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Fisher order measure and Petri's universe

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ABSTRACT

Given a closed system described via an amplitude function $\psi(\mathbf{x})$, what is its level of order? We consider here a quantity $\mathbb{R}$ defined by the property that it decreases (or stays constant) after the system is coarse grained. It was recently found that (i) this quantity exhibits a series of properties that make it a good order-quantifier and (ii) for a very simple model of the universe the Hubble expansion does not in itself lead to changes in the value of $\mathbb{R}$. Here we determine the value of the concomitant invariant for a somewhat more involved universe-model recently advanced by Petri. The answer is simply $\mathbb{R} = 2(\frac{r_H}{r_0} - \frac{r_0}{r_H})$, where r_H is a model's parameter and r_0 is the Planck length. Thus, curiously, the Petri-order seems to be a geometric property and not one of its mass-energy levels. Numerically, $\mathbb{R} = 26.0 \times 10^{60}$. This is a colossal number, which approximates other important cosmological constants such as the ratio of the mass of a typical star to that of the electron $\sim 10^{60}$, and microlevel constants such as $\exp(1/\alpha)$, where α is the fine structure constant.

1. Introduction

Order is a necessary condition for anything the human mind may be able to understand. Arrangements such as a display of merchandise or the verbal exposition of ideas are regarded as orderly when an observer is capable to grasp their overall structure and some of the details. One of the most relevant statements on the behavior of the physical world, namely, the Second Law of Thermodynamics, assures us that the material world moves from orderly states to an ever-increasing disorder. As stated by Planck in 1920 "…it is not the atomic distribution, but rather the hypothesis of elementary disorder, which forms the real kernel of the principle of increase of entropy and, therefore, the preliminary condition for the existence of entropy. Without elementary disorder there is neither entropy nor irreversible process" [1]. Thus, the concept of disorder, as measured by an entropic quantifier H is well established.

We will consider a complementary notion: that of "level of *order*", or just order, in the *continuous system* described by the probability density $p(\mathbf{x})$. The vocable itself is rarely quantified in mathematical fashion, as was the case with the term "information" before Shannon. What might we mean by a continuous system's order? Numerous authors have considered this question à la Planck, i.e., as equivalent to the statement of the second law of thermodynamics: "disorder must increase". On this basis order must decrease, where order is in some sense the "opposite" of disorder. However, to attempt to define a physical effect as being simply the opposite of some other effect cannot succeed, since the word "opposite" does not have a unique meaning. For example, with regards to the entropy H, do we speak of its negative (the "negentropy", see below) $-H$, or of its reciprocal $1/H$ or function $\exp(-H)$, or …? A way out of this dilemma is to observe that all other physical concepts

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are defined by distinct physical effects. For example, the concept of entropy arises as the measure that obeys the 2nd law of thermodynamics. Thus, the notion of order must likewise be defined by its own physical effect. What is that effect? A clue arises out of considering in this regard a discrete system, characterized by a probability law P_i, $i = 1, \ldots, N$, with discrete states *i*. There the measure of order is indeed the "negentropy",

$$\mathbb{R}_H \equiv +\sum_{i=1}^{N} P_i \log P_i. \tag{1}$$

This measure satisfies the non-increasing requirement after any regrouping of its finest microstates into larger (coarser) macrostates. *Such regrouping is precisely the physical operation defining the measure*. However, the probability density $p(\mathbf{x})$ above is continuous, *not discrete*. A form of the measure (1) valid for continuous probability densities can be formally obtained replacing the sum in (1) by an integral over the relevant configuration or phase space. Even if this new version of (1) plays a central role in several important scenarios (ranging from Boltzmann's celebrated interpretation of the second law of thermodynamics via his *H*-theorem [2], to Jaynes' maximum entropy principle and its multiple applications [3]) it is well-known that, to some extent, it is conceptually problematic. If one tries to "connect" the discrete and the continuous versions of (1) via a discretization of the continuous phase space one immediately runs into problems, since the concomitant continuous limit $P_i \to p(\mathbf{x})\mathrm{d}\mathbf{x}$ of the measure is not well-defined because $\log \mathrm{d}x \to -\infty$. Related to this difficulty is also the somehow "disturbing" fact that the continuous version of (1) does not have a definite sign: for some probability densities it adopts a positive value, while for others its value is negative.

Another limitation of (1) is found in its actual use. Its application for treating systems that are not in states at or near thermodynamic equilibrium has been questioned. But, of course, several important systems found in nature are not of this type (see Refs. [4,5] for comprehensive and interesting recent discussions of the inadequacy of (1) for the description of several out of equilibrium scenarios). Examples are very rarefied gases, where molecular collisions are infrequent; or systems of interacting fermions at very low temperature, where dissipative processes become ineffective. Last, but certainly not least, there are "living" systems [6]. These constitute the most important and most complex systems far from equilibrium. Living systems, in fact, die at thermodynamic equilibrium!

We ask then, is there any measure that works regardless of how far the system's state is from its equilibrium state? A clue is that the regrouping process described above is actually a special case (the discrete counterpart) of a generally continuous physical process called "coarse graining" (see below).

1.1. Coarse graining

A coarse-grained description refers to a model for which its finest details have been smoothed over or averaged out. There is a significant loss of resolution. Thus, a coarse-grained description of a system is limited to its grosser subcomponents. We wish to define the order-notion on the basis of regrouping processes. We will refer in abbreviate fashion to such order-concept as the regrouping order (RO).

An RO-example is where random noise from a generally shaped distribution $p(\mathbf{n})$ degrades the system. This produces a wider, smoother density law that is the convolution of $p(\mathbf{x})$ and $p(\mathbf{n})$. This operation effectively "coarsens" the granular nature of the system's density law $p(\mathbf{x})$. If $p(\mathbf{x})$ were a photographic picture, it would suffer an increased grain size. The RO notion is broader than that pertaining to measure (1), since the latter did not include the possibility of added noise from a *generally* shaped noise distribution $p(\mathbf{n})$: simply regrouping discrete states is roughly equivalent to convolution with a rectangle function.

1.2. Fisher order measure (FOM)

In a recent effort [7–9], a mathematical form for order was found from first principles, on the basis of coarse graining the system. Fisher's information measure (FIM) I [7] is found to enter into the answer,

$$\mathbb{R} \equiv 8^{-1} L^2 I, \quad \text{where } I = 4\int \mathrm{d}\mathbf{x} \nabla\psi^* \cdot \nabla\psi, \;\; \psi = \psi(\mathbf{x}), \qquad p \equiv |\psi|^2. \tag{2}$$

In comparison with (1), FOM does work for continuous distributions $p(\mathbf{x})$, and *regardless* of how far the system's state is from equilibrium. FOM is defined for a general state of a system. Parameter L is the maximum chord length connecting two surface points of the system. This communication concerns itself entirely with FOM, which originates out of a scenario of coarse graining.

FOM derives uniquely from the single physical requirement that, after the continuous distribution $p(\mathbf{x})$ is coarse grained, its level of order $\mathbb{R}$ may not increase, i.e.

$$\Delta\mathbb{R} \le 0 \quad \text{for } \Delta t > 0. \tag{3}$$

The coarse graining takes place over the small time interval Δt. Moreover, in contrast with the near-equilibrium state assumed in using (1), our PDF $p(\mathbf{x})$ is regarded as describing a *general* state, not necessarily at or near thermodynamic equilibrium. Order-measure (2) was derived for a one dimensional system in Ref. [7], and extended to any K dimensional system in Ref. [8].

1.3. Some properties of the FIM-associated order measure

Order-measure (2) is (a) unitless, which has the benefit of allowing completely different phenomena to be compared for their levels of order; (b) invariant under uniform stretching $y_k = a_k x_k$, $k = 1, \ldots, K$, with the $a_k = consts$, of all its coordinates; and (c) it measures *the number* of ordered "details" within the system, rather than their density of structure. E.g., for a one dimensional system $p(x) = (2/a)\sin^2(n\pi x/a),\ 0 \le x \le a$, the order $\mathbb{R} = 4\pi^2 n^2$. This is independent of the system extension a and, instead, is purely a rapidly increasing function of the *total number n* of sinusoidal waves within the system. This is in the spirit of the well-known Kolmogorov–Chaitin complexity measure [10], which is given by the *minimum length* of the computer program, or the shortest description in some fixed universal language, required to generate or to characterize a string of numbers.

1.4. Past applications

In Ref. [8] FOM was used to predict that a rodlike living creature such as an *E. coli* bacterium can be compressed into a disc-like shape with the same level of order, provided it undergoes mitosis while being deformed. This prediction was independently confirmed experimentally by another group (see Ref. [8] for details). It allows, e.g., life to exist and evolve even under great compression such as deep into the earth.

In a recent application to cosmology, it was found that, contrary to intuition, the Hubble expansion per se does not imply a FOM-decrease [9]. We are speaking of $\mathbb{R}$ for the mass-density in a Robertson–Walker universe. This is described by the metric

$$ds^2 = (cdt)^2 - a^2(t)(dx^2 + dy^2 + dz^2), \quad (x, y, z) \equiv \mathbf{r}. \tag{4}$$

FOM is, for such a universe, invariant. We emphasize that this invariance refers only to expansion-effects, without taking into account the interaction between the different elements of mass.

Present goal. Here we ask, for the same R–W universe, what is *the actual value* of this invariant level of order?

2. Application to a holographic universe with string structure

2.1. The holostar

Petri [11] has found the simplest, exact, spherically symmetric solution to the Einstein field equations with zero cosmological constant that (i) includes matter, but (ii) does not take into account structure-generation via gravitational effects. Such a solution, called a "holostar" by its discoverer, consists of a string-dominated system consisting of cold dark matter. In this model matter consists of densely packed strings pointing radially outward. Each string has a transverse extension of exactly one Planck area. The holographic solution [11] has a *real* spherical boundary membrane, which satisfies representation (2) for $\mathbb{R}$, which requires the system in question to be of finite size L. Such a real, physical membrane has the same properties as the (purely fictitious) "membrane" attributed to a *black hole* by the so-called membrane paradigm (in particular a Schwarzschild black hole). The interior matter-state is singularity-free, owing to the dense packing of the strings. The holographic solution suggests that string theory is relevant also on cosmological scales. That is, large scale phenomena in such an universe can be explained naturally in a string context.

The energy density p of the strings at any radial position r is the quantity in the 00 slot of the stress-energy tensor, i.e. $p(r) \propto 1/r^2$. Therefore, for the "stringy matter" of the model $p(r) \propto 1/r^2$ holds as well. This is for all radii on an interval (r_0, r_H), where r_0 is necessarily finite so as to avoid the singularity in $p(r)$ at the origin $r = 0$. Hence we assume that $p(\mathbf{r}) = 0$ at $|\mathbf{r}| = r \le r_0$. We take r_0 to be the radius of the spherical Planck volume, since for r smaller than this the concept of deterministic space breaks down. As was mentioned, the holostar solution admits of a real outer boundary membrane. Let us comment on its radius r_H (the subindex H refers to "holostar"). Although in our previous paper [9] we proved that the Universe's order R is constant even under Hubble expansion, we cannot here compute its actual value. Rather, the best we can do is to determine an upper bound to the level of order R. According to the Petri model [11], the radius r_H of the outer boundary membrane must be larger than the current local horizon (or Hubble) radius of the observable part of the universe. The so-called "junction conditions" at the radial position r_H of the outer boundary membrane require $r_H = 2M+$ [11], where M is the gravitational mass of the holostar (natural units $c = h = 1$ in use) as measured by an observer in the exterior Schwarzschild metric, and r_0 is a constant of order the Planck-length. Thus, since M is unknown, so is r_H. It is instead regarded as an upper bound to the radius of the holostar universe.

Now, the above reasoning satisfies the FOM-requirement that the system size must be finite. The universe is thereby modeled geometrically as a well-defined sphere of finite, albeit colossal, size with a material surface and a spherical "hole" of finite, albeit minuscule size, in it.

2.2. A simple model for the universe

The holostar's geometric properties suggest that *it might also serve as an alternative (simple) model for the universe*. The spherical boundary membrane mentioned above becomes the "edge" of the universe. Also, far away from its center the geodesic motion of massive particles is virtually indistinguishable from that of a uniformly stretched (expanding or contracting) Friedmann Robertson–Walker (FRW) universe. (Recall that in Ref. [9] we proved the FOM-invariance for the FRW universe, with the caveats mentioned at the end of Section 1.4). Although the holostar model would be straightforward to falsify, to the contrary it fits almost perfectly all experimental data that have a bearing on it. For example, the nearly unaccelerated expansion expected in such a string dominated universe is compatible with recent supernova measurements [11].

The expansion $a(t)$ in the FRW metric holostar solution is accelerated, with the proper acceleration falling off over time. The acceleration is not due to a cosmological constant, which as we mentioned is exactly zero in the holostar space–time. Rather, the proper acceleration in the co-moving frame can be traced to the radial dependence of the spherically symmetric gravitational potential, which falls off as $V(r) \propto 1/r$. The energy density p of the strings at any radial position r is the quantity in the 00 slot of the stress-energy tensor, i.e. $p(r) \propto 1/r^2$. Therefore, for the "stringy matter" of the model $p(r) \propto 1/r^2$ holds as well. This is for all radii on an interval (r_0, r_H), where r_0 is necessarily finite so as to avoid the singularity in $p(r)$ at the origin $r = 0$. Hence we assume that $p(\mathbf{r}) = 0$ at $|\mathbf{r}| = r \leq r_0$. We take r_0 to be the radius of the spherical Planck volume, since for r smaller than this the concept of deterministic space breaks down. As was mentioned, the holostar solution admits of a real outer boundary membrane of radius r_H. This satisfies the FOM-requirement that the system's size must be finite. The universe is thereby modeled geometrically as a well-defined sphere of finite, albeit colossal, size with a material surface and a spherical "hole" of finite, albeit minuscule size, in it.

What is the level of order in a universe obeying such a geometry? Since $p(r)$ is here a probability density, it must obey normalization. This requires the above proportionality $p(r) \propto 1/r^2$ to become the equality

$$\underset{r_0 \leq r \leq r_H}{p(r)} = \frac{C}{r^2}, \qquad C = [4\pi(r_H - r_0)]^{-1}. \tag{5}$$

Then from Eq. (2) the amplitude density function is, up to an unknown phase part,

$$\underset{r_0 \leq r \leq r_H}{\psi(r)} = \frac{\sqrt{C}}{r}. \tag{6}$$

By the radial symmetry $\nabla\psi = \mathrm{d}\psi/\mathrm{d}r$. Taking this derivative of Eq. (6), placing it in definition (2) of I and integrating gives simply

$$I = \frac{4}{r_0 r_H}. \tag{7}$$

Next, the longest chord length L connecting two surface points in this system is, by inspection, any chord tangent to the inside sphere of radius r_0. By the Pythagorean theorem, its length is

$$L = 2\sqrt{r_H^2 - r_0^2}. \tag{8}$$

Using results (6) and (7) in definition (2) gives

$$\mathbb{R} = 2\left(\frac{r_H}{r_0} - \frac{r_0}{r_H}\right) = 2\frac{r_H}{r_0}, \tag{9}$$

to an exceedingly good approximation since, of course, $r_H \gg r_0$. Using the current values $r_H = 13.76 \times 10^9$ lyr $= 1.302 \times 10^{26}$ m, and $r_0 = 1.00 \times 10^{-35}$ m, the order is value

$$\mathbb{R} = 26.0 \times 10^{60}. \tag{10}$$

3. Discussion

Interestingly, and keeping in mind the caveats at the end of Section 1.4, (9) predicts that the Fisher-order of space is a *purely geometrical* property. It depends only upon distances, so that regarding the universe as principally composed of dust and radiation (with zero cosmological constant) does not cause matter–energy properties to enter in. This is important because then (9) does not depend, e.g., upon knowledge of the amount of dark matter or dark energy that is present in the universe. This broadens the application of the answer (9).

Of course all local baryonic and non-baryonic matter–energy configurations such as clusters of galaxies and stars must inevitably contribute to total order, so that results (9) and (10) – which ignore these – actually give lower bounds to the figure. In fact the Robertson–Walker model regards these local mass effects as insignificant, merely "wrinkles" in the otherwise *uniform* cosmic dust of the model universe. The uniformity of the cosmic dust is presumed to dominate over all scales of matter. In turn, such dominance implies that the lower-bound value (10) of order is very close to the actual value. That FOM in (9) depends upon *both* boundary values r_0 and r_H has some interesting ramifications. It is actually consistent with loop gravity holographic theory, which requires the hologram to consist of the *entire boundary* of the universe. By our model this includes *both* the outer surface at r_H *and* the hole boundary at r_0. It is shown in Ref. [12] that the geometrical ratio r_H/r_0 in (9) also describes cosmological phenomena other than $\mathbb{R}$. The particular value 10^{60} of the numerical answer (10) is also of more general interest, occurring commonly in cosmology [12,13] and on the microlevel. Now, the cosmological constant Λ in the equations of gravitation is known to be factor $[10^{60}]^2$ larger than is observed [14]. The interesting work of Beck's in Ref. [15] should be also consulted on this issue. Also, $\exp(1/\alpha) \approx 10^{60}$, where α is the fine-structure constant, and clearly a constant on the microlevel (see also below). But are such correspondences to $\mathbb{R}$ mere numerology? In this regard we mention again the recent work of Beck's [15], in which an axiomatic approach to the cosmological constant is proposed. We suggest here that there may be connections between Beck's axioms and a putative axiomatics of order. Work is in progress on such regards.

Since quantity 10^{60} has the significance (10) of representing *the order* $\mathbb{R}$, we propose that these correspondences are all intrinsically expressions of order. For example, the ratio of the mass of a typical star to that of the electron $\sim 10^{60}$. Also, the order value (10) is *invariant in time* (as found in Ref. [9]), so that the correspondences should likewise exist for all time. Such predicted constancy was one of the postulates of physics and cosmology prior to the work of Dirac, who believed that they may change with time. Nowadays a more common belief [16] is that the fine structure constant α is central to the constancy of the physics of our universe. Thus, this constancy agrees with the above mentioned prediction that $\exp(1/\alpha) \approx 10^{60}$ is a constant.

The prediction that the universal order $\mathbb{R}$ has the very large value (10) is consistent with recent derivations [13,17] of the laws of science on the basis that they impart *maximum information I* to their data since, by (2), R is proportional to I. This verification also indicates that not only is maximum order a property common to all natural phenomena but, by the preceding, *the maximum possible value of order* is common to the *ratios*of many of their constants. Thus, the laws of science *and* the large cosmological ratios may all trace from a single requirement: that the level of order in the universe should have its maximum possible value, a number $\sim 10^{60}$.

We end with a question. By ordinary quantum mechanics, a general amplitude function $\psi(r)$ has a Fourier mate $\phi(\mu)$ in momentum space. After using the radial symmetry of the model this becomes

$$\psi(r) = 4\pi \int_{\mu_0}^{\infty} \mathrm{d}\mu \phi(\mu) \mathrm{sinc}(2\mu r), \quad \mu_0 > 0, \ \mathrm{sinc}(u) \equiv \sin(\pi u)/(\pi u). \tag{11}$$

Momentum values μ are relative to that of the origin of the holostar. Then what choice of $\phi(\mu)$ gives rise to our assumed $1/r$ form (6) for ψ? The mathematical answer is $\phi(\mu) = 1/\mu^2$, an inverse square law on momentum. From this we see that the finite lower limit μ_0 in (11) is necessary to avoid blowup of amplitude $\phi(\mu)$ at $\mu = 0$. Now, the scenario $\phi(\mu) = 1/\mu^2$ indicates a random spread of momentum values, with bias toward its minimum allowed value μ_0. But, what if this momentum value is required to obey $\phi(\mu) = \delta(\mu - \mu_0)$, i.e. to be *deterministically* attained? It turns out that, in comparison with the answer $\psi(r) \propto 1/r$, which monotonically decreases with r, these new answers $\psi(r)$ contain high-frequency oscillations in r that modulate the basic $1/r$ falloff. It might be expected these oscillations strongly increase the level of order $\mathbb{R}$, to values much exceeding the value $\sim 10^{60}$ found for the monotonic $\psi(r) \propto 1/r$. The calculation and its discussion are the subject of active research. We acknowledge that the present discussion is a much simplified one. Curved-space effects need to be explicitly included in a more complete investigation, to published in the future. Finally, it is worth stressing that, even if from the global point of view considered in the present work galaxies and other cosmological structures can be regarded as "mere wrinkles" in a uniform cosmological scenario, these "wrinkles" constitute the most important features of the universe, since they lead to our very existence. The analysis of the evolution of the level of order associated with these structures constitutes an important (and formidable) problem. Any new developments in this direction will be very welcome.

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References

[1] R. Arnheim, Entropy and Art: An Essay in Disorder and Order, University of California Press, Berkeley, 1971.
[2] O. Darrigol, From c-Numbers to q-Numbers, University of California Press, Berkeley, 1992.
[3] R.D. Rosenkrantz (Ed.), E.T. Jaynes: Papers on Probability, Statistics and Statistical Physics, D. Reidel, Dordrecht, 1987.
[4] M. Gell-Mann, C. Tsallis (Eds.), Nonextensive Entropy: Interdisciplinary Applications, Oxford University Press, Oxford, 2004.
[5] C. Tsallis, Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World, Springer, New York, 2009.
[6] E. Schrödinger, What Is Life? The Physical Aspect of the Living Cell and Mind and Matter, Cambridge University Press, New York, 1967.
[7] B.R. Frieden, R.J. Hawkins, Phys. Rev. E 82 (2010) 066117.
[8] B.R. Frieden, R.A. Gatenby, Phys. Rev. E 84 (2011) 011128.
[9] B.R. Frieden, A. Plastino, A.R. Plastino, Phys. A 391 (2011) 410.
[10] A.N. Kolmogorov, Problems inform, Transmission 1 (1) (1965) 1–7;
G.J. Chaitin, J. ACM 16 (1969) 407.
[11] M. Petri, The holographic solution—why general relativity must be understood in terms of strings, gr-qc/0405007, (2004). See also, Int. J. Modern Phys. E 16 (2007) 1603–1641.

[12] V.E Shemi-Zadeh, Coincidence of large numbers, exact value of cosmological parameters and their analytical representation, 21 July 2002. arXiv:gr-qc/0206084v3.
[13] B.R. Frieden, B.H. Soffer, in: B.R. Frieden, R.A. Gatenby (Eds.), Exploratory Data Analysis using Fisher Information, Springer-Verlag, London, UK, 2007, pp. 159–168.
[14] C.J. de Matos, C. Beck, Possible measurable effects of dark energy in rotating superconductors, arXiv: 0707.1797.
[15] C. Beck, Phys. A 388 (2009) 3384.
[16] J.D. Barrow, The Constants of Nature; From Alpha to Omega—The Numbers that Encode the Deepest Secrets of the Universe, Pantheon Books, 2002.
[17] B.R. Frieden, Science from Fisher Information, Cambridge University Press, Cambridge, UK, 2004.