of the affine quantum group $U_q(\widehat{sl(n)})$, where ω_k is the highest weigh of the representation π_{ω_k}. The second one, shows that the row-to-row transfer matrix of the quantum vertex model can be reconstructed from the fundamental $\mathcal{L}$-operators.

The quantum vertex model is stationary for $L_{12}(a, a^\dagger, k) = L_{12}(a', a^{\dagger'}, k')$, *i.e.*, for the case when the patterns for two consecutive time slices are identical so that the third dimension has range $n = 2$. In this case, the operator L_{vb} is:

$$L_{Vb}(u) = \begin{bmatrix} 1 + u\lambda_1\lambda_2 q^{h_1+h_2} & 0 & 0 \\ 0 & \mathcal{L}(\frac{1}{2}, \mu) & 0 \\ 0 & 0 & \mu_1\mu_2(q^{h_1+h_2} + q^{-2}u\lambda_i\lambda_2) \end{bmatrix}, \tag{3.51}$$

and

$$\mathcal{L}\left(\frac{1}{2}, u\right) = \begin{bmatrix} \mu_1(q^{h_1} - u\lambda_1\lambda_2 q^{h_2-1}) & -q^{-1}\lambda_1\mu_1 a_1 a_2^\dagger \\ -q^{-1}u\lambda_2\mu_2 a_1^\dagger a_2 & \mu_2(q^{h_2} - u\lambda_1\lambda_2 q^{h_1-1}) \end{bmatrix}. \tag{3.52}$$

For $h_1 = h_2 = 1/2$ we have:

$$q^{h_1} = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}, \quad q^{h_2} = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}, \quad a_1 a_2^\dagger = \begin{bmatrix} 0 & 0 \\ 1-q^2 & 0 \end{bmatrix}, \quad a_1^\dagger a_2 = \begin{bmatrix} 0 & 1-q^2 \\ 0 & 1 \end{bmatrix}.$$

If $\lambda_1 = \lambda_2 = 1$ and $\mu_1 = \mu_2 = 1$, the last operator is the R-matrix (3.26) of the *XXZ* spin chain (or Six-vertex R-matrix). Since is possible to project the quantum vertex model onto any lattice direction, any row or column should give rise to a *XXZ* spin chain, and one can expect that this system will be equivalent to a two dimensional *XXZ* model.

We are now ready to map among each other the original two-dimensional fermion model, the quantum vertex model and the two-dimensional *XXZ* spin model. The two-dimensional lattice fermion model describes charge density waves propagating on the lattice of the underlying electrons above (and below) the half-filling state. Viewed at a fixed time, the wave vectors of these charge density waves cross themselves at each lattice site, defining a 'Six-vertex model'. The half-filling condition means that the Fock space of the q-oscillator algebra must be restricted to: $\{|n\rangle = |1\rangle, |-1\rangle\}$. These states are in one-to-one correspondence with the states $|\uparrow\rangle|\downarrow\rangle$ of the two-dimensional spin model. Therefore, the quantum (Fock) space is also in correspondence with the representation space of a spin 1/2 particle. Introducing this information in Eqs. (3.45), (3.51) and (3.52), we see that for each line on the lattice we can define an *XXZ* (one dimensional) spin chain corresponding to the wave propagation of the CDW along this line. This fact allows us to identify the parameter $\Delta = U/t$ (representing the normalized Coulomb interaction) of the one-dimensional fermion model with $\Delta = -(q + q^{-1})/2$, where $q = \exp(i\gamma)$ is the deformation parameter of the quantum group.

Moreover, let us consider an interaction star, *i.e.*, the interaction among a central spin, labeled by the index α an its nearest neighbors, labeled by the index β. A direct calculation shows that:

$$\frac{i}{2}\frac{d}{du}\ln[1/a(u)Tr_{\alpha,\beta}\mathcal{L}(u)_{1/2}(\mathbf{r}_\alpha)\mathcal{L}(u)_{1/2}(\mathbf{r}_\beta)] = -H_{\alpha,\beta}, \tag{3.53}$$

where $a(u) = \sinh(u + i\gamma)$, and we have taken the parameters $u = v = 1$. $Tr_{\alpha,\beta}$ is the trace over the space $V_{i\alpha} \otimes V_{j\alpha} \otimes V_{i\beta} \otimes V_{j\beta}$ and $H_{a\beta}$ is the element (α, β) of the two-dimensional *XXZ* Hamiltonian. Now we take:

$$L_{ij}(\lambda_i, \mu_j) = \begin{cases} 1 & \text{if } |r_\beta - r_\beta| > 1 \\ L(a, a^\dagger, k, \mathbf{r}_\alpha) & \mathbf{r}_\alpha, \mathbf{r}_\beta \text{ are nearest neighbors} \end{cases} \tag{3.54}$$

and we define the 'link operator' $\hat{l}_{\alpha,\beta} = \hat{f}_\alpha \hat{f}_b$, such that the expectation value of a given path γ is $w(\gamma) = \langle \sum_{\alpha,\beta\in\gamma} \hat{l}_{\alpha,\beta} \rangle$. The partition function of the quantum vertex model can be therefore defined as:

$$\begin{aligned} Z(QVR) &= \sum_\gamma \prod_{\alpha,\mu\in\gamma} \langle n_\alpha n_{\alpha+e_\mu} | \hat{f}_\alpha \hat{f}_{\alpha+e_\mu} | n_\alpha n_{\alpha+e_\mu} \rangle \\ &= Tr_F^{nm} \prod_{\alpha\mu=x,y} [Tr_{V_\alpha \otimes V_{\alpha+e_\mu}} L(a_\alpha, a_\alpha^\dagger, k_\alpha, \mathbf{r}_\alpha) L(a_{\alpha+\mathbf{e}_\mu}, a_{\alpha+\mathbf{e}_\mu}^\dagger, k_{\alpha+\mathbf{e}_\mu}, \mathbf{r}_{\alpha+\mathbf{e}_\mu})] \end{aligned} \tag{3.55}$$

where $\hat{f}_\alpha$ is the weigth operator defined in the previous subsection, α runs over all the lattice sites, $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors pointing to the nearest neighbors of a given site, and $F^{nm} = F^{\otimes nn}$ is the complete

Fock space of the square lattice with n rows and m columns. Tracing over the vector space V_{y_0} (there is nothing special about the 'zero-th line', and we can also trace over any row space $V_{y_1}, V_{y_2} \ldots V_{y_n}$) and using Eq. (3.42), we have:

$$Z(QVM) = Tr_{F^{nm}} \prod_{\beta} [Tr_{V_\beta}(L_{vb}(\mathbf{y}_\beta) L_{vb}(\mathbf{y}_{\beta+1})^N)] \tag{3.56}$$

where now the index β labels the vector spaces along the 'vertical lines'. Taken into account the fact that we are using the two-dimensional (spin 1/2) representation of $U_q(\widehat{sl(2)})$ we have:

$$Z(QVM) = Tr_{F^m} \prod_{\beta} [Tr_{V_\beta}(R^{xxz}(\mathbf{y}_\beta) R^{xxz}(\mathbf{y}_{\beta+1}))^N] \tag{3.57}$$

$$= Tr_{F^m} \left[\prod_{\beta} e^{\sum_{\alpha} H_{\alpha,\beta}} \right] \tag{3.58}$$

$$= Tr_F \left[e^{\sum_{\alpha,\beta} H_{\alpha,\beta}} \right] \tag{3.59}$$

$$= Z(H^{2-D}_{XXZ}) \tag{3.60}$$

Therefore, we arrive at the interesting identity:

$$Z(QVM) = Z\left(H^{2d}_{xxz}\right), \quad H^{2d}_{xxz} = H_F. \tag{3.61}$$

This Eq. (3.61) shows that the two-dimensional XXZ spin lattice and the Lattice fermion system (2.1) have quantum group symmetry $U_q(\widehat{sl(2)})$.

Summing up: the two-dimensional lattice fermion model, viewed at two different times, can be considered as a two-layered three-dimensional system where the third direction coincides with the temporal axis (see Fig. 4). From an intuitive point of view, one can assign an arrow to the time propagation direction of the charge density waves of the underlying fermions. At the stationary point, where the pattern of arrows does not change with time, these arrows define a vertex model with

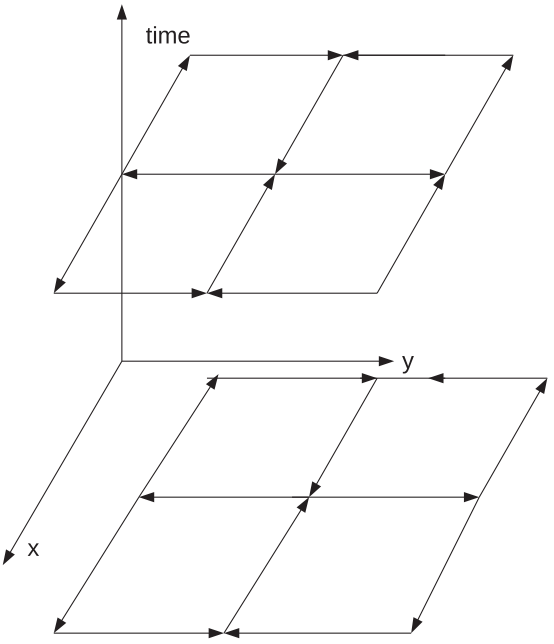

Fig. 4. Current patterns of the fermion model at two different times.

quantum group symmetry $U_q(\widehat{sl(2)})$. Alternatively, one can say that due the projection-like property (3.43), the two-dimensional *XXZ* spin system may be decomposed in a consistent way into two one-dimensional chains, so that the entire system will have a quantum group symmetry $((U_q(\widehat{sl(2)}))^{\otimes N})$. The property that (2 + 1)-dimensional quantum systems in square lattice with periodic boundary conditions reduce to (1 + 1)-dimensional quantum chains, implying that the $((U_q(\widehat{sl(2)}))^{\otimes N})$ symmetry reduces to the $U_q(\widehat{sl(2)})$ quantum group symmetry, was first noted in [6,25].

4. Construction of the effective field theory

As we have shown in the previous Section, the two-dimensional fermionic system defined by (2.1) has a (2 + 1) quantum-group symmetry $U_q(\widehat{sl(2)})$. Discrete time evolution is given by the transfer matrix $T_{mn}(\{\lambda_n\},\{\mu_m\})$. We have presented an explicit form for it in (3.28) which is a solution of the Zamolodchikov tetrahedron equation, rendering the model integrable in (2 + 1) dimensions. However, this solution is not completely transparent from the physical point of view. To further study the system and this solution, we will follow the path of writing down an Effective Field Theory (EFT) formulation for it (for a review, see [3]), valid for long-distance and low-energy domains. EFTs provide useful frameworks for analyzing the behavior of many-body systems, specially near to a phase transition point, were universal properties are vastly dominating the physical properties of the systems under study. The general scheme for applying the method of EFTs starts by identifying the effective degrees of freedom that dominate the low-energy regime of a given system and their characteristic symmetries. Note that these degrees of freedom are usually chosen by phenomenological reasons and could bear no resemblance with the microscopic degrees of freedom of any underlying model describing the system. One then proceeds to write down the most general local action in terms of second-quantized fields representing the selected degrees of freedom which are consistent with the noted symmetries. In the case at hand, the construction of the EFT is obtained by choosing fermionic fields as the degrees of freedom. By using the vertex model representation of the lattice fermion model (2.1),and its connection to the Chern–Simons theory (and analytical continuation in the Chern–Simons coupling constant given in [41]), we will now show that the corresponding effective field theory is a double Chern–Simons theory with gauge group $U(1)$ and quantum group symmetry $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$, which is the symmetry of the exact solution (3.28) and (3.34).

4.1. The one-dimensional case

For the sake of completeness and clarity of exposition, let us start by considering the EFT describing the one-dimensional Mott transition. As discussed in Section 1, the one dimensional counterpart of the lattice fermion system (2.1) can be bosonized. Its effective degrees of freedom are the charge density waves of the underlying strongly correlated electrons, and for a general coupling (Δ) the system has a $u(\widehat{1})$ Kac–Moody symmetry [28]. The Mott transition point ($\Delta = 1$) coincides with the Lutter-Emery point, where the degrees of freedom are two chiral non-interacting fermions. At this point, Eq. (2.20) can be written in coordinate space on a circle domain as:

$$H = \frac{v}{2}\int_0^{2\pi R} dx\left[\psi_r^\dagger(x)(-i\partial_x)\psi_r(x) + cc + \psi_l^\dagger(x)(-i\partial_x)\psi_l(x) + cc\right], \tag{4.1}$$

where $\psi_r(x)$ ($\psi_l(x)$) are the right (left) relativistic chiral fermions (Weyl fermions) moving on a circle of radius R with speed v. Each chiral branch can be bosonized independently. Let us focus on one chiral component,say the right one. This can system has a quadratic action on a circle (of radius $R = 1$) [29]:

$$S = -\frac{k}{4\pi}\int_\infty^\infty dt\int_0^{2\pi}(\partial_t + v\partial_x)\phi\partial_x\phi, \tag{4.2}$$

where k is the coupling constant, $\phi(x,t)$ is a real scalar field and x denotes the coordinate along the circle of length 2π and v is the 'speed of light'. By coupling this one-dimensional chiral sys-

tem to an external electromagnetic field E, the equation of motion of the bosonic field changes to [30]:

$$\partial_t Q = \frac{e}{k} E, \tag{4.3}$$

where $E = \partial_t A_x - \partial_x A_t$ is the electric field pointing along the circle and A is the $U(1)$ electromagnetic potential. Eq. (4.3) displays the chiral anomaly of the Weyl fermion. This description can be also obtained from the Abelian Chern–Simons theory with Lagrangian

$$L_{cs} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{4.4}$$

defined on the disc D, whose boundary is the circle of radius $R = 1$. By making a Gauge transformation $\delta A_\mu = e\partial_\mu \lambda$, the action undergoes a variation:

$$\delta S_{CS} = \frac{ke}{4\pi} \int \lambda(\partial_t A_\theta - \partial_\theta A_t), \tag{4.5}$$

showing that S_{CS} is not Gauge invariant on the boundary. Equivalently, there is a chiral current

$$J^r = \frac{\delta S_{cs}}{\delta A_r} = \frac{ke}{2\pi} E_\theta, \tag{4.6}$$

that shows again the presence of the chiral anomaly of the Weyl fermion for $k = 1$ [30]. The chiral anomaly of one chiral component is canceled by the corresponding anomaly of the anti-chiral sector of the complete theory, which is anomaly free. This implies that the Mott transition in a one-dimensional system is described by a double Chern–Simons theory:

$$S = \frac{k}{4\pi} \int_{DxR} d^3x \epsilon^{\mu\nu\rho} A^R_\mu \partial_\nu A^R_\rho - \frac{k}{4\pi} \int^R_{DxR} d^3x \epsilon^{\mu\nu\rho} A^L_\mu \partial_\nu A^L_\rho, \tag{4.7}$$

which contains two chiral Gauge fields (A^R, A^L) of opposite chiralities and where the value of the coupling constant is $k = 1$.

4.2. Two-dimensional effective field theory

Now let us consider the EFT for the Mott transition in the two-dimensional square lattice. As it has been remarked before, one must identify the correct degrees of freedom dominating the low-energy regime and their characteristic symmetries. The two-dimensional Jordan–Wigner transformation shows that the lattice fermionic system can be also considered as a bosonic system. We choose these bosonic degrees of freedom as the characteristic ones for the low-energy regime of the system. Furthermore, these can be viewed in more physical terms as charge density waves, which we choose to construct the EFT. We look for an EFT with three-dimensional quantum group symmetry that is consistent with the properties established in Section 2.2. Namely, that it should posses a kind of vertex model interpretation, that it should be parity-invariant and that it must be projection-able onto one-dimensional theories, each giving rise to a one-dimensional Mott transition. We recall that in these one-dimensional theories the degrees of freedom split naturally in two non-interacting chiral branches when the interaction parameter becomes $\Delta = 1$. We will now show that this EFT is a lattice double Chern–Simon theory with $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$ quantum group symmetry.

4.2.1. Vertex models and Chern–Simon theory

A direct connection between vertex models and Chern–Simons Gauge theory was first established in classical articles by Witten [41] [42] by using a non-Abelian Chern–Simons theory. Let us review this connection and consider a Gauge connection $A = A_i t_i^a$ that belongs to an a Lie group G, where t_i^a are the generators of the group. The non-Abelian Chern–Simons theory is defined by the action:

$$S_{CS} = \frac{k}{4\pi} tr_R \int_{\mathcal{M}} \left[\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \right], \tag{4.8}$$

where $\mathcal{M}$ is an orientated topological manifold, and tr_R denotes the trace on the representation R of the group G. The natural observables of this theory are Wilson loops: let us consider a link L as a disjoint

union of the circles C_i and pick up a representation R_i for each circle. The expectation value of the Link can be calculated as:

$$\langle L\rangle = \int DAe^{L_{cs}} \prod_i tr_{R_i} Pe^{i\int A_i}. \tag{4.9}$$

Here we can take $\mathcal{M} = \Sigma xR$, where R represents the temporal axis and Σ is a Riemann surface. In [41] it has been shown that it is possible to define vertex models by replacing the classical currents living on the links of a lattice by Wilson lines in a Chern–Simons gauge theory. Furthermore, it has been shown that the expectation values of this Wilson lines can be calculated from the data in the CS theory. More specifically, taking the Gauge group $SU(2)$ and projecting the three-dimensional knots onto the plane, the 'Boltzmann weights' for these vertex models are given by:

$$\begin{aligned} W_{up} &= q^{1/2}\delta_{s_1-s_3,0} - q^{-1/2}\delta_{s_1+s_2,0}q^{(s_2-s_3)},\\ W_{und} &= q^{-1/2}\delta_{s_1-s_3,0} - q^{1/2}\delta_{s_1+s_2,0}q^{(s_2-s_3)},\\ W_{pc} &= \epsilon_{s_1,s_2}q^{-s_1/2}, \end{aligned} \tag{4.10}$$

where $W_{up}(W_{und})$ denotes the Boltzmann weight for the vertex in which the line labeled by the representations of $SU(2)$ denoted by s_1 and s_2, are above (below) the line labeled by s_3 and s_4. Furthermore, W_{pc} denotes the Boltzmann weights for the pair-creation, and q is the deformation parameter of the quantum group, which is related to the coupling constant of the Chern–Simons theory by $q = \exp(i\pi/k)$. Note the a different factor in the definition of the deformation parameter of the quantum group with respect to [41]. Our definition match with the relations defining the quantum group (Eqs. (3.14) and (3.15)) within the conventions adopted by Alvarez-Gaume et al. [47], while the definition of the deformation parameter in [41] agrees with the convention adopted in [45]. To make contact with the statistical Six-vertex model, one needs to take the Gauge group $G = SU(2)$ and compute the 'Bob amplitude'. It was show using Skein theory that any four coupling (including over-crossing, under-crossing and pair creation) is equivalent to the Bob-amplitude:

$$A = u\cdot\delta_{s_1,s_3}\delta_{s_2,s_4} + v\epsilon_{s_1,s_2}\epsilon_{s_3,s_4}.q^{-(s_1+s_3)} \tag{4.11}$$

for some complex parameters u, v. Taking $u = (qx^{-1} - q^{-1}x)$ and $v = (x - x^{-1})$ the corresponding R-matrix R_{s_i,s_j} is given by:

$$R(x,q) = (qx^{-1} - q^{-1}x)I + (x - x^{-1})U(q), \tag{4.12}$$

$$U(q) = \begin{bmatrix} q & 1\\ 1 & q^{-1}\end{bmatrix} \text{ if } i<j, \tag{4.13}$$

$$U(q) = 0, \quad \text{if } i = j \tag{4.14}$$

which is a possible form of the R-matrix for the Six-vertex model [45]. For a Gauge group $G = SU(2)$, the Chern–Simons vertex models exhibit symmetry $U_q(\widehat{sl(2)})$ in the same way that the classical Six-vertex model does. Therefore, the mathematical structure of the quantum groups encodes the topology of planar Wilson loops. A little of caution must be taken with the above defined Boltzmann weights: first, note that since the Chern–Simons theory is well-defined for integer values of the coupling constant k, not all real values of the Boltzmann weights are well defined if we consider $q = \exp(\pi i/k)$. Second, the classical vertex model is defined in terms of classical degrees of freedom (the currents). However, the Chern–Simons action defines a non-dynamical theory (*i.e.*, there are no dynamical currents in CS theory). Both problems are solved by the following argument: let us impose boundary conditions to the EFT, *i.e.*, by compactifying the space domain onto a torus. Cutting down the torus along any cycle induces a loose of the Gauge symmetry, so that the Gauge fields become dynamical degrees of freedom, and as it is well-known that the CS theory becomes then equivalent to a chiral Weiss–Zumino–Witten (WZW) model defined on a circle [42])

$$S_{CWZW} = -k\int_{CxR} d\theta dttr[g^{-1}\partial_\theta g g^{-1}\partial_0 g] + \frac{k}{3}\int_{\Sigma xR}\epsilon^{\mu,\nu,\rho}\mathrm{Tr}[[g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g], \tag{4.15}$$

where $g(z)$ is the chiral WZW field living in the group manifold of G. This is a conformal field theory (CFT) with central charge $c = (k + |SU(2)|)/(k + c_v)$, where $|SU(2)|$ is the number of generators of the $SU(2)$ Lie algebra, and c_v is the dual coxeter number. The WZW model splits naturally into holomorphic and antiholomorphic pieces. The (holomorphic) energy–momentum tensor is given by the Sugawara form [43]:

$$T(z) = \frac{1}{2(k+c_v)}\sum_a :J_a(z)J_a(z):, \tag{4.16}$$

where k is the Chern–Simons coupling constant and c_v is the dual coxeter number, which for $SU(2)$ is $c_v = 2$. In other words, after quantization of the WZW model, there is a shift of the parameter $k \to k + c_v$ implied by the Sugawara construction. The currents $J_a(z)$ and the tensor $T(z)$ may be expanded in modes in the usual way (for a review see [44]):

$$J_a(z)\sum_n z^{-n-1}J_n^a, \tag{4.17}$$

$$T(z) = \sum_n z^{-n-2}L_n. \tag{4.18}$$

The WZW model has conformal and affine $\widehat{su(2)}_k$ symmetries which can be written in terms of the Fourier modes of the Virasoro and current operators:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}(n^3 - n), \tag{4.19}$$

$$\left[J_n^a, J_m^b\right] = i\sqrt{2}\epsilon^{abc}J_{n+m}^c + kn\delta^{ab}\delta_{n+m,0}, \tag{4.20}$$

$$\left[L_n, J_n^a\right] = -mJ_{n+m}^a. \tag{4.21}$$

The identification between the CS theory and WZW model was used in [41] to obtain an analytical continuation in k, which follows from the Knizhnik–Zamolodchikov equation:

$$\left[\partial_z - \frac{1}{k+g}\sum_{i\neq j}\frac{t_i^a t_j^a}{z_i - z_j}\right]\langle g(z_1,\bar{z}_1)\cdots g(z_N,\bar{z}_N\rangle = 0. \tag{4.22}$$

This can be used to compute the braiding matrices of this CFT. It has been shown that these braiding matrices correspond to the Boltzmann Weights of the associated 'interacting round a face' (IRF) model [47], which is equivalent to a vertex model (this is the so-called face-vertex equivalence). The Boltzmann Weights can now be defined by Eq. (4.10), with the redefinition $q = \exp(\pi i/(k+2))$. As it is known, the WZW model possesses a quantum group symmetry $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$ [47], with $q = \exp(\pi i/(k+2))$.

4.2.2. Abelian Chern–Simons theory and quantum groups

However, as it was pointed out by Witten [41], in the reduction from CS Gauge theory with group G to any vertex model, one losses the local and global G symmetry so that the vertex models retain only the maximal torus T symmetry of the group G. For the case at hand where $G = SU(2)$, the vertex model has a $U(1)$ symmetry which naturally corresponds to the Gauge symmetry of the charged fermions (charge density waves) propagating on the lattice. The extended Kac–Moody symmetry $\widehat{su(2)}_1$ present in the WZW model is obtained by taking the equivalent CFT of (two) chiral bosons at self-dual point *i.e.*, where the currents:

$$J^{\pm} = e^{\pm i\sqrt{2}\phi}, \tag{4.23}$$

$$J^z = i\partial\phi, \tag{4.24}$$

satisfy the algebra $\widehat{su(2)}_k$ (Eq. (4.20)) at level $k = 1$. As we have seen in the previous section, charge density waves are also naturally accounted by a *Abelian* Chern–Simons theory on the circle.

Imposing periodic boundary conditions on the square lattice amounts to consider the Abelian CS theory on a (spatial) torus, such that the square lattice on the plane defines the homology cycles on the torus. In this domain, new degrees of freedom associated to the global Gauge transformations arise. To be more precise, let us consider the Lagrangian of the Abelian CS theory:

$$L = \frac{k}{4\pi}\int_{\mathbf{T}} d^2x\epsilon_{ij}(\dot{A}_i A_j + A_o F_{ij}), \tag{4.25}$$

where **T** denotes a torus with modular parameter τ and homology cycle basis C_α C_β. As it is known ([33,32]), the Gauge field A can be parametrized in this domain by using the Hodge decomposition, which incorporates the windings around the non-contractible loops on the torus. In holomorphic coordinates $z = x + iy$, $\bar{z} = x - iy$ it is given by

$$A = \partial_{\bar{z}}\chi + i\frac{\pi}{Img(\tau)}\bar{\omega}(z)a, \tag{4.26}$$

where the 1-form ω satisfies $\int_{C_a}\omega = 1$ and $\int_{C_b}\omega = \tau$ and $a = a(t)$ is a complex (space independent) function on time. The it can be shown that the Lagrangian, in the Gauge, $A_0 = 0$ becomes [32]:

$$L_{cs} = iB_{eff}(\dot{a}a^* - \dot{a}^* a) + ik\int_\sigma (\partial_z\dot{\chi}\partial_z\chi^* - \partial_z\dot{\chi}^*\partial_z\chi). \tag{4.27}$$

The second term in the Lagrangian corresponds to Abelian CS theory on the plane with coupling constant k. The first one shows that the degree of freedom labeled by the function 'a' behaves as the coordinate of a quantum mechanical particle moving into an effective magnetic field $B_{eff} = \pi k/Img(\tau)$ restricted to the lowest Landau level. The quantization of this theory is well known and can be done in the Schödinger picture [31]. The wave functional is:

$$\Psi[A] = \psi(\chi)\psi(a), \tag{4.28}$$

where $\psi(\chi) = e^{-\int\chi\frac{\partial_+}{\partial^-}\chi}e^{-\int|\chi|}$, and $\chi = \sqrt{\frac{k}{4\pi}}(\chi_1 + i\chi_2)$, $\partial_\pm = (\partial_1 \pm i\partial_2)$. Small Gauge transformations do not affect the wave functional. However, due to the existence of non-contractible loops on the torus, the global Gauge transformations defined by Bos [34]:

$$a \to a + n_1 + \tau n_2, \quad \chi \to \chi, \tag{4.29}$$

$$A_{\bar{z}} \to -i\partial_{\bar{z}}RR^{-1}, \tag{4.30}$$

$$R(z,\bar{z}) = exp\left(-\frac{n_1\pi}{Im\tau}\int_{z_0}^{z}(\bar{\omega} - \omega) - \frac{n_2\pi}{Im\tau}\int_{z_0}^{z}\tau\bar{\omega} - (\bar{\tau}\omega)\right) \tag{4.31}$$

affect both the zero and a modes. These large Gauge transformations are precisely the magnetic translations across a parallelogram unit cell:

$$T_R\psi(a) = e^{i\frac{B}{2}|a\wedge R|}\psi(a + R), \tag{4.32}$$

where

$$T_R = e^{(\nabla + ieA)R}, \quad T_aT_b = e^{iB/2|a\wedge b|}T_{a+b}. \tag{4.33}$$

The exponential factor involves the flux through the parallelogram defined by $\vec{a}$ and $\vec{b}$. Now, following [35] [36] we take the combination of the magnetic translations:

$$E = \frac{1}{q - q^{-1}}[T(a,a) - T(-a,a)], \tag{4.34}$$

$$F = \frac{1}{q - q^{-1}}[T(-a,-a) - T(a,-a)], \tag{4.35}$$

$$K = T(a,0), \tag{4.36}$$

where the translations are made from the elemental square plaquette of side a. These operators satisfy the relations:

$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}, \tag{4.37}$$

$$kE = q^2 EK \quad KF = q^{-2} FK, \tag{4.38}$$

that define the quantum group $U_q(sl(2)$. Here $q = \exp(i\, B/2\, a^2) = \exp(i\Phi/2)$ is the deformation parameter of the quantum group, and Φ is the flux per plaquette. Note that for $\Phi = 2\pi$ we have $q = -1$ and then

$$T_{(a,0)}\psi(x) = -\psi(x + a), \tag{4.39}$$

$$T_{(2a,0)}\psi(x) = \psi(x + 2a). \tag{4.40}$$

This is reminiscent to the staggered flux phase [37], where the fluxes are antiferromagnetically ordered. Alternatively, the quantum group symmetry can be also analyzed directly at the level of the Gauge transformations acting on the wave functionals and the resulting quantum group is also $U_q(sl2)$ where the deformation parameter is identified directly in terms of the coupling constant k of the CS theory as $q = e^{i\pi/k}$ [38]. Summarizing, the Gauge invariance of the Chern–Simons theory on the torus implies the existence of a Quantum group symmetry hidden in the theory. Now following [19] we can define the generators:

$$E_0 = e^u E, \quad E_1 = e^u E, \tag{4.41}$$

$$F_0 = e^{-u} E, \quad F_1 = e^{-u} E, \tag{4.42}$$

$$K_0 = K^{-1}, \quad K_1 = K, \tag{4.43}$$

where $x = e^u$ is an affinization parameter. These operators define a representation $(e^u,1/2)$ of the affine quantum group $U_q(\widehat{sl2})$. So that the double Abelian Chern–Simons theory on the torus possesses a quantum group symmetry $U_q(\widehat{sl2}) \otimes U_q(\widehat{sl2})$, which is identified with the quantum group symmetry of the Six-vertex model or, equivalently, with the symmetry Group of the two-dimensional lattice fermion model. To write down the corresponding EFT on the square lattice with periodic boundary conditions all we need to do is to take the modular parameter $\tau = i$ and to restrict the motion of the effective degrees of freedom (charge density waves) to the links on the lattice by replacing the Chern–Simons term with the Lattice Chern–Simons term (which posses lattice differential operators). Therefore, the EFT is defined by the action:

$$S_{DCS} = \frac{k}{4\pi} \int d^3x a^R_\mu K_{\mu,\nu} a^R_\nu - \frac{k}{4\pi} \int d^3x a^L_\mu K_{\mu,\nu} a^L_\nu, \tag{4.44}$$

with $K_{\mu,\nu} = S_{mu}\epsilon_{\mu,\alpha,\nu} d_\alpha$, $S_\mu f(x) = f(x + a\epsilon_\mu)$, $d_\mu f(x) = (f(x + a\epsilon_\mu) - f(x))/a$, (where a is the lattice spacing), which can also be written as a mixed Chern–Simons theory [39,40].

4.2.3. Identification of the Mott point ($q = -1$)

We are now ready to identify the Mott point in the two-dimensional case. As we have seen in Section 3.2 the two dimensional fermion model (2.1) which is equivalent to the *XXZ* spin lattice, can be split into one dimensional systems for each row or column of the square lattice, with Hamiltonian

$$H^{1d}_{XXZ} = \sum_{i=1}^{L} \left(S^x_i S^x_{i+1} + S^y_i S^y_{i+1} - \frac{q + q^{-1}}{2} S^z_i S^z_{i+1} \right) + H_b, \tag{4.45}$$

where $H_b = \alpha(S^z_1 - S^z_L)$ and $\alpha = (q - q^{-1})/2$.

On the one hand the critical point of this system, which has been analyzed in Section 1, allows us to identify the Mott transition point in the square lattice with the one-dimensional transition at $\Delta = 1$. Since we have defined $\Delta = -(q + q^{-1})/2$, it implies $q = -1$. Besides the critical point in the one-dimensional system was identified with the WZW model or a double Chern Simons theory on the circle. On the other hand we have identified the critical point of the two dimensional (double periodic) lattice fermion system (2.1) with the Double Chern Simon theory on the (space) torus with quantum group symmetry $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$. As we discussed in the Section 4.2.1, cutting down the torus the Gauge field become dynamical currents represented by a WZW model, which

naturally splits its degrees of freedom in chiral components) with quantum group symmetry $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$ [47] with $q = \exp(i\pi/(k+2))$.However, since the identification of the deformation parameter q is done at the classical (Hamiltonian) level, there is no Sugawara shift in the WZW coupling constant k due vacuum renormalization, an we should use $q = \exp(i\pi/k)$. Therefore, since we can cut down the torus on any cycle, the affine Quantum group symmetry of the double Chern–Simons theory on the domain $T \times R$ implies the quantum Group symmetry of each *XXZ* spin system cut down in any circle. We therefore conclude that $q = -1$ is the correct value of the deformation parameter characterizing the Mott point.

The above statement is the EFT formulation of the statement discussed in [6] and reviewed in Section 3: the Quantum TE implies the standard Yang–Baxter equation, so that the integrability in the (2 + 1)-dimensional quantum system implies the integrability of the (1 + 1)-dimensional systems obtained from it by the projection-like character of the solution (3.34). Sumarizing, the EFT of the lattice fermion system (2.1) at the Mott transition point is represented by a double Chern–Simons theory at coupling constant $k = 1$, in agreement with the analysis based on the magnetic algebra presented in Section 4.2.1.

4.3. The order parameter

We would now like to discuss the emergence of the order parameter characterizing the transition. We can do this either from the point of view of the double-Chern–Simons theory or from the point of view of the equivalent CFT. First, we recall that, as we stated in Section 3.2, each row and column of the lattice system (2.1) can be considered as a one-dimensional system, which exhibit charge density wave order as we have stated in Section 1. At $\Delta = 1 + \epsilon$ the one dimensional system in the row is in gap-full state which correspond to the antiferromagnetic phase of the equivalent *XXZ* spin chain. For $\Delta = 1$, the one-dimensional fermion system is in a gapless phase and is described by a CFT with $\widehat{su(2)}_1 \otimes \widehat{su(2)}_1$ symmetry, which is encoded in the WZW action or in a chiral-antichiral bosonic system compactified on a circle at the self-dual radius. Let us now consider the transition in the square lattice: since we can cut along any cycle of the torus (defined in Section 4.2.2, we can obtain the above bosonic CFT at the self-dual radius in any row or column in the lattice. Invariance under surgery of the states in the CS theory, implies by consistency that the fermion system on the lattice should be described by a CFT with $c = 1$. This is interpreted as the theory of the free boson on the (space) torus, with partition function:

$$Z = \frac{1}{\eta(\tau)} \sum_{e,m} q^{1/2(e/R+mR/2)} \bar{q}^{1/2(e/R-mR/2)}, \tag{4.46}$$

where $\eta(\tau)$ is the Dedekind function, τ is the modular parameter of the torus and e and m are the electric and magnetic charges. Under R-duality (*i.e.*, the interchange $R \leftrightarrow 2/R$) the partition function is invariant if one exchanges the electric and magnetic charges ($e \leftrightarrow m$).[2] This is also the partition function of an equivalent two-dimensional Coulomb gas described by the action:

$$S_{CG} = \frac{1}{2} \sum_{jk} \left[\left(\frac{e_j}{\sqrt{g}} + m_j \sqrt{g} \right) G(R_j - R_k) \left(\frac{e_k}{\sqrt{g}} + m_k \sqrt{g} \right) \right], \tag{4.47}$$

where G is the two-dimensional lattice Green function. Note that the above action describes the charge–charge, vortex–vortex and charge–vortex interactions. The Mott transition on the tours (square lattice with periodic boundary conditions) is given by the above partition function at the self-dual radius $R = \sqrt{2}$, such that the system exhibits electric–magnetic duality.

We would now like to discuss the behavior of the EFT away from the Mott critical point. We consider first the one-dimensional theory describing one of the orientations on the lattice. For

[2] By including all possible boundary conditions (periodic and anti-periodic) one obtains the S_1/Z_2 orbifold partition function that is also related to the Six-vertex model at the critical line [46]. We are not focusing in this case because we are working at half-filling states, which fixes the boundary conditions to be periodic.

$\Delta > 1$, the massive antiferromagnetic phase is represented in the continuous limit by a Sine Gordon theory:

$$S_{SG} = \int d^2x[\partial_\mu\phi\partial_\mu\phi + 2\alpha_0\cos\beta\phi]. \tag{4.48}$$

It is well-known that the partition function may be written as [48]:

$$Z = \lim_{\epsilon\to 0}\sum_n \frac{\alpha^{2n}}{(2n)!^2}\int\prod_{i=1}^{2n} d^2z_i \exp\left(\frac{\beta^2}{8\pi}\sum_{i\neq j} q_iq_jLn|z_i - z_j|^2 + \epsilon^2\right). \tag{4.49}$$

Here $z = x + iy$ denotes the position of the charges q_i in complex coordinates, that take values ±1. The renormalized coupling constant $\alpha = \alpha_0(\epsilon^2)^{\frac{\beta^2}{8\pi}}$ plays the role of a fugacity (ϵ is a short distance regulator). This describes the antiferroelectric phase of the Coulomb gas model.

Moreover, the staggered flux phase revealed by the quantum group analysis of the previous Section can be described by a CS theory defined on a torus with punctures, by defining pseudo-spins on the dual lattice representing these fluxes. If we define the dual system as the two-dimensional *XXZ* model made from these pseudo-spins with coupling constant $\Delta' = \Delta^{-1}$, then the self-dual point is defined by $\Delta = 1$. Here the self-duality in defined by invariance under exchange among the spins with coupling constant Δ in the direct lattice, and spins in the dual lattice with coupling constant Δ'. Just below the Mott transition,*i.e.*, for $\Delta = 1 - \epsilon$, the dual system is in the frozen (Mott) state with the pseudo-spins in an Neel state, so that the fluxes are also antiferromagnetically ordered. In this a case the EFT is given by a:

$$S_{CS} = \frac{k}{4\pi}\int d^3x a^\mu K_{\mu,\nu}a_\lambda + \sum_p{}' \phi^0[\delta(x_d, y_d) - \delta(x_d + 1, y) - \delta(x_d, y_d + 1) + \delta(x_d + 1, y_d + 1)], \tag{4.50}$$

where a_λ is an Abelian CS field and $\sum_p'$ means that the sum is taken over all fundamental domains. Each domain has period $2a$ and contains four vortices in antiferromagnetic array. The emergence of only one CS term reflects the breakdown of the chiral symmetry (as can be seen comparing Eq. (4.50) with Eq. (4.44) which contain two chiral fields), and the classical low-lying states reproduce an staggered-flux phase current pattern. This fact coincides with the quantum group analysis at $q = -1$ presented in Section 4.2. At the quantum level, Gauss law selects the physical states from the lattice CS Gauge theory on the torus with punctures. Therefore, the quantum order of the ground state of this theory is characterized as a staggered flux phase.

The EFT that we have presented above allows one to study some properties of the system under doping. By analogy with the CS theory of the quantum Hall effect, we could expect a ground state stable against small doping. In that case, for the simplest inverse filling fractions $k = m$ (m odd integer), the ground state is described as a droplet of incompressible quantum liquid [49] (however, other phases with more exotic quantum orders, like Nematic phases are also possible in other regimes ([52]) ([53]) ([54]) and is stable under small perturbations away from the center of a given plateau in the conductivity. In the Mott system, we have already assumed that the dynamically generated vortices act as external statistical fields for the new electrons injected in the system by doping (this can be considered as an extension of the *R*-duality). At the self-dual point, statistical magnetic fields can be interchanged with statistical electric fields (on a torus). After imposing the lattice symmetries, the low-lying effective Hamiltonian for the injected electrons (in first quantization) is:

$$H = \sum_i\left[-\hbar^2\frac{1}{2m}\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}\right) + \lambda_i(x_i^2 - y_i^2)\right], \tag{4.51}$$

where λ_i can take the values $\pm\lambda$. Therefore, the electric potential changes sign in $x = \pm y$, producing domain walls between regions with different electron densities. Similar results can be obtained using the

W_4 symmetry, which is related to the relevant perturbations of the Ashkin–Teller and Six-vertex models away from the critical point [50,51].

One consequence of having discussed the EFT is that, *a posteriori*, the behavior of the electrons can be more easily understood. It can be shown that the interaction term in the Hamiltonian (2.1) in the continuum limit contains a chemical potential term of the form $-\mu\rho$, with $\mu = \Delta$, which ensures the half-filling condition. Therefore, changing the chemical potential by doping in $\delta\mu$ modifies the Hamiltonian (in the spin representation (2.6) by:

$$H(\Delta) \to H(\Delta) + \delta\mu \sum_{\langle ij \rangle} S_i^z S_j^z. \tag{4.52}$$

For $\Delta = 1$, the dynamics of the electron system is given by the double CS theory (4.44), whose Hamiltonian can be defined as the temporal component of the stress-energy tensor $H_{cs} = T_{00}$, where $T_{\mu\nu} = \delta S_{CS}/\delta g_{\mu\nu}$ and $g_{\mu\nu}$ is the metric tensor. However, the *CS* action is topological and, therefore, independent of the metric implying $H_{cs} = 0$ for each chiral component, which leads to $H(\Delta = 1) = 0$. This means that doping the system away from the critical point, the dynamics is controlled by an effective Ising Hamiltonian.

5. Conclusions

In this paper, we have studied the Mott transition in an interacting electron system with a hopping term and nearest neighbors density–density coupling defined on a square lattice (2.1). We have first reviewed the one-dimensional case with periodic boundary conditions (*i.e.* when the system lives on a circle) in Section 1, and we have written down a conformal field theory description leading to the identification of the Mott transition point (*i.e.*, when the coupling constant is $\Delta = 1$) as the Lutter–Emery point in the bosonic formulation, and the degrees of freedom with charge density waves. We have also pointed out that the low-energy dynamics at the transition is described by a Wess–Zumino–Witten model, as it was already implicit in the literature.

We have also discussed the two-dimensional Mott transition, starting with the study of the integrability of the two-dimensional fermion system (2.1). To do that, we have used a two-dimensional Jordan–Wigner transformation and a new solution of the Zamolodchikov tetrahedron equation, which allowed us to identify the fermion system (2.1) as a 'quantum vertex model', and shown that the fermion system is characterized by an affine quantum group symmetry. As a consequence of the projection-like property of the new solution, we concluded that the two-dimensional lattice system factorizes into two one-dimensional systems, one for any row and one for any column of the two-dimensional square lattice. This fact allowed for the identification of the two-dimensional Mott transition with the one-dimensional one, which occurs when the coupling parameter is $\Delta = -(q + q^{-1})/2 = 1$. The identification of the symmetry and of the effective degrees of freedom (charge density waves) led to the construction of the Effective Field Theory at the critical point, which is a double (Abelian) Chern–Simons theory with quantum group symmetry $U_q(\widehat{sl(2)}) \otimes U_q(\widehat{sl(2)})$ and deformation parameter given by $q = \exp(-i\pi/k)$. Furthermore, this effective theory may be considered as the broken phase of a non-Abelian Chern–Simons theory associated to the vertex models (which has been already pointed out by Witten).

Finally, the behavior of the system near the Mott point has also been investigated using the ideas of EFT. We have found that the transition is of the Kosterlitz–Thouless class, characterized by an array of Chern–Simons vortices in a anti-ferromagnetic order. This description corresponds to a *d*-density-wave order parameter for the matter currents. Upon doping with electrons, the magnetic-electric duality of the KT transition implies the appearance of domain walls between region of different densities.

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