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Coherent states, vacuum structure and infinite component relativistic wave equations

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It is commonly claimed in the recent literature that certain solutions to wave equations of positive energy of Dirac-type with internal variables are characterized by a non-thermal spectrum. As part of that statement, it was said that the transformations and symmetries involved in equations of such type corresponded to a particular representation of the Lorentz group. In this paper, we give the general solution to this problem emphasizing the interplay between the group structure, the corresponding algebra and the physical spectrum. This analysis is completed with a strong discussion and proving that: (i) the physical states are represented by coherent states; (ii) the solutions in [Yu. P. Stepanovsky, *Nucl. Phys. B* (*Proc. Suppl.*) **102** (2001) 407–411; **103** (2001) 407–411] are not general, (iii) the symmetries of the considered physical system in [Yu. P. Stepanovsky, *Nucl. Phys. B* (*Proc. Suppl.*) **102** (2001) 407–411; **103** (2001) 407–411] (equations and geometry) do not correspond to the Lorentz group but to the fourth covering: the Metaplectic group $Mp(n)$.

Keywords: Group theory; relativistic wave equations; geometry and topology; coherent states.

1. Introduction and Results

For the last 50 years, there has been an increased interest with respect to two fundamental points of theoretical physics: the new representations of algebras with variables of the harmonic oscillator and the study of relativistic wave equations. These two points were developed placing great attention on the condition of positive energy and the role of the spin in these representations. The main motives were the theoretical problems of optics, the positive energy spectrum of physical states and the close relation between the spin and the generalized statistics. Despite the recent interest and the continuous efforts into the study on compact groups and their relationship to physics, there was no major progress on the issue.

For example, in a recent reference it was claimed that certain solutions to wave equations of positive energy of the Dirac type with internal variables have as the main characteristic a non-thermal spectrum. As part of that statement, it was said that the transformations and symmetries involved in a such an equation would correspond to a particular representation of the Lorentz group.

In this work, we will demonstrate that both claims and the same state solutions in [1, 2] are unfortunately not fully correct. Calculating the solutions of the physical system of [1, 2], we show that:

(i) the solutions are coherent states, as described before (e.g. [3, 6–15]);

(ii) we show that the transformations and symmetries involved into the relativistic wave equation of [1, 2] do not belong to the group of Lorentz but to the double cover of the groups $Sp(2)$ and $SU(1,1)$: the Metaplectic group $Mp(2)$ [3, 12, 13] and;

(iii) that these solutions have, in general, a thermal spectrum going under certain conditions, to the non-classical behavior (squeezed) [1, 2, 6–11].

Regarding the theoretical basis of the problem, we start as follows:

Let such a spinor such that it can be described schematically by the chain:

$$A_\alpha :\in Mp(2) \supset Sp(2\mathbb{R}) \sim SU(1,1) \supset SO(1,2) \approx L(3) \tag{1}$$

(take note of the above structure that will be important into the analysis that follows) that is defined as:

$$A_\alpha = \begin{pmatrix} a \\ a^+ \end{pmatrix}_\beta \Rightarrow [A_\alpha, A_\beta] = \epsilon_{\alpha\beta}, \tag{2}$$

where a and a^+ are standard annihilation and creation operators, respectively. As we will see soon, there exists a close relation with the squeezed vacuum structure. The equation to solve has the typical structure of the positive energy equation with internal variables, as proposed by Majorana [5] and Dirac [6–11], and is explicitly written as

$$(\sigma^i \partial_i - m)_\alpha^\beta A_\beta |\psi\rangle = 0. \tag{3}$$

In [1, 2], similar to the case of the Dirac positive energy equation, a wave solution was proposed as:

$$|\psi\rangle = e^{ip\cdot x}|u\rangle. \tag{4}$$

The first wrong fact in [1, 2] is to assume *a priori* that momentum p and x in the exponent of the proposed wave equation (4) commute with the annihilation and creation operators a and a^+. Consequently, in our analysis we will consider the phase space coordinates p and x in the exponent of the proposed wave equations as constants or as if the annihilation and creation operators a and a^+ act in an internal or auxiliary space.

Only under these conditions, we can insert (4) in (3) obtaining:

$$(ip_i\sigma_i - m)_\alpha^\beta \begin{pmatrix} a \\ a^+ \end{pmatrix}_\beta |u\rangle = 0, \tag{5}$$

$$\begin{pmatrix} ip_3 - m & ip_1 - p_2 \\ ip_1 + p_2 & -ip_3 - m \end{pmatrix} \begin{pmatrix} a \\ a^+ \end{pmatrix}_\beta |u\rangle = 0. \tag{6}$$

At this point, the second wrong fact in [1, 2] is evident: the author remains with only one component of the spinor solution. In fact, if we impose the same conditions as in [1, 2] namely $p_i = (0, p, i\varepsilon)$, we have

$$\begin{pmatrix} \varepsilon + m & p \\ -p & m - \varepsilon \end{pmatrix} \begin{pmatrix} a \\ a^+ \end{pmatrix}_\beta |u\rangle = 0, \tag{7}$$

$$\begin{pmatrix} (\varepsilon + m)a + pa^+ \\ -pa + (m - \varepsilon)a^+ \end{pmatrix}_\beta |u\rangle = 0. \tag{8}$$

Note that there are two different and simultaneous conditions that $|u\rangle$ must satisfy. If we put now $p = 0$ as in [1, 2], then

$$\begin{pmatrix} (\varepsilon + m)a \\ (m - \varepsilon)a^+ \end{pmatrix}_\beta |u_0\rangle = 0. \tag{9}$$

Here we clearly see that $|u_0\rangle$ cannot be the Fock vacuum $|0\rangle$ as stated in [1, 2] (it can only be if $m = \varepsilon$). Through the next sections, we will find the true vacuum, the spectrum and the solution of the problem.

2. Relation with the Squeezed Vacuum

Looking at expressions (7), (8), it is not difficult to see that these can be obtained from a similar form as the squeezed vacuum. The squeezed vacuum is generated by the $Mp(2)$ transformation $U = S(\xi)$

$$A_\alpha \to S(\xi) \begin{pmatrix} a \\ a^+ \end{pmatrix}_\alpha S^\dagger(\xi) = \begin{pmatrix} \lambda a + \mu a^+ \\ \lambda^* a^+ + \mu^* a \end{pmatrix}_\alpha, \tag{10}$$

where $\lambda(\xi)$ and $\mu(\xi)$ satisfy $|\lambda|^2 + |\mu|^2 = 1$, e.g. $SU(1,1)$ elements.

We must note that the right-hand side of Eq. (10) is governed by the operators $S(\xi) \in Mp(2)$ being the right side affected by a matrix representation of $SU(1,1)$ as follows

$$S(\xi) \begin{pmatrix} a \\ a^+ \end{pmatrix}_\alpha S^\dagger(\xi) = \begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda^* \end{pmatrix} \begin{pmatrix} a \\ a^+ \end{pmatrix}_\alpha. \tag{11}$$

Clearly, the above equivalence is only local (infinitesimal) since at the level of the group structure (see the chain (1)) there is a homomorphism relationship.

The homomorphisms between $Mp(2)$ and $SU(1,1)$ (or $Sp(2\mathbb{R})$), which are two to one and four to one in the case of $SO(1,2)$, can be expressed in α (polar)-parameterization [12, 13] in the usual way:

$$S(\alpha_\perp, \alpha_3) \in Mp(2) \to s(\alpha_\perp, [\alpha_3]_{4\pi}) \in Sp(2\mathbb{R}) \tag{12}$$

$$\to s(\alpha_\perp, [\alpha_3]_{4\pi}) \in SU(1,1) \tag{13}$$

$$[\alpha_3]_{4\pi} \in (-2\pi, 2\pi] \to [\alpha_3]_{4\pi} \bmod 4\pi, \alpha_\perp \in \mathbb{R}^2, \alpha_3 \in (-4\pi, 4\pi]$$

$$\text{for the } SU(1,1)(\text{or } Sp(2\mathbb{R})) \text{ case} \tag{14}$$

$$\to [\alpha_3]_{8\pi} \bmod 8\pi, \alpha_\perp \in \mathbb{R}^2, \alpha_3 \in (-8\pi, 8\pi] \tag{15}$$

$$\text{for the } SO(1,2) \text{ case.}$$

Consequently, it is clear that the "two to one" and the "two to four" nature are involved in the reduction of the range of the parameter α_3. This is the main reason why, for the physical scenarios of current interest, the above parameterization is better than the Iwasawa (KAN) one.

The most general expression for an element of the Metaplectic group can be computed, with the following result:

$$e^{A(aa^+ + a^+a) + Ba^{+2} + Ca^2} = e^{-A/2} \exp\left(\frac{Ba^{+2}}{\Delta \coth \Delta - A}\right)$$

$$\times \exp\left[H \ln\left(\frac{\Delta \sec h\Delta}{\Delta - A \tanh \Delta}\right)\right] \exp\left(\frac{Ca^2}{\Delta \coth \Delta - A}\right),$$

$$\Delta \equiv \sqrt{A^2 - 4BC}, \quad H \equiv \left(\frac{aa^+ + a^+a}{2}\right), \quad \widehat{N} \equiv a^+a, \tag{16}$$

where the Baker–Haussdorf–Campbell formula was used. A, B, C are arbitrary in principle only linked by expression (16) (all group theoretical properties of the non-compact groups involved, were assumed there). Therefore, with the parameters as given by expressions (7)–(9), $S(\xi)$ takes a concrete form as follows

$$S(\xi) = \exp\left(\frac{p}{m+\epsilon} a^{+2}\right) \left(\frac{1}{\sqrt{m^2 - \epsilon^2}}\right)^{1/2}$$

$$\times \left\{\sum_{n=0}^{\infty} \frac{1}{n!} \left[\ln\left(\frac{1}{\sqrt{m^2 - \epsilon^2}}\right) \widehat{N}\right]^n\right\} \exp\left(-\frac{p}{m-\epsilon} a^2\right),$$

thus, the unitary (squeezed) operator acting on the true vacuum (fiducial vector) defines the following general state

$$|\xi\rangle \equiv S(\xi)|z_0\rangle. \tag{17}$$

3. The Solution

We arrive at the construction of coherent states on a general vacuum: $A|0\rangle + B|1\rangle$ with A and B depending on initial and boundary conditions. If $|z_0\rangle \equiv A|0\rangle + B|1\rangle$, then

$$|\xi\rangle \equiv S(\xi)|z_0\rangle = \frac{\exp(\alpha a^{+2})}{(m^2-\epsilon^2)^{1/4}}\left[A|0\rangle + \frac{B}{(m^2-\epsilon^2)^{1/2}}|1\rangle\right] \tag{18}$$

$$= (m^2-\epsilon^2)^{-1/4}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(a^{+2})^n\left[A + \frac{B}{(m^2-\epsilon^2)^{1/2}}a\right]|0\rangle, \tag{19}$$

$$\text{with } \alpha \equiv \frac{p/2}{m+\epsilon}. \tag{20}$$

Note that a^2 annihilates $|z_0\rangle$ but a does not. After standard normalization, the constants in the "thermal" (photon) case reach the critical point. When the quantum state solution is simultaneously eigenstate of a and a^2, they take the particular fashion

$$A = (|m^2-\epsilon^2| + p^2\mathrm{sign}(\epsilon^2-m^2))^{1/4}, \tag{21}$$

$$B = (|m^2-\epsilon^2| + p^2\mathrm{sign}(\epsilon^2-m^2))^{3/4} = A^3. \tag{22}$$

We have a standard coherent state (eigenstate of the operator a) as a linear combination of two states belonging to $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$, respectively (that are independent coherent states as eigenvalues of a^2). In this particular case, we have

$$|z_0\rangle_{th} = A(1 + A^2 a^+)|0\rangle. \tag{23}$$

Note that this vacuum is not singular at $m \to \epsilon$ but is analytically continued into the complex plane where it is defined:

$$|\xi\rangle_{\mathrm{th}} \equiv S(\xi)|z_0\rangle_{\mathrm{th}} = \left(1 + \frac{p^2\mathrm{sign}(\epsilon^2-m^2)}{|m^2-\epsilon^2|}\right)^{1/4} \times e^{\frac{p/2}{m+\epsilon}a^{+2}}\left[1 + \left(1 + \frac{p^2\mathrm{sign}(\epsilon^2-m^2)}{|m^2-\epsilon^2|}\right)^{1/2} a^+\right]|0\rangle. \tag{24}$$

4. Bargmann Representation: Analytical Versus Geometrical Viewpoint

4.1. *The Bargmann representation*

We have so far worked mainly with the photon-number description of the Hilbert space $\mathcal{H}$ and the operators a, a^+. In this section we analyze the misunderstanding pointed out previously, introducing the Bargmann representation.

The Bargmann representation of $\mathcal{H}$ associates an entire analytic function $f(z)$ of a complex variable z, with each vector $|\varphi\rangle \in \mathcal{H}$ in the following manner:

$$|\varphi\rangle \in \mathcal{H} \rightarrow f(z) = \sum_{n=0}^{\infty} \langle n \mid \varphi\rangle \frac{z^n}{\sqrt{n!}}, \tag{25}$$

$$\langle \varphi \mid \varphi\rangle \equiv \|\varphi\|^2 = \sum_{n=0}^{\infty} |\langle n \mid \varphi\rangle|^2 \tag{26}$$

$$= \int \frac{d^2 z}{\pi} e^{-|z|^2} |f(z)|^2, \tag{27}$$

where the integration is over the entire complex plane. The above association can be compactly written in terms of the normalized coherent states of the Barut–Girardello type, namely, (right) eigenstates of the annihilation operator a:

$$a|z\rangle = z|z\rangle, \tag{28}$$

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{29}$$

$$\langle z' \mid z\rangle = e^{(-|z|^2/2 - |z'|^2/2 + z'^* z)} \tag{30}$$

then, we have

$$f(z) = e^{-|z|^2/2} \langle z^* | \varphi\rangle. \tag{31}$$

However, $f(z)$ must be as $|z| \rightarrow \infty$ so that $\|\varphi\|$ is finite. In this particular representation, the actions of a and a^+, and the functions representing $|n\rangle$ are as follows:

$$(a^+ f)(z) = z f(z), \tag{32}$$

$$(af)(z) = \frac{df(z)}{dz}, \tag{33}$$

$$|n\rangle \rightarrow \frac{z^n}{\sqrt{n!}}. \tag{34}$$

4.2. $Mp(2)$ generalized coherent states in the Bargmann representation

Having introduced the necessary ingredients, we can now describe the physical states of the system under consideration.

(i) The $\mathcal{H}_{1/4}$ states occupy the sector even of the full Hilbert space $\mathcal{H}$ and we may describe them as follows

$$f^{(+)}(z, \omega) = (1 - |\omega|^2)^{1/4} e^{\omega z^2/2} \tag{35}$$

$$= (1 - |\omega|^2)^{1/4} \sum_{m=0,1,2,\ldots} \frac{(\omega/2)^m}{m!} z^{2m} \tag{36}$$

then, in the vector representation we have:

$$|\Psi^{(+)}(\omega)\rangle = (1-|\omega|^2)^{1/4} \sum_{m=0,1,2,\ldots} \frac{(\omega/2)^m}{m!}\sqrt{2m!}|2m\rangle \tag{37}$$

consequently, the number representation is obtained as:

$$\langle 2m \,|\, \Psi^{(+)}(\omega)\rangle = (1-|\omega|^2)^{1/4} \frac{(\omega/2)^m}{m!}\sqrt{2m!} \tag{38}$$

$$\langle 2m+1 \,|\, \Psi^{(+)}(\omega)\rangle \equiv 0. \tag{39}$$

(ii) The $\mathcal{H}_{3/4}$ states occupy the odd sector of the full Hilbert space $\mathcal{H}$ and we may describe them as before:

$$\begin{aligned} f^{(-)}(z,\omega) &= (1-|\omega|^2)^{3/4} z e^{\omega z^2/2} \\ &= (1-|\omega|^2)^{3/4} \sum_{m=0,1,2,\ldots} \frac{(\omega/2)^m}{m!} z^{2m+1} \end{aligned} \tag{40}$$

then, in vector representation we have:

$$|\Psi^{(-)}(\omega)\rangle = (1-|\omega|^2)^{3/4} \sum_{m=0,1,2,\ldots} \frac{(\omega/2)^m}{m!}\sqrt{(2m+1)!}|2m+1\rangle. \tag{41}$$

The number representation is consequently:

$$\langle 2m+1 \,|\, \Psi^{(-)}(\omega)\rangle = (1-|\omega|^2)^{3/4} \frac{(\omega/2)^m}{m!}\sqrt{(2m+1)!}, \tag{42}$$

$$\langle 2m \,|\, \Psi^{(-)}(\omega)\rangle \equiv 0. \tag{43}$$

(iii) The full Hilbert space, defined by the direct sum $\mathcal{H} = \mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$, is trivially described as follows

$$f(z,\omega) = f^{(+)}(z,\omega) + f^{(-)}(z,\omega) \tag{44}$$

$$= (1-|\omega|^2)^{1/4} \sum_{m=0,1,2,\ldots} \frac{(\omega/2)^m}{m!} z^{2m}[1 + (1-|\omega|^2)^{1/2} z]. \tag{45}$$

Then, in complete analogy as their even and odd subspaces, the corresponding states are described by:

$$\Psi(\omega) = \Psi^{(+)}(\omega) + \Psi^{(-)}(\omega) \tag{46}$$

$$= (1-|\omega|^2)^{1/4} \sum_{m=0,1,2,\ldots} \frac{(\omega/2)^m}{m!}\sqrt{2m!}[1 + (1-|\omega|^2)^{1/2} a^+]|2m\rangle, \tag{47}$$

$$\langle m \,|\, \Psi(\omega)\rangle \begin{cases} (1-|\omega|^2)^{1/4} \dfrac{(\omega/2)^m}{m!}\sqrt{2m!}, & m \text{ even}, \\[2ex] (1-|\omega|^2)^{3/4} \dfrac{(\omega/2)^m}{m!}\sqrt{(2m+1)!}, & m \text{ odd}, \end{cases} \tag{48}$$

where the link between the physical observables and the group parameters is given by the following expression (measure):

$$\left(1+\frac{p^2\mathrm{sign}(\epsilon^2-m^2)}{|m^2-\epsilon^2|}\right)^{1/4} \to (1-|\omega|^2)^{1/4}. \tag{49}$$

5. The Limit $\epsilon \to m$

This is precisely the limit $|\omega|^2 \to 1$ from the point of view of the Metaplectic analysis that corresponds to the edge of the complex disc. As we could see easily, the state solutions are generally thermalized (full spectrum corresponding to $\mathcal{H}$). What happens is that in the limit $\epsilon \to m$ the density of states corresponding to $\mathcal{H}_{1/4}$ is greater than that of the odd states belonging to $\mathcal{H}_{3/4}$. It is for this reason that states belonging to $\mathcal{H}_{1/4}$ will survive in this limit. As we will see in a separate publication, there is a particular case of the two-dimensional electron transport with a magnetic field in the plane whose states belong to metaplectic group.

6. Complete Equivalence Between Sannikov's Representation and Metaplectic One

The main characteristics of the particular representation introduced in [3] is the following commutation relation that defines the generators L_i:

$$[L_i, a^\alpha] = \frac{1}{2}a^\beta(\sigma_i)_\beta{}^\alpha. \tag{50}$$

The above representation which corresponds to a non-compact Lie algebra with the following matrix form [6–11, 13] is:

$$\sigma_i = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{51}$$

$$\sigma_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{52}$$

$$\sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{53}$$

that fulfills evidently:

$$\sigma_i \wedge \sigma_j = -i\sigma_k, \tag{54}$$

$$\sigma_k \wedge \sigma_i = i\sigma_j, \tag{55}$$

$$\sigma_j \wedge \sigma_k = i\sigma_i. \tag{56}$$

The equivalence that we want to remark is manifested by the following:

Proposition 1. *The generators in the representation of* [3] *fulfill*:

$$L_i = \frac{1}{2}a^\beta(\sigma_i)_\beta{}^\alpha a_\alpha = T_i, \tag{57}$$

where T_i are the Metaplectic generators namely [12, 13]:

$$T_1 = \frac{i}{4}(a^{+2} - a^2), \tag{58}$$

$$T_2 = \frac{-1}{4}(a^{+2} + a^2), \tag{59}$$

$$T_3 = \frac{-1}{4}(aa^+ + a^+a). \tag{60}$$

Proof. Explicitly in matrix form we can write the generators proposed in [3] (and for instance in [1, 2]) as

$$L_i = \overline{u}\mathbb{M}_i v, \tag{61}$$

$$\overline{u} \equiv (a^+ a), \tag{62}$$

$$v \equiv \begin{pmatrix} a \\ a^+ \end{pmatrix}. \tag{63}$$

In the representation (50), that is faithful, and taking into account that σ_k enter as "metric" in the sense given by Sannikov [3], we have

$$M_1 = \frac{i}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{4}\sigma_k\sigma_i, \tag{64}$$

$$M_2 = -\frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{1}{4}\sigma_k\sigma_j, \tag{65}$$

$$M_3 = -\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{4}\sigma_k^2, \tag{66}$$

consequently and by inspection (50) coincides with (61): thus, the equivalence (57) is proved. □

7. Concluding Remarks

In this paper, we have studied from the physical and group-theoretical points of view, the close relation between the Metaplectic group, the Lorentz group and its covering the $SL(2,C)$ ones. The main emphasis was to clarify the existent confusion between the representations of the considered non-compact groups. To this end, using a typical example, a recently posed problem in [1, 2], we solved exactly the corresponding equations to the physical scenario given in [1, 2], highlighting consequently the common errors and misunderstandings that appear to confuse representations: namely, the Metaplectic one with the other non-compact (Lorentz and Special Linear) ones. The analysis was made easier using the group generators written with the Harmonic oscillator variables, arriving at the following conclusions and results:

(i) the solutions are coherent states, coinciding with previous theoretical descriptions (e.g. [13–15]);

(ii) the transformations and symmetries involved in the equation of [1, 2] do not belong to the group of Lorentz but to the double cover of $Sp(2)$ and $SU(1,1)$: the Metaplectic group Mp(2) [3, 6–11, 13]; and

(iii) that these solutions are generally thermal going under certain conditions to the non-classical condition (squeezed), as was verified before [6–11].

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