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2012 Class. Quantum Grav. 29 095017

(http://iopscience.iop.org/0264-9381/29/9/095017)

IOP Publishing | Classical and Quantum Gravity

Class. Quantum Grav. **29** (2012) 095017 (20pp) doi:10.1088/0264-9381/29/9/095017

Unstable fields in Kerr spacetimes

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Received 24 November 2011, in final form 24 February 2012
Published 16 April 2012
Online at stacks.iop.org/CQG/29/095017

Abstract
We show that both the interior region $r < M - \sqrt{M^2 - a^2}$ of a Kerr black hole and the $a^2 > M^2$ Kerr naked singularity admit unstable solutions of the Teukolsky equation for any value of the spin weight. For every harmonic number, there is at least one axially symmetric mode that grows exponentially in time and decays properly in the radial directions. These can be used as Debye potentials to generate solutions for the scalar, Weyl spinor, Maxwell and linearized gravity field equations on these backgrounds, satisfying appropriate spatial boundary conditions and growing exponentially in time, as shown in detail for the Maxwell case. It is suggested that the existence of the unstable modes is related to the so-called time machine region, where the axial Killing vector field is timelike, and the Teukolsky equation, restricted to axially symmetric fields, changes its character from hyperbolic to elliptic.

PACS numbers: 04.50.+h, 04.20.−q, 04.70.−s, 04.30.−w

1. Introduction

Kerr's solution [1] of the vacuum Einstein's equations in the Boyer–Lindquist coordinates is

$$\mathrm{d}s^2 = \frac{(\Delta - a^2 \sin^2\theta)}{\Sigma}\,\mathrm{d}t^2 + 2a\sin^2\theta\,\frac{(r^2+a^2-\Delta)}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi - \left[\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right]\sin^2\theta\,\mathrm{d}\phi^2 - \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 - \Sigma\,\mathrm{d}\theta^2, \qquad (1)$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. We use the metric signature $+ - - -$ to match the formulas in the original Newman–Penrose null tetrad formulation [2] and Teukolsky perturbation treatment [3]. Kerr's metric has two integration constants: the mass M and the angular momentum per unit mass a. They can be obtained as the Komar integrals [4] using

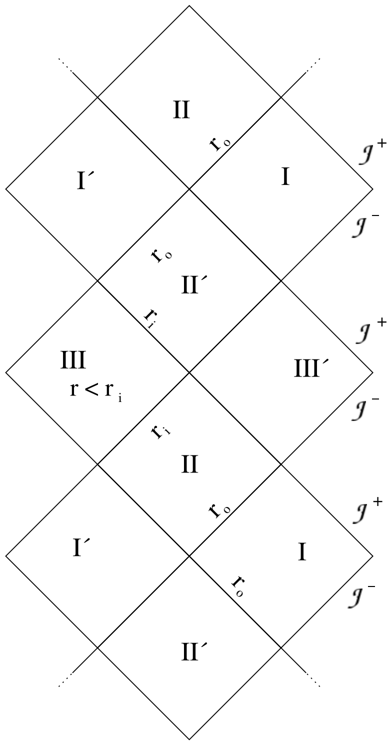

Figure 1. The Penrose diagram for the maximal analytic extension of Kerr's spacetime. Regions labeled I and I′ are isometric, and so are II and II′, and III and III′.

the time translation Killing vector field K^a and the axial Killing vector field ζ^a (in the above coordinates, these are $\partial/\partial t$ and $\partial/\partial\phi$, respectively). We will only consider the case $M > 0$, and we will take $a > 0$ without loss of generality, since for $a < 0$ we can always change coordinates $\phi \to -\phi$, under which $a \to -a$. If $0 < a < M$ (sub-extreme case), the $\Sigma = 0$ ring curvature singularity at $r = 0, \theta = \pi/2$ is hidden behind the black hole inner and outer horizons located at the zeros of Δ: $r_i = M - \sqrt{M^2 - a^2}$ and $r_o = M + \sqrt{M^2 - a^2}$. As is well known that r_i and r_o are just coordinate singularities in (1), Kerr's spacetime can be extended through these horizons and new regions isometric to I: $r > r_o$, II: $r_i < r < r_o$ and III: $r < r_i$ arise *ad infinitum* and give rise to the Penrose diagram displayed in figure 1. In the extreme case, $M = a$, $r_i = r_o$ and region II is absent; however, we will still call region I (III) that for which $r > r_i = r_o$ ($r < r_i = r_o$). In the 'super-extreme' case $a > M$, there is no horizon at all, the ring singularity being causally connected to future null infinity. This is not a black hole, but a naked singularity.

In this paper, we study the Kerr naked singularity (KNS), and region III of (sub-extreme and extreme) Kerr black holes. We will refer to these solutions of the vacuum Einstein equations as KIII and KNS from now on. These have a number of undesirable properties, among which we mention (i) the timelike curvature singularity as we approach the ring boundary at $r = 0,\ \theta = \pi/2$; (ii) the fact that *any* two events can be connected with a future timelike curve (in particular, there are closed timelike curves through any point), making KIII and KNS 'totally vicious sets' in the terminology of [5], and causing a number of puzzling causality problems [6]; and (iii) the violation of cosmic censorship.

Two conjectures are known under the name of cosmic censorship: the *weak* cosmic censorship conjecture establishes that the collapse of *ordinary* matter cannot *generically* lead to a naked singularity, while the *strong* cosmic censorship is the assertion that the maximal development of data given on a Cauchy surface cannot be *generically* continued in a smooth way [7]. The words in italics above signal aspects of the conjectures that need to be properly specified to turn them into a well-defined statement. In any case, the KNS violates weak cosmic censorship, and KIII violates strong cosmic censorship, since it is a smooth extension of the development of an initial data surface extending from spatial infinity of region I to spatial infinity of region I′ in figure 1, whose Cauchy horizon agrees with the inner horizon at r_i.

According to the Carter–Robinson theorem [8] and further results by Hawking and Wald [9], if $(\mathcal{M}, g_{ab})$ is an asymptotically flat stationary vacuum black hole that is non-singular on and outside an event horizon, then it must be an $a^2 < M^2$ member of the two-parameter Kerr family. The spacetime outside a black hole formed by gravitational collapse of a star is, independent of the characteristics of the collapsing body, modeled by region I of a Kerr black hole solution, placing this among the most important of the known exact solutions of Einstein's equations. Since this region is stationary (outside the ergosphere), there is a well-defined notion of modal linear stability, under which it has been shown to be stable [3, 10]. It is our opinion, however, that proving the instability of those stationary solutions of Einstein's equations that display undesirable features is as relevant as proving the stability of the physically interesting stationary solutions. In this line of thought, we are carrying out a program to analyze the linear stability of the most salient naked singularities ($M < 0$ Schwarzschild spacetime [11], $Q^2 > M^2$ Reissner–Nordström spacetime [12] and $a^2 > M^2$ Kerr spacetime [13, 14]; and also of those regions lying beyond the Cauchy horizons in the Reissner–Nordström and Kerr black holes [14, 12]. As is well known, the linear perturbations of the spherically symmetric Einstein–Maxwell spacetimes are much easier to deal with than those of the axially symmetric Kerr spacetime, for which the only separable equations known to date are not directly related to the metric perturbation. They are equations satisfied by perturbations of the components of the Weyl tensor

$$\psi_0 := -C_{abcd} l^a m^b l^c m^d, \quad \psi_4 := -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d \tag{2}$$

along a complex null tetrad l^a, n^a, m^a, $\bar{m}^a$, among which the only nonzero dot products are [2]

$$l^a n_a = 1, \quad m^a \bar{m}_a = -1. \tag{3}$$

We use a bar for complex conjugation, l^a and n^a are real vector fields, whereas m^a is complex. The null tetrad we use is that introduced by Kinnersley [15], given in equation (4.4) in [3]. If we take

$$\varepsilon_{abcd} = i\,4! \quad l_{[a} n_b m_c \bar{m}_{d]} \tag{4}$$

as a right-handed volume element, we find that the following two-forms are self-dual:

$$\bar{m}_{[a} n_{b]}, \quad n_{[a} l_{b]} + m_{[a} \bar{m}_{b]}, \quad l_{[a} m_{b]}. \tag{5}$$

A complex electromagnetic field can be written as

$$\begin{aligned} F_{ab} := {} & 2\phi_1 (n_{[a} l_{b]} + m_{[a} \bar{m}_{b]}) + 2\phi_2 l_{[a} m_{b]} + 2\phi_0 \bar{m}_{[a} n_{b]} \\ & + 2\tilde{\phi}_1 (n_{[a} l_{b]} + \bar{m}_{[a} m_{b]}) + 2\tilde{\phi}_2 l_{[a} \bar{m}_{b]} + 2\tilde{\phi}_0 \, m_{[a} n_{b]}. \end{aligned} \tag{6}$$

If F_{ab} is real, then $\tilde{\phi}_j = \bar{\phi}_j$, and if F_{ab} is self-dual (anti-self-dual), then $\tilde{\phi}_j$ (ϕ_j) vanish. Teukolsky [3] found that Maxwell, (Weyl) spinor and scalar fields on a Kerr background can be treated in a similar way as the gravitational perturbations $\delta\psi_0$ and $\delta\psi_4$. If we take the components of the Maxwell fields (see (3))

$$\phi_0 = F_{ab} l^a m^b, \quad \phi_1 = \tfrac{1}{2} F_{ab} (l^a n^b + \bar{m}^a m^b), \quad \phi_2 = F_{ab} \bar{m}^a n^b, \tag{7}$$

and those of the two component spinors χ_A

$$\chi_0 = \chi_A o^A \quad \chi_1 = \chi_A \iota^A, \tag{8}$$

and weight them with an appropriate power of the spin coefficient $\rho = m^a \bar{m}^b \nabla_b l_a = (\mathrm{i}a\cos\theta - r)^{-1}$:

$$\Psi_{\frac{1}{2}} := \chi_0, \quad \Psi_{-\frac{1}{2}} := \rho^{-1}\chi_1, \quad \Psi_1 := \phi_0, \quad \Psi_{-1} := \rho^{-2}\phi_2, \quad \Psi_2 := \delta\psi_0, \quad \Psi_{-2} := \rho^{-4}\delta\psi_4, \tag{9}$$

then the (source-free) Maxwell, spinor and linearized gravity equations can all be encoded in Teukolsky's master equation [3]:

$$\begin{aligned} T_s[\Psi_s] := &\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\Psi_s}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\Psi_s}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\Psi_s}{\partial\phi^2} \\ &- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\Psi_s}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi_s}{\partial\theta}\right) \\ &- 2s\left[\frac{a(r-M)}{\Delta} + \frac{\mathrm{i}\cos\theta}{\sin^2\theta}\right]\frac{\partial\Psi_s}{\partial\phi} \\ &- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - \mathrm{i}a\cos\theta\right]\frac{\partial\Psi_s}{\partial t} + (s^2\cot^2\theta - s)\Psi_s = 0. \end{aligned} \tag{10}$$

The index s in Ψ_s gives the spin weight under tetrad rotations. The above equation also gives the massless scalar field equation $\Box\,\Psi_0 = 0$ if we set $s = 0$.

In [13], we found numerical evidence that there are solutions of the $s = -2$ Teukolsky equation in the KNS that grow exponentially in time while satisfying appropriate boundary conditions. In [14], we confirmed this fact by proving that there are *infinitely many* axially symmetric unstable (meaning, behaving as e^{kt} for some positive k) solutions of the $s = -2$ equation in the KNS, and also in KIII.

In this paper, we extend further this result to other linear fields. We show that there are infinitely many unstable solutions of the Teukolsky equation for any s value ($|s| = 0, 1/2, 1, 2$). The proof is given in section 3, with some calculations relegated to the appendix. The existence of the unstable modes is shown in section 4 to be related to the time machine region near the ring singularity, which produces a change of character of the Teukolsky PDE—restricted to axial modes—from hyperbolic to elliptic. It is suggested that the emergency of an instability when a PDE changes from hyperbolic to elliptic is generic, and this is illustrated with a simple toy model in 1 + 1 dimensions.

The use of unstable solutions of Teukolsky's equation as 'Debye' potentials [16] for constructing unstable spinor, Maxwell and linear gravitational fields is illustrated in section 5, where the reconstruction process is explained in detail for Maxwell fields. All the reconstructed fields decay properly along spatial directions while growing exponentially in time. This is the notion of instability used in this work. The lack of a sensible initial value formulation due to the fact that there are no partial Cauchy surfaces in KIII and KNS forbids a more traditional approach to the stability issue. This is quite different from what happens for the Reissner–Nordström and negative mass Schwarzschild timelike naked singularities, for which a unique evolution of data given on a partial Cauchy surface can be defined, and instability proven afterward [11, 12].

The following section contains information on the Teukolsky equation that is used in the proof of existence of unstable modes.

2. Separated Teukolsky's equations

Introducing

$$\Psi_s = R_{\omega,m,s}(r)S^m_{\omega,s}(\theta)\exp(\mathrm{i}m\phi)\exp(-\mathrm{i}\omega t), \tag{11}$$

the linearized Teukolsky PDE (10) is reduced to a coupled system for S and R:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right)+\left(a^2\omega^2\cos^2\theta-2a\omega s\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+E-s^2\right)S=0, \tag{12}$$

$$\Delta\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+(s+1)\frac{\mathrm{d}\Delta}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left\{\frac{K^2-2\mathrm{i}s(r-M)K}{\Delta}+4\mathrm{i}r\omega s-[E-2am\omega+a^2\omega^2-s(s+1)]\right\}R=0, \tag{13}$$

where $K=(r^2+a^2)\omega-am$. The system (12)–(13) is coupled by their common eigenvalue E, whose relation with the separation constant A in [3, 17] is given by $A=E-s(s+1)$. Suppose s, m and ω are given; then, E in (12) has to be chosen so that S is regular on the sphere. This gives a denumerable set of eigenvalues that we label as $E^{\mathrm{ang}}_\ell(s,m,a\omega)$, with $\ell=0,1,2,\ldots$ and E increasing with ℓ. In a similar way, E in (13) is chosen such that R decays properly as $|r|\to\infty$ for the KNS ($r\to-\infty$ and $r\to r_i^-$ for KIII), and this also gives a denumerable set of increasing values $E^{\mathrm{rad}}_n(s,m,a\omega)$, $n=0,1,2,\ldots$. A solution of the system (12)–(13) is obtained whenever $E^{\mathrm{ang}}_\ell(s,m,a\omega)=E^{\mathrm{rad}}_n(s,m,a\omega)=:E$. Thus, for given (s,m), we may regard a solution as an intersection of the curves E^{ang}_ℓ versus $a\omega$ and E^{rad}_n versus $a\omega$, the allowed frequencies being those at which the curves intersect. This point of view is the one used in the proof of instability below, for which we restrict our search to $a\omega=\mathrm{i}k$, where k a real positive number (so that (11) gives an $\exp(kt/a)$ behavior), and show that there are intersections for every ℓ and, at least, the fundamental radial mode $n=0$.

One way of finding the radial and angular spectra consists in reducing the regularity conditions to a continued fraction equation involving E [18]. This equation arises when solving a three term recursion relation on the coefficients of a series solution for S and R in (12) and (13) [18]. An alternative way to obtain the angular spectrum, which is well suited to the case we are interested, $m=0$ and $a\omega=\mathrm{i}k$, is discussed in section 2.1 of [14]. This approach is used in the numerical computations leading to figure 2.

2.1. Angular equation: spin-weighted spheroidal harmonics

The solutions of (12) that are regular on the sphere are called *spin-weighted spheroidal harmonics* (SWSH). The spectrum of E values is discrete, and we will use the notation $E^{\mathrm{ang}}_\ell(s,m,a\omega)$, with $\ell=0,1,2,\ldots$, to label the eigenvalues in increasing order for a given set of parameters $(s,m,a\omega)$ (the notation is not unified in the literature, note that we use $\ell=0$ for the lowest eigenvalue *independently* of the spin weight s). Equation (12) exhibits some interesting symmetries: if $(S(\theta),E)$ is a regular solution of (12) for some given (s,m,ω) values, it is easy to check that $(S(\pi-\theta),E)$ is a solution for $(s,-m,-\omega)$ *and* for $(-s,m,\omega)$, and that $(\overline{S(\theta)},\overline{E})$ is a solution for $(s,m,\bar{\omega})$. This implies that

$$E^{\mathrm{ang}}_\ell(s,m,a\omega)=E^{\mathrm{ang}}_\ell(s,-m,-a\omega)=E^{\mathrm{ang}}_\ell(-s,m,a\omega)=\overline{E^{\mathrm{ang}}_\ell(s,m,a\bar{\omega})}. \tag{14}$$

We are interested in axially symmetric ($m=0$) solutions with purely imaginary frequencies ($a\omega=\mathrm{i}k$, $k\in\mathbb{R}$). In this case, from a solution $S(\theta)$, we can extract solutions with

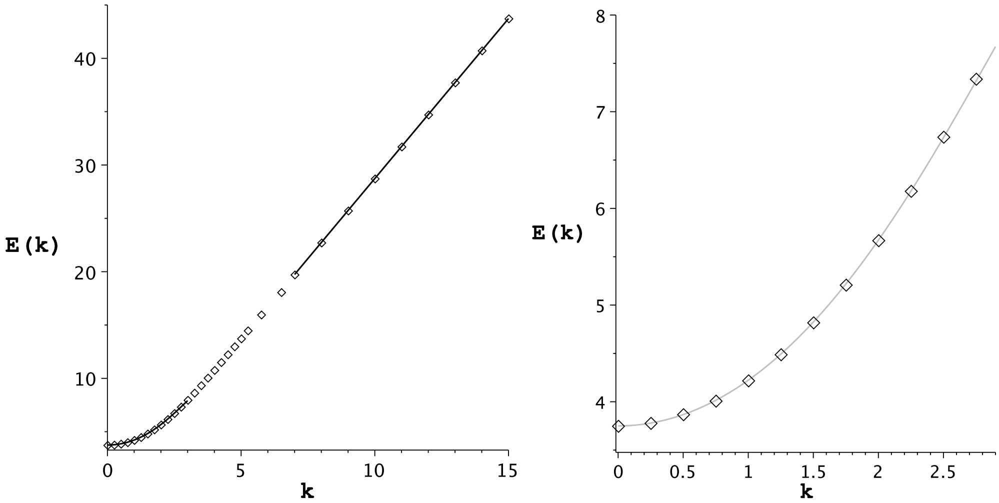

Figure 2. (Left) $E^{\rm ang}_{\ell=1}(s = 1/2, m = 0, a\omega = {\rm i}k)$ obtained numerically by solving equation (9) in [14] for $s = 1/2$ and different values of $k \in [0, 15]$, together with a least-squares linear fit using the large k data points, which gives $E = 2.998\,43k$+constant, in excellent agreement with the expected asymptotic expansion equation (20). (Right) The values obtained in this way for $0 < k < 2.5$. The solid line is the low-frequency approximation of $E^{\rm ang}_{\ell=1}(s = 1/2, m = 0, a\omega = {\rm i}k)$ in [19, 17] to order k^6.

real and imaginary parts of opposite parities by taking the linear combinations $S(\theta) \pm \overline{S(\pi - \theta)}$. Also, the eigenvalues are real, as follows from (14):

$$E^{\rm ang}_{\ell}(s, m = 0, {\rm i}k) = \overline{E^{\rm ang}_{\ell}(s, m = 0, {\rm i}k)}. \tag{15}$$

The behavior of $E^{\rm ang}_{\ell}(s, m = 0, a\omega = {\rm i}k)$ in the limits $k \to 0^+$ and $k \to \infty$ will be relevant in what follows, since, as explained above, an intersection of the curve $E^{\rm ang}_{\ell}(s, m = 0, a\omega = {\rm i}k)$, $k \in \mathbb{R}^+$, with any of the radial curves implies the existence of an unstable (i.e. $\sim \exp(kt/a)$) mode. To study these limits, we will assume that $s \geqslant 0$ without loss of generality (see (14)). Setting $x := \cos(\theta)$, $m = 0$, $a\omega = {\rm i}k$ in (12), we find that, for any k, this equation has regular singular points at $x = \pm 1$, the possible behavior of local solutions around these points being $(1 - |x|)^{s/2}$ or $(1 - |x|)^{-s/2}$.

For $k = 0$, we expand the regular solutions as

$$S(x) = (1 - x^2)^{(s/2)} \sum_{j=0}^{\infty} a_j (1 - x)^j, \tag{16}$$

and find that (12) implies the recursion relation

$$2(j + 1)(j + 1 + s)a_{j+1} - ((j + s)(j + s + 1) - E)\, a_j = 0, \tag{17}$$

which, for large j, gives $a_{j+1} \sim a_j/2$. The series in (16) will thus diverge at $x = -1$ unless we cut it to down to a polynomial of degree $\ell = 0, 1, \ldots$ by choosing $E = (\ell + s)(\ell + s + 1)$ for some $\ell = 0, 1, 2, \ldots$. Repeating the calculation for negative s, or just using (14), we obtain

$$E = E^{\rm ang}_{\ell}(s, m = 0, a\omega = 0) = (\ell + |s|)(\ell + |s| + 1). \tag{18}$$

A more detailed analysis of (12) using continued fraction techniques gives a Taylor expansion for $E^{\rm ang}_{\ell}(s, m, a\omega)$ for complex ω near $\omega = 0$ The expansion up to order $(a\omega)^6$ is available in the literature (see ([17, 19] and references therein), exhibits the symmetries (14) and has (18) as the leading order term:

$$E = E_\ell^{\text{ang}}(s, m = 0, a\omega) = (\ell + |s|)(\ell + |s| + 1) + \mathcal{O}(\omega). \tag{19}$$

Asymptotic expansions for $a\omega = \mathrm{i}k, k \to \infty$ can be found in [20–22, 17]. Particularly useful to our purposes is that, in our notation [17],

$$E_\ell^{\text{ang}}(s, m, a\omega = \mathrm{i}k) = (2\ell + 1)k + \mathcal{O}(k^0), \quad \text{as } k \to \infty. \tag{20}$$

We should warn the reader, however, that a complete proof of the above formula is not available for $s \neq 0$. Although arguments suggesting the validity of (20) are given in [17], where the formula was also numerically checked for $s = 1, 2$, the case $s = 1/2$ has not been reported as tested there. Given that these equations are key in the proof of instability for spinor fields, we numerically tested their validity using the method developed in section 2.1 of [14]. We have found and excellent agreement with both (18) and (20) for $s = 0, 1/2, 1, 2$. As an illustration, we give in figure 2 the results for the only case not dealt with in the literature, that of $s = 1/2$.

2.2. Radial equation: reduction to a Schrödinger form

Equation (13) is of the form $\Delta\ddot{R} + Q\dot{R} + (Z - E)R = 0$, with dots denoting derivatives with respect to r. If we introduce an integrating factor L, $\psi := R/L$, and change the radial variable to r^*, where $\frac{\mathrm{d}r^*}{\mathrm{d}r} := \frac{1}{f}$, with f being an unspecified positive-definite function of r, (13) gives the following equation for ψ:

$$-\psi'' + \left(\frac{f'}{f} - \frac{2L'}{L} - \frac{fQ}{\Delta}\right)\psi' + \left(\frac{L'f'}{Lf} - \frac{L''}{L} - \frac{QfL'}{L\Delta} - \frac{f^2 Z}{\Delta}\right)\psi = \frac{f^2}{\Delta}E\psi, \tag{21}$$

where primes denote derivatives with respect to r^*. By choosing $f = \sqrt{\Delta}$ (note that Δ is strictly positive for KIII and KNS) and L such that the coefficient of ψ' vanishes, i.e. $L = \Delta^{-\frac{(2s+1)}{4}}$, (21) reduces to a stationary Schrödinger equation with the energy eigenvalue $-E$:

$$\mathcal{H}\psi := -\psi'' + V\psi = -E\psi. \tag{22}$$

In the case $m = 0$, $a\omega = \mathrm{i}k$, the potential is

$$V = -\left(\frac{\Delta\ddot{L}}{L} + \frac{Q\dot{L}}{L} + Z\right) = \left[\frac{r(r^3 + ra^2 + 2Ma^2)}{a^2(r^2 - 2Mr + a^2)}\right]k^2 + 2s\left[\frac{(r^3 - 3r^2M + ra^2 + Ma^2)}{a(r^2 - 2Mr + a^2)}\right]k$$
$$+ \frac{1}{4}\left[1 + \frac{(M^2 - a^2)(4s^2 - 1)}{r^2 - 2Mr + a^2}\right] =: k^2 V_2 + kV_1 + V_0. \tag{23}$$

We will show that the spectrum of (22) is entirely discrete and use the notation $-E_n^{\text{rad}}(s, m = 0, a\omega = \mathrm{i}k), n = 0, 1, 2, \ldots,$ for its eigenvalues, to be consistent with our previous conventions.

For k large enough, V is negative in a region $r_n < r < 0$, and the resulting bound states lead to unstable modes of the Teukolsky PDE. This is explained in detail in the following section.

3. Unstable linear fields on Kerr' s spacetime

The results from the previous section can be summarized as follows. There are solutions of the Teukolsky equations behaving as $\mathrm{e}^{kt/a}$ if and only if solutions can be found of equations (12) and (13) with the same E value for $\omega = \mathrm{i}k/a$, i.e. $E_\ell^{\text{ang}}(s, m, \omega = \mathrm{i}k/a) = E_n^{\text{rad}}(s, m, \omega = \mathrm{i}k/a)$ for some ℓ and n. Since we restrict our attention to the axially symmetric case, we will drop the m index from now on and use (22)–(23) instead of (13). Given that the instability is a consequence of the intersection of spectral lines of the angular and radial operators for purely imaginary frequencies, we need to gather information on the spectra of theses operators. Since we are interested in spotting intersections for $\omega = \mathrm{i}k/a, k \in (0, \infty)$, we will gather information on the asymptotic expressions for these spectra in the limits, where $k \to 0^+$ and $k \to \infty$. The strategy of the proof consists in showing that in one of these limits, $E_\ell^{\mathrm{ang}} > E_0^{\mathrm{rad}}$, whereas in the other $E_\ell^{\mathrm{ang}} < E_0^{\mathrm{rad}}$; thus, the intersection follows from continuity on the spectral lines on k. For the angular equation, these limits are given in (19) and (20). For the radial equation, we need to work them out and, since the analysis depends on the domain of r and boundary conditions, we will consider separately KIII and the KNS.

3.1. Unstable modes on a KNS

In this section, we consider the extreme case $a^2 > M^2$, for which $-\infty < r < \infty, t, \theta$ and ϕ are the global coordinates, and $\Delta > 0$ everywhere. The choice $\frac{\mathrm{d}r^*}{\mathrm{d}r} := \frac{1}{f} = 1/\sqrt{\Delta}$ made above gives an adimensional r^*:

$$r^* = \ln\left(\frac{r - M + \sqrt{r^2 - 2Mr + a^2}}{M}\right) \simeq \begin{cases} \ln\left(\frac{2r}{M}\right) & r \to \infty \\ \ln\left(\frac{a^2 - M^2}{2M|r|}\right) & r \to -\infty \end{cases} \tag{24}$$

that grows monotonically with r and can easily be inverted in terms of elementary functions:

$$r = \frac{M\exp(r^*)}{2} + M + \frac{M^2 - a^2}{2M\exp(r^*)}. \tag{25}$$

Note that (22) defines a quantum mechanical problem in the entire r^* line, with $\mathcal{H}$ in (21) being a self-adjoint operator in $L^2(\mathbb{R}, \mathrm{d}r^*)$. The asymptotic form of the potential (23) for large $|r^*|$ does not depend on the value of s and is given by

$$V \sim \begin{cases} \left(\frac{Mk}{2a}\right)^2 \mathrm{e}^{2r^*}, & r^* \to \infty, \\ \left(\frac{(M^2 - a^2)k}{2Ma}\right)^2 \mathrm{e}^{-2r^*}, & r^* \to -\infty. \end{cases} \tag{26}$$

From the above equation and the fact that V is smooth, we conclude that V reaches a minimum and $\mathcal{H}$ in (21) has an entirely discrete, bounded from below spectrum $-E_n^{\mathrm{rad}}(s, k), n = 0, 1, 2.\ldots$ As explained above, we will need information on this spectrum in both $k \to 0^+$ and $k \to \infty$ limits.

3.1.1. Asymptotic behavior of the radial equation spectrum as $k \to \infty$. The large real k limit is simple to deal with, because the behavior of V in this limit does not depend on the value of s, and therefore, the analysis in [14] for $s = -2$ applies with only minor modifications. The cubic polynomial $r^3 + a^2 r + 2Ma^2$ in the numerator of V_2 (see (23)) has a unique real root at $r = r_n(M) < 0$; thus, V_2 is negative in the interval $r_n(M) < r < 0$, and non-negative elsewhere. Note that since $a^2 = -{r_n}^3/(2M + r_n)$, then $r_n(M)$ goes from $-M$ to $-2M$ as a^2 goes from M^2 to infinity.

Let ψ be a properly normalized ($\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} |\psi|^2 \,\mathrm{d}r^* = 1$) function supported in the interval $(r_n(M), 0)$; then,

$$\langle\psi|\mathcal{H}|\psi\rangle = \langle\psi| - (\partial/\partial r^*)^2|\psi\rangle + \sum_{j=0}^{2} k^j \langle\psi|V_j|\psi\rangle \tag{27}$$

with

$$\langle\psi|V_2|\psi\rangle = \int_{r_n(M)}^{0} |\psi|^2 \, V_2 \, \frac{\mathrm{d}r}{\sqrt{\Delta}} < 0. \tag{28}$$

Take k_c to be the largest among zero and the real roots (if any) of

$$p(k) := \langle\psi| - (\partial/\partial r^*)^2|\psi\rangle + \langle\psi|V_0|\psi\rangle + k\langle\psi|V_1|\psi\rangle + \frac{k^2}{2}\langle\psi|V_2|\psi\rangle$$

(note the one half factor in the k^2 term!), and then $p(k) < 0$ for $k > k_c$ and if $-E_0^{\mathrm{rad}}(s, a\omega = \mathrm{i}k)$ is the lowest eigenvalue of $\mathcal{H}$ (see equation (22)):

$$-E_0^{\mathrm{rad}}(s, a\omega = \mathrm{i}k) \leqslant \langle\psi|\mathcal{H}|\psi\rangle = \frac{k^2}{2}\langle\psi|V_2|\psi\rangle + p(k) < \frac{k^2}{2}\langle\psi|V_2|\psi\rangle, \quad \text{if } k > k_c. \tag{29}$$

meaning that the absolute value of the fundamental radial level grows at least quadratically in k

$$E_0^{\mathrm{rad}}(s, a\omega = \mathrm{i}k) > \frac{k^2}{2}|\langle\psi|V_2|\psi\rangle|, \text{ if } k > k_c. \tag{30}$$

3.1.2. Asymptotic behavior of the radial equation spectrum as $k \to 0^+$. A quick inspection to the potential (23) gives the minima for $k = 0$ and different spin weights:

$$\min\{V(r, k = 0, s), r \in \mathbb{R}\} = \begin{cases} \frac{1}{4}, & s = 0, \\ \frac{1}{4}, & |s| = 1/2, \\ -\frac{1}{2}, & |s| = 1, \\ -\frac{7}{2}, & |s| = 2. \end{cases} \tag{31}$$

A few lengthy calculations show however that

$$\lim_{k\to 0^+} \min\{V(r, k, s, a > M), r \in \mathbb{R}\} = \tfrac{1}{4} - s^2, \tag{32}$$

so that there is a discontinuity at $k = 0$ for higher spin values. Since we will be using continuity arguments in our proof of instability, we will consider the fundamental energy $-E_0^{\mathrm{rad}}(s, a\omega = \mathrm{i}k)$ of the quantum Hamiltonian in (22)–(23) as a function of $k \in (0, \infty)$, for which

$$\lim_{k\to 0^+} -E_o^{\mathrm{rad}}(s, a\omega = \mathrm{i}k) > \lim_{k\to 0^+} \min\{V(r, k, s), r \in \mathbb{R}\} \tag{33}$$

given in (32). The appendix contains the details of the calculations leading to (32).

3.1.3. Proof of the existence of unstable modes for every s. Let us gather the relevant results of the previous sections for the axially symmetric ($m = 0$) modes. For $k \to 0^+$ and $\ell = 0, 1, 2\ldots$,

$$E_\ell^{\mathrm{ang}}(s, a\omega = \mathrm{i}k)\,|_{k=0^+} = (\ell + |s|)(\ell + |s| + 1) > s^2 - \tfrac{1}{4} > E_0^{\mathrm{rad}}(s, a\omega = \mathrm{i}k)\,|_{k=0^+}, \tag{34}$$

whereas for large enough positive real k

$$E_\ell^{\mathrm{ang}}(s, a\omega = \mathrm{i}k) = 2(\ell + 1)k + \mathcal{O}(k^0) < \frac{k^2}{2}|\langle\psi|V_2|\psi\rangle| < E_0^{\mathrm{rad}}(s, a\omega = \mathrm{i}k). \tag{35}$$

By continuity, we must have, *for every* ℓ and s, a $k_{(\ell,s)}$, such that $E_\ell^{\mathrm{ang}}(s, a\omega = \mathrm{i}k_{(\ell,s)}) = E_0^{\mathrm{rad}}(s, a\omega = \mathrm{i}k_{(\ell,s)})$. This proves that there is an axially symmetric unstable solution of the Teukolsky equation for the fundamental radial level and every harmonic number ℓ. For higher excited radial level, the arguments in [14] for $s = -2$ suggesting that there also are intersections generalize to arbitrary s. In any case, we have shown that there are infinitely many unstable modes for every spin weight. These solutions of the Teukolsky equations decay exponentially with $|r|$ as $|r| \to \infty$, so that they are initially bounded and grow exponentially in time.

The calculations above can be adapted to perturbations in the *interior* region $r < r_i := M - \sqrt{M^2 - a^2}$ of an $a \leqslant M$ Kerr *black hole*. There are some subtle differences between the extreme $a = M$ and sub-extreme $a < M$ cases, as shown in the following sections.

3.2. Unstable modes on region III of an extreme Kerr black hole

For the extreme black hole, the solution of $\mathrm{d}r^*/\mathrm{d}r = 1/\sqrt{\Delta}$ in the interior region $r < r_i = r_o = M$ is

$$r^* = -\ln\left(\frac{M-r}{M}\right), \qquad r < r_i, \tag{36}$$

with inverse

$$r = M(1 - \mathrm{e}^{-r^*}), \qquad -\infty < r^* < \infty. \tag{37}$$

Using the integration factor $\Delta^{-\frac{2s+1}{4}}$ as before, we are led back to (22) and (23), with r given in (37). Note that

$$V \sim \begin{cases} 4k^2 \exp(2r^*), & r^* \to \infty, \\ k^2 \exp(-2r^*), & r^* \to -\infty, \end{cases} \tag{38}$$

then for any $k > 0$ the spectrum of the self-adjoint operator $\mathcal{H}$ is again fully discrete and has a lower bound. The argument leading to (30) goes through in the super-extreme case without modifications, because the test function in (27) is supported in the $r < 0$ region. Thus, the fundamental energy of the radial Hamiltonian is negative and there is an ℓ_o, such that $E_o^{\mathrm{rad}}(s, a\omega = \mathrm{i}k)|_{k=0^+} < (\ell_o + |s|)(\ell_o + |s| + 1)$, from where it follows that there is an unstable mode for every $\ell \geqslant \ell_o$. The radial decay of these modes as $r \to r_i^-$ and $r \to -\infty$ is

$$\psi \sim \begin{cases} \left(\frac{M}{M-r}\right)^{2k-s-\frac{1}{2}} \exp\left[-2k\left(\frac{M}{M-r}\right)\right]\left(1 + \mathcal{O}\left(\frac{M-r}{r}\right)\right), & r \to M^-, \\ \left(\frac{M}{r}\right)^{\frac{1}{2}-2k-s} \exp\left[\frac{rk}{M}\right]\left(1 + \mathcal{O}(M/r)\right), & r \to -\infty. \end{cases} \tag{39}$$

3.3. Unstable modes on region III of a sub-extreme Kerr black hole

In the sub-extreme case, $\mathrm{d}r^*/\mathrm{d}r = 1/\sqrt{\Delta}$ and $r < r_i$ give

$$r^* = \ln\left(\frac{r_i + r_o - 2r - 2\sqrt{(r_o - r)(r_i - r)}}{r_i + r_o}\right) \tag{40}$$

so that r^* has an upper bound:

$$-\infty < r^* < r_i^* := \ln\left(\frac{r_o - r_i}{r_o + r_i}\right). \tag{41}$$

Near the domain boundaries,

$$r^* \simeq \begin{cases} r_i^* - \frac{2\sqrt{r_i - r}}{\sqrt{r_o - r_i}}, & r \to r_i^-, \\ \ln\left(-\left(\frac{r_o - r_i}{r_o + r_i}\right)^2 \frac{1}{4r}\right), & r \to -\infty, \end{cases} \tag{42}$$

and

$$V \simeq \begin{cases} \left(\nu(k)^2 - \frac{1}{4}\right) / \left(r_i^* - r^*\right)^2, & r^* \to r_i^{*-}, \\ [k(M^2 - a^2)/(2aM)]^2 \exp(-2r^*), & r^* \to -\infty, \end{cases} \tag{43}$$

where we have defined

$$\nu(k) := 2\sqrt{\frac{r_i}{r_o}}\left(\frac{r_o + r_i}{r_o - r_i}\right) k - s. \tag{44}$$

The sub-extreme case is essentially different from the extreme and super-extreme cases because (22) is a Schrödinger equation *on the half-axis* $r^* < r_i^*$, with a potential that is singular at the r_i^* boundary. This situation and type of singularity is well known [23, 24]. Any local solution of (22), in particular those which are square integrable for r^* near $-\infty$, behave as

$$\psi \sim a\left[\left(r_i^* - r^*\right)^{\frac{1}{2}+\nu} + \cdots\right] + b\left[\left(r_i^* - r^*\right)^{\frac{1}{2}-\nu} + \cdots\right], \tag{45}$$

near the horizon. Thus, if $\nu > 1$, these are not square integrable near the horizon, unless $b = 0$, and this is precisely the condition that selects a discrete set of possible E values as the spectrum of $\mathcal{H}$ and that defines the space of functions where $\mathcal{H}$ is self-adjoint. This case is called *limit point* in [24]. It is quite different from the *limit circle* case $\nu < 1$, for which *for any E* the eigenfunction behaving properly at minus infinity will be square integrable in $r^* \in (-\infty, r_i^*)$, and a choice of boundary condition needs to be imposed to define a set of allowed perturbations D_{phys}, in order that $\mathcal{H}$ be a self-adjoint operator on D_{phys} (i.e. $D_{\text{phys}} = D^*_{\text{phys}}$, see [24] for more details), and thus have a complete set of eigenfunctions. This is done by requiring a behavior like (45) with a fixed (possibly infinite) b/a ratio [24]. Since we are ultimately interested in the large k case, in view of (44), we do not have to deal with this ambiguity. In any case, regardless of our choice of boundary conditions, the test function used in (27)–(28), being supported in a $r < 0$ region, will belong to the chosen space of perturbations, and the argument of instability used for the nakedly singular Kerr spacetime will go through in the sub-extreme black hole case if, as done for the extreme case, we restrict the harmonics to $\ell > \ell_o$ with ℓ_o the smallest non-negative integer satisfying $E_o^{\text{rad}}(s, a\omega = \mathrm{i}k)|_{k=0^+} < (\ell_o + |s|)(\ell_o + |s| + 1)$.

3.4. Consistency with modal stability outside the black hole horizon

We would like to note that our results do not contradict the well-established modal stability of the *outer* stationary, region I of Kerr black holes (see [10] and references therein). Our arguments break down in this case since the interval where V_2 is negative lies outside the domain of interest.

Consider the extreme case $a = M$ and switch to the adimensional variable $x = r/M$; then,

$$V_{\text{ext}} = \left[\frac{x(x+1)(x^2 - x + 2)}{(x-1)^2}\right]k^2 + 2s\left[\frac{x^2 - 2x - 1}{x - 1}\right]k + \frac{1}{4}. \tag{46}$$

For $s = 0$, $V_{\text{ext}} > 1/4$ outside the horizon ($x > 1$), and thus, there is no instability. For $s \neq 0$, $\partial V_{\text{ext}}/\partial x = 0$ at x_o gives

$$-\frac{k}{s} = \frac{(x_o - 1)\left(x_o{}^2 - 2x_o + 3\right)}{\left(x_o{}^2 + 1\right)\left(x_o{}^2 - 2x_o - 1\right)}. \tag{47}$$

The function on the rhs above decreases monotonically from zero to minus infinity for $x_o \in (1, 1 + \sqrt{2})$, and then from infinity to zero if $x \in (1 + \sqrt{2}, \infty)$. Thus, for any $s \neq 0$, V_{ext} has a unique critical point, which (by inserting (47) in $\partial^2 V_{\text{ext}}/\partial x^2$) we find that is a local, and then absolute (using that $V_{\text{ext}} \to \infty$ for $x \to 1^+$ and $x \to \infty$) minimum V^o_{ext} of V_{ext} in the domain of interest. This absolute minimum V^o_{ext} can easily be seen to decrease with k, with a lower bound $\frac{1}{4} - s^2$ as $k \to 0^+$. Summarizing

$$V_{\text{ext}}(k, x, s) > \tfrac{1}{4} - s^2, \quad |s| = 0, 1/2, 1, 2; \quad k > 0, \quad x > 1. \tag{48}$$

Then, for the radial equation, we obtain

$$\begin{aligned} E_0^{\text{rad}}(s, a\omega = \mathrm{i}k) &< s^2 - \tfrac{1}{4} < s^2 + |s| \leqslant (\ell + |s|)(\ell + |s| + 1) \\ &\leqslant E_\ell^{\text{ang}}(s, a\omega = \mathrm{i}k), \quad k > 0, \quad \ell \geqslant 0, \end{aligned}$$

and thus, the intersection argument implying the existence of modes that grow exponentially in time breaks down.

The reasoning for the sub-extreme case is similar, although the calculations are more complicated. Instead of (47), we obtain a two branched solution for k, only one of which corresponds to local minima. The bounds (48) are obtained again, and thus, there is no instability.

4. Instabilities and time machine

The axial Killing vector field ζ^a of the Kerr spacetime ($\zeta = \partial/\partial\phi$ in the Boyer–Lindsquit coordinates) becomes timelike in what is called the *time machine* region $\mathscr{T}$ of the Kerr spacetime [5], the region where

$$(r^2 + a^2\cos^2\theta)(r^2 + a^2) + 2Mra^2\sin^2\theta < 0. \tag{49}$$

The character of the restriction of the Teukolsky PDE (10) to the space of functions satisfying $\pounds_\zeta\Psi = \mathrm{i}m\Psi$) changes from hyperbolic to elliptic within $\mathscr{T}$. To see this, note that the second-order terms of the Teukolsky equation are independent of s, and so, for any s value, they equal those of the scalar wave equation

$$0 = g^{ab}\nabla_a\nabla_b\Psi = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}g^{ab}\partial_b\Psi) \sim g^{ab}\partial_a\partial_b\Psi \sim \left(g^{ab} - \frac{\zeta^a\zeta^b}{\zeta^c\zeta_c}\right)\partial_a\partial_b\Psi, \tag{50}$$

where $\sim$ means 'equal up to lower order terms', and $\pounds_\zeta\Psi = \mathrm{i}m\Psi$ was used in the last step. The proof of existence of unstable modes in the Teukolsky equation in the previous section is based on the fact that the piece V_2 of the potential, which is dominant for large k, becomes negative in the region $r(r^3 + ra^2 + 2Ma^2)$. This region is precisely the intersection of $\mathscr{T}$ with the equatorial plane $\theta = \pi/2$ (see (49)).

The existence of instabilities seems to be related to this change of character of the Teukolsky PDE for axial modes from hyperbolic to elliptic. To illustrate this point, consider the simple toy model (a, ω_o positive):

$$\frac{\omega_o^2(x^2 - a^2)}{4}\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} = 0, \tag{51}$$

which is hyperbolic for $|x| > a$, elliptic otherwise. Unstable solutions $\Phi(t, x) = \mathrm{e}^{kt/a}\psi(x)$ will exist if

$$-\frac{\partial^2\psi}{\partial x^2} + \left(\frac{k\omega_o}{2}\right)^2(x^2 - a^2)\psi = 0. \tag{52}$$

The above equation is that of a quantum harmonic oscillator; it has square integrable solutions if

$$E = k\omega_0\left(n + \frac{1}{2}\right) - \left(\frac{k\omega_o a}{2}\right)^2 = 0, \quad n = 0, 1, 2, \ldots. \tag{53}$$

Thus, there are no instabilities if $a = 0$ (i.e. (51) is hyperbolic everywhere). Otherwise, there will be infinitely many unstable modes, with

$$k = \frac{4}{\omega_o a^2}\left(n + \frac{1}{2}\right).$$

One of the consequences of the existence of the time machine region is that it allows to construct a future directed timelike curve connecting *any ordered pair* of events in either KIII or KNS (see [5] and references therein). This causes a number of difficulties when trying to define notions, such as 'evolution' and 'instability', as discussed in the following section.

5. Unstable modes as the Debye potentials

Solutions of the transposed Teukolsky equations of different spin weights can be used as 'Debye potentials' to generate Maxwell, spinor and linearized gravity fields [16]. An explanation of why this is so was first given by Wald in [25] and is reviewed in detail in [26, appendix C]. It is based on a notion of transpose of a linear differential operator $\mathcal{O}$ acting on tensor fields of rank k, under the inner product $(U, V) := \int_M U_{a_1\cdots a_k} V^{a_1\cdots a_k}$, where M is the spacetime manifold and indices are raised and lowered using the metric. The transpose is defined as usual by $(U, \mathcal{O}V) = (\mathcal{O}^T U, V)$, and assumes a proper decay of the fields in the domain of $\mathcal{O}$, so that integration by parts is allowed. Suppose f is the tensor field we are interested in (e.g., the Maxwell potential A_b, or the metric perturbation h_{ab}) and $\mathcal{E}(f) = 0$ is the linear differential equation that it satisfies. In the Teukolsky formalism, with the exception of the scalar field equation, one does not work with the field f of interest, but with a derived field $\psi_s := \mathcal{T}_s(f)$. Here, $\mathcal{T}_s$ is a linear differential operator that projects out a null tetrad component with spin weight s of a tensor derived from f, e.g, a perturbed Weyl component associated with a metric perturbation h_{ab}. The Teukolsky perturbative treatment can be summarized as follows [25]: there exist linear differential operators $\mathcal{S}_s$ and $\mathcal{O}_s$, such that

$$\mathcal{S}_s\mathcal{E}(f) = \mathcal{O}_s\mathcal{T}_s(f) = \mathcal{O}_s(\psi_s). \tag{54}$$

The field equation $\mathcal{E}(f) = 0$ then implies the Teukolsky equation $\mathcal{O}_s(\psi_s) = 0$. The operators $\mathcal{O}_s$ are defined by the left-hand sides of the following equations in [3] (refer also to equations (2)–(8)): (2.12) for $s = 2$ ($\psi_2 = \delta\psi_0$), (2.14) for $s = -2$ ($\psi_{-2} = \delta\psi_4$), (3.5) for $s = 1$ ($\psi_1 = \phi_0$), (3.7) for $s = -1$ ($\psi_{-1} = \phi_2$), (B.4) for $s = 1/2$ ($\psi_{1/2} = \chi_0$) and (B.5) for $s = -1/2$ ($\psi_{-1/2} = \chi_1$). Using the information in [3, table I] (and calculating for the spinor case), we find that the relations between the $\mathcal{O}_s$ and the operator T_s in the master Teukolsky equation (10) are

$$\mathcal{O}_s = \begin{cases} (2\Sigma)^{-1} \circ T_s, & s \geqslant 0, \\ (2\Sigma)^{-1}\rho^{-2s} \circ T_s \circ \rho^{2s}, & s < 0. \end{cases} \tag{55}$$

Thus, $\mathcal{O}_s(\psi_s) = 0$ reduces to $T_s\Psi_s = 0$, where $\Psi_s = \psi_s$ for $s \geqslant 0$, and $\Psi_s = \rho^{2s}\psi_s$ for $s < 0$ (cf equation (9)). The Teukolsky master equation $T_s\Psi_s = 0$ is spelled out in the Boyer–Lindquist coordinates in (10).

As Wald noted in [25], for spinor, Maxwell and linear gravitational fields, $\mathcal{E}^T = \mathcal{E}$, and the transpose of $\mathcal{S}_s\mathcal{E} = \mathcal{O}_s\mathcal{T}_s$ (equation (54)) then gives $\mathcal{T}_s{}^T\mathcal{O}_s{}^T = \mathcal{E}\mathcal{S}_s{}^T$. Thus, if $\hat{\psi}_s$ is a solution of the transposed Teukolsky equation, $\mathcal{O}_s{}^T\hat{\psi}_s = 0$, then $\mathcal{E}\mathcal{S}_s{}^T\hat{\psi}_s = 0$. In other words, $\hat{\psi}_s$ is a 'potential' for a solution $f = \mathcal{S}_s{}^T\hat{\psi}_s$ of the field equation $\mathcal{E}(f) = 0$.

A straightforward computation shows that there is a close relation between transpose and spin weight flip:

$$\rho^{2|s|} \circ \mathcal{O}_{(\pm s)}{}^T \circ \rho^{-2|s|} = \mathcal{O}_{(\mp s)}. \tag{56}$$

Solutions for the transpose of the spin weight s source-free Teukolsky equations can then be readily obtained by multiplying a solution of spin weight $-s$ times an appropriate power of ρ. Unstable solutions of the Teukolsky equations will therefore produce unstable spinor, Maxwell or gravitational fields, since the $\exp(kt/a)$ factors in the potential go through the differential operators $\mathcal{S}_s{}^T$.

When analyzing the linear stability of a super-extreme Reissner–Nordström spacetime (or the interior static region of a Reissner–Nordström black hole) one is faced with the problem that the unperturbed spacetime is non-globally hyperbolic due to the timelike singular boundary [12]. The evolution of fields on this spacetime is *a priori* not well defined, and the curvature singularity poses the additional problem of deciding what should be considered as a 'reasonable' behavior for linearized perturbations. The way around these problems is hinted by the observation that there is a *unique* choice of boundary condition at the singularity that guarantees that the perturbed curvature invariants will not diverge faster than the unperturbed ones as the singularity is approached [12]. By choosing this particular boundary condition, we make sure that the perturbation treatment is self-consistent, as perturbations can be *uniformly* bounded on an 'initial time' partial Cauchy surface Σ_o that meets the singularity (any hypersurface orthogonal to the timelike Killing vector field). At the same time we solve the issue of uniqueness of evolution from data given at Σ_o. Furthermore, this evolution preserves the chosen boundary condition [12]. The KNS, as well as KIII, also has a timelike curvature singularity, the ring singularity, located at $r = 0, \theta = \pi/2$ in the Boyer–Lindquist coordinates. Note, however, that $r \in (-\infty, \infty)$ for the super-extreme case ($r \in (-\infty, r_i)$ for the black hole interior), as one can enter the $r < 0$ region avoiding the singularity. The character of the singularity and the chosen fields Ψ_s is such that the separated Teukolsky equations (12) and (13) *are not singular*, i.e. $r = 0$ is a regular point of the ODE (13), and similarly for $\theta = \pi/2$ in (12). When solving (13), which is a second-order equation, one can impose that $R(r)$ vanish as $r \to -\infty$ and $r \to \infty$ ($r \to r_i$), but that leaves out any further choice, such as a selecting a specific behavior as $r \to 0$. This implies that although (super-extreme or black hole interior) Reissner–Nordström and Kerr spacetimes share some properties, such as the lack of a Cauchy surface and the existence of a time-like singularity, the issue of field propagation on those spacetimes is technically rather different, the Kerr ring singularity being milder. On the other hand, the causality issues are much worse in the Kerr case. This is because, as mentioned in the introduction, any two events in KIII or KNS can be connected with a future-directed timelike curve (in particular, there is a closed timelike curve through *any* point.) There is no partial Cauchy surfaces and thus no clear notion of 'initial time slice' that allows to pose the stability problem as an initial value problem. The $t =$ constant slices are spacelike outside a compact set, and our notion of instability is limited to the observation that there exist solutions to the linear field equations behaving as $\exp(kt/a)$, $k > 0$, decaying exponentially or faster as $|r| \to \infty$ in the KNS (vanishing at the inner horizon in KIII), and behaving 'properly'—as defined below—near the ring singularity.

To check how unstable fields behave near the ring singularity of Kerr spacetime, we may use unstable solutions of the Teukolsky master equation (proved to exist for every s in the previous section) as Debye potentials for unstable spinor, Maxwell or gravitational fields. Note, however, that, in contrast to our previous instability results for the Reissner–Nordström or naked Schwarzschild case [11, 12], we lack *explicit* expressions for the unstable solutions of the Teukolsky master equations. We can still get information on the behavior of unstable fields near the ring singularity by analyzing the Frobenius series solutions of the ODE (13) near $r = 0$ and the ODE (12) near $\theta = \pi/2$. These series, however, are independent of the stable or unstable character of the solution, since ω does not show up at leading order. Thus, whatever criterion we adopt to disregard field solutions from Debye potentials based on their behavior near the singularity, it will overrule *every* field (unstable or not) that can be constructed using the potential method outlined above.

5.1. Maxwell fields

For Maxwell fields, $s = \pm 1$, and the operators and fields in (54) are

$$f = A_b, \quad [\mathcal{E}(A_b)]_a = \nabla^c \nabla_c A_a - \nabla^c \nabla_a A_c, \tag{57}$$

and, for $s = 1$,

$$\mathcal{T}_1(A_b) = l^a m^b(\nabla_a A_b - \nabla_b A_a), \tag{58}$$

$$S_1(J_a) = \tfrac{1}{2}(\delta - \beta - \bar{\alpha} - 2\tau + \bar{\pi})(j_c l^c) - \tfrac{1}{2}(D - 2\rho - \bar{\rho})(j_c m^c), \tag{59}$$

$$\mathcal{O}_1(\psi_1) = (D - 2\rho - \bar{\rho})(\Delta + \mu - 2\gamma)\psi_1 - (\delta - \beta - \bar{\alpha} - 2\tau + \bar{\pi})(\bar{\delta} + \pi - 2\alpha)\psi_1, \tag{60}$$

where the standard null tetrad formulation notation [2, 3] is used (D, Δ, δ, $\bar{\delta}$ are the derivatives along the tetrad vectors, and the other symbols represent spin coefficients.) Since $\mathcal{T}_1$ projects out a self-dual piece of F_{ab} (see (58)), the complex potential $\mathcal{S}_1{}^T\hat{\psi}_1$ constructed from a solution $\mathcal{O}_1{}^T(\hat{\psi}_1) = 0$ of the transpose Teukolsky equation [25],

$$\left[\mathcal{S}_1{}^T\hat{\psi}_1\right]_b = [-l_b(\delta + 2\beta + \tau) + m_b(D + \rho)]\hat{\psi}_1, \tag{61}$$

will produce a self-dual Maxwell field $G_{ab} = F_{ab} + \mathrm{i}\,{}^*F_{ab}$ [25]. In fact, the easiest way to check that the exterior derivative G_{ab} of the potential (61) satisfies the source-free Maxwell equations is by checking that it is self-dual, which amounts to checking that the contractions of G_{ab} with any of the three anti-self-dual two-forms obtained by complex conjugation of (5) vanishes as a consequence of $\mathcal{O}_1{}^T(\hat{\psi}_1) = 0$.

To evaluate the strength of the real Maxwell field F_{ab} near the ring singularity, we compute the algebraic invariants $I_1 = F_{ab}F^{ab}, I_2 = F_{ab}{}^*F^{ab}$ (any other algebraic invariant will be a polynomial on these). Note that, since $\frac{1}{2}G_{ab}G^{ab} = I_1 + \mathrm{i}I_2 =: I$, we can compute the invariants of F_{ab} more efficiently without even taking the real part of G_{ab}. For generic separable solutions $\hat{\Psi}_1 = \mathrm{e}^{\mathrm{i}\omega t}R(r)S(\theta)$ of the $s = -1$ Teukolsky mater equation, (61) gives a field whose invariants admit an expression that can be simplified near the ring singularity by applying iteratively the equation $T_{-1}(\hat{\psi}_1) = 0$ to

$$I \simeq \frac{2\left(-\mathrm{i}S(\theta)a\frac{\mathrm{d}R(r)}{\mathrm{d}r} + R(r)\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\right)^2\mathrm{e}^{2\mathrm{i}\omega t/a}}{(r + \mathrm{i}a\cos(\theta))^4}. \tag{62}$$

As already explained, this leading order term (omitting the $\exp(2\mathrm{i}\omega t/a)$ factor) will be the same for any complex ω. This behavior near the ring singularity is universal and thus independent of the un/stable character of the field.

To evaluate weather or not the above divergency is 'reasonable', we may compare with the static Maxwell field on Kerr that we obtain from the Kerr–Newman solution

$$F = \mathrm{d}A, \quad A = \frac{Qr}{\Sigma}(\mathrm{d}t - a\sin^2(\theta)\,\mathrm{d}\phi). \tag{63}$$

Note that since the Kerr–Newman metric is quadratic in Q, this field is a *first order in* Q solution of Maxwell equations on a fixed Kerr metric and, being Maxwell equations linear, (63) is also an *exact* solution on the Kerr background. For this static field, a straightforward calculation shows that

$$I_{\text{static}} = \frac{-Q^2}{2(r - \mathrm{i}a\cos\theta)^4}, \tag{64}$$

which exhibits the same degree of divergency as (62), the latter being possibly even milder along selected directions, or for some particular solutions. Note that the unstable solutions of the $s = -1$ Teukolsky master equation, which evolve as $\mathrm{e}^{kt/a}$, decay in the KNS as $\mathrm{e}^{-k|r|/a}$ for large $|r|$, as opposed to the slow, r^{-4} decay of the invariants of the static field above. The unstable modes of KIII decay exponentially as $r \to \infty$, and as a power of $r - r_i$ toward the inner horizon.

In conclusion, we have shown that there are solutions of the Maxwell equations that behave in a similar way as the static field from the Kerr–Newman solution near the ring singularity, decay much faster away of along in spacelike directions and grow exponentially with time.

6. Conclusions

We have proved the existence of instabilities in both Kerr naked singularities (KNS) and the region beyond the inner horizon of sub-extreme and extreme Kerr black holes (KIII). Our notion of instability is given by the existence of solutions of linear massless field equations that behave as e^{kt}, $k > 0$, with a fast decay to zero as $|r| \to \infty$ in the case of KNS (as $r \to -\infty$ and $r \to r_i^-$ in the case of KIII), where $\{t, r, \theta, \phi\}$ are the Boyer–Lindquist coordinates. We have shown that there exist massless scalar fields, Weyl spinors, Maxwell fields and linear gravity perturbations with these properties.

Since KIII and KNS do not admit a partial Cauchy surface, there is no natural notion of evolution from initial data and therefore of instability in the usual way (bounded data grows unbounded). However, these spacetimes are time orientable, since the never vanishing vector field

$$V = (r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \phi}$$

is everywhere timelike. The integral lines of V give a congruence of timelike curves filling the entire KIII (KNS) space, which may be regarded as worldlines of (accelerated) observers. Since the unstable fields we have found are axially symmetric (independent of ϕ), they grow boundless along these curves. That a congruence of observers measure a boundless growth of these linear fields (more properly, of any scalar made out from them) is for us an appropriate notion of instability in spacetimes lacking a partial Cauchy surface (for examples of evolution and stability notion in non-globally hyperbolic spacetimes admitting a partial Cauchy surface; see the Schwarzschild and Reissner–Nordström cases [11, 12])).

A modification of the KNS metric based on alternative string motivated theories, or just the excision of a region around the ring singularity, gives rise to ‘superspinars’, compact rotating objects violating the black hole $a \leqslant M$ bound [27]. The gravitational stability of these objects (more concretely, the region $r > r_0$ of KNS, assuming different boundary conditions at $r = r_0$) was studied in detail in [28] and references therein. In section 3.B of that paper, the results of superspinar gravitational instabilities are confronted with those in our works [13] and [14], generalized in this paper. This is done by studying the $r_0 \to -\infty$ limit of unstable axially symmetric perturbations of a superspinar with a perfectly reflecting ‘string horizon’ at $r_0 \to -\infty$. This gives a result that agrees perfectly with our previous numerical calculations in [11].

The mathematical origin of the instability in both cases (KNS and superspinars) is, however, quite different. In the KNS case, it is due to the negative portion of the effective potential (23) for small $r < 0$ values, a region that is usually excised in a superspinar. The superspinar stability problem reduces also to a Schrödinger equation problem. However, in the KNS case, the Schrödinger equation is posed on the entire real axis, whereas in the superspinar case is a quantum mechanics problem on a half-axis (which may well exclude the region where our effective potential is negative), with non-trivial boundary conditions that are responsible of the instability. For KIII in the sub-extreme case, we have also reduced the field equations to a QM problem on a half-axis; however, the boundary conditions are trivial, and the negative portion of the effective potential belongs to this half-axis.

We should also comment on the connection found in [28] (see also [29]) between circular geodesics with negative energies and superspinar instabilities. We note that those results do not seem to apply in a straightforward way to our case, since the arguments in [28] apply for $\ell = m \gg 1$ modes, whereas unstable modes in this work have $m = 0$. We should also stress that contrary to what is found in [28], the unstable modes whose existence is proved in this work exist for any value of a/M as long as $a/M > 1$.

Acknowledgments

This work was supported in part by grants from CONICET and Universidad Nacional de Córdoba. GD is supported by CONICET and Universidad Nacional de Córdoba. RJG is partially supported by CONICET. IFRS is a fellow of CONICET.

Appendix. Lower bounds to the radial potentials for the KNS as $k \to 0^+$

This appendix gathers the calculations leading to the bounds given in (32) for the global minima of the potential (23) in the $k \to 0^+$ limit. These are used in section 3 to obtain a bound if the radial spectrum in this limit.

A.1. Case $s = 0$

Introducing the adimensional variables $x = r/M$, $\alpha = a/M$, the critical points of the potential (23) for $s = 0$ are given by

$$4k^2\,(\alpha^2 + x^2)(x^3 - 3x^2 + \alpha^2 x + \alpha^2) + \alpha^2(\alpha^2 - 1)(1 - x) = 0. \tag{A.1}$$

For k small enough, three out of the five roots are real, and we order them as $x_1 < x_2 < x_3$. Given the asymptotic behavior (26), it is clear that the absolute minimum of V is reached at some of these points (x_1 or x_3 if there are no inflection points). Inspection of the numerical solutions of (A.1) for increasingly smaller k values suggests that $x_1 \to -\infty$, $x_2 \to 1$ and $x_3 \to \infty$ as $k \to 0^+$. Guided by this observation, we propose a solution of (A.1) in the form of a power series in k taking the value $x_2 = 1$ at $k = 0$ and obtain by iteration the successive corrections in increasing powers of k, the result being

$$x_2 = 1 + \frac{8(1+\alpha^2)}{\alpha^2}k^2 + \frac{32(3\alpha^4 - 5\alpha^2 + \alpha^6 - 7)}{\alpha^4(\alpha^2 - 1)}k^4 + \mathcal{O}(k^6). \tag{A.2}$$

Then, we assume $x = x_1$ in (A.1), solve for k, and expand the resulting expression for $x_1 \to \infty$ (which we know that corresponds to taking $k \to 0^+$) to extract the leading order behavior of the relation between x_1 and k, which is $x_1 = -((\alpha^2 - 1)/4)^{1/4} k^{-1/2}$. Inserting this expression plus a correction (to be obtained) back into (A.1), we can iteratively obtain as many higher order terms as we wish. The first few of them are

$$x_1 = -\left(\frac{\alpha^2 - 1}{4}\right)^{1/4} k^{-1/2} + \frac{1}{2} + \left[\frac{\sqrt{2}(4\alpha^2 - 7)}{8(\alpha^2 - 1)^{1/4}}\right] k^{1/2} + \mathcal{O}(k). \tag{A.3}$$

Proceeding in a similar way leads to

$$x_3 = \left(\frac{\alpha^2 - 1}{4}\right)^{1/4} k^{-1/2} + \frac{1}{2} - \left[\frac{\sqrt{2}(4\alpha^2 - 7)}{8(\alpha^2 - 1)^{1/4}}\right] k^{1/2} + \mathcal{O}(k). \tag{A.4}$$

The values of the potential at these points are $V_1 = 1/4 + \mathcal{O}(k)$, $V_2 = 1/2 + \mathcal{O}(k^2)$, $V_3 = 1/4 + \mathcal{O}(k)$. It then follows that

$$\lim_{k \to 0^+} \min\{V(r, k, s = 0, \alpha), r \in \mathbb{R}\} = 1/4, \tag{A.5}$$

in agreement with (32).

A.2. Case $|s| = 1/2$

For $s = 1/2$, there are three real critical points $x_1 < x_2 < x_3$ (x_1 and x_3 are local minima, x_2 is a maximum) only if k is bigger than an a-dependent critical value, below which x_2 and x_3 coalesce into a single (inflection) point. In any case, the absolute minimum is reached at x_1, for which

$$x_1 = -\frac{\alpha}{2k} - 1 - \frac{4k}{\alpha} - \frac{16(\alpha^2-4)}{\alpha^2}k^2 + \mathcal{O}(k^3), \quad V_1 = -\frac{2}{\alpha}k + \frac{7}{\alpha^2}k^2 + \mathcal{O}(k^3).$$

For $s = -1/2$, the analysis is similar, with $x_1 \to x_2$ and a global minimum at $x_3 \to \infty$ as $k \to 0^+$:

$$x_3 = \frac{\alpha}{2k} - 1 + \frac{4k}{\alpha} - \frac{16(\alpha^2-4)}{\alpha^2}k^2 + \mathcal{O}(k^3), \quad V_3 = \frac{2}{\alpha}k + \frac{7}{\alpha^2}k^2 + \mathcal{O}(k^3).$$

It follows that

$$\lim_{k\to 0^+} \min\{V(r, k, s = \pm 1/2, \alpha), r \in \mathbb{R}\} = 0, \tag{A.6}$$

in agreement with (32).

A.3. Case $|s| = 1$

For $s = \pm 1$, the critical points and potential values at these points are

$$x_1 = 1 \mp \frac{4}{3}\frac{(\alpha^2-3)k}{\alpha} - \frac{8}{9}\frac{(11\alpha^2-21)k^2}{\alpha^2} + \mathcal{O}(k^3),$$
$$x_2 = \mp\frac{\alpha}{k} - 1 \pm \frac{1}{4}\frac{(-11+3\alpha^2)k}{\alpha} - \frac{1}{2}\frac{(-41+17\alpha^2)k^2}{\alpha^2} + \mathcal{O}(k^3),$$
$$x_3 = \frac{1}{2}\sqrt[3]{6}\sqrt[3]{\mp\frac{(-1+\alpha^2)\alpha}{k}} + 1 + \mathcal{O}(k^{1/3}),$$

with the values of the potential at these points being

$$V_1 = -\frac{1}{2} \pm \frac{4}{\alpha}k + \mathcal{O}(k^2),$$
$$V_2 = -\frac{3}{4} \mp \frac{4}{\alpha}k + \mathcal{O}(k^2),$$
$$V_3 = \frac{1}{4} - \frac{3^{4/3}}{2^{2/3}}\frac{\alpha^2-1}{(\alpha(\alpha^2-1))^{2/3}}k^{2/3} + \mathcal{O}(k^{4/3}).$$

Thus,

$$\lim_{k\to 0^+} \min\{V(r, k, s = \pm 1, \alpha), r \in \mathbb{R}\} = -3/4, \tag{A.7}$$

in agreement with (32).

A.4. Case $|s| = 2$

For $s = \pm 2$, the critical points and potential values are

$$x_1 = 1 \mp \frac{8}{15}\frac{(\alpha^2-3)k}{\alpha} - \frac{8}{225}\frac{(47\alpha^2-81)k^2}{\alpha^2} + \mathcal{O}(k^3)$$
$$x_2 = \mp 2\frac{\alpha}{k} - 1 \pm \frac{1}{32}\frac{(-47+15\alpha^2)k}{\alpha} - \frac{1}{32}\frac{(-173+77\alpha^2)k^2}{\alpha^2} + \mathcal{O}(k^3)$$
$$x_3 = \left(\pm\frac{15}{8}(1-\alpha^2)\alpha\right)^{1/3} k^{-1/3} + 1 + \mathcal{O}(k^{1/3}),$$

and the values of V at these points are

$$V_1 = -\frac{7}{2} \pm \frac{8}{\alpha}k + \mathcal{O}(k^2),$$

$$V_2 = -\frac{15}{4} \mp \frac{8}{\alpha}k + \mathcal{O}(k^2),$$

$$V_3 = \frac{1}{4} - \frac{315^{1/3}(\alpha^2 - 1)}{(\alpha(\alpha^2 - 1))^{2/3}}k^{2/3} + \mathcal{O}(k^{4/3}).$$

It follows that

$$\lim_{k\to 0^+} \min\{V(r, k, s = 0, \alpha), r \in \mathbb{R}\} = -15/4. \tag{A.8}$$

This completes the verification of (32).

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